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APPLICATION ARTICLE

Joint module importance measures for multistate systems

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Abstract
In reliability engineering, the use of importance and joint importance measures to identify the weak components or module of a system and signify the roles of components/modules in either causing or contributing to proper functioning of the system, is crucial. Systems are made up of different modules. This paper introduces, Joint Module Reliability Achievement Worth, Joint Module Reliability Reduction Worth, Joint Module Reliability Fussell-Vesely measure and analogues joint risk importance measures for three multistate components of a multistate system. A steady state performance level distribution with restriction to the component's states is used to evaluate the proposed measures. Universal generating function technique is applied for the evaluation of proposed joint importance measures. An illustrative example is provided.

Keywords Multistate system · Reliability · Joint importance measure · Universal generating function

1 Introduction

In early reliability literature, components and systems are considered binary, Barlow and Proschan [1]. But this is an oversimplification, while a plenty of reliability analysis method is now available for complex multistate system(MSS)s, a system having more than two levels of performance. But MSS approach invites more mathematical complexity in reliability analysis. If we consider a power generation system, which produces 100MW in first stage, 75 MW in second stage and 50MW in third stage, we can model it as a MSS. A detailed presentation useful for the analysis on MSSs can be seen in Lisnianski and Levitin [13] and Natvig [14].

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This is more realistic than binary systems. One of the important research problems in the analysis of complex systems is the determination of *most important component or group of components* with regard to the variation in values of various system performance measures like reliability or availability or unreliability/risk or unavailability etc with respect to the variation of corresponding measures of components or group of components. From the traditional analysis, the determination of most important component or group of components is carried out by inspecting improvement in system performance measure due to the improvement in component performance measure. As we know, for a coherent system, the improvement of component performance through repair or maintenance activities enhances the system performance. In reliability study, the system is usually considered to be made up of various modules, in which each module consists of two or more component. A computer system contains various modules like Central Processing Unit (CPU), Keyboard, Mouse, Monitor etc. Each of them is module. An assembly of the modules is exactly a computer. In defense research, a missile system consists of Propulsion module, Control & Guidance module, warhead module, wings module etc. In most of the engineering systems, system is made up of various modules. One of the important strategies in reliability engineering is the investigation of reliability and importance of module in functioning of system consists of different modules, [12].

The investigation of variation in system performance via the variation in joint performance of components in modules is very useful in system performance improvement activities. It should be carried out before launching the system. In order to order the components of the system in reliability engineering, there are several importance measures available in literature (Birnubaum (1969) and Barlow and Proschan [1]. Barlow and Wu [2] and Bueno [5] discussed the importance measures of multistate systems, which helps the system designers in improving system performance. Interaction importance of groups of components, with respect to various output performance measure(OPM)s, reliability, expected output performance, risk etc. is more helpful to the designers, engineers and managers to arrive at a decision Wu [16]. Chacko [8], measured the change in system performance measure based on the change in first component performance, then change in second component and then change in third component. Chacko [9] investigated joint importance of three components of MSS by investigating change in joint importance of two components by keeping third component below and above some specified levels. More details on fundamental developments of joint importance measures can be seen in [6] and [7].

In risk informed applications of the nuclear industry, Cheok, Parry, and Sherry [11] and Borst and Shoonakker [4]. Borgonovo and Apostolakis [3] studied the role of importance measures. Joint risk importance measures are proposed by Chacko [9]. The joint importance measures of three components for MSS with respect to various OPMs like reliability, expected output performance and risk to the existing measures of importance are discussed in literature in the Birnbaum sense Chacko [9, 10].

But, measuring the role of interaction of components in a module is an unexplored one. In this paper, for module consisting of three components of binary and MSS, Joint Module Reliability Achievement Worth (JMRAW), Joint Module Reliability Reduction Worth (JMRRW), and Joint Module Reliability Fussel-Vesely (JMRFV) importance measures are introduced by considering modules and generalized to expected output performance measure, availability etc. Joint Module Risk Achievement Worth (JMrAW), Joint Module Risk Reduction Worth (JMrRW), and Joint Module Risk-Fussel-Vesely (JMrFV) importance measures are also introduced for modules consisting of three components. JMRAW measures the reliability achievement when interaction effect of three components in a module change from lower level to higher level, JMRRW measures the reliability reduction of system when interaction effect of three components in a module change from higher level to lower level and JMRFV measures the fractional contribution in improving reliability of system by interaction effect of three components of module.

A steady state performance level distribution for the system is considered for obtaining the proposed measures. The information derived by these joint importance measures allows the analyst to judge, based on their interaction effect of three components of a module for system OPM improvement, how to give reliability operations to the module.

