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THEORETICAL ARTICLE

Optimal inventory policies for deteriorating items with expiration date and dynamic demand under two-level trade credit

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Abstract
In this paper, we build an inventory model for deteriorating items with expiration time which incorporates both quantity and quality losses under two-level trade credit. The demand is dynamic and varies simultaneously with the length of credit period offered to customers and product freshness condition. In addition, the risk of default increases with the credit period length. First, we investigate the retailer's inventory system for deteriorating items as a profit maximization problem to determine the optimal inventory policies. In order to obtain the optimal ordering policies, we propose some lemmas to help the retailer in accurately and quickly determine the optimal replenishment decisions under maximizing the annual total profit. Finally, we have used some numerical examples to illustrate the proposed models and study the sensitivity analysis on the optimal solution with respect to each parameter and provide some managerial insights.

Keywords Inventory · EOQ · Expiration date · Deterioration · Trade credit

1 Introduction

Maintaining inventory is necessary for any company dealing with deteriorating products. Deterioration means loss of utility, or loss of marginal value of commodity, which decreases its usefulness. Ghare and Schrader [1] made the first attempt to describe the optimal ordering policies for such items having constant rate of deterioration. After that, Philip [2] developed the optimal inventory policies with a three parameter weibull distribution rate and no shortages. Shah [3] generalized Philip's model (1974) by considering shortages. Dave [4] presented an inventory model for deteriorating items with time proportional demand. Later,

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Aggarwal and jaggi [5] developed the retailer's optimal ordering policies for deteriorating items under permissible delay in payments. Sarkar et al. [6] presented an order-level lot size inventory model with inventory level dependent demand for deteriorating items. Chu et al. [7] developed an EOQ model of deteriorating items under permissible delay in payments. Manna and Chaudhuri [8] extended an EOQ model by taking unit production cost, time dependent deterioration rate and shortages. Liao et al. [9] formulated an optimal order policy for deteriorating items under inflation and permissible delay in payments. Chang et al. [10] discussed the optimal cycle time for exponentially deteriorating products under trade credit financing. Later, a fuzzy EPQ model for deteriorating items under permissible delay in payments was introduced by Mahata and Goswami [11]. Shah et al. [12] studied on optimizing inventory and marketing policy for non-instantaneous deteriorating items with generalized type deterioration and holding cost rates. After, Giri and Bardhan [13] considered a supply chain coordination for deteriorating item with stock and price dependent demand under revenue sharing contract. Teng et al. [14] presented inventory lot-size policies for deteriorating items with expiration dates and advance payments. Recently, Sharma et al. [15] developed an inventory model for deteriorating items with expiry date and time-varying holding cost. There are many articles related to deterioration as like Mahata and Goswami [16], Sana [17], Skouri et al. [18], and Sarkar et al. [19] etc.

All the above articles are based on the assumption that the retailer must pay for the items within the fixed time period given by the supplier. This fixed time period is known as trade credit period. Interest is charged if the retailer is unable to pay for the items within that time. If supplier offers trade credit to the customers but the retailer does not it to his customers then it is known as One-level trade credit. Based on this idea Goyal [20] first established an EOQ model under trade credit. Later, Khouja and Mehrez 1996) [21] developed this model under different credit policies. Chung 1998) [22] considered the DCF (discounted cash flow) approach for the analysis of the optimal ordering policy under trade credit. After that, Teng [23] developed an EOQ model under the conditions of permissible delay in payments. Huang and Chung [24] obtained an optimal retailer's ordering policies in the EOQ model under trade credit financing. Abad and Jaggi [25] discussed a joint approach for setting unit-price and the length of credit period for a seller when end customer's demand is price sensitive. Thangam and UthayKumar [26] developed Teng's (2002) model from EOQ model to EPQ model with considering a partial trade credit policy. Different discount rates on purchasing cost offered by the supplier in a single level trade credit policy are discussed by Sarkar et al. [27]. Khanra et al. [28] extended the optimal order policies by taking time dependent demand and shortages under trade credit. Chen et al. [29] brought the strategy that supplier offers retailer a fully permissible delay of some periods if retailer's order more than or equal to a predetermined items. Mahata and Mahata [30] obtained a finite replenishment model with trade credit and variable deterioration for fixed lifetime products. After that, Kaur et al. [31] developed an optimal ordering policy with non increasing demand for time dependent deterioration under fixed life time production and permissible delay in payments. There are several articles relevant to trade credit such as Jamal et al. [32], Sarkar et al. [33], Chung [34], Chung et al. [35], Huang [36] etc.

All the above mentioned articles are based on the single level trade credit policy where supplier would offer the trade credit to the retailer but retailer does not extend it to his customer. If both supplier and retailer offer their customers then it is called two level trade credit. To hedge against negative impacts of expiration date towards the retailer's order incentive, two-level trade credit (including upstream and downstream) has long been employed to adjust the retailer's order quantity (Wu et al. [14]; Mahata et al. [37]). That is, upstream credit period without paying any interest is usually offered by the supplier, which makes it possible for the retailer to generate additional opportunity income and to adopt an excessive ordering policy. In addition, downstream credit period is often granted by the retailer, aiming to directly promote the market demand. Due to their positive affects towards the retailer's optimal order policy, upstream/downstream trade credit can be widely observed in grocery offerings and e-Commerce, such as Walmart, Amazon, etc. In addition, it has been estimated that the median levels of trade credit in industrialized nations (such as US, Canada and Japan) range from 13 to 40% from 1988 to 2007 (Seifert et al. [38]).

However, in the context of the deteriorating items with expiration date, two-level trade credit may lead to double effects towards the retailer's order incentive. That is, benefiting from two-level trade credit, customers may be stimulated to purchase more products, resulting in higher values of the retailer's order cycle (Shi et al. 2018). Nevertheless, in the context of expiration date, higher order cycle may derive more deteriorated quantity (cost), and lead the retailer to be responsible for more opportunity cost of capital. For example, in grocery offerings, Walmart/Amazon is often allowed to delay the payment by the upstream suppliers, while customers can obtain credit period when purchasing from retailer. Nevertheless, it is still unclear whether two-level trade credit is effective for the retailer to hedge against quantity loss and quality loss derived from deteriorating items with expiration date. However, the strategy of granting credit terms adds not only an additional cost but also an additional dimension of default risk to the retailer.

