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SHORT RESEARCH COMMUNICATION

Application of Kudryashov method for high-order nonlinear Schrödinger equation

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Abstract: Higher-order nonlinear Schrödinger equation for describing the propagation of femtosecond pulses in optical fibers is studied. Kudryashov method is used for obtaining exact soliton solutions of this equation.

Keywords: Kudryashov method; Solitons; Partial differential equations; Higher-order nonlinear Schrödinger equation

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1. Introduction

Over the last few decades, search for exact solutions of nonlinear partial differential equations has become a more attractive topic in physical and nonlinear sciences. Investigation of the traveling wave solutions for nonlinear partial differential equations plays an important role in study of nonlinear physical phenomena [1–7]. Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, fluid dynamics, etc. In order to understand these nonlinear phenomena, many mathematicians and physical scientists have made efforts to seek more exact solutions of them. Several powerful methods have been proposed to obtain exact solutions of nonlinear evolution equations, such as Ansatz method and topological solitons [8–12], tanh method [13, 14], multiple exp-function method [15], Hirotas direct method [16, 17], transformed rational function method [18] and so on.

The higher-order nonlinear Schrödinger equation [19, 20] is as follows:

$$q_z = ia_1 q_{tt} + ia_2 q|q|^2 + a_3 q_{ttt} + a_4 (q|q|^2)_t + a_5 q(|q|^2)_t, \tag{1}$$

which describes propagation of ultrashort pulses in nonlinear optical fibers, where the complex function $q = q(z, t)$ is slowly varying envelop of the electric field, the subscripts z and t are spatial and temporal partial derivative in retard time coordinates. a_1, a_2, a_3, a_4 and a_5 are the real parameters related to group velocity, self-phase modulation, third order dispersion, self-steepening and self-frequency shift arising from stimulated Raman scattering respectively. More details are presented in [21–30].

The powerful and effective method for finding exact solutions of nonlinear ordinary differential equations (ODEs) has been proposed by Kudryashov and hence called the Kudryashov method [31]. The most complete description of this method is given in [32]. The successful application of this method to nonlinear differential equations has been performed in several works [33–35].

The aim of this paper is to find exact solutions of the higher-order nonlinear Schrödinger equation by using the Kudryashov method [31–36].

2. Modification of truncated expansion method

We consider a general nonlinear partial differential equation (PDE) in the form

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$$E_1(u, u_t, u_x, u_{xx}, \ldots) = 0. \tag{2}$$

Using traveling wave $u(x, t) = y(\xi)$, $\xi = kx - \omega t$ carries Eq. (2) into the following ODE:

$$E_2(y, -\omega y_\xi, ky_\xi, k^2 y_{\xi\xi}, \ldots) = 0. \tag{3}$$

Modification of the truncated expansion method contains the following steps.

Step I: Determination of the dominant term with highest order of singularity. To find dominant terms, we substitute

$$y = \xi^{-p}, \tag{4}$$

to all terms of Eq. (3). Then we compare degrees of all terms of Eq. (3) and choose two or more with the lowest degree. The maximum value of p is the pole of Eq. (3) and we denote it as N. This method can be applied when N is integer. If the value N is noninteger, one can transform the equation studied.

Step II: We look for exact solution of Eq. (3) in the form

$$y = \sum_{i=0}^{N} b_i Q^i(\xi), \tag{5}$$

where $b_i (i = 0, 1, \ldots, N)$ are constants to be determined later, such that $b_N \neq 0$, while $Q(\xi)$ has the form

$$Q(\xi) = \frac{1}{1 + c\exp(\xi)}, \tag{6}$$

which is a solution to the Riccati equation

$$Q'(\xi) = Q^2(\xi) - Q(\xi),$$

where c is arbitrary constant.