When the components $_i$, j and q are restricted in their performance with respect to performance thresholds α, β and δ respectively, let, $OPM_{i,j,q}^{>\alpha,>\beta,>\delta}$, $OPM_{i,j,q}^{>\alpha,>\beta,<\delta}$, $OPM_{i,j,q}^{>\alpha,<\beta,>\delta}$, $OPM_{i,j,q}^{>\alpha,<\beta,<\delta}$, $OPM_{i,j,q}^{<\alpha,>\beta,>\delta}$, $OPM_{i,j,q}^{<\alpha,>\beta,<\delta}$, $OPM_{i,j,q}^{<\alpha,<\beta,>\delta}$, and $OPM_{i,j,q}^{<\alpha,<\beta,<\delta}$ arestate space restricted OPMs. In some complex systems like power generation, oil transportation systems etc., the performance measure of series system will be sum performance measure of components, hence UGF method is found to be useful to evaluate system performance.

In the following Sect. 2, the performance measures of the MSS are reviewed. In Sect. 3, new joint importance measures for three components of the binary and MSS are introduced. In Sect. 4, an approach of element performance restriction for the evaluation of performance measures is given. Also, a technique for joint importance measures evaluation based on the UGF method, is given. Results and discussion are given in Sect. 5. Numerical example is provided in Sect. 6. Conclusion is given in Sect. 7.

2 Performance measures of a multistate system

Consider the structure function of a MSS at time t. Let $\varphi(X(t)) = i$, $i \in \{0, 1, 2, ..., M\}$, where $X(t) = (X_1(t), X_2(t), ..., X_n(t)), X_i(t) \in \{0, 1, 2, ..., M_i\}$, and $M = \max_{1 \le i \le n}\{M_i\}$. The output performance of the MSS at time t is denoted by $W(t)$, where $W(t) \in \{w_i, i = 0, 1, ..., M\}$ and w_i is the performance corresponding to the system state $\varphi(X(t)) = i$. Let

$$p_i = \lim_{t\to\infty}\Pr\{W(t) = w_i\} = \lim_{t\to\infty} \Pr\{\varphi(X(t)) = i\}, 0 \le i \le M.$$

Clearly p_i represent the steady-state probability distribution of the MSS states. Then the steady state performance distribution of the output performance of system, $\mathbf{w} = \{w_i, 0 \le i \le M\}$ is represented by $\mathbf{p} = \{p_i, 0 \le i \le M\}$. With the steady state distribution, expected value is

$$E(W) = \sum_{i=0}^{M} p_i w_i. \tag{1}$$

and expected system state is

$$E_s(\varphi(X)) = \sum_{i=0}^{M} i p_i. \tag{2}$$

MSS reliability for constant demand D_k, to state k of the system is

$$R(t) = Pr\{W(t) \ge D_k\} = Pr\{\varphi(t) \ge k\}. \tag{3}$$

From (2.1) and (2.3), the stationary reliability is

$$R(D_k) = \sum_{i=0}^{M} p_i 1(w_i - D_k). \tag{4}$$

The system risk, F, at time t, is

$$F(t) = Pr\{W(t) \le D_k\} = Pr\{\varphi(X(t)) \le k\} \tag{5}$$

which represents the unreliability or unavailability. At steady state, the risk metric is

$$F = \lim_{t \to \infty} Pr\{W(t) \le D_k\}. \tag{6}$$

3 New joint importance measures for three components in the MSS

To understand the role of interaction effect of three components in a module, three joint importance measures are proposed below. Suppose now the components are statistically independent and reliabilities are known.

3.1 Joint module reliability achievement worth (JMRAW)

Reliability achievement worth (RAW) is useful in characterizing reliability properties of components and helpful for improvement in system reliability. RAW measures the worth in achieving present level of reliability. With reference to module, to improve the existing level of reliability, the modules having highest reliability achievement worth will be most important. To measure the increase in reliability by the presence or functioning or switching to functioning states of a module, the reliability achievement worth has to be measured. Apart from measuring RAW of components, the knowledge of contributions of interaction of components in a module of two or more components will be helpful to the designers and engineers.

