Indian J Phys (March 2014) 88(3):283–292
DOI 10.1007/s12648-013-0403-4

ORIGINAL PAPER

Supersymmetry quantum mechanics to Dirac equation with a modified Yukawa potential and a Yukawa tensor term

A N Ikot[1]*, E Maghsoodi[2], S Zarrinkamar[3] and H Hassanabadi[2]

[1]Theoretical Physics Group, Department of Physics, University of Uyo, Uyo, Nigeria

[2]Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran

[3]Civil Engineering Group, Alaodoleh Semnani Institute of Higher Education, Garmsar, Iran

Received: 25 August 2013 / Accepted: 03 October 2013 / Published online: 26 October 2013

Abstract: Relativistic symmetries of Dirac equation, i.e. spin and pseudospin symmetries are investigated for a modified Yukawa potential including a Yukawa tensor interaction. Using supersymmetry quantum mechanics and a proper approximation to the inverse square centrifugal term, arbitrary-state solutions are reported in an analytical approach. We have included some useful numerical data and illustrative figures to represent a better understanding of the solutions.

Keywords: Dirac equation; Spin symmetry; Pseudospin symmetry; Yukawa tensor interaction

PACS Nos.: 03.65.Ge; 03.65.Pm; 03.65.Ca

1. Introduction

For decades, Dirac equation has been a promising basis to study relativistic spin-1/2 fermions in nuclear and particle physics. The problem in this case, just as that of linear Schrödinger equation, appears as an ordinary second-order differential equation, which has been extensively discussed in the literature by various analytical and numerical techniques. Numerical approaches, despite their reliability, are less clear in comparison with their analytical counterpart in which the explicit forms of the wavefunction and the energy relation are reported. Some of analytical tools are Nikiforov–Uvarov (NU) technique [1], asymptotic iteration method (AIM) [2], shape invariant approach [3] and supersymmetric quantum mechanics (SUSYQM) [4–6], factorization method [7] and others [8] and references therein.

The concept of SUSYQM has been introduced many years ago [9, 10] and it provides theoretical physicists with a powerful tool to deal with non-relativistic Schrödinger equation. However, some years later, it has been applied to other wave equations of quantum mechanics such as Dirac, Klein–Gordon, Duffin–Kemmer Petiau (DKP) and spinless Salpeter equations [11–22].

Within present work, we have considered the so-called spin and pseudospin symmetries of Dirac equation. These symmetries, due to their applications in hadronic and nuclear spectroscopy, have renewed interests in the study of the equation [23]. In more precise words, these symmetries could explain the experimental observation of the quasi-degeneracy in single-nucleon doublets between normal parity orbitals $\left(n, l, j = l + \frac{1}{2}\right)$ and $\left(n-1, l+2, j = l + \frac{3}{2}\right)$, where n, l and j represent the radial, orbital and total angular momentum quantum numbers, respectively. These symmetries have been successfully used in the study of nuclear shell model [24] as well as many other concepts such as nuclear deformation, super-deformation, magnetic moment and identical bands [25]. It is now understood that pseudo-orbital angular momentum $\tilde{l} = l + 1$ in pseudospin limit is in fact the usual orbital angular momentum l of lower component of Dirac spinor [26]. It should be mentioned that spin symmetry refers to the case where difference of repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant; $\Delta(r) = V(r) - S(r) = const.$ other jargon, i.e. pseudospin symmetry, corresponds to $\Sigma(r) = V(r) + S(r) = const$. Until now, these symmetries have been investigated for different interactions by various techniques [27, 28]. In our work, we consider a modified Yukawa potential of the form [29]

*Corresponding author, E-mail: ndemikotphysics@gmail.com

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$$V(r) = -V_0\left(1-\frac{1}{r}e^{-\alpha r}\right)^2 = -\frac{A}{r^2}e^{-2\alpha r}+\frac{B}{r}e^{-\alpha r}-C,$$
$$A = C = V_0, \quad B = 2V_0 \tag{1}$$

where α is screening parameter and V_0 is coupling strength of the potential. Mie-type, inversely quadratic Yukawa, Yukawa and Coulomb potentials are all special cases of this interaction. For $\alpha \to 0$, this potential reduces to Mie-type potential. When $B = C = 0$ the potential changes into inversely quadratic Yukawa potential, which is widely used in nuclear, particle, atomic, condensed matter and chemical physics [30]. For $A = C = 0, B \to -\frac{B}{2}$, we recover the quite famous Yukawa potential. In addition, when $A = C = 0$, $\alpha \to 0$, Coulomb potential is obtained. Different forms of Yukawa potentials have also been investigated [31].

2. Supersymmetry quantum mechanics (SUSYQM)

In SUSUQM, we deal with the partner Hamiltonians [6, 32]

$$H_{\pm} = \frac{p^2}{2m} + V_{\pm}(x), \tag{2}$$

where

$$V_{\pm}(x) = \Phi^2(x) \pm \Phi'(x). \tag{3}$$

In case of unbroken supersymmetry, i.e. $E_0 = 0$ ground state of the system is obtained via

$$\phi_0^-(x) = Ce^{-U(x)}, \tag{4}$$

where C is a normalization constant and

$$U(x) = \int_{x_0}^{x} dz\Phi(z). \tag{5}$$

Next, we check whether the shape invariant condition holds. The shape-invariancy implies

$$V_+(a_0,x) = V_-(a_1,x) + R(a_1), \tag{6}$$

where a_1 is a new set of parameters uniquely determined from old set a_0 via mapping $F: a_0 \mapsto a_1 = F(a_0)$ and $R(a_1)$ does not include x. If condition in Eq. (6) is satisfied, higher state solutions are obtained via

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{7}$$

$$\phi_n^-(a_0,x) = \prod_{s=0}^{n-1}\left(\frac{A^{\dagger}(a_s)}{[E_n-E_s]^{1/2}}\right)\phi_0^-(a_n,x), \tag{8}$$

$$\phi_0^-(a_n,x) = C\exp\left\{-\int_0^x dz\Phi(a_n,z)\right\}, \tag{9}$$

where

$$A_s^{\dagger} = -\frac{\partial}{\partial x} + \Phi(a_s,x). \tag{10}$$

Therefore, this condition determines the spectrum of bound states of Hamiltonian

$$H_s = -\frac{\partial^2}{\partial x^2} + V_-(a_s,x) + E_s. \tag{11}$$

The energy eigenfunctions of

$$H_s\phi_{n-s}^-(a_s,x) = E_n\phi_{n-s}^-(a_s,x), \quad n \geq s \tag{12}$$

are related via [6, 32]

$$\phi_{n-s}^-(a_s,x) = \frac{A^{\dagger}}{[E_n-E_s]^{1/2}}\phi_{n-(s+1)}^-(a_{s+1},x). \tag{13}$$