Based on this concept, Huang [39] developed the retailer's optimal ordering policies in the EOQ model under a two level trade credit policy. After that Huang [40] modified Huang's (2003) model to incorporate a retailer's storage space limitation. Teng [41] presented optimal manufacturer's replenishment policies in the EPQ model under two level trade credit policy. Chung [42] pointed out the simplified solution procedure for the optimal replenishment decision under two level trade credit policy. Ho [43] developed an optimal integrated inventory policy with price and credit linked demand under two level of trade credit policy. After that, Mahata [44] introduced an EPQ inventory model for exponentially deteriorating item under retailer's partial trade credit policy in supply chain. Wu et al. [14] discussed Inventory models for deteriorating items with maximum lifetime under downstream partial trade credits to credit-risk customers by discounted cash-flow analysis. The effect of preservation technology investment on a non instantaneous deteriorating inventory model was discussed by Dye [45]. Sarkar [46] studied on two level trade credit policy with time dependent deterioration rate and demand. A comprehensive extension of optimal replenishment decisions under two level trade credit depending on order quantity was developed by Ouyang et al. [47]. Wu et al. [48] obtained an optimal credit period and lot size for deteriorating items with expiration dates under two level trade credit financing.

Under trapezoidal type demand, Wu et al. [49] addressed inventory policies for deteriorating items with maximum lifetime and two-level trade credit. Li et al. [50] derived different inventory models under two level trade credit linked to order quantity. Mahata and Mahata [51] extended an EOQ model under two level partial trade credit by taking time varying deteriorating items. Besides these articles there are many articles related to optimal inventory policies under two level trade credit like as Thangam and Uthayakumar [52], Jaggi et al. [53], Soni [54], and Sarkar et al. [55] etc.

In this paper, we propose the retailer's optimal credit period and cycle time in a EOQ model in which (1) the supplier offers retailer a trade credit period (M) and the retailer in turn offers a trade credit period (N) to his/her customers, (2) deteriorating items with expiration date m where the replenishment cycle time T is not more than m, (3) Demand rate is dependent on both credit period offered by retailer and product freshness condition and (4) Replenishment rate is instantaneous, and shortages are not allowed. Considering these conditions, we construct the retailer's inventory model as a profit maximization problem. In order to obtain the optimal ordering policies, we propose some lemmas to help the retailer in accurately and quickly determine the optimal replenishment decisions under maximizing the annual total profit. Finally, we have used software MATLAB to study the sensitivity analysis on the optimal solution with respect to each parameter to illustrate the model and provide some managerial insights.

2 Assumptions and notations

The following assumptions and notations are adopted to formulate the new proposed models throughout the paper.

2.1 Assumptions

(i) The inventory system involves one type of deteriorating items with expiration date m, where the replenishment cycle time T is not more than m, i.e., $T \leq m$. And both quantity and quality losses are involved in this paper.

(ii) By referring to Sarkar [46] and Mahata et al. [56], during the expiration date m, the quantity loss rate of the items can be defined as follows:

$$\theta(t) = \frac{1}{1+m-t}, 0 \leq t \leq T \leq m$$

Apparently, $\theta(t)$ is closed to 1 when time is approaching to the expiration date.

(iii) Regarding quality loss, the freshness index decreases with t. By referring to Chen et al. [57] and Li and Teng [58], the freshness of products can be defined as follows:

$$f(t) = \frac{m-t}{m}, 0 \le t \le T \le m$$

From the above Eq., $f(t)$ would reach 0 when the product approach its expiration date, i.e., quality of deteriorating items reduces to zero.

(iv) The retailer settles the account at time M and pay for the interest charges on items in stock with rate I_c over the interval $[M, T]$ when $T \ge M$. Alternatively, the retailer settles the account at time M and is not required to pay any interest charge for items in stock during the whole cycle when $T \le M$. On the other hand, the retailer can accumulate revenue and earn interest during the period from N to M (when $M > N$) with rate I_e under the trade credit conditions.

(v) Since consumers prefer a deteriorating item that is further from its expiration date, implying that the demand for deteriorating items is influenced by product freshness perceived by the expiration date. As a result, quality loss impairs the customer willingness to purchase deteriorating items throughout its expiration date, leading to the decrease of market demand due to its instantaneous freshness. On the other hand, it is observed that trade credit offered by the Retailer to customers has a positive impact on demand. Because credit trade allows customers to enjoy the benefits of delayed payments, lengthening the period will stimulate sales. The longer the credit period is, the higher is the demand. Hence, demand strictly increases in the credit period.

Combining above two relations, demand rate $D(N, t)$ is dependent on both credit period offered by retailer and product freshness condition. Trade credit has a positive impact on demand while demand for product decreases with losses its freshness with time. Here we assumed the functional representation of demand rate as follows:

$$D(N, t) = Ke^{aN} f(t) = Ke^{aN} \frac{m-t}{m}, 0 \le t \le T \le m.$$

where K and a are positive constants. For convenience, $D(N, t)$ and D will be used interchangeably. This type of demand is seen to occur in the case of product such as fresh food, fresh fruits, vegetables, chemicals and medicines which may deteriorate when they are stored in warehouse. These deteriorating items may lose their utility with time due to decay, damage or spoilage.

(vi) A 30-year mortgage has a higher default risk than a 15-year mortgage. Likewise, the longer the credit period is, the higher the percentage that the buyer will not be able to pay off the debt. Although sales can be stimulated by trade credit, longer credit period increases the probability of a customer default. Therefore, we assume without loss of generality that the rate of default risk giving the credit period N is

$$F(N) = 1 - e^{-bN},$$

where b is the coefficient of the default risk, which is a positive constant. This default risk pattern is used in some studies (Teng and Lou [59], Mukherjee and Mahata [60]).

(vii) Replenishment rate is instantaneous, and shortages are not allowed.

2.2 Notations

A	The retailer's fixed order cost per order.
c	The retailer's purchase cost per unit.
p	The retailer's selling price per unit, where $c < p$.
h	The retailer's unit holding cost per unit time (excluding interest charge when involving upstream trade credit), where $h < p$.
r	Annual compound interest paid per dollar per year.
N	The customer's credit period granted by the retailer.
$D(N, t)$	The retailer's market demand rate, which is a function of N and t.
m	The expiration date of deteriorating items.
$\theta(t)$	The time-varying quantity loss rate at time $t \in [0, T]$, where $0 \le \theta(t) \le 1$.
$f(t)$	The time-varying quality loss rate at $t \in [0, T]$, which is a decreasing function within [0, 1].
$I(t)$	The retailer's inventory level at time $t \in [0, T]$.
T	The retailer's order cycle time (decision variable), where $T \le m$.
Q	The retailer's order quantity.
Q_d	The retailer's total sales volume during the order cycle.
$TP(N, T)$	The retailer's average profit per unit time.

Given the above notation and assumptions, the retailer's aim is to determine credit period N and replenishment cycle time T such that the profit per unit time is maximized.

