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ORIGINAL PAPER

Soliton solutions to coupled nonlinear wave equations in (2 + 1)-dimensions

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Abstract: This paper implemented the tanh method to solve a few coupled nonlinear wave equations in (2 + 1)-dimensions. They are the Konopelchenko–Dubrovsky equation, dispersive long wave equation and the Riemann wave equation. Additionally, the traveling wave hypothesis is used to extract a few more solutons to some of these equations. Finally, the numerical simulations supplement these analytical results.

Keywords: Solitons; Integrability; Tanh method; Traveling waves

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1. Introduction

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of Mathematical and Physical sciences such as physics, biology, chemistry, biochemistry, applied and pure mathematics, applied and pure physics, just to name a few [1–28]. In particular soliton solutions of such NLEEs play a vital role in the dynamics of pulse propagation through optical fibers for trans-continental and trans-oceanic distances [27, 28]. These analytical solutions of such equations are of fundamental importance since a lot of mathematical and physical models are described by NLEEs.

The nonlinear wave phenomena observed in the above mentioned scientific fields, are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution. The investigation of exact solutions of NLPDEs plays an important role in the study of these phenomena [29–33]. In the past several decades, many effective methods for obtaining exact solutions of NLPDEs have been presented. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions, such as the Backlund transformation method [1], Hirota's direct method by Hirota [2], tanh-sech method by author(s) [1, 3, 4], extended tanh method [5–8], hyperbolic function method [9], sine cosine method [10, 11], F-expansion method [12], the transformed rational function method [13] and ansatz method [14].

This paper outlines the implementation of efficient and reliable technique which is called Tanh method for solving system of coupled equations which are very important in applied sciences. The hyperbolic tangent (tanh) method is a powerful technique to symbolically compute traveling wave solutions of one-dimensional nonlinear wave and evolution equations. In particular, the method is well suited for problems where dispersion, convection, and reaction diffusion phenomena play an important role.

2. Description of the tanh method

The tanh method will be introduced as presented by Malfliet [15] and by Wazwaz [16]. The tanh method is

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based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function to solve the coupled KdV equations.

The tanh method is developed by Malfliet [15]. The method is applied to find out exact solutions of a coupled system of nonlinear differential equations with three unknowns:

$$P_1(u, v, u_x, v_x, u_t, v_t, u_y, v_y, u_{xx}, v_{xy}, \ldots) = 0$$
$$P_2(u, v, u_x, v_x, u_t, v_t, u_y, v_y, u_{xx}, v_{xy}, \ldots) = 0 \tag{1}$$

where P_1, P_2 are polynomials of the variable u, v and its derivatives. If we consider $u(x,t) = u(\xi), v(x,t) = v(\xi)$, $\xi = kx + \alpha y + \omega t + \theta_0$, so that $u(x,t) = U(\xi)$, $v(x,t) = V(\xi)$, we can use the following changes:

$$\frac{\partial}{\partial t} = k\lambda\frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = k\frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = k^2\frac{d^2}{d\xi^2}, \quad \frac{\partial^3}{\partial x^3} = k^3\frac{d^3}{d\xi^3},$$

and so on, then Eq. (1) becomes an ordinary differential equation

$$Q_1(U, U', U'', U''', \ldots) = 0,$$
$$Q_2(U, U', U'', U''', \ldots) = 0 \tag{2}$$

with Q_1, Q_2 being another polynomials form of there argument, which will be called the reduced ordinary differential equations (ODEs) of Eq. (2). Integrating Eq. (2) as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well. Now finding the traveling wave solutions to Eq. (1) is equivalent to obtaining the solution to the reduced ordinary differential equation given in Eq. (2). For the tanh method, we introduce the new independent variable

$$Y(x,t) = \tanh(\xi) \tag{3}$$

that leads to the change of variables:

$$\frac{d}{d\xi} = (1 - Y^2)\frac{d}{dY},$$
$$\frac{d^2}{d\xi^2} = -2Y(1 - Y^2)\frac{d}{dY} + (1 - Y^2)^2\frac{d^2}{dY^2},$$
$$\frac{d^3}{d\xi^3} = 2(1 - Y^2)(3Y^2 - 1)\frac{d}{dY} - 6Y(1 - Y^2)^2\frac{d^2}{dY^2} + (1 - Y^2)^3\frac{d^3}{dY^3} \tag{4}$$