To measure the role of interaction of components in a module consisting k components, in increasing reliability of system, define the following:

Let R^+_{m1} = The increased reliability level by the high level of interaction effect of components of module and

$$R_0 = \mathit{The\ present\ reliability\ level.}$$

Joint Module Reliability achievement worth (JMRAW) of the interaction effect of components of module $m1$ is defined as: $JMRAW_{m1} = \frac{R^+_{m1}}{R_0}$

Let $c{i_j}^+$ represent the event that, i_j th component is in functioning states or up states and $c{i_j}^-$ represent the event that, i_j th component is in unreliable states or down states. Let, $I_i = \left(ci^+ - ci^-\right), i = 1, 2, \dots, n$ and $I_{12} = \left(c1^+ - c1^-\right)\left(c2^+ - c2^-\right)$, that is

$I_{12} = \left(c1^+ - c1^-\right)c2^+ - \left(c1^+ - c1^-\right)c2^- = I^+_{12} - I^-_{12}$ which represent the contrast of interaction effect of the components 1 and 2, while they switch from reliable states to down states, where contrasts

$$I^+_{12} = \left(c1^+ - c1^-\right)c2^+ \text{represent is the high level interaction of component 1 and 2}$$

and

$I^-_{12} = \left(c1^+ - c1^-\right)c2^-$represent is the low level interaction of component 1 and 2. Similarly, define contrasts for higher order interaction events,

$$I_{123} = \left(c1^+ - c1^-\right)\left(c2^+ - c2^-\right)\left(c3^+ - c3^-\right)$$

$$= \left[\left(c1^+ - c1^-\right)c2^+ - \left(c1^+ - c1^-\right)c2^-\right]c3^+ - [\left(c1^+ - c1^-\right)c2^+ - \left(c1^+ - c1^-\right)c2^-]c3^-$$

$$= I^+_{123} - I^-_{123}$$

where $I^+_{123} = \left[\left(c1^+ - c1^-\right)c2^+ - \left(c1^+ - c1^-\right)c2^-\right]c3^+$ and

$$I^-_{123} = \left[\left(c1^+ - c1^-\right)c2^+ - \left(c1^+ - c1^-\right)c2^-\right]c3^-.$$

Now let, $B_i = \left(ci^+ - ci^-\right), i = 1, 2, \dots, n.$ Then, $I_{12} = B_1B_2$, where $I^+_{12} = B_1c2^+$, $I^-_{12} = B_1c2^-$, and $I_{123} = B_1B_2B_3 = I^+_{123} - I^-_{123}$, where $I^+_{123} = B_1B2c3^+$, and $I^-_{123} = B_1B2c3^-$.

Proceeding like this, we can represent $I_{\{123\ldots n\}} = B_1B_2 \dots .B_n = I^+_{\{123\ldots n-1\}} - I^-_{\{123\ldots n-1\}}$

where $I^+_{\{123\ldots n-1\}} = (B_1B_2 \dots .B_{n-1})cn^+, I^-_{\{123\ldots n-1\}} = (B_1B_2 \dots .B_{n-1})cn^-$

Again, let $I^+_{\{123\ldots i\ldots n\}} = (B_1B_2 \dots B_{i-1.}B_{i+} \dots ..B_n)ci^+$, and $I^-_{\{123\ldots i\ldots n\}} = (B_1B_2 \dots B_{i-1.}B_{i+} \dots ..B_n)ci^-$

For any k integers, $i_1, i_2, \dots, i_k$ define,

$I^+_{\{i_1,i_2,\ldots,i_k\}} = (B_{i_1}B_{i_2} \dots B_{i_{k-1}.})c{i_k}^+$ and $I^-_{\{i_1,i_2,\ldots,i_k\}} = (B_{i_1}B_{i_2} \dots B_{i_{k-1}.})c{i_k}^-$

Define the following $\partial R_\varphi(i) = P\left(\varphi(X(t)) = 1, {I_i}^+\right) - P\left(\varphi(X(t)) = 1, {I_i}^-\right)$

$$= P\left(\varphi(X(t)) = 1, X_i(t) = 1\right) - P\left(\varphi(X(t)) = 1, X_i(t) = 0\right)$$

i = 1,2,…,n, which is the change in reliability of component *i*. For two components, i and j, define, $\partial R_{\varphi}(i,j) = \partial R\left(\varphi(X(t)) = 1, I_{ij}{}^{+}\right) - \partial R\left(\varphi(X(t)) = 1, I_{ij}{}^{-}\right)$

$$= \left[P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 1\right) - P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 1\right)\right]$$

$$-\left[P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 0\right) - P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 0\right)\right],$$

Which is the joint Birnbaum reliability importance of components *i* and *j.*

$$\partial R_{\varphi}(i,j,k) = \partial R\left(\varphi(X(t)) = 1, I_{ijk}{}^{+}\right) - \partial R\left(\varphi(X(t)) = 1, I_{ijk}{}^{-}\right) =$$

$$\left[P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 1, X_k(t) = 1\right) - P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 1, X_k(t) = 1\right)\right]$$

$$-\left[P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 0, X_k(t) = 1\right) - P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 0, X_k(t) = 1\right)\right]$$

$$-\left[P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 1, X_k(t) = 0\right) - P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 1, X_k(t) = 0\right)\right]$$

$$+\left[P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 0, X_k(t) = 0\right) - P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 0, X_k(t) = 0\right)\right],$$

Which is the joint Birnbaum reliability importance of components *i, j* and k.