3. Dirac equation with a tensor coupling

Dirac equation with a tensor potential $U(r)$ in relativistic unit ($\hbar = c = 1$) is written as [27–29]

$$[\vec{\alpha}\cdot\vec{p} + \beta(M + S(r) - i\beta\vec{\alpha}\cdot\hat{r}U(r)]\psi(r) = [E - V(r)]\psi(r), \tag{14}$$

where E is relativistic energy of the system, $\vec{p} = -i\vec{\nabla}$ is three-dimensional momentum operator and M is mass of the fermionic particle. $\vec{\alpha}$, β are 4×4 Dirac matrices given as

$$\vec{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{15}$$

where I is a 2×2 unit matrix and $\vec{\sigma}_i$ are Pauli three-vector matrices defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

Eigenvalues of spin–orbit coupling operator are $\kappa = (j+\frac{1}{2}) > 0, \kappa = -(j+\frac{1}{2}) < 0$ for unaligned $j = l - \frac{1}{2}$ and aligned spin $j = l + \frac{1}{2}$ cases respectively. The set (H, K, J^2, J_z) forms a complete set of conserved quantities. Thus, we can write the spinors as [27–29]

$$\psi_{n\kappa}(r) = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)Y_{jm}^{l}(\theta,\varphi) \\ iG_{n\kappa}(r)Y_{jm}^{\tilde{l}}(\theta,\varphi) \end{pmatrix} \tag{17}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ represent upper and lower components of Dirac spinors. $Y_{jm}^{l}(\theta,\varphi), Y_{jm}^{\tilde{l}}(\theta,\varphi)$ are spin and pseudospin spherical harmonics and m is projection on the z-axis. With other known identities given by [27, 28]

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right)=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right),$$
$$\vec{\sigma}\cdot\vec{p}=\vec{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\vec{p}+i\frac{\vec{\sigma}\cdot\vec{L}}{r}\right) \tag{18}$$

as well as

$$\begin{aligned}
&\left(\vec{\sigma}\cdot\vec{L}\right)Y_{jm}^{\tilde{l}}(\theta,\varphi)=(\kappa-1)Y_{jm}^{\tilde{l}}(\theta,\varphi)\\
&\left(\vec{\sigma}\cdot\vec{L}\right)Y_{jm}^{l}(\theta,\varphi)=-(\kappa-1)Y_{jm}^{l}(\theta,\varphi)\\
&(\vec{\sigma}\cdot\hat{r})Y_{jm}^{l}(\theta,\varphi)=-Y_{jm}^{\tilde{l}}(\theta,\varphi)\\
&(\vec{\sigma}\cdot\hat{r})Y_{jm}^{\tilde{l}}(\theta,\varphi)=-Y_{jm}^{l}(\theta,\varphi)
\end{aligned} \tag{19}$$

we obtain the following coupled equations [27–31],

$$\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)=(M+E_{n\kappa}-\Delta(r))G_{n\kappa}(r), \tag{20}$$

$$\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)=(M-E_{n\kappa}+\Sigma(r))F_{n\kappa}(r), \tag{21}$$

where

$$\Delta(r)=V(r)-S(r), \tag{22}$$

$$\Sigma(r)=V(r)+S(r). \tag{23}$$

After a simple decoupling, we obtain second-order Schrödinger-like equations

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa U(r)}{r}-\frac{dU(r)}{dr}\\ \quad -U^2(r)-(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))\\ +\frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)}{(M+E_{n\kappa}-\Delta(r))}\end{array}\right\}F_{n\kappa}(r)=0, \tag{24}$$

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa U(r)}{r}+\frac{dU(r)}{dr}-U^2(r)\\ \quad -(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))\\ +\frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)}{(M-E_{n\kappa}+\Sigma(r))}\end{array}\right\}G_{n\kappa}(r)=0, \tag{25}$$

with $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1), \kappa(\kappa+1)=l(l+1)$.

3.1. Pseudospin symmetry limit

In pseudospin symmetry limit, $\frac{d\Sigma(r)}{dr}=0$ or $\Sigma(r)=C_{ps}=$ *const* [23–26]. As previously mentioned, we intent to study the potential

$$\Delta(r)=-\frac{A^{ps}}{r^2}e^{-2\alpha r}+\frac{B^{ps}}{r}e^{-\alpha r}-C^{ps}, \tag{26}$$

besides Yukawa tensor interaction [33]

$$U(r)=-V_1\left(\frac{e^{-\alpha r}}{r}\right), \tag{27}$$

where V_1 and α are strength and range of nucleon force, respectively [33]. The corresponding equation is not exactly solvable. Consequently, to provide an analytical solution, we have to proceed on an approximate basis. Therefore, we introduce the approximations [34]

$$\frac{1}{r^2}\approx\frac{4\alpha^2e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}, \tag{28}$$

$$\frac{1}{r^2}\approx\frac{4\alpha^2e^{-\alpha r}}{(1-e^{-2\alpha r})^2}, \tag{29}$$

Substituting Eqs. (26)–(29) into Eq. (25), we obtain

$$\begin{aligned}&\left\{\frac{d^2}{dr^2}-\frac{4\alpha^2\kappa(\kappa-1)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-\frac{8\alpha^2\kappa V_1e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right.\\
&\left.\quad+\frac{2\alpha^2V_1e^{-2\alpha r}}{(1-e^{-2\alpha r})}+\frac{4\alpha^2V_1e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-\frac{4\alpha^2V_1^2e^{-4\alpha r}}{(1-e^{-2\alpha r})^2}\right\}G_{n\kappa}^{ps}(r)\\
&\quad+\left(\begin{array}{l}\frac{-4\alpha^2A^{ps}(M-E_{n\kappa}+C_{ps})e^{-4\alpha r}}{(1-e^{-2\alpha r})^2}\\ +\frac{2\alpha B^{ps}(M-E_{n\kappa}+C_{ps})e^{-2\alpha r}}{(1-e^{-2\alpha r})}\\ -C^{ps}(M-E_{n\kappa}+C_{ps})\end{array}\right)G_{n\kappa}^{ps}\\
&=(M+E_{n\kappa})(M-E_{n\kappa}+C_{ps})G_{n\kappa}^{ps}\end{aligned} \tag{30}$$

or, more neatly,

$$-\frac{d^2G_{n\kappa}^{ps}}{dr^2}+V_{eff}(r)G_{n\kappa}^{ps}=\tilde{E}_{n\kappa}^{ps}G_{n\kappa}^{ps}, \tag{31}$$