The next crucial step is that the solution we are looking for is expressed in the form

$$u(x,t) = U(\xi) = \sum_{i=0}^{m} a_i Y^i, \quad v(x,t) = V(\xi) = \sum_{i=0}^{n} b_i Y^i \tag{5}$$

where the parameters m, and n can be found by balancing the highest-order linear term with the nonlinear terms in Eq. (2), and $k, \lambda, a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_m$ are to be determined. Substituting Eq. (5) into Eq. (2) will yield a set of algebraic equations for $k, \lambda, a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_m$ because all coefficients of Y^i have to vanish. From these relations, $k, \lambda, a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_m$ can be obtained. Having determined these parameters, knowing that m, n are positive integers in most cases, and using Eq. (5) we obtain analytic solutions $u(x,t), v(x,t)$ in a closed form [16]. The tanh method seems to be powerful tool in dealing with coupled nonlinear physical models.

3. Applications

The tanh method is generalized on three examples that will now be discussed in the following sub-sections.

3.1. Konopelchenko–Dubrovsky equation

$$u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv = 0 \tag{6}$$

$$u_y = v_x \tag{7}$$

This system was studied by Taghizadeh and Mirzazadeh [17] by the first integral method. By using tanh method and using the traveling wave transformations in Eqs. (3) and (5) with

$$\xi = kx + \alpha y + \omega t + \theta_0 \tag{8}$$

The nonlinear system of partial differential equations given in Eqs. (6) and (7) is carried to a system of ordinary differential equations

$$\omega U' - k^3U''' - 6bkUU' + \frac{3}{2}a^2kU^2U' - 3\alpha V' + 3akU'V = 0 \tag{9}$$

$$\alpha U' = kV' \tag{10}$$

Integrating Eq. (10) once with zero constant and we postulate the tanh series, Eq. (10) reduces to

$$V = \frac{\alpha}{k}U \tag{11}$$

Subtitute Eq. (11) in Eq. (9), then

$$\omega U' - k^3U''' + \frac{3}{2}a^2kU^2U' - \frac{3}{k}\alpha^2U' + \left(\frac{3a\alpha - 6bk}{2}\right)2U'U = 0 \tag{12}$$

Integrating Eq. (12) once with zero constant, it reduces to

$$\omega U - k^3U'' + \frac{1}{2}a^2kU^3 - \frac{3}{k}\alpha^2U + \left(\frac{3a\alpha - 6bk}{2}\right)U^2 = 0 \tag{13}$$

Now, to determine the parameters m, and n we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq. (13) we balance U''' with U^3, to obtain $2 + m = 3m$, so that $m = 1$, while in Eq. (11) we balance V with U, then $n = 1$. The tanh method admits the use of the finite expansion for both:

$$u(x,t) = U(Y) = a_0 + a_1 Y, \quad a_1 \neq 0 \tag{14}$$

and

$$v(x,t) = V(Y) = b_0 + b_1 Y, \quad b_1 \neq 0 \tag{15}$$

Substituting U', U'' in Eq. (14), then equating the coefficient of Y^i, i = 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations:

$$\begin{aligned}
&2k^3 Y(1-Y^2)a_1 + \frac{1}{2}a^2 k\left(a_0^3 + 3a_0^2 a_1 Y + 3a_0 a_1^2 Y^2 + a_1^3 Y^3\right)\\
&\quad + \omega(a_0 + a_1 Y) - \frac{3}{k}\alpha^2(a_0 + a_1 Y)\\
&\quad + \left(\frac{3a\alpha - 6bk}{2}\right)\left(a_0^2 + 2a_0 a_1 Y + a_1^2 Y^2\right) = 0,\\
&Y^0 : 2\omega k + a^2k^2a_0^2 - 6\alpha^2 + 3k[a\alpha - 2bk]a_0 = 0,\\
&Y^1 : \omega + 2k^3 + \frac{3}{2}a^2 k a_0^2 - \frac{3}{k}\alpha^2 + 3[a\alpha - 2bk]a_0 = 0,\\
&Y^2 : a^2 k a_0 + a\alpha - 2bk = 0, \quad Y^3 : -4k^2 + a^2 a_1^2 = 0
\end{aligned} \tag{16}$$