3 Mathematical model formulation

According to above assumptions, during the time interval $[0, T]$, the inventory level $I(t)$ decreases with the combined effects of the quantity loss $\theta(t)$ and market demand $D(N, t)$, which can be expressed by the following differential equation:

$$\frac{dI(t)}{dt} = -D(N,t) - \theta(t)I(t) = -Ke^{aN}\frac{m-t}{m} - \frac{1}{1+m-t}I(t), 0 \le t \le T. \quad (1)$$

Solution of Eq. (1) with the boundary condition $I(T) = 0$ yields,

$$I(t) = \frac{Ke^{aN}}{m}(1+m-t)(T-t) + \frac{Ke^{aN}}{m}(1+m-t)\log\left(\frac{1+m-T}{1+m-t}\right) \quad (2)$$

The order quantity Q is obtain by substituting $t = 0$, i.e.,

$$Q = I(0) = \frac{Ke^{aN}}{m}(1+m)T + \frac{Ke^{aN}}{m}(1+m)\log\left(\frac{1+m-T}{1+m}\right) \quad (3)$$

Then, during the order cycle, the retailer's total sales volume

$$Q_d = \int_0^T D(N,t)dt = KTe^{aN}\frac{2m-T}{2m}. \quad (4)$$

Thus, the elements comprising the retailer's average profit function are listed below.

The discounted sales revenue after the default risk during the replenishment period $[0, T]$ is,

$$pKe^{-rN}e^{aN}(1-F(N))\int_0^T \frac{m-t}{m}dt = pKe^{[a-(b+r)]N}\frac{2m-T}{2m}. \quad (5)$$

$$\text{The fixed order cost per cycle is} = \frac{A}{T}. \quad (6)$$

$$\text{Purchase cost per cycle is} = c\frac{Ke^{aN}}{m}(1+m) + c\frac{Ke^{aN}(1+m)}{mT}\log\left(\frac{1+m-T}{1+m}\right). \quad (7)$$

The holding cost per cycle (including interest charges without trade credit)

$$= \frac{h}{T}\int_0^T I(t)dt = \frac{h}{T}\left[\frac{Ke^{aN}}{m}\left\{\frac{(1+m)T^2}{2} - \frac{T^3}{6}\right\}\right. \left. + \frac{Ke^{aN}}{m}\left\{\frac{(1+m)^2}{2}\log\left(\frac{1+m-T}{1+m}\right) + \frac{T(1+m)}{2} - \frac{T^2}{4}\right\}\right]. \quad (8)$$

The two cases may arise to calculate the annual capital opportunity cost i.e. (i) and (ii).

Case 1 $\boldsymbol{N} < M$.

In this case, there are two possible cases arise: (1) $T + N \le M$ and (2) $T + N \ge M$. Now, let us discuss the detailed formulation of each sub-case.

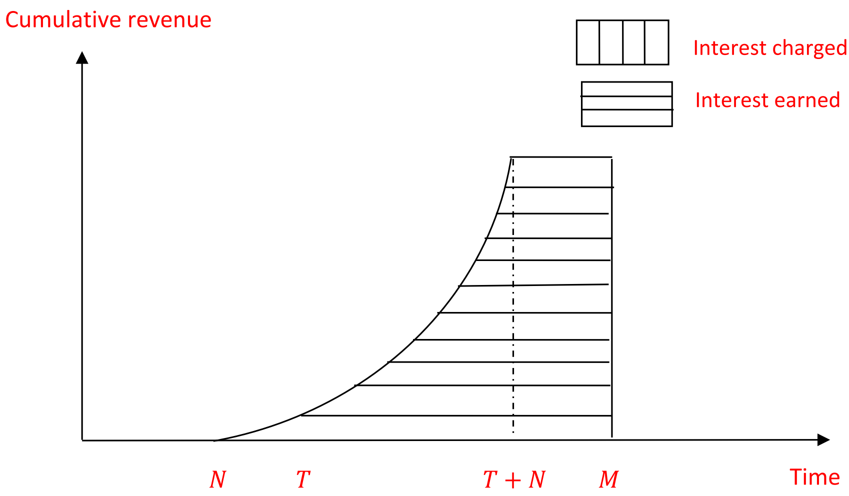

Fig. 1 The retailer's interest earned and interest charged when $N \le T + N \le M$

Sub-case 1.1 $\boldsymbol{T + N \le M}$ (see Fig. 1).

With $T + N \le M$, the retailer receives sales revenue of all items at time $T + N$ and is able to pay off total purchasing cost by M. Therefore, there is no interest charged. On the other hand, during the period $[N, T + N]$ retailer can earned interest on the sale revenues received from customers and on full sales revenue during the period $[T + N, M]$. Therefore, annual interest earned is,

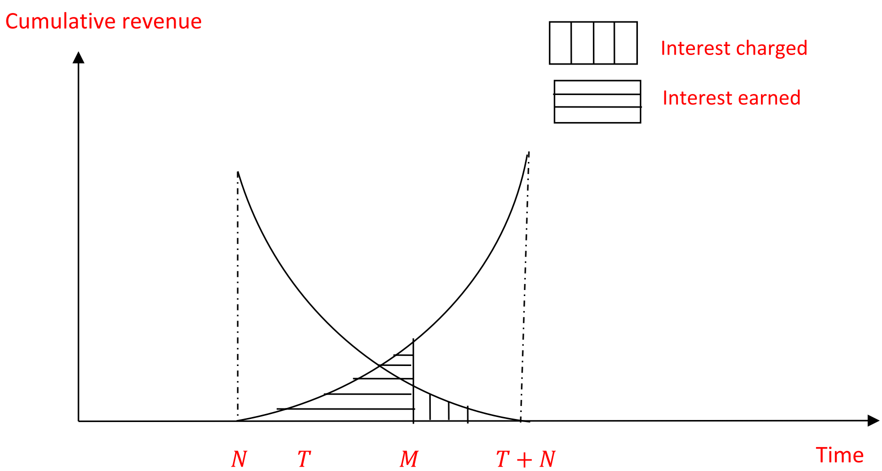

Fig. 2 The retailer's interest earned and interest charged when $N \le M \le T + N$

$$\frac{sI_e}{T}\left[\int_N^{T+N}\int_N^{t+N} D(N,u-N)dudt + (M-T-N)Q_d\right]$$
$$= pI_eKe^{aN}\left[\frac{T}{2} + N - \frac{T^2}{6m} - \frac{TN}{2m} - \frac{N^2}{2m} + \frac{(M-T-N)(2m-T)}{2m}\right] \tag{9}$$