where

$$V_{eff}(r)=\frac{\eta_1^{ps}e^{-4\alpha r}+\eta_2^{ps}e^{-2\alpha r}+\eta_3^{ps}}{(1-e^{-2\alpha r})^2}, \tag{32}$$

$$\tilde{E}_{n\kappa}^{ps}=-(M+E_{n\kappa})(M-E_{n\kappa}+C_{ps}), \tag{33}$$

$$\begin{aligned}\eta_1^{ps}=&\left[4\alpha^2A^{ps}(M-E_{n\kappa}+C_{ps})+4\alpha^2V_1\left(V_1+\frac{1}{2}\right)\right.\\&\left.+2\alpha(M-E_{n\kappa}+C_{ps})B^{ps}+C^{ps}(M-E_{n\kappa}+C_{ps})\right],\end{aligned} \tag{34}$$

$$\begin{aligned}\eta_2^{ps}=&\left[4\alpha^2\kappa(\kappa-1)+8\alpha^2V_1\left(\kappa-\frac{3}{4}\right)\right.\\&\left.-2C^{ps}(M-E_{n\kappa}+C_{ps})-2\alpha B^{ps}(M-E_{n\kappa}+C_{ps})\right],\end{aligned} \tag{35}$$

Table 1 Energies in pseudospin symmetry limit for $M = 1\ \text{fm}^{-1}$, $\alpha = 0.01$ fm

$\tilde{\ell}$	n	κ	(ℓ, j)	$E^{ps}_{n\kappa}(fm^{-1})$ $\begin{pmatrix} A^{ps} = -0.2,\ B^{ps} = -0.4 \\ C^{ps} = -0.2,\ V_1 = 0 \\ C_{ps} = -5\ \text{fm}^{-1} \end{pmatrix}$	$E^{ps}_{n\kappa}(fm^{-1}) = [16]$ $\begin{pmatrix} A^{ps} = 0,\ B^{ps} = 1 \\ C^{ps} = 0,\ V_1 = 0 \\ C_{ps} = 0\ \text{fm}^{-1} \end{pmatrix}$	$E^{ps}_{n\kappa}(fm^{-1})$ $\begin{pmatrix} A^{ps} = -0.2,\ B^{ps} = -0.4 \\ C^{ps} = -0.2,\ V_1 = 0.5 \\ C_{ps} = -5\ \text{fm}^{-1} \end{pmatrix}$	$E^{ps}_{n\kappa}(fm^{-1}) = [16]$ $\begin{pmatrix} A^{ps} = 0,\ B^{ps} = 1 \\ C^{ps} = 0,\ V_1 = 0.5 \\ C_{ps} = 0\ \text{fm}^{-1} \end{pmatrix}$
1	1	−1	$1S_{\frac{1}{2}}$	−0.942576757	−0.955227812	−0.942502609	−0.931659752
2	1	−2	$1P_{\frac{3}{2}}$	−0.942710712	−0.978280583	−0.942603069	−0.968839010
3	1	−3	$1d_{\frac{5}{2}}$	−0.942911663	−0.988854451	−0.942770516	−0.984284996
4	1	−4	$1f_{\frac{7}{2}}$	−0.943179636	−0.994345276	−0.943004969	−0.991903836
1	2	−1	$2S_{\frac{1}{2}}$	−0.946996080	−0.978280583	−0.946921128	−0.968839010
2	2	−2	$2P_{\frac{3}{2}}$	−0.947132339	−0.988854451	−0.947023317	−0.984284996
3	2	−3	$2d_{\frac{5}{2}}$	−0.947336747	−0.994345276	−0.947193644	−0.991903836
4	2	−4	$2f_{\frac{7}{2}}$	−0.947609326	−0.997356164	−0.947432128	−0.996004056
1	1	2	$0d_{\frac{3}{2}}$	−0.942576757	−0.955227812	−0.942603069	−0.968839010
2	1	3	$0f_{\frac{5}{2}}$	−0.942710712	−0.978280583	−0.942770516	−0.984284996
3	1	4	$0g_{\frac{7}{2}}$	−0.942911663	−0.988854451	−0.943004969	−0.991903836
4	1	5	$0h_{\frac{9}{2}}$	−0.943179636	−0.994345276	−0.943306456	−0.996004056
1	2	2	$1d_{\frac{3}{2}}$	−0.946996080	−0.978280583	−0.947023317	−0.984284996
2	2	3	$1f_{\frac{5}{2}}$	−0.947132339	−0.988854451	−0.947193644	−0.991903836
3	2	4	$1g_{\frac{7}{2}}$	−0.947336747	−0.994345276	−0.947432128	−0.996004056
4	2	5	$1h_{\frac{9}{2}}$	−0.947609326	−0.997356164	−0.947738796	−0.998264813

$$\eta_3^{ps} = C^{ps}\left(M - E_{n\kappa} + C_{ps}\right) \tag{36}$$

According to Eq. (4), the lower component is

$$G^{ps}_{0,\kappa}(r) = \exp\left(-\int \phi(r)dr\right), \tag{37}$$

where in integrand is determined from Riccati equation

$$\phi^2 - \phi' = V_{eff}(r) - \tilde{E}^{ps}_{0,\kappa}, \tag{38}$$

After a change of variable of the form $y = e^{-2\alpha r}$, superpotential is found as

$$\phi(y) = \frac{\mu^{ps} y}{(1-y)} + \lambda^{ps}, \tag{39}$$

which rewrites Eq. (36) as

$$\frac{(\mu^{ps})^2 y^2}{(1-y)^2} + \frac{2\mu^{ps}\lambda^{ps} y}{(1-y)} + (\lambda^{ps})^2 + \frac{2\alpha y \mu^{ps}}{(1-y)^2} = \frac{\eta_1^{ps} y^2 + \eta_2^{ps} y + \eta_3^{ps}}{(1-y)^2} - \tilde{E}^{ps}_{0,\kappa}, \tag{40}$$

Equating the corresponding powers on both sides of Eq. (40), we find

$$\tilde{E}^{ps}_{0,\kappa} = \eta_3^{ps} - (\lambda^{ps})^2, \tag{41}$$

$$\mu^{ps} = -\alpha \pm \sqrt{\alpha^2 + \left(\eta_1^{ps} + \eta_3^{ps} + \eta_2^{ps}\right)} \tag{42}$$

$$\lambda^{ps} = \frac{(\mu^{ps})^2 + \eta_3^{ps} - \eta_1^{ps}}{2\mu^{ps}}, \tag{43}$$