Remark 1: This Riccati equation also admits the following exact solutions [37]:

$$Q_1(\xi) = \frac{1}{2}\left(1 - \tanh\left[\frac{\xi}{2} - \frac{\varepsilon \ln \xi_0}{2}\right]\right), \quad \xi_0 > 0,$$

$$Q_2(\xi) = \frac{1}{2}\left(1 - \coth\left[\frac{\xi}{2} - \frac{\varepsilon \ln \xi_0}{2}\right]\right), \quad \xi_0 < 0,$$

more general solutions are presented in [37].

Step III: We calculate necessary number of derivatives of function y. It has been done using Maple or Mathematica package. Using case $N = 1$, we have some derivatives of function $y(\xi)$ in the form

$$\begin{aligned} y &= b_0 + b_1 Q, \\ y_\xi &= -b_1 Q + b_1 Q^2, \\ y_{\xi\xi} &= b_1 Q - 3b_1 Q^2 + 2b_1 Q^3, \\ y_{\xi\xi\xi} &= -b_1 Q + 7b_1 Q^2 - 12b_1 Q^3 + 6b_1 Q^4. \end{aligned} \tag{7}$$

Step IV: We substitute expressions given by Eqs. (5)–(7) in Eq. (3). Then we collect all terms with the same powers of function $Q(\xi)$ and equate expressions to zero. As a result we obtain algebraic system of equations. Solving this system we get the values of unknown parameters.

3. Higher-order nonlinear Schrödinger equation

Since $q = q(z, t)$ in Eq. (1) is a complex function, we suppose that

$$q(z, t) = y(\xi)e^{i(\alpha z + \beta t)}, \quad \xi = i(kz - \omega t), \tag{8}$$

where α, β, k and ω are constants, all of them are to be determined.

Substituting these into Eq. (1) yields

$$q_z = i(ky_\xi + \alpha y)e^{i(\alpha z + \beta t)}, \tag{9}$$

$$q_{tt} = -(\omega^2 y_{\xi\xi} - 2\beta\omega y_\xi + \beta^2 y)e^{i(\alpha z + \beta t)}, \tag{10}$$

$$q_{ttt} = i(\omega^3 y_{\xi\xi\xi} - 3\beta\omega^2 y_{\xi\xi} + 3\beta^2\omega y_\xi - \beta^3 y)e^{i(\alpha z + \beta t)}, \tag{11}$$

$$(q|q|^2)_t = i(-3\omega y^2 y_\xi + \beta y^3)e^{i(\alpha z + \beta t)}, \tag{12}$$

$$q(|q|^2)_t = -2i\omega y^2 y_\xi e^{i(\alpha z + \beta t)}. \tag{13}$$

Substituting Eqs. (8)–(13) into Eq. (1), we have

$$\begin{aligned} &a_3\omega^3 y_{\xi\xi\xi} - (3\beta a_3\omega^2 + a_1\omega^2)y_{\xi\xi} \\ &+ (-k + 3\beta^2\omega a_3 + 2\beta\omega a_1)y_\xi - (a_3\beta^3 + a_1\beta^2 + \alpha)y \\ &+ (a_2 + a_4\beta)y^3 - (3\omega a_4 + 2\omega a_5)y^2 y_\xi = 0. \end{aligned} \tag{14}$$

The pole order of Eq. (14) is $N = 1$. So we look for solution of Eq. (14) in the following form

$$y(\xi) = b_0 + b_1 Q(\xi). \tag{15}$$

Substituting Eq. (15) into Eq. (14), we obtain the system of algebraic equations in the following form