Now define JMRAW of three components. Let.

$$R_{\{i+,j+,k+\}} = P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 1, X_k(t) = 1\right),$$

$$R_{\{i-,j+,k+\}} = P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 1, X_k(t) = 1\right),$$

$$R_{\{i+,j-,k+\}} = P\left(\varphi(X(t)) = 1, X_i(t) = 1, X_j(t) = 0, X_k(t) = 1\right),$$

and

$$R_{\{i-,j-,k+\}} = P\left(\varphi(X(t)) = 1, X_i(t) = 0, X_j(t) = 0, X_k(t) = 1\right).$$

Then, for a Module consisting of three components, *i, j* and k,

$JMRAW_{i,j,k}$ = Maximum Reliability due to high level interaction effect of three components of a module/The present reliability level

Let $R_0 = Present\ reliability\ level$

$$JMRAW_{i,j,k} = \frac{\left[R_{\{i+,j+,k+\}} - R_{\{i-,j+,k+\}}\right] - \left[R_{\{i+,j-,k+\}} - R_{\{i-,j-,k+\}}\right]}{R_0}.$$

The JMRAW measure quantifies the maximum possible achievement of reliability due to interaction effect of component *i, j* and *k* which switches from lower level to higher.

level. Similarly define JMRAW of *k* binary components $i_1, i_2, \dots, i_{k,}$ as

$$JMRAW_{i_1,i_2,\ldots,i_k} = \frac{\left[R_{\{i_1+,i_2+,\ldots,i_k+\}} - R_{\{i_1-,i_2+,\ldots,i_k+\}} - \ldots - R_{\{i_1+,i_2+,\ldots,i_{k-1}-,i_k+\}} + R_{\{i_1-,i_2-,\ldots,i_{k-1}-,i_k+\}}\right]}{R_0}$$

For *i*th multistate component with performance threshold α, let $k_{i\alpha}$ be the state in the ordered set of states of component i such that $x_{ik_{i\alpha}} \le \alpha < x_{ik_{i\alpha}+1.}$ For a constant demand D_k, to define Multistate Joint Module Reliability Achievement Worth (JMRAW) of components *i, j* and q, let,

$$R_{\{i^{\ge\alpha},j^{\ge\beta},q^{\ge\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) \ge x_{ik_{i\alpha}}, X_j(t) \ge x_{jk_{j\beta}}, X_q(t) \ge x_{qk_{q\delta}}\Big),$$

$$R_{\{i^{<\alpha},j^{\ge\beta},q^{\ge\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) < x_{ik_{i\alpha}}, X_j(t) \ge x_{jk_{j\beta}}, X_q(t) \ge x_{qk_{q\delta}}\Big),$$

$$R_{\{i^{\ge\alpha},j^{<\beta},q^{\ge\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) \ge x_{ik_{i\alpha}}, X_j(t) < x_{jk_{j\beta}}, X_q(t) \ge x_{qk_{q\delta}}\Big)$$

$$R_{\{i^{\ge\alpha},j^{\ge\beta},q^{<\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) \ge x_{ik_{i\alpha}}, X_j(t) \ge x_{jk_{j\beta}}, X_q(t) < x_{qk_{q\delta}}\Big),$$

$$R_{\{i^{<\alpha},j^{\ge\beta},q^{<\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) < x_{ik_{i\alpha}}, X_j(t) \ge x_{jk_{j\beta}}, X_q(t) < x_{qk_{q\delta}}\Big),$$

$$R_{\{i^{\ge\alpha},j^{<\beta},q^{<\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) \ge x_{ik_{i\alpha}}, X_j(t) < x_{jk_{j\beta}}, X_q(t) < x_{qk_{q\delta}}\Big),$$

$$R_{\{i^{<\alpha},j^{<\beta},q^{\ge\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) < x_{ik_{i\alpha}}, X_j(t) < x_{jk_{j\beta}}, X_q(t) \ge x_{qk_{q\delta}}\Big)$$

and

$$R_{\{i^{<\alpha},j^{<\beta},q^{<\delta}\}} = P\Big(\varphi(X(t)) \ge k, X_i(t) < x_{ik_{i\alpha}}, X_j(t) < x_{jk_{j\beta}}, X_q(t) < x_{qk_{q\delta}}\Big).$$

where α is the performance threshold and $x_{ik_{i\alpha}}$ performance of component i in state $k_{i\alpha}$, β is the performance threshold and $x_{jk_{j\beta}}$ is the performance of component j in the state $k_{j\beta}$,, δ is the performance threshold and $x_{qk_{q\delta}}$ is the performance of component q in the state $k_{q\delta}$, *i, j, k* = *1,2,…,n*. Thus, following this, JMRAW of three components *i, j* and *k* can be defined as,

$$JMRAW_{i,j,k} = \frac{\left[R_{\{i\ge,j\ge,k\ge\}} - R_{\{i<,j\ge,k\ge\}}\right] - \left[R_{\{i\ge,j<,k\ge\}} - R_{\{i<,j<,k\ge\}}\right]}{R_0} \tag{7}$$

JMRAW measures the reliability achievement worth of joint effect of three components in a module.