By virtue of Eq. (3), we can construct the partner Hamiltonian as

$$V_{eff+}(r) = \phi^2 + \frac{d\phi}{dr} = \frac{\mu^{ps}(\mu^{ps} - 2\alpha)y^2}{(1-y)^2} + \frac{((\mu^{ps})^2 + \eta_3^{ps} - \eta_1^{ps} - 2\alpha\mu^{ps})y}{(1-y)} + \left(\frac{(\mu^{ps})^2 + \eta_3^{ps} - \eta_1^{ps}}{2\mu^{ps}}\right)^2 \tag{44}$$

$$V_{eff-}(r) = \phi^2 - \frac{d\phi}{dr} = \frac{\mu^{ps}(\mu^{ps} + 2\alpha)y^2}{(1-y)^2} + \frac{((\mu^{ps})^2 + \eta_3^{ps} - \eta_1^{ps} + 2\alpha\mu^{ps})y}{(1-y)} + \left(\frac{(\mu^{ps})^2 + \eta_3^{ps} - \eta_1^{ps}}{2\mu^{ps}}\right)^2 \tag{45}$$

Thus, it is not difficult to show that $V_+(r)$ and $V_-(r)$ are shape invariant via

$$\mu^{ps} \to \mu^{ps} - 2\alpha, \tag{46}$$

with $a_0 = \mu^{ps}$, $a_1 = f(a_0) = a_0 - 2\alpha$, $a_n = f(a_0) = a_0 - 2n\alpha$. Thus, from Eq. (6), we have

Table 2 Energies in spin symmetry limit for $M = 1\,\text{fm}^{-1}$, $\alpha = 0.01$ fm

n	ℓ	κ	(ℓ, j)	$E^s_{n\kappa}(fm^{-1})$ $\begin{pmatrix} A^s = 0.2,\ B^s = 0.4 \\ C^s = 0.2,\ V_1 = 0 \\ C_s = 5\ \text{fm}^{-1} \end{pmatrix}$	$E^s_{n\kappa}(fm^{-1}) = [16]$ $\begin{pmatrix} A^s = 0,\ B^s = -1 \\ C^s = 0,\ V_1 = 0 \\ C_s = 0\ \text{fm}^{-1} \end{pmatrix}$	$E^s_{n\kappa}(fm^{-1})$ $\begin{pmatrix} A^s = 0.2,\ B^s = 0.4 \\ C^s = 0.2,\ V_1 = 0.5 \\ C_s = 5\ \text{fm}^{-1} \end{pmatrix}$	$E^s_{n\kappa}(fm^{-1}) = [16]$ $\begin{pmatrix} A^s = 0,\ B^s = -1 \\ C^s = 0,\ V_1 = 0.5 \\ C_s = 0\ \text{fm}^{-1} \end{pmatrix}$
0	0	−1	$0S_{\frac{1}{2}}$	0.857444434	0.607950247	0.857436394	0.004987562
1	0	−1	$1S_{\frac{1}{2}}$	0.860271262	0.891565680	0.860263027	0.808888053
2	0	−1	$2S_{\frac{1}{2}}$	0.863164719	0.955227812	0.863156287	0.932381312
3	0	−1	$3S_{\frac{1}{2}}$	0.866125068	0.978280583	0.86611644	0.969190359
0	1	−2	$0P_{\frac{3}{2}}$	0.857508755	0.891565680	0.857468554	0.808888053
1	1	−2	$1P_{\frac{3}{2}}$	0.860337143	0.955227812	0.860295967	0.932381312
2	1	−2	$2P_{\frac{3}{2}}$	0.863232169	0.978280583	0.863190012	0.969190359
3	1	−2	$3P_{\frac{3}{2}}$	0.866194095	0.988854451	0.866150953	0.984479831
0	2	−3	$0d_{\frac{5}{2}}$	0.857637405	0.955227812	0.857565038	0.932381312
1	2	−3	$1d_{\frac{5}{2}}$	0.860468912	0.978280583	0.860394791	0.969190359
2	2	−3	$2d_{\frac{5}{2}}$	0.863367072	0.988854451	0.863291189	0.984479831
3	2	−3	$3d_{\frac{5}{2}}$	0.866332151	0.994345276	0.866254495	0.992018477
0	3	−4	$0f_{\frac{7}{2}}$	0.857830397	0.978280583	0.857725857	0.969190359
1	3	−4	$1f_{\frac{7}{2}}$	0.860666578	0.988854451	0.860559507	0.984479831
2	3	−4	$2f_{\frac{7}{2}}$	0.863569437	0.994345276	0.863459821	0.992018477
3	3	−4	$3f_{\frac{7}{2}}$	0.866539239	0.997356164	0.866427066	0.996072265
0	1	1	$0P_{\frac{1}{2}}$	0.857508755	0.891565680	0.857565038	0.932381312
1	1	1	$1P_{\frac{1}{2}}$	0.860337143	0.955227812	0.860394791	0.969190359
2	1	1	$2P_{\frac{1}{2}}$	0.863232169	0.978280583	0.863291189	0.984479831
3	1	1	$3P_{\frac{1}{2}}$	0.866194095	0.988854451	0.866254495	0.992018477
0	2	2	$0\,d_{\frac{3}{2}}$	0.857637405	0.955227812	0.857725857	0.969190359
1	2	2	$1d_{\frac{3}{2}}$	0.860468912	0.978280583	0.860559507	0.984479831
2	2	2	$2d_{\frac{3}{2}}$	0.863367072	0.988854451	0.863459821	0.992018477
3	2	2	$3d_{\frac{3}{2}}$	0.866332151	0.994345276	0.866427066	0.996072265
0	3	3	$0f_{\frac{5}{2}}$	0.857830397	0.978280583	0.857951028	0.984479831
1	3	3	$1f_{\frac{5}{2}}$	0.860666578	0.988854451	0.860790128	0.992018477
2	3	3	$2f_{\frac{5}{2}}$	0.863569437	0.994345276	0.86369592	0.996072265
3	3	3	$3f_{\frac{5}{2}}$	0.866539239	0.997356164	0.866668671	0.998303763

$$R(a_1) = \left(\frac{(a_0)^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_0}\right)^2 - \left(\frac{(a_1)^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_1}\right)^2, \tag{47}$$

$$R(a_2) = \left(\frac{(a_1)^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_1}\right)^2 - \left(\frac{(a_2)^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_2}\right)^2, \tag{48}$$

$$R(a_n) = \left(\frac{(a_{n-1})^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_{n-1}}\right)^2 - \left(\frac{(a_n)^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_n}\right)^2, \tag{49}$$