Therefore, the total annual profit function is

$$\begin{aligned}
TP_{11}(N,T) &= pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - \frac{A}{T} - \left\{\frac{cKe^{aN}}{m}(1+m) + \frac{cKe^{aN}(1+m)}{mT}\log\left(\frac{1+m-T}{1+m}\right)\right\} \\
&- \frac{h}{T}\left[\frac{Ke^{aN}}{m}\left\{\frac{(1+m)T^2}{2} - \frac{T^3}{6}\right\} + \frac{Ke^{aN}}{m}\left\{\frac{(1+m)^2}{2}\log\left(\frac{1+m-T}{1+m}\right) + \frac{T(1+m)}{2} - \frac{T^2}{4}\right\}\right] \\
&+ pI_eKe^{aN}\left[\frac{T}{2} + N - \frac{T^2}{6m} - \frac{TN}{2m} - \frac{N^2}{2m} + \frac{(M-T-N)(2m-T)}{2m}\right] \\
&= pKe^{[a-(b+r)]N} - \frac{pTKe^{[a-(b+r)]N}}{2m} - \frac{A}{T} - \frac{cK(1+m)e^{aN}}{m} + \frac{cKe^{aN}(2+2m+T)}{2m(1+m)} \\
&- \frac{hKe^{aN}(1+m)T}{2m} + \frac{hKe^{aN}T^2}{6m} + \frac{hKe^{aN}(2+2m+T)}{4m} - \frac{hKe^{aN}(1+m)}{2m} + \frac{hKe^{aN}T}{4m} \\
&+ pI_eKe^{aN}\left[\frac{T}{2} + N - \frac{T^2}{6m} - \frac{TN}{2m} - \frac{N^2}{2m} + \frac{(M-T-N)(2m-T)}{2m}\right]
\end{aligned} \tag{10}$$

(Approximately).

Sub-Case 1.2 $\boldsymbol{T + N \geq M}$ (see Fig. 2).

With $T + N \geq M$, the retailer does not receive the last payment before the permissible delay period M. As a result, the retailer must finance all items sold after time $(M - N)$ at time M, and pay off the loan until $T + N$ at an interest rate I_c per dollar per year. Therefore, we can have the interest charged in the following:

$$\begin{aligned}
&\frac{cI_c}{T}\int_M^{T+N} I(t-N)dt = \frac{cI_c}{T}\int_{M-N}^{T} I(t)dt \\
&= \frac{Ke^{aN}cI_c}{mT}\left[\frac{(1+m)T^2}{2} - \frac{T^3}{6} + \frac{T}{2}\left\{(M-N)^2 - 2(1+m)(M-N)\right\}\right. \\
&+ \frac{(M-N)^2(1+m)}{2} - \frac{(M-N)^3}{3} \\
&\quad + \frac{(1+m-M+N)^2}{2}\log\left(\frac{1+m-T}{1+m-M+N}\right) - \frac{T^2}{4} + \frac{(m+1)T}{2} \\
&\quad \left. + \left\{\frac{(M-N)^2 - 2(1+m)(M-N)}{4}\right\}\right]
\end{aligned}$$

On the other hand, during the period $[N, M]$ retailer can earned interest on the sale revenues received from the delayed payment during the period $[N, M]$. Therefore, annual interest earned is,

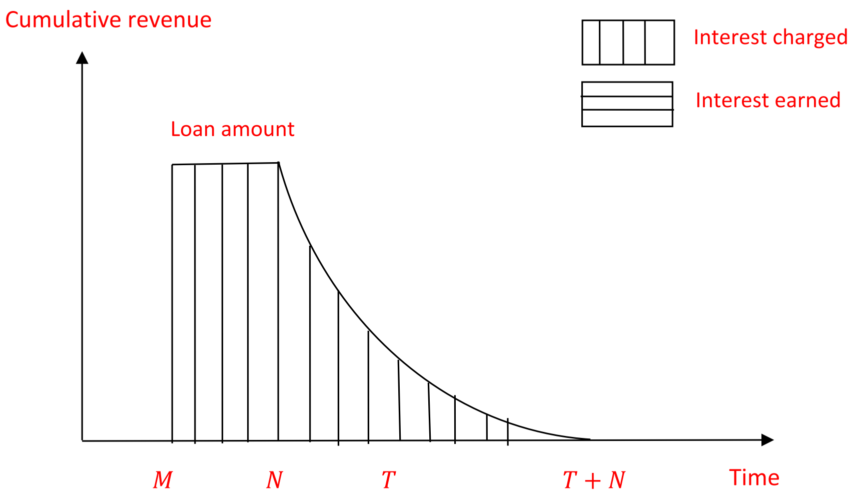

Fig. 3 The retailer's interest earned and interest charged when $M \le N \le T + N$

$$\frac{pI_e}{T}\int_N^M\int_N^{t+N} D(N, u-N)dudt = \frac{pI_eKe^{aN}}{T}\left[\frac{\left(M^2-N^2\right)}{2} - \frac{\left(M^3-N^3\right)}{6m}\right]$$

Hence, the retailer's annual total profit function is

$$\begin{aligned}
TP_{12}(N,T) &= \text{annual sales revenue} - \text{annual ordering cost} - \text{annual purching cost}\\
&\quad - \text{annual holding cost} - \text{annual capital opportunity cost}\\
&= pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - \frac{A}{T} - \left\{\frac{cKe^{aN}}{m}(1+m) + \frac{cKe^{aN}(1+m)}{mT}\log\left(\frac{1+m-T}{1+m}\right)\right\}\\
&\quad - \frac{h}{T}\left[\frac{Ke^{aN}}{m}\left\{\frac{(1+m)T^2}{2} - \frac{T^3}{6}\right\} + \frac{Ke^{aN}}{m}\left\{\frac{(1+m)^2}{2}\log\left(\frac{1+m-T}{1+m}\right) + \frac{T(1+m)}{2} - \frac{T^2}{4}\right\}\right]\\
&\quad - \frac{cI_cKe^{aN}}{mT}\left[\frac{(1+m)T^2}{2} - \frac{T^3}{6} + \frac{T}{2}\left\{(M-N)^2 - 2(1+m)(M-N)\right\} + \frac{(M-N)^2(1+m)}{2} - \frac{(M-N)^3}{3}\right.\\
&\quad \left. + \frac{(1+m-M+N)^2}{2}\log\left(\frac{1+m-T}{1+m-M+N}\right) - \frac{T^2}{4} + \frac{(m+1)T}{2} + \left\{\frac{(M-N)^2 - 2(1+m)(M-N)}{4}\right\}\right]\\
&\quad + \frac{pI_eKe^{aN}}{T}\left[\frac{\left(M^2-N^2\right)}{2} - \frac{\left(M^3-N^3\right)}{6m}\right]\\
&= pKe^{[a-(b+r)]N} - \frac{pTKe^{[a-(b+r)]N}}{2m} - \frac{A}{T} - \frac{cK(1+m)e^{aN}}{m} + \frac{cKe^{aN}(2+2m+T)}{2m(1+m)}\\
&\quad - \frac{hKe^{aN}(1+m)T}{2m} + \frac{hKe^{aN}T^2}{6m} + \frac{hKe^{aN}(2+2m+T)}{4m} - \frac{hKe^{aN}(1+m)}{2m} + \frac{hKe^{aN}T}{4m}\\
&\quad - \frac{cI_cKe^{aN}}{mT}\left[\frac{(1+m)T^2}{2} - \frac{T^3}{6} + \frac{T}{2}\left\{(M-N)^2 - 2(1+m)(M-N)\right\} + \frac{(M-N)^2(1+m)}{2} - \frac{(M-N)^3}{3}\right.\\
&\quad \left. - \frac{(T+N-M)(2+2m-3M+3N+T)}{4} - \frac{T^2}{4} + \frac{(m+1)T}{2} + \left\{\frac{(M-N)^2 - 2(1+m)(M-N)}{4}\right\}\right]\\
&\quad + \frac{pI_eKe^{aN}}{T}\left[\frac{\left(M^2-N^2\right)}{2} - \frac{\left(M^3-N^3\right)}{6m}\right]
\end{aligned} \tag{11}$$