3.2 Joint module reliability reduction worth (JMRRW)

To measure the role of interaction effect of components of a module in reducing the present reliability, Joint Module Reliability Reduction Worth (JMRRW) is introduced in this section. To examine how the decrease in reliability happens by the decreased level or low level of interaction effect of components in a module, JMRRW of a module can be defined as follows.

Let

$$R_{m1}^{-} = \textit{The decreased reliability level by the low level of interaction of components}$$
$$\textit{in a module or when module is perfectly not reliable},$$

$$\textit{and}$$

$R_0 = \textit{Present reliability level}$. The JMRRW of a module is defined as:

$$JMRRW_{m1} = \frac{R_0}{R_{m1}^{-}}$$

JMRRW of three binary components *i, j* and *k* is

$$JMRRW_{i,j,k} = \frac{\textit{Present Reliability Level}}{\textit{Reliability when interaction of module is in low level or module is perfectly not reliable}}$$

$$= \frac{R_0}{\left[R_{\{i+,j+,k-\}} - R_{\{i-,j+,k-\}}\right] - \left[R_{\{i+,j-,k-\}} - R_{\{i-,j-,k-\}}\right]}$$

The *JMRRW* measure of a module consisting of components *i, j* and *k,* quantifies the maximum possible reduction of reliability due to low level of interaction effect of component *i, j* and *k.*

Similarly define *JMRRW* of k binary components $i_1, i_2, \ldots, i_k$ as

$$JMRRW_{i_1,i_2,\ldots,i_k} = \frac{R_0}{\left[R_{\{i_1+,i_2+,\ldots,i_k-\}} - R_{\{i_1-,i_2+,\ldots,i_k-\}} - \ldots - R_{\{i_1+,i_2-,\ldots,i_{k-1}-,i_k-\}} + R_{\{i_1-,i_2-,\ldots,i_{k-1}-,i_k-\}}\right]}$$

For a constant demand D_k, Multistate Joint Module Reliability Reduction Worth (JMRRW) of a module consisting of three components *i, j* and *k* is defined as,

$$= \frac{R_0}{\left[R_{\{i\geq\alpha,j\geq\beta,k<\delta\}} - R_{\{i<\alpha,j\geq\beta,k<\delta\}}\right] - [R_{\{i\geq\alpha,j<\beta,k<\delta\}} - R_{\{i<\alpha,j<\beta,k<\delta\}}]}.$$

JMRRW measures the reliability reduction worth of a module consisting of three components i, j and k.

3.3 Joint module reliability Fussel–Vesely (JMRFV) measure

To measure the fractional contribution of interaction effect of components of module m1 to the increase of reliability, Fussel-Vesly (FV) measure of reliability importance can be used. JMRFV measure can be expressed as, $FV_{m1} = \frac{R_0 - R^-_{m1}}{R_0}$. If there are k components in a module,

$$JMRFV_{i_1,i_2,\ldots,i_k} = \frac{Present\ Reliability\ Level - Reliability\ when\ interaction\ of\ module\ is\ in\ low\ level}{Present\ Reliability\ Level}$$

$$= \frac{R_0 - \left[R_{\{i+,j+,k-\}} - R_{\{i-,j+,k-\}}\right] - \left[R_{\{i+,j-,k-\}} - R_{\{i-,j-,k-\}}\right]}{R_0}$$

Similarly define JMRFV of k binary components $i_1, i_2, \ldots, i_k$ as

$$JMRFV_{i_1,i_2,\ldots,i_k} = \frac{R_0 - \left[R_{\{i_1+,i_2+,\ldots,i_k-\}} - R_{\{i_1-,i_2+,\ldots,i_k-\}} - \ldots. - R_{\{i_1+,i_2-,\ldots,i_{k-1}-,i_k-\}} + R_{\{i_1-,i_2-,\ldots,i_{k-1}-,i_k-\}}\right]}{R_0}$$

For a constant demandD_k, Multistate Joint Module Reliability Fussel-Vesely (JBRFV) of module consisting of three components *i, j*, and *k* is defined as,

$$= \frac{R_0 - \left[R_{\{i^{\geq\alpha},j^{\geq\beta},k^{<\delta}\}} - R_{\{i^{<\alpha},j^{\geq\beta},k^{<\delta}\}}\right] - [R_{\{i^{\geq\alpha},j^{<\beta},k^{<\delta}\}} - R_{\{i^{<\alpha},j^{<\beta},k^{<\delta}\}}]}{R_0}$$

Multistate JMRFV measures the reliability FV of a module consisting of three components.