$$Q^4 : 6a_3\omega^3 b_1 - \omega(3a_4 + 2a_5)b_1^3 = 0, \tag{16}$$

$$Q^3: -2\omega^2(a_1+3\beta a_3)b_1+(a_2+a_4\beta)b_1^3-12a_3\omega^3 b_1 -2\omega(3a_4+2a_5)b_0b_1^2+\omega(3a_4+2a_5)b_1^3=0,$$
$$Q^2: 7a_3\omega^3 b_1+3\omega^2(a_1+3\beta a_3)b_1-\omega(3a_4+2a_5)b_0^2b_1 +2\omega(3a_4+2a_5)b_0b_1^2+(2a_1\omega\beta+3a_3\omega\beta^2-k)b_1+3(a_2+a_4\beta)b_0b_1^2=0,$$
$$Q^1: -a_3\omega^3b_1+\omega(3a_4+2a_5)b_0^2b_1-\omega^2(a_1+3\beta a_3)b_1 -(\alpha+a_1\beta^2+a_3\beta^3)b_1-(2a_1\omega\beta+3a_3\omega\beta^2-k)b_1+3(a_2+a_4\beta)b_0^2b_1=0,$$
$$Q^0: -(\alpha+a_1\beta^2+a_3\beta^3)b_0+(a_2+a_4\beta)b_0^3=0.$$

With the aid of Maple, we find the special solutions of the above system

Case i

$$b_0=\mp\omega\sqrt{\frac{6a_3}{3a_4+2a_5}},\quad b_1=\pm\omega\sqrt{\frac{6a_3}{3a_4+2a_5}},$$
$$\beta=\frac{-3a_1a_4-2a_1a_5+3a_2a_3+9\omega a_3a_4+6\omega a_3a_5}{6a_3(a_4+a_5)},$$
$$\begin{aligned}\alpha=&-\frac{1}{216a_3^2(a_4+a_5)^3}[(162a_3^3a_4^2a_5+324a_3^3a_4a_5^2\\&+81a_3^3a_4^3+216a_3^3a_5^3)\omega^3\\&+(108a_1a_3^2a_4^2a_5-27a_1a_4^3a_3^2+108a_1a_4a_3^2a_5^2\\&-324a_3^3a_2a_4a_5+81a_2a_3^3a_4^2-324a_3^3a_2a_5^2)\omega^2\\&+(-108a_1a_4a_3^2a_2a_5-270a_3a_4^2a_1^2a_5-252a_3a_5^2a_1^2a_4\\&-162a_1a_4^2a_3^2a_2-72a_3a_5^3a_1^2+243a_3^3a_2^2a_4\\&-81a_3a_4^3a_1^2+162a_3^3a_2^2a_5)\omega-72a_3a_2a_1^2a_4a_5\\&+72a_1^3a_4^2a_5+60a_1^3a_4a_5^2+16a_1^3a_5^3+27a_1^3a_4^3+27a_3^3a_2^3\\&-27a_1a_3^2a_2^2a_4-36a_1^2a_2a_3a_5^2-27a_1^2a_2a_3a_4^2],\end{aligned}$$
$$\begin{aligned}k=&\frac{\omega}{12a_3(a_4+a_5)^2}[(3a_3^2a_4^2+6a_3^2a_4a_5+12a_3^2a_5^2)\omega^2\\&+(6a_1a_3a_4a_5-18a_3^2a_2a_5)\omega\\&-3a_1^2a_4^2-8a_1^2a_4a_5-6a_3a_2a_1a_4-4a_1^2a_5^2+9a_3^2a_2^2],\end{aligned}$$

where ω is arbitrary constant.

Using Ansatz given by Eq. (15), we obtain the following traveling-wave solution of Eq. (14)

$$y_1(\xi)=\mp\omega\sqrt{\frac{6a_3}{3a_4+2a_5}}\left(1-\frac{1}{1+ce^{\xi}}\right). \tag{17}$$