(Approximately).

Case 2 $\boldsymbol{N \geq M}$ (see Fig. 3).

Since $N \geq M$, there is no interest earned. The retailer must finance all the purchasing cost from.

$[M, N]$ and pay off the loan from $[N, T+N]$. Therefore, the interest charged per cycle is

$$\frac{cI_c}{T}\left[(N-M)Q_d + \int_0^T I(t)dt\right]$$
$$= \frac{cI_cKe^{aN}}{2mT}\left[(N-M)T(2m-T) + \left\{(1+m)T^2 - \frac{T^3}{3} + (1+m)^2\log\left(\frac{1+m-T}{1+m}\right) - \frac{T^2}{2} + (1+m)T\right\}\right]$$

Consequently, the retailer's annual total profit function is

$$\begin{aligned} TP_2(N,T) = {} & pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - \frac{A}{T} - \left\{\frac{cKe^{aN}(1+m)}{m} + \frac{cKe^{aN}(1+m)}{mT}\log\left(\frac{1+m-T}{1+m}\right)\right\} \\ & - \frac{h}{T}\left[\frac{Ke^{aN}}{m}\left\{\frac{(1+m)T^2}{2} - \frac{T^3}{6}\right\} + \frac{Ke^{aN}}{m}\left\{\frac{(1+m)^2}{2}\log\left(\frac{1+m-T}{1+m}\right) + \frac{T(1+m)}{2} - \frac{T^2}{4}\right\}\right] \\ & - \frac{cI_cKe^{aN}}{2mT}\left[(N-M)T(2m-T) + \left\{(1+m)T^2 - \frac{T^3}{3} + (1+m)^2\log\left(\frac{1+m-T}{1+m}\right)\right.\right. \\ & \left.\left. - \frac{T^2}{2} + (1+m)T\right\}\right] \\ = {} & pKe^{[a-(b+r)]N} - \frac{pTKe^{[a-(b+r)]N}}{2m} - \frac{A}{T} - \frac{cK(1+m)e^{aN}}{m} + \frac{cKe^{aN}(2+2m+T)}{2m(1+m)} \\ & - \frac{hKe^{aN}(1+m)T}{2m} + \frac{hKe^{aN}T^2}{6m} + \frac{hKe^{aN}(2+2m+T)}{4m} - \frac{hKe^{aN}(1+m)}{2m} + \frac{hKe^{aN}T}{4m} \\ & - \frac{cI_cKe^{aN}}{2m}\left[(N-M)(2m-T) + \left\{(1+m)T - \frac{T^2}{3} - \frac{(2+2m+T)}{2} - \frac{T}{2} + (1+m)\right\}\right] \end{aligned} \tag{12}$$

(Approximately).

Hence our problem is,

$$MaximizeTP(N,T) = \begin{cases} TP_1(N,T), & \text{if } N \leq M \\ TP_2(N,T), & \text{if } N \geq M \end{cases} \tag{13}$$

where $TP_1(N,T) = \begin{cases} TP_{11}(N,T), & \text{if } T+N \leq M \\ TP_{12}(N,T), & \text{if } T+N \geq M \end{cases}$ and $TP_{11}(N,T)$, $TP_{12}(N,T)$, and $TP_2(N,T)$ are given by (10), (11), and (12) respectively.

4 Optimal solution

Case 1 $\boldsymbol{N < M}$.

Sub-Case 1.1 $\boldsymbol{T + N \leq M}$

Taking 1st and 2nd partial derivatives of $TP_{11}(N,T)$ in Eq. (10) with respect to T keeping N as fixed, we get

$$\frac{\partial TP_{11}(N,T)}{\partial T} = \frac{A}{T^2} - \frac{pKe^{[a-(b+r)]N}}{2m} + \frac{cKe^{aN}}{2m(1+m)} - \frac{hKe^{aN}}{m}\left(\frac{m}{2} - \frac{T}{3}\right) + pI_eKe^{aN}\left(-\frac{1}{2} - \frac{M}{2m} + \frac{2T}{3m}\right), \tag{14}$$

$$\frac{\partial^2 TP_{11}(N,T)}{\partial T^2} = -\frac{2A}{T^3} + \frac{\left(h+2pI_e\right)Ke^{aN}}{3m} = \frac{1}{3mT^3}\left\{\left(h+2pI_e\right)Ke^{aN}T^3 - 6mA\right\} < 0, \tag{15}$$

provided that $\left(h+2pI_e\right)Ke^{aN}T^3 - 6mA < 0$. Based on it, we have the following lemma.

Lemma 1 *For fixed value of N, the retailer's profit function $TP_{11}(N,T)$ is a concave function of T, provided that $\left(h+2pI_e\right)Ke^{aN}T^3 - 6mA < 0$.*