Instead of reliability, when the performance measure changes, like expected output performance, availability etc., define Multistate Joint Module Output Performance Measure Achievement Worth(MJMOPMAW), Multistate Joint Module Output Performance Measure Reduction Worth (MJMOPMRW) and Multistate Joint Module Output Performance Measure Fussel-Vesely (MJMOPFV) measures as below.

$$MJMOPMAW_{i,j,k} = \frac{\left[OPM_{\{i\geq,j\geq,k\geq\}} - OPM_{\{i<,j\geq,k\geq\}}\right] - \left[OPM_{\{i\geq,j<,k\geq\}} - OPM_{\{i<,j<,k\geq\}}\right]}{OPM}$$

$$MJMOPMRW_{i,j,k} = \frac{OPM}{\left[OPM_{\{i^{\geq\alpha},j^{\geq\beta},k^{<\delta}\}} - OPM_{\{i^{<\alpha},j^{\geq\beta},k^{<\delta}\}}\right] - [OPM_{\{i^{\geq\alpha},j^{<\beta},k^{<\delta}\}} - OPM_{\{i^{<\alpha},j^{<\beta},k^{<\delta}\}}]}$$

$$MJMOPMFV_{i,j,k} = \frac{OPM - \left[OPM_{\{i^{\geq\alpha},j^{\geq\beta},k^{<\delta}\}} - OPM_{\{i^{<\alpha},j^{\geq\beta},k^{<}\}}\right] - \left[OPM_{\{i^{\geq\alpha},j^{<\beta},k^{<\delta}\}} - OPM_{\{i^{<\alpha},j^{<\beta},k^{<\delta}\}}\right]}{OPM}$$

3.4 Joint module risk achievement worth (JMrAW)

A measure of risk importance is risk achievement worth. It measures the worth in achieving present level of risk. To keep the existing level of risk, the modules having highest risk achievement worth will be of most important. To measure the increase in risk by the absence or failure or switching to failed state of a module, the risk achievement worth has to be measured. To measure the role of interaction effect of *k* components in a module, in increasing risk of system, define the following:

Let F^{-}_{m1} = *The increased* risk *level by down of interaction effect of components in module* $m1$ *or by the low level of interaction effect of module* $m1$, and F_0 = *The present risk level.*

Risk achievement worth due to the interaction effect of components of module *m1* is defined as: $JMrAW_{m1} = \frac{F^{-}_{m1}}{F_0}$

For three components i, j and k,

$$JMrAW_{i,j,k} = \frac{\left[F_{\{i-,j-,k-\}} - F_{\{i+,j-,k-\}}\right] - \left[F_{\{i-,j+,k-\}} - F_{\{i+,j+,k-\}}\right]}{F_0}$$

For k components $i_1, i_2, \dots, i_k$

$$JMrAW_{i_1,i_2,\dots,i_k} = \frac{\left[F_{\{i_1-,i_2-,\dots,i_k-\}} - F_{\{i_1+,i_2-,\dots,i_k-\}}\right] - \cdots - \left[F_{\{i_1-,i_2-,\dots,i_{k-1}+,i_k-\}} - F_{\{i_1+,i_2+,\dots,i_k-\}}\right]}{F_0}$$

The Multistate JMrAW measure quantifies the maximum possible achievement of risk due to low level interaction effect of component *i, j* and *k* which switches from high level to low level.

3.5 Joint module risk reduction worth (JMrRW)

To measure the role of interaction effect of components of a module in reducing the present risk, **Joint Module Risk Reduction Worth (JMrRW) is** proposed. By making the module highly reliable, one can reduce the risk. To examine how the decrease in risk, risk reduction worth of a module can be defined as follows.