$$\tilde{E}^-_{0,\kappa} = 0 \tag{50}$$

Consequently, energy eigenvalues can be obtained from Eqs. (47)-(49) as

$$\tilde{E}^{ps-}_{n\kappa} = \sum_{k=1}^{n} R(a_k) = \left(\frac{(a_0)^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_0}\right)^2 - \left(\frac{(a_n)^2 + \eta_3^{ps} - \eta_1^{ps}}{2a_n}\right)^2, \tag{51}$$

and

$$\tilde{E}^{ps}_{n\kappa} = \tilde{E}^{ps-}_{n\kappa} + \tilde{E}^{ps}_{0,\kappa} = -\left(\frac{(a_0 - 2n\alpha)^2 + \eta_3^{ps} - \eta_1^{ps}}{2(a_0 - 2n\alpha)}\right)^2 + \eta_3^{ps}, \tag{52}$$

which, from Eqs. (33)–(36), (41)–(43) and (52), yields

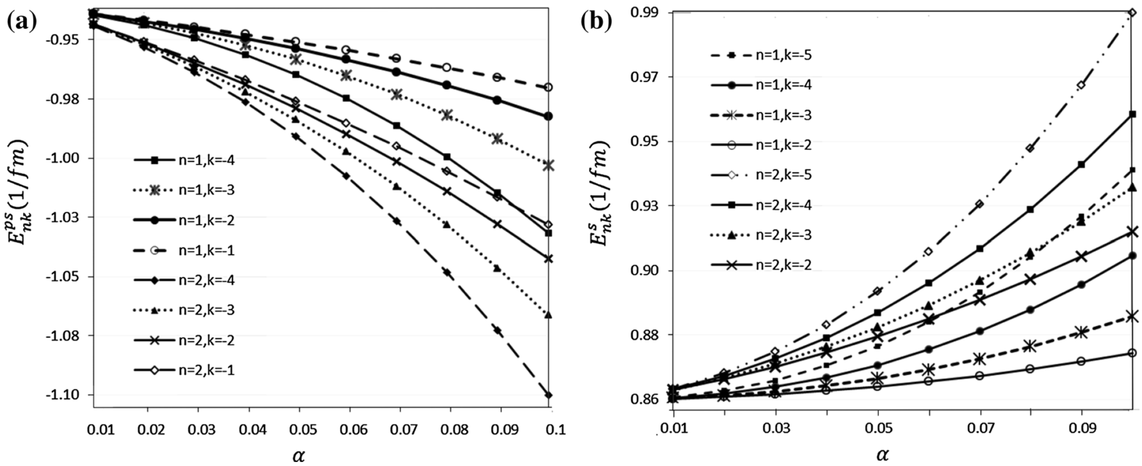

Fig. 1 (**a**) Energy versus α for pseudospin symmetry limit for $A^{ps}=-0.2,\ B^{ps}=-0.4,\ C^{ps}=-0.2,$, $V_1=0.5\ \mathrm{fm}^{-1}$, $M=1\ \mathrm{fm}^{-1}$, $C_{ps}=-5\ \mathrm{fm}^{-1}$ and (**b**) energy versus α for spin symmetry limit for $A^s=0.2,\ B^s=0.4,\ C^s=0.2,\ V_1=0.5\ \mathrm{fm}^{-1}$, $M=1\ \mathrm{fm}^{-1},\ C_s=5\ \mathrm{fm}^{-1}$

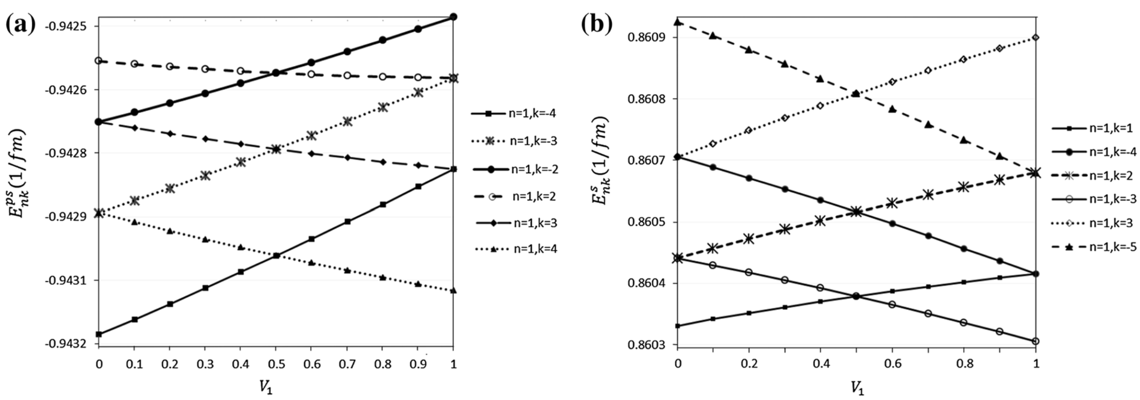

Fig. 2 (**a**) Energy versus V_1 for pseudospin symmetry limit for $A^{ps}=-0.2,\ B^{ps}=-0.4,\ C^{ps}=-0.2,\ \alpha=0.01,\ M=1\ \mathrm{fm}^{-1}$, $C_{ps}=-5\ \mathrm{fm}^{-1}$and (**b**) energy versus V_1 for spin symmetry limit for $A^s=0.2,\ B^s=0.4,\ C^s=0.2,\ \alpha=0.01,M=1\ \mathrm{fm}^{-1},\ C_s=5\ \mathrm{fm}^{-1}$