Substituting U, V in Eq. (11), then equating the coefficient of Y^i, i = 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$\begin{aligned}
b_0 &= \frac{\alpha a_0}{k},\\
b_1 &= \frac{\alpha a_1}{k}
\end{aligned} \tag{17}$$

Solving the nonlinear systems of Eq. (16) we can get:

<u>Case 1</u> :

$$a_0 = a_1 = \frac{2k}{a}, \quad b_0 = b_1 = \frac{4k}{a^2}[b - ak],$$

$$\alpha = \frac{2k}{a}[b - ak],$$

$$\begin{aligned}
u_{1,2}(x,y,t) = \frac{2k}{a}\Bigg[1 + \tan h\Bigg\{&kx + \frac{2k}{a}(b-ak)y\\
&+ \frac{4k}{a^2}(3b^2 \pm 6abk + 4a^2k^2)t + \theta_0\Bigg\}\Bigg]
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
v_{1,2}(x,y,t) = \frac{4k}{a^2}[b-ak]\Bigg[1 + \tanh\Bigg\{&kx + \frac{2k}{a}(b-ak)y\\
&+ \frac{4k}{a^2}(3b^2 \pm 6abk + 4a^2k^2)t + \theta_0\Bigg\}\Bigg]
\end{aligned} \tag{19}$$

for $a = b = k = 1$ and $\theta_0 = 0$

$$\begin{aligned}
&u_1(x,t) = 2[1 + \tanh\{x + 52t\}],\\
&u_2(x,t) = 2[1 + \tanh\{x + 4t\}],\\
&v(x,y,t) = 0
\end{aligned}$$

<u>Case 2</u> :

$$a_0 = a_1 = \frac{-2k}{a}$$

$$b_0 = b_1 = -4[b + ak]\frac{k}{a^2}$$

$$\alpha = \frac{2k}{a}[b + ak]$$

$$\begin{aligned}
u(x,y,t) = -\frac{2k}{a}\Bigg[1 + \tanh\Bigg\{&kx + \frac{2k}{a}(b+ak)y\\
&+4k\left(\frac{3b^2}{a^2} \pm \frac{6bk}{a} + 4k^2\right)t + \theta_0\Bigg\}\Bigg]
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
v(x,y,t) = -\frac{4k}{a^2}[b+ak]\Bigg[1 + \tanh\Bigg\{&kx + \frac{2k}{a}(b+ak)y\\
&+4k\left(\frac{3b^2}{a^2} \pm \frac{6bk}{a} + 4k^2\right)t + \theta_0\Bigg\}\Bigg]
\end{aligned} \tag{21}$$

for, $a = b = k = 1$ and $\theta_0 = 0$

$$u(x,y,t) = -2\{1 + \tanh(x + 4y + 52t)\}$$

or

$$u(x,y,t) = -2\{1 + \tanh(x + 4y + 4t)\}$$

and

$$v(x,y,t) = -8\{1 + \tanh(x + 4y + 4t)\}.$$

3.2. Dispersive long wave equation

This coupled system of equation is given by

$$u_{yt} + v_{xx} + \frac{1}{2}(u^2)_{xy} = 0 \tag{22}$$

$$v_t + (uv + u + u_{xy})_x = 0 \tag{23}$$

Using the traveling wave transformations in Eqs. (3) and (5) with

$$\xi = kx + \alpha y + \omega t + \theta_0 \tag{24}$$

the nonlinear system of partial differential equations given in Eqs. (22) and (23) is transformed to the system of ODEs given by

$$k\omega U'' + k^2 V'' + k\alpha(UU')' = 0 \tag{25}$$

$$\omega V' + k^2 UV + kU + k^2\alpha U'' = 0 \tag{26}$$

Integrating Eq. (25) twice with zero constant and we postulate the tanh series, Eq. (22) reduces to

$$\omega U + kV + \frac{\alpha}{2}U^2 = 0 \tag{27}$$

we postulate the tanh series, Eq. (27) reduces to

$$\omega\left\{(1-Y^2)\frac{dV}{dY}\right\}+k^2UV+kU$$
$$+k^2\alpha\left\{-2Y(1-Y^2)\frac{dU}{dY}+(1-Y^2)^2\frac{d^2U}{dY^2}\right\}=0 \tag{28}$$