Then the exact solution to Eq. (1) is written as

$$\begin{aligned}q_1(z,t)=&\mp\omega\sqrt{\frac{6a_3}{3a_4+2a_5}}\left(1-\frac{1}{1+ce^{i\omega(Hz-t)}}\right)\\&\times\exp\Bigg[i\Bigg(\Bigg(-\frac{1}{216a_3^2(a_4+a_5)^3}\\&[(162a_3^3a_4^2a_5+324a_3^3a_4a_5^2+81a_3^3a_4^3+216a_3^3a_5^3)\omega^3\\&+(108a_1a_3^2a_4^2a_5-27a_1a_4^3a_3^2+108a_1a_4a_3^2a_5^2\\&-324a_3^3a_2a_4a_5+81a_2a_3^3a_4^2-324a_3^3a_2a_5^2)\omega^2\\&+(-108a_1a_4a_3^2a_2a_5-270a_3a_4^2a_1^2a_5\\&-252a_3a_5^2a_1^2a_4-162a_1a_4^2a_3^2a_2-72a_3a_5^3a_1^2+243a_3^3a_2^2a_4\\&-81a_3a_4^3a_1^2+162a_3^3a_2^2a_5)\omega-72a_3a_2a_1^2a_4a_5\\&+72a_1^3a_4^2a_5+60a_1^3a_4a_5^2+16a_1^3a_5^3+27a_1^3a_4^3+27a_3^3a_2^3\\&-27a_1a_3^2a_2^2a_4-36a_1^2a_2a_3a_5^2-27a_1^2a_2a_3a_4^2])z\\&+\left(\frac{-3a_1a_4-2a_1a_5+3a_2a_3+9\omega a_3a_4+6\omega a_3a_5}{6a_3(a_4+a_5)}\right)t\Bigg)\Bigg],\end{aligned}$$

where $H=\frac{1}{12a_3(a_4+a_5)^2}[(3a_3^2a_4^2+6a_3^2a_4a_5+12a_3^2a_5^2)\omega^2+(6a_1a_3a_4a_5-18a_3^2a_2a_5)\omega-3a_1^2a_4^2-8a_1^2a_4a_5-6a_3a_2a_1a_4-4a_1^2a_5^2+9a_3^2a_2^2]$.

Case ii

$$b_0=\mp\frac{\sqrt{6}(3\omega a_3a_4+a_1a_4-3a_2a_3)}{6a_4\sqrt{a_3(3a_4+2a_5)}},$$
$$b_1=\pm\frac{\sqrt{6}\omega\sqrt{a_3(3a_4+2a_5)}}{3a_4+2a_5},\quad \beta=-\frac{a_2}{a_4},$$
$$\alpha=-\frac{a_2^2(a_1a_4-a_2a_3)}{a_4^3},$$
$$k=-\frac{\omega(a_1^2a_4^2+3\omega^2a_3^2a_4^2+6a_1a_2a_3a_4-9a_3^2a_2^2)}{6a_3a_4^2}$$

where ω is arbitrary constant.

Using Ansatz given by Eq. (15), we obtain the following exact solution of Eq. (14)

$$y_2(\xi)=\mp\frac{\sqrt{6}(3\omega a_3a_4+a_1a_4-3a_2a_3)}{6a_4\sqrt{a_3(3a_4+2a_5)}}\pm\frac{\sqrt{6}\omega\sqrt{a_3(3a_4+2a_5)}}{(3a_4+2a_5)(1+ce^{\xi})}. \tag{18}$$

Then the complex solution to Eq. (1) is written as

$$q_2(z,t)=\mp\left(\frac{\sqrt{6}(3\omega a_3a_4+a_1a_4-3a_2a_3)}{6a_4\sqrt{a_3(3a_4+2a_5)}}-\frac{\sqrt{6}\omega\sqrt{a_3(3a_4+2a_5)}}{(3a_4+2a_5)(1+ce^{i\omega(Hz-t)})}\right)\times e^{-i\left(\left(\frac{a_2^2(a_1a_4-a_2a_3)}{a_4^3}\right)z+\frac{a_2}{a_4}t\right)},$$

where $H = -\frac{a_1^2 a_4^2 + 3\omega^2 a_3^2 a_4^2 + 6a_1 a_2 a_3 a_4 - 9a_3^2 a_2^2}{6a_3 a_4^2}$.