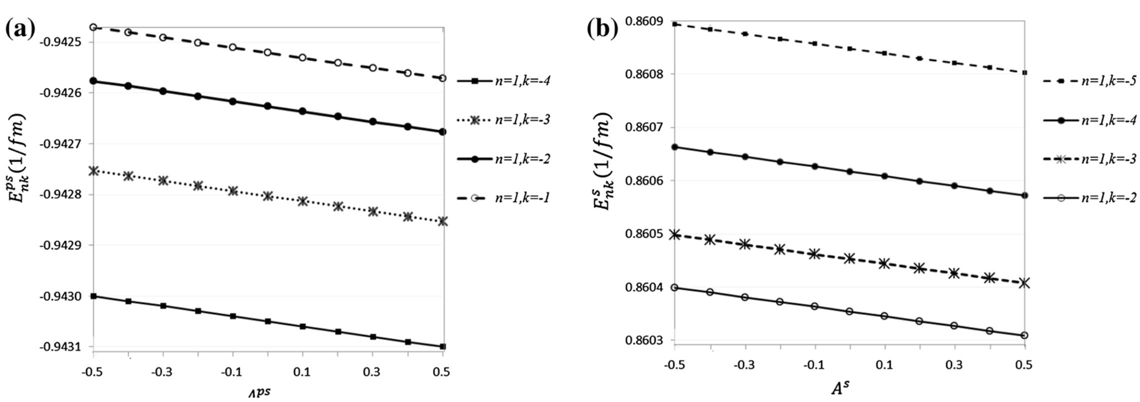

Fig. 3 (**a**) Energy versus A^{ps} for pseudospin Symmetry limit for $B^{ps}=-0.4,\ C^{ps}=-0.2,\ \alpha=0.01,\ V_1=0.5\ \mathrm{fm}^{-1}$, $M=1\ \mathrm{fm}^{-1}$, $C_{ps}=-5\ \mathrm{fm}^{-1}$ and (**b**) energy versus A^s for spin symmetry limit for $B^s=0.4,\ C^s=0.2,\ \alpha=0.01,\ V_1=0.5\ \mathrm{fm}^{\{-\}1}$, $M=1\ \mathrm{fm}^{-1},\ C_s=5\ \mathrm{fm}^{-1}$

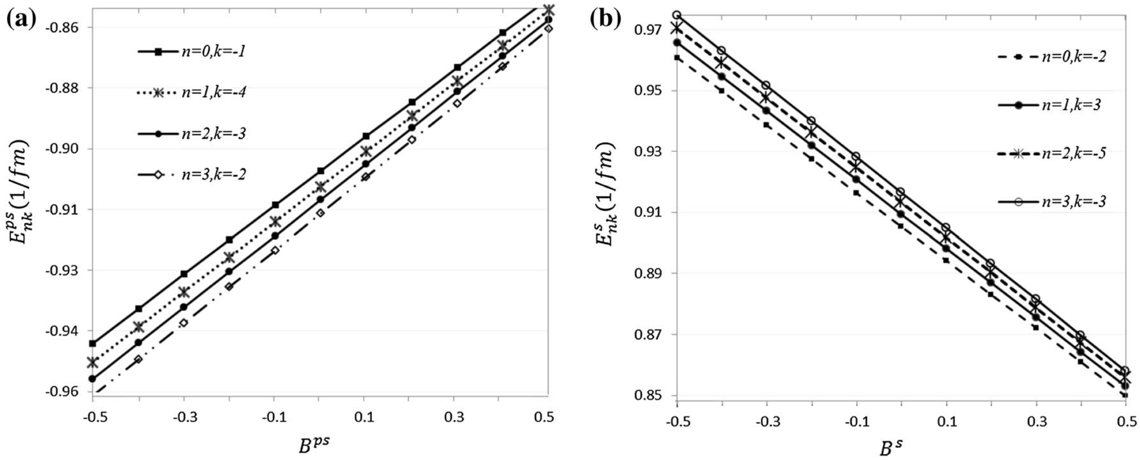

Fig. 4 (**a**) Energy versus B^{ps} for pseudospin symmetry limit for $A^{ps} = -0.2$, $C^{ps} = -0.2$, $\alpha = 0.01$, $V_1 = 0.5$ fm^{-1} $M = 1$ fm^{-1}, $C_{ps} = -5$ fm^{-1} and (**b**) energy versus B^s for spin symmetry limit for $A^s = 0.2$, $C^s = 0.2$, $\alpha = 0.01$, $V_1 = 0.5$ fm^{-1}, $M = 1$ fm^{-1}, $C_s = 5$ fm^{-1}

$$- (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}) + \left(\frac{\left(-\alpha \pm \sqrt{\alpha^2 + \left(\eta_1^{ps} + \eta_3^{ps} + \eta_2^{ps}\right)} - 2n\alpha\right)^2 + \eta_3^{ps} - \eta_1^{ps}}{2\left(-\alpha \pm \sqrt{\alpha^2 + \left(\eta_1^{ps} + \eta_3^{ps} + \eta_2^{ps}\right)} - 2n\alpha\right)}\right)^2 - \eta_3^{ps} = 0, \tag{53}$$

For completeness description of potential model under investigation, let us find the corresponding wave functions for pseudospin symmetry limit. By using transformation $y = e^{-2\alpha r}$, we obtain lower component of the wave function as

$$G_{n\kappa}(r) = N_{n\kappa}\left(e^{-2\alpha r}\right)^{\sqrt{\zeta_3^{ps}}}\left(1 - e^{-2\alpha r}\right)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \zeta_1^{ps} + \zeta_3^{ps} - \zeta_2^{ps}}} \times {}_1F_2\left(-n; n + 2\sqrt{\zeta_3^{ps}} + \sqrt{\frac{1}{4} + \zeta_1^{ps} + \zeta_3^{ps} - \zeta_2^{ps}} + 1; 2\sqrt{\zeta_3^{ps}} + 1; e^{-2\alpha r}\right), \tag{54}$$

where

$$\zeta_1^{ps} = \left[\begin{array}{l} A(M - E_{n\kappa} + C_{ps}) + V_1\left(V_1 + \dfrac{1}{2}\right) \\ + \dfrac{1}{2\alpha}(M - E_{n\kappa} + C_{ps})B + \dfrac{C(M - E_{n\kappa} + C_{ps})}{4\alpha^2} + \\ \dfrac{(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})}{4\alpha^2} \end{array}\right], \tag{55}$$

$$\zeta_2^{ps} = \frac{2(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})}{4\alpha^2} - \left[\begin{array}{l} \kappa(\kappa - 1) + 4V_1\left(\kappa - \dfrac{3}{4}\right) - \dfrac{2C(M - E_{n\kappa} + C_{ps})}{4\alpha^2} \\ - \dfrac{B(M - E_{n\kappa} + C_{ps})}{2\alpha} \end{array}\right], \tag{56}$$

$$\zeta_3^{ps} = \frac{C\left(M - E_{n\kappa} + C_{ps}\right)}{4\alpha^2} + \frac{(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})}{4\alpha^2}, \tag{57}$$