Now, to determine the parameters m, and n we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq. (28) we balance U'' with UV, to obtain $m+2=m+n$, so that $n=2$, while in Eq. (26) we balance V with U^2, then $2m=n$ and $m=1$. The tanh method admits the use of the finite expansion for both:

$$u(x,t)=U(Y)=a_0+a_1Y \quad a_1\neq 0 \tag{29}$$

and

$$v(x,t)=V(Y)=b_0+b_1Y+b_2Y^2 \quad b_2\neq 0 \tag{30}$$

Substituting U', U'' into Eq. (28), then equating the coefficient of Y^i, i = 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations:

$$\begin{aligned}&Y^0: \omega b_1+k^2a_0b_0+ka_0=0,\\ &Y^1: 2\omega b_2+k^2[a_0b_1+a_1b_0]+ka_1-2k^2\alpha a_1=0,\\ &Y^2: -\omega b_1+k^2[a_0b_2+a_1b_1]=0,\\ &Y^3: -2\omega b_2+k^2a_1b_2+2k^2\alpha a_1=0\end{aligned} \tag{31}$$

Substituting U, V in Eq. (28), then equating the coefficient of Y^i, i = 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$\begin{aligned}&\omega a_0+kb_0+\frac{\alpha}{2}a_0^2=0,\\ &\omega a_1+kb_1+\alpha a_0a_1=0,\\ &kb_2+\frac{\alpha}{2}a_1^2=0\end{aligned} \tag{32}$$

Solving the nonlinear system of Eq. (32) we get:

$$\begin{aligned}&a_0=0,\quad b_0=0,\quad b_1=0,\\ &b_2=-\frac{k^3}{2\omega^2\alpha}(1-2k\alpha)^2,\\ &a_1=\frac{k^2}{\omega\alpha}(1-2k\alpha),\\ &\omega=\frac{k^2}{\sqrt{2\alpha}}(1-2k\alpha)\end{aligned}$$

Then:

$$u_1(x,y,t)=\frac{k^2}{\omega\alpha}(1-2k\alpha)\tanh\left\{kx+\alpha y+\frac{k^2}{\sqrt{2\alpha}}(1-2k\alpha)t+\theta_0\right\} \tag{33}$$

$$v_1(x,y,t)=-\frac{k^3}{2\omega^2\alpha}(1-2k\alpha)^2\times\tanh^2\left\{kx+\alpha y+\frac{k^2}{\sqrt{2\alpha}}(1-2k\alpha)t+\theta_0\right\} \tag{34}$$

3.3. Riemann wave equation

The Riemann wave equation is given by

$$u_t+\beta u_{xxy}+4\beta uv_x+4\beta u_xv=0 \tag{35}$$

$$u_y=v_x \tag{36}$$

where β is a known constant. Eqs. (35) and (36) describe the (2 + 1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. In the past years, many authors have studied Eqs. (35) and (36). For instance, the Painleve' property was examined and localized coherent structures were presented [20, 21]. Some soliton-like solutions were obtained by the generalized expansion method of Riccati equation [22]. Recently, a class of periodic wave solutions was obtained by the mapping method [23]. Two classes of new exact solutions were obtained by the singular manifold method [24]. Very recently, Jacobi elliptic function solutions and their degenerate solutions are obtained by a generalized extended F-expansion method [25]. In this section, many new and more general exact solutions by tanh method proposed in Section 2 were introduced. Using the traveling wave transformations in Eqs. (3) and (5) with

$$\xi=kx+\alpha y+\omega t+\theta_0, \tag{37}$$

the nonlinear system of partial differential equations given in Eqs. (35) and (36) is transformed to a system of ODEs as follows

$$\omega U'+\beta k^2\alpha U'''+4\beta kUV'+4\beta kU'V=0 \tag{38}$$

$$\alpha U'=kV' \tag{39}$$

Integrating Eq. (38) once with zero constant and we postulate the tanh series, Eq. (38) reduces to

$$\omega U+\beta k^2\alpha\left\{-2Y(1-Y^2)\frac{dU}{dY}+(1-Y^2)^2\frac{d^2U}{dY^2}\right\}+4\beta kUV=0 \tag{40}$$