Case iii

$$b_0 = \mp \frac{\sqrt{6}a_3\omega}{2\sqrt{a_3(3a_4 + 2a_5)}}, \quad b_1 = \pm \frac{\sqrt{6}a_3\omega}{\sqrt{a_3(3a_4 + 2a_5)}},$$

$$\beta = -\frac{-3a_2a_3 + 3a_1a_4 + 2a_1a_5}{6a_3(a_4 + a_5)},$$

$$\alpha = -\frac{1}{216a_3^2(a_4 + a_5)^3}[(-162a_3^3a_2a_4^2 + 54a_1a_4^3a_3^2 + 54a_1a_4a_3^2a_5^2 + 108a_1a_4^2a_3^2a_5 - 324a_2a_4a_5a_3^3 - 162a_3^3a_2a_5^2)\omega^2 - 27a_2a_3a_1^2a_4^2 + 27a_1^3a_4^3 + 72a_1^3a_4^2a_5 + 60a_1^3a_5^2a_4 + 16a_1^3a_5^3 - 27a_1a_4a_3^2a_2^2 + 27a_2^3a_3^3 - 72a_3a_2a_1^2a_4a_5 - 36a_2a_3a_1^2a_5^2],$$

$$k = -\frac{\omega}{12a_3(a_4 + a_5)^2}[(6a_3^2a_4^2 + 6a_3^2a_5^2 + 12a_3^2a_4a_5)\omega^2 + 3a_1^2a_4^2 + 6a_1a_2a_3a_4 + 4a_1^2a_5^2 + 8a_1^2a_4a_5 - 9a_2^2a_3^2],$$

where ω is arbitrary constant.

Using Ansatz given by Eq. (15), we obtain the following traveling-wave solution of Eq. (14)

$$y_3(\xi) = \mp \frac{\sqrt{6}a_3\omega}{2\sqrt{a_3(3a_4 + 2a_5)}}\left(1 - \frac{2}{1 + ce^{\xi}}\right). \tag{19}$$

Then the exact solution to Eq. (1) can be written as

$$q_3(z,t) = \mp \frac{\sqrt{6}a_3\omega}{2\sqrt{a_3(3a_4+2a_5)}}\left(1 - \frac{2}{1 + ce^{i\omega(Hz-t)}}\right) \times \exp\left[i\left(\left(\left(-\frac{1}{216a_3^2(a_4+a_5)^3}[(-162a_3^3a_2a_4^2 + 54a_1a_4^3a_3^2 + 54a_1a_4a_3^2a_5^2 + 108a_1a_4^2a_3^2a_5 - 324a_2a_4a_5a_3^3 - 162a_3^3a_2a_5^2)\omega^2 - 27a_2a_3a_1^2a_4^2 + 27a_1^3a_4^3 + 72a_1^3a_4^2a_5 + 60a_1^3a_5^2a_4 + 16a_1^3a_5^3 - 27a_1a_4a_3^2a_2^2 + 27a_2^3a_3^3 - 72a_3a_2a_1^2a_4a_5 - 36a_2a_3a_1^2a_5^2]\right)z - \left(\frac{-3a_2a_3 + 3a_1a_4 + 2a_1a_5}{6a_3(a_4+a_5)}\right)t\right)\right],$$

where $H = -\frac{1}{12a_3(a_4+a_5)^2}[(6a_3^2a_4^2 + 6a_3^2a_5^2 + 12a_3^2a_4a_5)\omega^2 + 3a_1^2a_4^2 + 6a_1a_2a_3a_4 + 4a_1^2a_5^2 + 8a_1^2a_4a_5 - 9a_2^2a_3^2]$.

4. Results and discussion

It is well known that propagation of picosecond pulses in optical fibres is described by the nonlinear Schrödinger equation (NLS). However, for ultrashort femtosecond pulses, NLS is invalid and is described by higher order nonlinear Schrödinger equation. In this paper, Eq. (1) is studied for describing the propagation of femtosecond pulses in optical fibers. Kudryashov method is used for constructing exact soliton solutions of this equation. Solutions obtained are potentially significant and important for the explanation of some practical physical problems. The results show that this method is efficient in finding the exact solutions of nonlinear differential equations.

5. Conclusions

Kudryashov method is applied successfully for solving higher order nonlinear Schrödinger equation. This proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

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