3.2. Spin symmetry limit

In spin symmetry limit $\frac{d\Delta(r)}{dr} = 0$ or $\Delta(r) = C_s = \text{const}$ [23–26]. As in previous section, we consider

$$\Sigma(r) = -\frac{A^s}{r^2}e^{-2\alpha r} + \frac{B^s}{r}e^{-\alpha r} - C, \tag{58}$$

$$U(r) = -V_1\left(\frac{e^{-\alpha r}}{r}\right), \tag{59}$$

Substitution of Eqs. (58) and (59) into Eq. (24) gives

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{2\kappa V_1 e^{-\alpha r}}{r^2} - \frac{\alpha V_1 e^{-\alpha r}}{r} - \frac{V_1 e^{-\alpha r}}{r^2} - \frac{V_1^2 e^{-2\alpha r}}{r^2}\right\}F_{n\kappa}(r) - (M + E_{n\kappa} - C_s)\left(-\frac{A^s}{r^2}e^{-2\alpha r} + \frac{B^s}{r}e^{-\alpha r} - C^s\right)F_{n\kappa}(r) = (M + E_{n\kappa} - C_s)(M - E_{n\kappa})F_{n\kappa}(r), \tag{60}$$

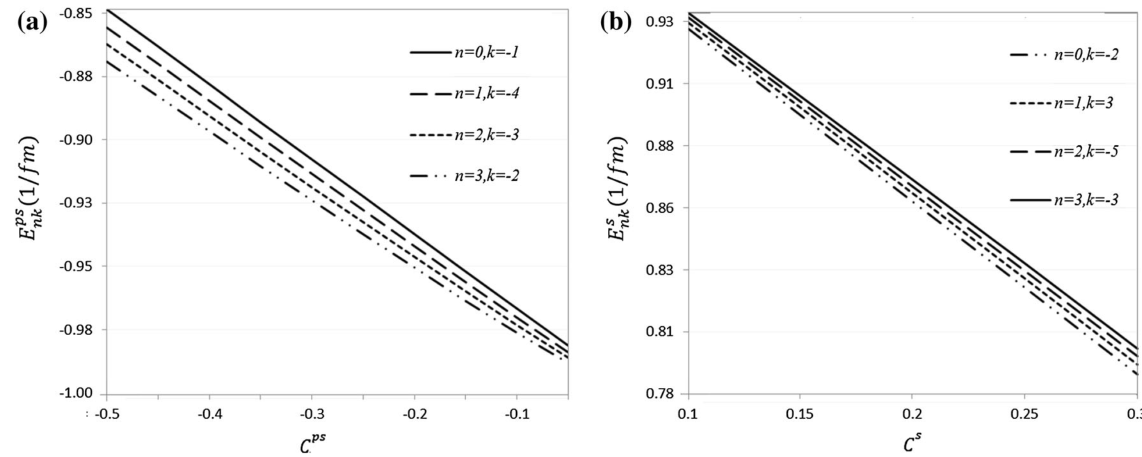

Fig. 5 (**a**) Energy versus C^{ps} for pseudospin symmetry limit for $A^{ps} = -0.2$, $B^{ps} = -0.4$, $\alpha = 0.01$, $V_1 = 0.5\ \text{fm}^{-1}$, $M = 1\ \text{fm}^{-1}$, $C_{ps} = -5\ \text{fm}^{-1}$ and (**b**) energy versus C^s for spin symmetry limit for $A^s = 0.2$, $B^s = 0.4$, $\alpha = 0.01$, $V_1 = 0.5\ \text{fm}^{-1}$, $M = 1\ \text{fm}^{-1}$, $C_s = 5\ \text{fm}^{-1}$

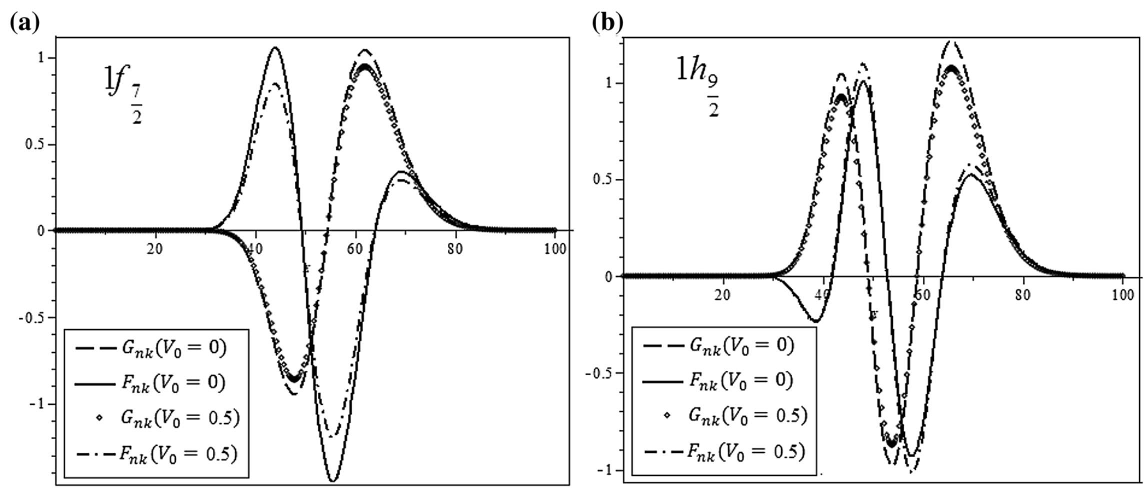

Fig. 6 Lower and upper radial wave functions in view of the pseudospin symmetry under condition (**a**) $\kappa < 0$ and (**b**) $\kappa > 0$ for $A^{ps} = -0.2$, $B^{ps} = -0.4$, $C^{ps} = -0.2$, $V_1 = 0.5\ \text{fm}^{-1}$, $\alpha = 0.01$, $M = 1\ \text{fm}^{-1}$, $C_{ps} = -5\ \text{fm}^{-1}$

By using approximations given by Eqs. (28) and (29) in Eq. (60), we arrive at

$$-\frac{d^2F_{n\kappa}}{dr^2} + V_{eff}(r)F_{n\kappa} = \tilde{E}^s_{n\kappa}F_{n\kappa}, \tag{61}$$

where

$$V_{eff}(r) = \frac{\chi_1^s e^{-4\alpha r} + \chi_2^s e^{-2\alpha r} + \chi_3^s}{(1-e^{-2\alpha r})^2}, \tag{62}$$

$$\tilde{E}^s_{n\kappa} = -(M + E_{n\kappa} - C_s)(M - E_{n\kappa}), \tag{63}$$

$$\chi_1^s = 4\alpha^2 V_1\left(V_1 - \frac{1}{2}\right) - 4\alpha^2 A(M + E_{n\kappa} - C_s) - 2\alpha B(M + E_{n\kappa} - C_s) - C(M + E_{n\kappa} - C_s), \tag{64}$$

$$\chi_2^s = 4\alpha^2\kappa(\kappa+1) + 4\alpha^2 V_1\left(2\kappa + \frac{3}{2}\right) + 2\alpha B(M + E_{n\kappa} - C_s) + 2C(M + E_{n\kappa} - C_s), \tag{65}$$

$$\chi_3^s = -C(M + E_{n\kappa} - C_s), \tag{66}$$

Following same steps of previous section, we find energy equation in this case as

$$-(M + E_{n\kappa} - C_s)(M - E_{n\kappa}) + \left(\frac{\left(-\alpha \pm \sqrt{\alpha^2 + (\chi_1^s + \chi_3^s + \chi_2^s)} - 2n\alpha\right)^2 + \chi_3^s - \chi_1^s}{2\left(-\alpha \pm \sqrt{\alpha^2 + (\chi_1^s + \chi_3^s + \chi_2^s)} - 2n\alpha\right)}\right)^2 - \chi_3^s = 0 \tag{67}$$