Proof The proof is immediately follows from the above discussion.□

Now, for fixed values of T, differentiating $TP_{11}(N,T)$ partially with respect to N, we get

$$\begin{aligned}\frac{\partial TP_{11}(N,T)}{\partial N} = {}& [a-(b+r)]pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - acKe^{aN}\left\{\frac{(1+m)}{m} - \frac{(2+2m+T)}{2m(1+m)}\right\} \\ &+ \frac{ahKe^{aN}}{m}\left\{\frac{T^2}{6} - \frac{(1+m)T}{2} + \frac{(2+2m+T)}{4} - \frac{(1+m)}{2} + \frac{T}{4}\right\} \\ &+ apI_eKe^{aN}\left[\frac{T}{2} + N - \frac{T^2}{6m} - \frac{TN}{2m} - \frac{N^2}{2m} + \frac{(M-T-N)(2m-T)}{2m}\right] - \frac{pI_eKe^{aN}N}{m},\end{aligned} \tag{16}$$

and

$$\begin{aligned}\frac{\partial^2 TP_{11}(N,T)}{\partial N^2} = {}& [a-(b+r)]^2pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - a^2cKe^{aN}\left\{\frac{(1+m)}{m} - \frac{(2+2m+T)}{2m(1+m)}\right\} \\ &+ \frac{a^2hKe^{aN}}{m}\left\{\frac{T^2}{6} - \frac{(1+m)T}{2} + \frac{(2+2m+T)}{4} - \frac{(1+m)}{2} + \frac{T}{4}\right\} \\ &+ a^2pI_eKe^{aN}\left[\frac{T}{2} + N - \frac{T^2}{6m} - \frac{TN}{2m} - \frac{N^2}{2m} + \frac{(M-T-N)(2m-T)}{2m}\right] - \frac{2apI_eKe^{aN}N}{m} - \frac{pI_eKe^{aN}}{m} \\ = {}& -(b+r)[a-(b+r)]pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - \frac{apI_eKe^{aN}N}{m} - \frac{pI_eKe^{aN}}{m} < 0.\end{aligned} \tag{17}$$

Based on it, we have the following lemma.

Lemma 2 *For fixed value of T, the retailer's profit function $TP_{11}(N,T)$ is a concave function of N.*

Proof The proof is immediately follows from the above discussion.□

Sub-Case 1.2 $\boldsymbol{T+N \geq M}$.

Taking 1st and 2nd partial derivatives of $TP_{12}(N,T)$ in Eq. (11) with respect to T keeping N as fixed, we get

$$\begin{aligned}\frac{\partial TP_{12}(N,T)}{\partial T} &= \frac{A}{T^2} - \frac{pKe^{[a-(b+r)]N}}{2m} + \frac{cKe^{aN}}{2m(1+m)} - \frac{hKe^{aN}}{m}\left(\frac{m}{2}-\frac{T}{3}\right)\\ &- \frac{cI_cKe^{aN}}{m}\left\{\frac{(1+m)}{2} - \frac{T}{3} - \frac{(M-N)^2(1+m)}{2T^2} + \frac{(M-N)^3}{3T^2} - \frac{1}{2} - \left(\frac{(M-N)^2-2(1+m)(M-N)}{4T^2}\right)\right\}\\ &- \frac{pI_eKe^{aN}}{T^2}\left\{\frac{\left(M^2-N^2\right)}{2} - \frac{\left(M^3-N^3\right)}{6m}\right\}\end{aligned} \tag{18}$$

and

$$\begin{aligned}\frac{\partial^2 TP_{12}(N,T)}{\partial T^2} = -\frac{2A}{T^3} - \frac{Ke^{aN}}{6mT^3}\big[cI_c\{6(M-N)^2(1+m) + 3(M-N)^2 - 6(1+m)(M-N)\\ - 4(M-N)^3 - 2T^3\} - 2hT^3 + 2pI_e\left\{\left(M^3-N^3\right) - 3m\left(M^2-N^2\right)\right\}\big] < 0,\end{aligned} \tag{19}$$

provided that $\begin{aligned}\Delta_1 &= cI_c\{6(M-N)^2(1+m) + 3(M-N)^2 - 6(1+m)(M-N) - 4(M-N)^3 - 2T^3\}\\ &- 2hT^3 + 2pI_e\left\{\left(M^3-N^3\right) - 3m\left(M^2-N^2\right)\right\} > 0\end{aligned}$. Based on it, we have the following lemma.

Lemma 3 *For fixed value of N, the retailer's profit function, $TP_{12}(N,T)$, is a concave function of T provided that $\Delta_1 > 0$.*

Proof The proof is immediately follows from the above discussion.□

Next, for fixed value of T, differentiating $TP_{12}(N,T)$ in Eq. (11) partially with respect to N, we have

$$\begin{aligned}\frac{\partial TP_{12}(N,T)}{\partial N} &= [a-(b+r)]pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - acKe^{aN}\left\{\frac{(1+m)}{m} - \frac{(2+2m+T)}{2m(1+m)}\right\}\\ &+ \frac{ahKe^{aN}}{m}\left\{\frac{T^2}{6} - \frac{(1+m)T}{2} + \frac{(2+2m+T)}{4} - \frac{(1+m)}{2} + \frac{T}{4}\right\}\\ &- \frac{acI_cKe^{aN}}{mT}\Bigg[\frac{(1+m)T^2}{2} - \frac{T^3}{6} + \frac{T}{2}\left\{(M-N)^2 - 2(1+m)(M-N)\right\} + \frac{(M-N)^2(1+m)}{2} - \frac{(M-N)^3}{3}\\ &- \frac{(T+N-M)(2+2m-3M+3N+T)}{4} - \frac{T^2}{4} + \frac{(m+1)T}{2} + \left\{\frac{(M-N)^2-2(1+m)(M-N)}{4}\right\}\Bigg]\\ &- \frac{cI_cKe^{aN}(M-N-m)(M-N-T)}{mT} + \frac{apI_eKe^{aN}}{T}\left[\frac{\left(M^2-N^2\right)}{2} - \frac{\left(M^3-N^3\right)}{6m}\right] - \frac{pI_eKe^{aN}N(2m-N)}{2mT},\end{aligned} \tag{20}$$

and

$$\begin{aligned}\frac{\partial^2 TP_{12}(N,T)}{\partial N^2} &= -(b+r)[a-(b+r)]pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - \frac{acI_cKe^{aN}(M-N-m)(M-N-T)}{mT}\\ &- \frac{cI_cKe^{aN}(m-2M+2N+T)}{mT} - \frac{apI_eKe^{aN}N(2m-N)}{2mT} - \frac{pI_eKe^{aN}(m-N)}{mT} < 0.\end{aligned} \tag{21}$$

Based on it, we have the following lemma.

Lemma 4 For fixed value of T, the retailer's profit function $TP_{12}(N,T)$ is a concave function of N.

Proof The proof is immediately follows from the above discussion.☐

Case 2 $\boldsymbol{N \geq M}$.

Taking 1st and 2nd partial derivatives of $TP_2(N,T)$ in Eq. (12) with respect to T keeping N as fixed, we get

$$\begin{aligned}\frac{\partial TP_2(N,T)}{\partial T} =& \frac{A}{T^2} - \frac{pKe^{[a-(b+r)]N}}{2m} + \frac{cKe^{aN}}{2m(1+m)} \\ & - \frac{hKe^{aN}}{m}\left(\frac{m}{2} - \frac{T}{3}\right) - \frac{cI_cKe^{aN}}{2m}\left\{(M-N+m) - \frac{2T}{3}\right\},\end{aligned} \tag{22}$$

and

$$\frac{\partial^2 TP_2(N,T)}{\partial T^2} = -\frac{2A}{T^3} + \frac{\left(h+cI_c\right)Ke^{aN}}{3m} = \frac{1}{3mT^3}\left\{\left(h+cI_c\right)Ke^{aN}T^3 - 6mA\right\} < 0, \tag{23}$$

provided that $\left(h+cI_c\right)Ke^{aN}T^3 - 6mA < 0$. Based on it, we have the following lemma.