Let

F^{+}_{m1} = The decreased risk level by the presence of high level of interaction of components in a module

Let F_0 be the present risk level. The risk reduction worth of a module is defined as:

$$RRW_{m1} = \frac{F_0}{F^{+}_{m1}}$$

For three components *i, j* and *k,*

$$JMrRW_{i,j,k} = \frac{F_0}{\left[F_{\{i-,j-,k+\}} - F_{\{i+,j-,k+\}}\right] - \left[F_{\{i-,j+,k+\}} - F_{\{i+,j+,k+\}}\right]}$$

For k components $i_1, i_2, \dots, i_k$

$$JMrRW_{i_1,i_2,\dots,i_k} = \frac{F_0}{\left[F_{\{i_{1-},i_{2-},\dots,i_{k+}\}} - F_{\{i_1+,i_{2-},\dots,i_k+\}}\right] - \dots - \left[F_{\{i_1-,i_2+,\dots,i_k+\}} - F_{\{i_1+,i_2+,\dots,i_k+\}}\right]}$$

The Multistate JMrRW is an index measuring the maximum decrease in system risk when interaction effect of components *i*, *j* and *k* of a module switch from high level to low level.

3.6 Joint module reliability Fussel–Vesely (JMrFV) measure

To measure the fractional contribution of interaction effect of components of module m1 to the increase of risk, Fussel-Vesly measure of risk importance can be expressed as, $\mathrm{FV_{m1}} = \frac{\mathrm{F_0 - F_{m1}^-}}{\mathrm{F_0}}$.

$$JMrFV_{i_1,i_2,\dots,i_k} = \frac{Present\ Risk\ Level - Risk\ when\ interaction\ of\ module\ is\ in\ low\ level}{Present\ Risk\ Level}$$

Risk FV Worth due to the interaction effect of components of module *m1* is defined as: $JMrFV_{m1} = \frac{\mathrm{F_0 - F_{m1}^-}}{\mathrm{F_0}}$

For three components *i, j* and *k*,

$$JMrFV_{i,j,k} = \frac{F_0 - \left[F_{\{i-,j-,k-\}} - F_{\{i+,j-,k-\}}\right] - \left[F_{\{i-,j+,k-\}} - F_{\{i+,j+,k-\}}\right]}{F_0}$$

For k components $i_1, i_2, \dots, i_k$

$$JMrFV_{i_1,i_2,\dots,i_k} = \frac{F_0 - \left[F_{\{i_{1-},i_{2-},\dots,i_{k-}\}} - F_{\{i_1+,i_2-,\dots,i_k-\}}\right] - \dots - \left[F_{\{i_1-,i_2-,\dots,i_{k-1}+,i_k-\}} - F_{\{i_1+,i_2+,\dots,i_k-\}}\right]}{F_0}$$

The Multistate *JMrFV* importance measure quantifies the fractional increase of risk due to interaction effect of components of reliability when interaction effect of component *i, j* and *k* switches from high level to low level.

4 Evaluation procedure

A component's performance restriction approach is useful for computation of the joint importance measures and for the evaluation procedure UGF method can be adopted. The coefficients of UGFs are used for the evaluation of joint importance measures using various OPMs, see Chacko [9].

5 Result discussion

In binary and multistate context, the proposed measures quantifies the RAW, RRW and FV measures of a module consisting of three components and generalized to *k* components.

Many of the complex systems are made up of different modules having two or more components. JMRAW measures the reliability achievement when interaction effect of three components in a module change from lower level to higher level, JMRRW measures the reliability reduction of system when interaction effect of three components in a module change from higher level to lower level and JMRFV measures the fractional contribution of interaction effect of three components of module. Using the information of JMRAW, it is easy to understand and identify the module with highest contribution to system reliability improvement. JMRRW provides the information regarding the module which induce lowest reduction in system reliability with lower level of module performance. The fractional contribution in reliability improvement of a module can be measured using JMRFV.

MJMOPMAW, MJMOPMRW and MJMOMPFV measures are useful when a researcher use output performance measures like expected output performance measure, reliability, availability etc. In order to apply in risk informed applications, Multistate JMrAW, JMrRW and JMrFV measures are proposed. The proposed measures can be used to apply reliability improvement activities in order on engineering systems.

6 Illustrative example

Consider a system made up of n = 4 multi-state components in series logic, see Chacko [9]. Component states are 0, 1, 2, 3 and 4, with corresponding values of performance $x_{j0} = 0$, $x_{j1} = 25$, $x_{j2} = 50$, $x_{j3} = 75$, $x_{j4} = 100$, j = 1, 2, 3,4 (see Fig. 1).

The probability distribution of component j in state k,p_{jk}, are given in Table 1. Let 0, 1 and 2 are un-reliable states for $< \alpha$ or $< \beta$ or.

$\delta < \delta$ and 3 and 4 ate reliable states for $\geq \alpha$ or $\geq \beta$, $\geq \beta$ or $\geq \delta$.

Multistate joint importance measures are computed and given in Table 2 and plotted in Fig. 2. A numerical comparison can be made using the sign and size of the value of joint importance measure with regard to their impact on expected system output performance.