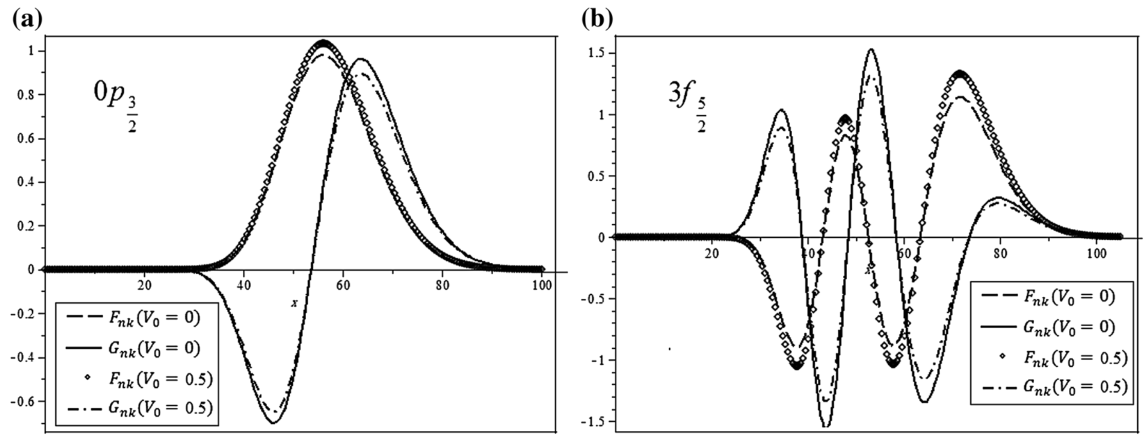

Fig. 7 *Upper* and *lower* radial wave functions in view of the spin symmetry under condition (**a**) $\kappa<0$ and (**b**) $\kappa>0$ for $A^s=0.2$, $B^s=0.4$, $C^s=0.2$, $V_1=0.5\ \text{fm}^{-1}$, $\alpha=0.01$, $M=1\ \text{fm}^{-1}$, $C_s=5\ \text{fm}^{-1}$

$$F_{n\kappa}(r)=N_{n\kappa}\left(e^{-2\alpha r}\right)^{\sqrt{\omega_3^s}}\left(1-e^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\omega_1^s+\omega_3^s-\omega_2^s}} \times {}_1F_2\left(-n;n+2\sqrt{\omega_3^s}+\sqrt{\frac{1}{4}+\omega_1^s+\omega_3^s-\omega_2^s}+1;2\sqrt{\omega_3^s}+1;e^{-2\alpha r}\right), \tag{68}$$

where

$$\omega_1^s=V_1\left(V_1-\frac{1}{2}\right)-A(M+E_{n\kappa}-C_s)-\frac{1}{2\alpha}B(M+E_{n\kappa}-C_s)-\frac{C(M+E_{n\kappa}-C_s)}{4\alpha^2}+\frac{(M+E_{n\kappa}-C_s)(M-E_{n\kappa})}{4\alpha^2}, \tag{69}$$

$$\omega_2^s=\frac{2(M+E_{n\kappa}-C_s)(M-E_{n\kappa})}{4\alpha^2}-\kappa(\kappa+1)-V_1\left(2\kappa+\frac{3}{2}\right)-\frac{1}{2\alpha}B(M+E_{n\kappa}-C_s)-\frac{2C(M+E_{n\kappa}-C_s)}{4\alpha^2}, \tag{70}$$

$$\omega_3^s=-\frac{C(M+E_{n\kappa}-C_s)}{4\alpha^2}+\frac{(M+E_{n\kappa}-C_s)(M-E_{n\kappa})}{4\alpha^2}, \tag{71}$$

and the other component can be simply found via,

$$G_{n\kappa}(r)=\frac{1}{M+E_{n\kappa}-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r). \tag{72}$$

4. Numerical results

We have listed some numerical results given in Tables 1 and 2 for the set $A^{ps}=-0.2$, $B^{ps}=-0.4$, $C^{ps}=-0.2$, $V_1=0$, $C_{ps}=-5$, $M=1$, $\alpha=0.01$ in pseudospin symmetry and $A^s=0.2$, $B^s=0.4$, $C^s=0.2$, $V_1=0$, $C_s=5$, $M=1$, $\alpha=0.01$ for spin symmetry. For values of $A^{ps}=0$, $B^{ps}=1$, $C^{ps}=0$, $V_1=0,0.5$, $C_{ps}=0$ for pseudospin symmetry case and for $A^s=0$, $B^s=-1$, $C^s=0$, $V_1=0,0.5$, $C_s=0$ for spin symmetry case, numerical results are compared with [16]. In Tables 1 and 2 numerical results of energy eigenvalue equations are presented in presence and absence of Yukawa potential as tensor interaction. In addition, it can be seen that all degeneracies between two states in spin and pseudospin doublets are removed by tensor interaction ($V_1\neq 0$). In Fig. 1, we have plotted the energy versus α for both pseudospin and spin symmetry limits. In Fig. 2, we present the effects of tensor interaction on bound-states. Figures. 3, 4 and 5 present dependence of bound-state energy levels on potential parameters. Also, Figs. 6 and 7 respectively show the wavefunctions for pseudospin and symmetry limits without and with tensor interaction.

5. Conclusions

In this paper, we have considered the spin and pseudospin symmetries of Dirac equation with these interactions with successful prediction of Yukawa potential and modified Yukawa potential in particle and nuclear physics. We have shown that using supersymmetry quantum mechanics, shape-invariance condition and a physical approximation to centrifugal term, problem can be analytically solved for any arbitrary states. Various parameters obtained from analysis give a better insight to their effect on spectrum and wavefunction of system. Our results, for special cases of parameters reduce to well-known Coulomb, Mie-type, Yukawa and inversely-quadratic Yukawa potential and therefore provide more general results than reported [29, 30].

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