Lemma 5 For fixed value of N, the retailer's profit function, $TP_2(N,T)$, is a concave function of T provided that $\left(h+cI_c\right)Ke^{aN}T^3 - 6mA < 0$.

Proof The proof is immediately follows from the above discussion.☐

Now, keeping T as fixed, differentiating $TP_2(N,T)$ in Eq. (12) with respect to N, we have

$$\begin{aligned}\frac{\partial TP_2(N,T)}{\partial N} =& [a-(b+r)]pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - acKe^{aN}\left\{\frac{(1+m)}{m} - \frac{(2+2m+T)}{2m(1+m)}\right\} \\ & + \frac{ahKe^{aN}}{m}\left\{\frac{T^2}{6} - \frac{(1+m)T}{2} + \frac{(2+2m+T)}{4} - \frac{(1+m)}{2} + \frac{T}{4}\right\} \\ & - \frac{acI_cKe^{aN}}{2m}\left[(N-M)(2m-T) + \left\{(1+m)T - \frac{T^2}{3} - \frac{(2+2m+T)}{2} - \frac{T}{2} + (1+m)\right\}\right] \\ & - \frac{cI_cKe^{aN}(2m-T)}{2m},\end{aligned} \tag{24}$$

and

$$\frac{\partial^2 TP_2(N,T)}{\partial N^2} = -(b+r)[a-(b+r)]pKe^{[a-(b+r)]N}\frac{2m-T}{2m} - \frac{acI_cKe^{aN}(2m-T)}{2m} < 0. \tag{25}$$

Based on it, we have the following lemma.

Lemma 6 For fixed value of T, the retailer's profit function $TP_2(N, T)$ is a concave function of N.

Proof The proof is immediately follows from the above discussion.□

To determine the optimal values of the cycle time (T) and the credit period offered by retailer to the customers (N), differentiate the profit function $TP(N, T)$ partially with respect to N and T and equating to zero, we obtain

$$\frac{\partial TP(N,T)}{\partial N} = 0, \tag{26}$$

and

$$\frac{\partial TP(N,T)}{\partial T} = 0. \tag{27}$$

Solving the Eqs. (26) and (27) simultaneously we obtain the optimal value of N and T. The sufficient conditions for the profit maximization are as follows

$$\frac{\partial^2 TP(N,T)}{\partial N^2} < 0 \text{ and } \frac{\partial^2 TP(N,T)}{\partial T^2} < 0. \tag{28}$$

$$\frac{\partial^2 TP(N,T)}{\partial N^2} . \frac{\partial^2 TP(N,T)}{\partial T^2} - \left(\frac{\partial^2 TP(N,T)}{\partial N \partial T}\right)^2 > 0 \tag{29}$$

The profit functions of the present problem under various cases are highly non-linear and too complicated to solve. Further also it is not easy to show mathematically their concavity jointly, alternatively, we have shown the concavity of these profit functions graphically for all the cases and sub cases with the help of computer software MATLAB (see Figs. 4, 5, 6).

5 Numerical examples

In this section, we consider the following examples to illustrate our proposed model.

Example 1 (*Case 1* $N < M$, Sub-case 1.2: $N + T < M$): Following parameters are presented to obtain the retailer's optimal solutions:

$K = 3000$ units/year, $a = 0.4$ units/year,$b = 0.05$, $p = \$40$/unit, $c = \$4$/unit, $r = 0.06$, $A = \$200$/order, $h = \$2$/unit/year, $m = 0.7$ year, $M = 0.3$ year, $I_c = \$0.1/\$/$ year, $I_e = \$0.15/\$/$ year. Substituting these values in Eqs. (26) and (27), we obtain the optimum solutions for $T = T^* = 0.0393$ year, $N = N^* = 0.2606$ year and corresponding optimum total annual profit $TP = TP^*_{11} = 111957.00$.

The sensitivity analysis of different parameters involved is carried out with the help of using same data as in Example 1. It will be helpful in decision making to

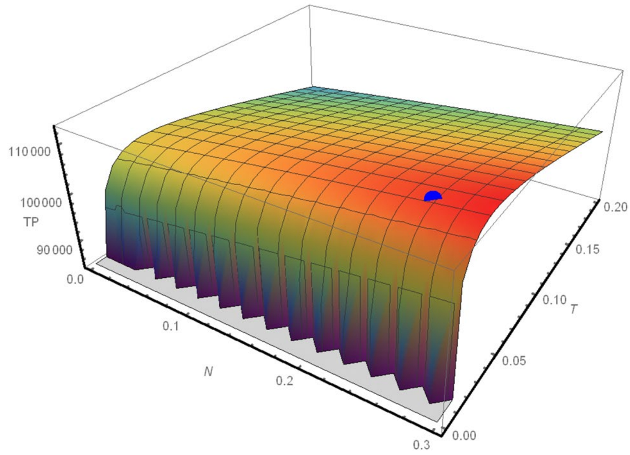

Fig. 4 The concave property of the profit function $TP_{11}(N, T)$ in N and T

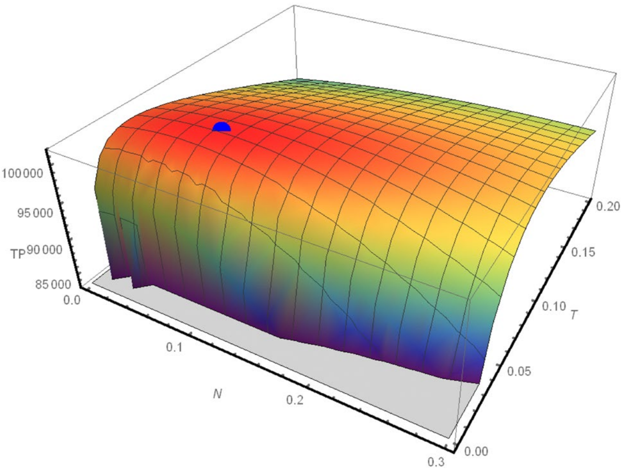

Fig. 5 The concave property of the profit function $TP_{12}(N, T)$ in N and T

analyze the effect of change of these variations. We then study the effect of the variations in a parameter on the optimal solutions keeping other system parameters same.