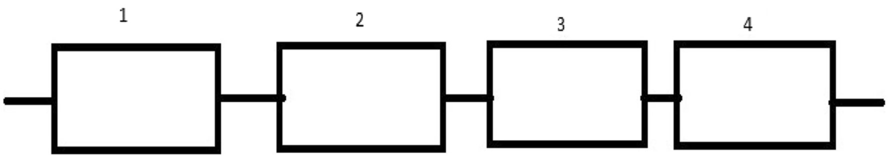

Fig. 1 Series system

Table 1 Probability distributions of components 1, 2, 3 and 4

Probability distribution	Component number							
	1		2		3		4	
P(Xi0 = 0)	p10	= 0.5	p20	= 0.45	p30	= 0.4	p40	= 0.45
P(Xi1 = 25)	p11	= 0	p21	= 0	p31	= 0.1	p41	= 0
P(Xi2 = 50)	p12	= 0	p22	= 0.1	p32	= 0	p42	= 0.1
P(Xi3 = 75)	p13	= 0	p23	= 0	p33	= 0.1	p43	= 0
P(Xi4 = 100)	p14	= 0.5	p24	= 0.45	p34	= 0.4	p44	= 0.45

Table 2 Multistate joint importance measures

For components 1, 2, 3	For components 2, 3, 4
MJMEOPAW= 0.2424	MJMEOPAW=− 1.73171
MJMEOPRW= 1.5107	MJMEOPRW= 2.141827
MJMEOPFV = 0.9596	MJMEOPFV = 0.533109

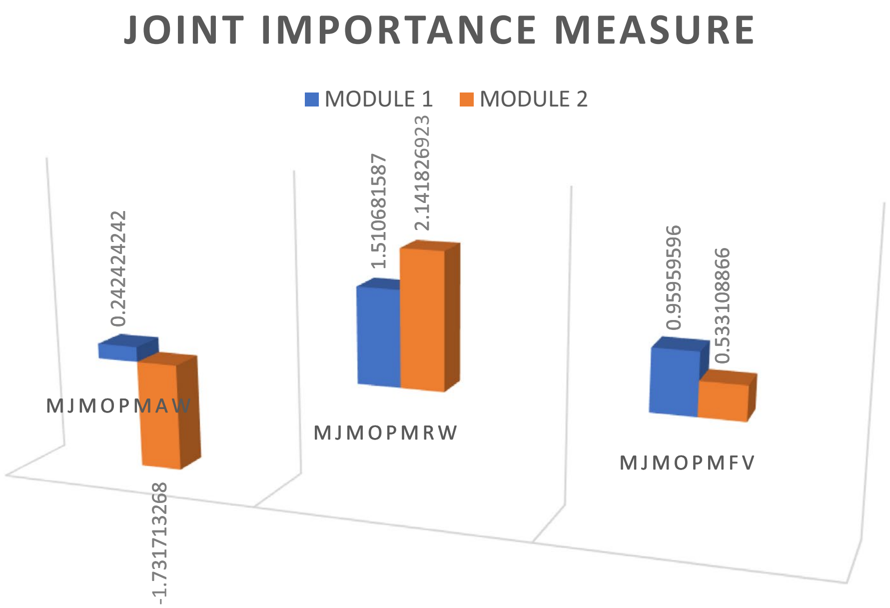

Fig. 2 Multistate joint importance measures of Module 1 and Module 2

Consider two modules, Module 1 with components 1, 2 and 3 and Module 2 with components 2, 3 and 4. Highest value for MJMOPMRW is attained for the module 2 of components 2, 3 & 4 and highest value for MJMOPMFVis attained for the module 1 of components 1, 2 & 3 while the MJMOPMAWis in opposite sign for Module 1 and Module 2. Highest values of MJMOPRW and MJMOPFV are due to highest influence of those groups in reducing system reliability and lowest influence of fractional contribution to system reliability improvement, respectively.

This information can be used to provide more reliability operations for different modules. Highest values in various importance measures clearly emphasize the need of special care. A researcher needs to understand the dynamics of system reliability via module reliability to adopt reliability improvement activities.

7 Conclusion

This paper introduced three module joint importance measures for MSSs with reference to the OPMs reliability, expected system performance and risk. The joint importance measures JMRAW, JMRRW, and JMRFV for three components are introduced and generalized to various output performance measures like expected output performance measure, availability etc. Multistate JMrAW, JMrRW, and JMrFV importance measures are also introduced for three components. The new joint importance measures are useful for giving priority for reliability improvement activities. The UGF method is used to evaluate the joint importance measures, in which the system performance is measured in terms of productivity or capacity. Joint importance measure values will be useful for reliability engineering. The value and size of the importance measure provides the guidelines for reliability operations.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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