Integrating Eq. (39) once with zero constant and we postulate the tanh series, Eq. (39) reduces to

$$\alpha U=kV \tag{41}$$

Now, to determine the parameters m and n we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq. (38) we balance U'' with UV, to obtain $2+m=m+n$, then $n=2$ while in Eq. (39) we balance V with U, then $m=2$. The tanh method admits the use of the finite expansion for both:

$$u(x,t)=U(Y)=a_0+a_1Y+a_2Y^2, \quad a_2\neq 0 \tag{42}$$

and

$$v(x,t)=V(Y)=b_0+b_1Y+b_2Y^2, \quad b_2\neq 0 \tag{43}$$

Substituting U', U'' in Eq. (38), then equating the coefficient of Y^i, i = 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations:

$$\begin{aligned}
&Y^0: \omega a_0+\beta k^2\alpha 2a_2+4\beta a_0b_0=0,\\
&Y^1: \omega a_1-2\beta k^2\alpha a_1+4\beta k[a_0b_1+a_1b_0]=0,\\
&Y^2: \omega a_2-8\beta k^2\alpha a_2+4\beta k[a_0b_2+b_1a_1+a_2b_0]=0,\\
&Y^3: \beta k^2\alpha a_1+2\beta k[a_1b_2+b_2b_1Y^3]=0,\\
&Y^4: 3\beta k^2\alpha a_2+2\beta kb_2^2=0
\end{aligned} \tag{44}$$

Substituting U, V in Eq. (39), then equating the coefficient of Y^i, i = 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$\begin{aligned}
&\alpha a_0=kb_0,\\
&\alpha a_1=kb_1,\\
&\alpha a_2=kb_2
\end{aligned} \tag{45}$$

Solving the nonlinear systems of Eqs. (42) and (43) we get:

Case 1 :

$$a_0=k^2-\frac{\omega}{8\beta\alpha}, \quad a_1=0, \quad a_2=\frac{-3}{2\alpha}k^3,$$
$$b_0=\alpha k-\frac{\omega}{8k\beta}, \quad b_1=0, \quad b_2=\frac{-3}{2}k^2$$

Then:

$$u(x,t)=k^2-\frac{\omega}{8\beta\alpha}-\frac{3}{2\alpha}k^3\tanh^2(kx+\alpha y+\omega t+\theta_0) \tag{46}$$

$$v(x,t)=\alpha k-\frac{\omega}{8k\beta}-\frac{3}{2}k^2\tanh^2(kx+\alpha y+\omega t+\theta_0) \tag{47}$$

Case 2 :

$$a_0=k^2-\frac{\omega}{8\beta\alpha}, \quad a_1=0, \quad a_2=-\frac{k^3}{2(k+\alpha)},$$
$$b_0=\alpha k-\frac{\omega}{8k\beta}, \quad b_1=0, \quad b_2=-k^2\frac{\alpha}{2(k+\alpha)},$$

Then:

$$u(x,t)=k^2-\frac{\omega}{8\beta\alpha}-\frac{k^3}{2(k+\alpha)}\tanh^2(kx+\alpha y+\omega t+\theta_0) \tag{48}$$

$$v(x,t)=\alpha k-\frac{\omega}{8k\beta}-k^2\frac{\alpha}{2(k+\alpha)}\tanh^2(kx+\alpha y+\omega t+\theta_0) \tag{49}$$

4. Traveling wave solutions

In this section the traveling wave hypothesis will be applied to integrate dispersive long wave equation and Riemann wave equation to obtain the soliton solutions. The focus in this section is going to be on the soliton solutions only. The tool of integration in this section is the traveling wave hypothesis.

4.1. Dispersive long wave equation

The (2 + 1)-dimensional dispersive long wave equations, in shallow water, are given by

$$q_{yt}+r_{xx}+\frac{1}{2}(q^2)_{xy}=0, \tag{50}$$

$$r_t+(qr+q+q_{xy})_x=0, \tag{51}$$

where q and r are functions of the spatial variables x, y and the temporal variable t, and subscripts denote partial derivatives. The traveling wave hypothesis are now given by

$$q(x,y,t)=g(B_1x+B_2y-vt)=g(s) \tag{52}$$

$$r(x,y,t)=h(B_1x+B_2y-vt)=h(s) \tag{53}$$

where g and h represent the respective wave profiles while v is the velocity of the wave and

$$s=B_1x+B_2y-vt \tag{54}$$

The parameters B_1 and B_2 are inverse width of the wave in the x - and y -directions respectively. By introducing these traveling wave hypothesis equations given by Eqs. (50) and (51) reduce to

$$-vB_2g''+B_1^2h''+\frac{1}{2}B_1B_2(g^2)''=0, \tag{55}$$

$$-vh'+B_1(gh+g+B_1B_2g'')'=0, \tag{56}$$

Integrating Eq. (55) once gives

$$-vB_2g'+B_1^2h'+\frac{1}{2}B_1B_2(g^2)'=C, \tag{57}$$

where C is an integration constant. Integrating Eq. (57) once again, we obtain

$$-vB_2g+B_1^2h+\frac{1}{2}B_1B_2g^2=C_1, \tag{58}$$

where C_1 is again an integration constant. From Eq. (58), we find that

$$h(s)=\frac{C_1}{B_1^2}+\frac{vB_2}{B_1^2}g-\frac{B_2}{2B_1}g^2. \tag{59}$$

Substituting Eq. (59) into Eq. (56) yields

$$\left[\frac{C_1}{B_1}-\frac{B_2v^2}{B_1^2}+B_1\right]g'+\frac{3vB_2}{B_1}gh'-\frac{3B_2^2}{2}g^2g'+B_1^2B_2g'''=0, \tag{60}$$

which can be rewritten as

$$g''' - \frac{3}{2B_1^2}g^2g' + \frac{3v}{B_1^3}gh' + \frac{1}{B_2B_1^2}\left[\frac{C_1}{B_1} - \frac{B_2v^2}{B_1^2} + B_1\right]g' = 0, \tag{61}$$

Eq. (61) can be integrated with respect to s directly to yield

$$g'' - a_1g + a_2g^2 - a_3g^3 = 0 \tag{62}$$

where the integration constant is taken to be zero and

$$a_1 = \frac{1}{B_2B_1^2}\left[\frac{B_2v^2}{B_1^2} - \frac{C_1}{B_1} - B_1\right], \tag{63}$$

$$a_2 = \frac{3v}{B_1^3}, \tag{64}$$

$$a_3 = \frac{3}{2B_1^2}. \tag{65}$$

Multiplying Eq. (62) by g' and integrating once and then separating variables leads to

$$B_1x + B_2y - vt = \int \frac{dg}{g\sqrt{a_1 - \frac{2a_2}{3}g + \frac{2a_3}{5}g^3}} \tag{66}$$

which after simplification yields

$$B_1x + B_2y - vt = \frac{1}{\sqrt{a_1 - \frac{2}{3}a_2g + \frac{2}{5}a_3g^3}}\frac{H}{Q}\sqrt{\frac{J}{K}}|\frac{M}{N}| \times \prod\left(1 - \frac{P}{Q}; \sin^{-1}\sqrt{-\frac{S}{T}}|\frac{U}{K}\right) \tag{67}$$

where

$$H = 2i\sqrt{3}\left(\frac{4a_2}{3r\sqrt[3]{12}} - \frac{5r}{2a_3\sqrt[3]{18}}\right) \tag{68}$$

$$J = \frac{4a_2}{3r\sqrt[3]{12}} - g + \frac{5r}{2a_3\sqrt[3]{18}} \tag{69}$$

$$K = \frac{2(3 - i\sqrt{3})a_2}{3r\sqrt[3]{12}} + \frac{5(3 + i\sqrt{3})r}{4a_3\sqrt[3]{18}} \tag{70}$$

$$M = \frac{2(1 + i\sqrt{3})a_2}{3r\sqrt[3]{12}} + g + \frac{5(1 - i\sqrt{3})r}{4a_3\sqrt[3]{18}} \tag{71}$$

$$N = -\frac{4a_2\sqrt{3}}{3r\sqrt[3]{12}} + \frac{5\sqrt{3}r}{2a_3\sqrt[3]{18}} \tag{72}$$

$$P = \frac{2(1 + i\sqrt{3})a_2}{3r\sqrt[3]{12}} + \frac{5(1 - i\sqrt{3})r}{4a_3\sqrt[3]{18}} \tag{73}$$

$$Q = \frac{2(1 - i\sqrt{3})a_2}{3r\sqrt[3]{12}} + \frac{5(1 + i\sqrt{3})r}{4a_3\sqrt[3]{18}} \tag{74}$$

$$S = \frac{2(1 - i\sqrt{3})a_2}{3r\sqrt[3]{12}} + g + \frac{5(1 + i\sqrt{3})r}{4a_3\sqrt[3]{18}} \tag{75}$$

$$T = \frac{4ia_2\sqrt{3}}{3r\sqrt[3]{12}} + \frac{5r\sqrt{3}}{2a_3\sqrt[3]{18}} \tag{76}$$

$$U = -\frac{4ia_2\sqrt{3}}{3r\sqrt[3]{12}} + \frac{5ir\sqrt{3}}{2a_3\sqrt[3]{18}} \tag{77}$$

$$r = \left[4\sqrt{\frac{81}{625}a_1^2a_3^4 - \frac{2}{9}a_2^3a^3} - \frac{36}{25}a_1a_3^2\right]^{\frac{1}{3}} \tag{78}$$

and Π is the incomplete elliptic integral of the third kind that is defined as

$$\Pi(n;\phi|k) = \int_0^{\sin\phi} \frac{dt}{(1 - nt^2)\sqrt{(1 - t^2)(1 - k^2t^2)}} \tag{79}$$

or as

$$\Pi(n;\phi|\alpha) = \int_0^{\phi} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{(1 - \sin^2\theta\sin^2\alpha)}} \tag{80}$$

4.2. Riemann wave equation

The coupled Riemann wave equation is rewriten in this subsection as

$$q_t + aq_{xxy} + bqr_x + cq_xr = 0 \tag{81}$$

$$q_y = r_x \tag{82}$$

In order to solve Eqs. (81) and (82), the traveling wave hypothesis is taken as in the previous subsection. Therefore, subsituting these hypotheses into Eqs. (81) and (82) leads to the ODEs given by

$$-vg' + aB_1^2B_2g''' + bB_2gh' + cB_1g'h = 0 \tag{83}$$

$$B_2g' = B_1h' \tag{84}$$

Eliminating h between Eqs. (83) and (84) and integrating once leads to the following ODE for g

$$g'' = \frac{v}{aB_1^2B_2}g - \frac{bB_2^2 + cB_1B_2}{2aB_1^2B_2}g^2 \tag{85}$$

On multiplying both sides of Eq. (85) by g' and taking the integration constant to be zero gives

$$g' = \sqrt{\frac{bB_2^2 + cB_1B_2}{3aB_1^3B_2}}g\sqrt{\lambda - g} \tag{86}$$

where

$$\lambda = \frac{3vB_1}{bB_2^2 + cB_1B_2} \tag{87}$$

Separating variables in Eq. (86) leads to

$$\int \frac{dg}{g\sqrt{\lambda - g}} = \sqrt{\frac{bB_2^2 + cB_1B_2}{3aB_1^3B_2}}(B_1x + B_2y - vt) \tag{88}$$

which upon integration yields the soliton solutions

$$g(x,y,t) = A_1 sech^2[B(B_1x + B_2y - vt)] \tag{89}$$

$$h(x,y,t) = A_2 sech^2[B(B_1x + B_2y - vt)] \tag{90}$$

where the amplitudes A_1 and A_2 are respectively

$$A_1 = \lambda = \frac{3vB_1}{bB_2^2 + cB_1B_2} \tag{91}$$

$$A_2 = \frac{\lambda B_2}{B_1} = \frac{3vB_2}{bB_2^2 + cB_1B_2} \tag{92}$$

and the parameter B is given by

$$B = \frac{1}{2B_1}\sqrt{\frac{v}{\lambda B_2}} \tag{93}$$

Eq. (93) requires the constraint condition

$$\lambda v > 0 \tag{94}$$

and in terms of original notations as

$$v(bB_2 + cB_1) > 0 \tag{95}$$

to hold in order for the soliton solutions to exist.

5. Conclusions

This paper integrates a few of the nonlinear wave equations in (2 + 1)-dimensions by the aid of tanh method as well as using the traveling wave hypothesis. These lead to several kinds of solutions including the soliton solutions, Jacobi's elliptic function of the third kind as well as several other solutions. These results are going to be extremely useful in various areas of applied mathematics and theoretical physics wherever there is a study of soliton theory.

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