Example 2 (*Sub-case 1.2* $N + T > M$): Following parameters are presented to obtain the retailer's optimal solutions: $K = 3000$ units/year, $a = 0.4$ units/year,$b = 0.05$, $p = \$40$/unit, $c = \$4$/unit, $r = 0.06$, $A = \$200$/order, $h = \$2$/unit/year, $m = 0.7$ year, $M = 0.06$ year, $I_c = \$0.1/\$/$ year, $I_e = \$0.15/\$/$ year Substituting these values in the Eqs. (26) and (27), we get, the optimum solutions for

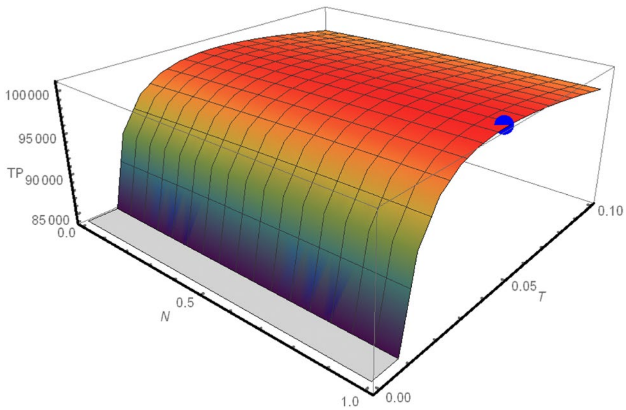

Fig. 6 The concave property of the profit function $TP_2(N, T)$ in N and T

$T = T^* = 0.0512$ year,$N = N^* = 0.0803$ year and corresponding optimum total annual profit $TP = TP^*_{12} = 101605.80$.

The sensitivity analysis of different parameters involved is carried out with the help of using same data as in Example 2. It will be helpful in decision making to analyze the effect of change of these variations. We then study the effect of the variations in a parameter on the optimal solutions keeping other system parameters same.

Example 3 (*Case* 2 $N \geq M$): Following parameters are presented to obtain the retailer's optimal solutions: $K = 3000$ units/year, $a = 0.251$ units/year,$b = 0.145$, $p = \$40$/unit, $c = \$4$/unit, $r = 0.06$, $A = \$200$/order, $h = \$2$/unit/year, $m = 0.7$ year, $M = 0.3$ year, $I_c = \$0.1/\$/$ year, $I_e = \$0.15/\$/$ year Substituting these values in the Eqs. (26) and (27), we get the optimum solutions for $T = T^* = 0.0480$ year,$N = N^* = 0.99$ year and corresponding optimum total annual profit $TP = TP^*_2 = 100811.10$.

The sensitivity analysis of different parameters involved is carried out with the help of using same data as in Example 3. It will be helpful in decision making to analyze the effect of change of these variations. We then study the effect of the variations in a parameter on the optimal solutions keeping other system parameters same.

Above observations can be summed up as follows: From Tables 1, 2 and 3, following inference can be made.

The sensitivity analysis reveals that: (1) a higher value of a, K, and p causes higher values of N^* and $TP^*(N^*, T^*)$ while a lower value of T^*; (2) in contrast, a higher value of b and A causes lower values of N^* and $TP^*(N^*, T^*)$ while a higher value of T^*; (3) a higher value of c causes lower values of all N^* and $TP^*(N^*, T^*)$ while a higher value of T^*; and (4) conversely, a higher value of m

Table 1 Sensitivity analysis on parameters of Example 1

Changing parameters	Change	T^*	N^*	TP^*_{11}
K	2250	0.0455	0.2544	82,790.64
	2500	0.0431	0.2568	92,490.32
	2750	0.0411	0.2588	102,213.50
	3000	0.0393	0.2606	111,957.00
a	0.25	0.0432	0.2567	107,510.40
	0.30	0.0418	0.2581	108,964.10
	0.35	0.0405	0.2594	110,446.30
	0.40	0.0393	0.2606	111,957.00
b	0.05	0.0393	0.2606	111,957.00
	0.10	0.0406	0.2593	110,333.60
	0.15	0.0420	0.2579	108,741.40
	0.20	0.0435	0.2564	107,180.20
p	25	0.0509	0.2490	63,369.19
	30	0.0460	0.2539	79,497.94
	35	0.0423	0.2576	95,699.29
	40	0.0393	0.2606	111,957.00
c	4	0.0393	0.2606	111,957.00
	5	0.0398	0.2601	108,684.00
	6	0.0402	0.2597	105,412.10
	7	0.0407	0.2592	102,141.60
A	200	0.0393	0.2606	111,957.00
	225	0.0418	0.2581	111,341.30
	250	0.0440	0.2559	110,759.30
	275	0.0462	0.2537	110,206.00
M	0.15	0.0400	0.1099	105,119.20
	0.20	0.0398	0.1601	107,417.40
	0.25	0.0396	0.2103	109,697.70
	0.30	0.0393	0.2606	111,957.00
m	0.4	0.0326	0.2673	108,858.50
	0.5	0.0352	0.2647	110,252.90
	0.6	0.0375	0.2624	111,230.10
	0.7	0.0393	0.2606	111,957.00

causes higher values of all T^*, and $TP^*(N^*, T^*)$ and lower values of N^*. A simple economic interpretation of (1) is as follows: if a is higher, then the trade credit N makes demand (as well as annual profit) increase higher. Hence, a higher value of a causes higher values of trade credit N^* and annual profit $TP^*(N^*, T^*)$ while a lower value of T^* to reduce holding cost. For the given parameters (i.e., c, p and K), the retailer would definitely place higher order cycle when deteriorating items

Table 2 Sensitivity analysis on parameters of Example 2

Changing parameters	Change	T^*	N^*	TP^*_{12}
K	2250	0.0603	0.0933	75,308.32
	2500	0.0568	0.0884	84,055.18
	2750	0.0539	0.0841	92,821.75
	3000	0.0512	0.0803	101,605.80
a	0.34	0.0484	0.0598	101,164.40
	0.36	0.0492	0.0662	101,297.10
	0.38	0.0502	0.0731	101,444.00
	0.40	0.0512	0.0803	101,605.80
b	0.05	0.0512	0.0803	101,605.80
	0.10	0.0486	0.0608	101,191.10
	0.15	0.0468	0.0441	100,884.30
	0.20	0.0457	0.0291	100,671.00
p	25	0.0647	0.0857	57,584.76
	30	0.0592	0.0847	72,215.27
	35	0.0549	0.0827	86,892.15
	40	0.0512	0.0803	101,605.80
c	4	0.0512	0.0803	101,605.80
	5	0.0508	0.0753	98,564.68
	6	0.0505	0.0707	95,530.93
	7	0.0503	0.0663	92,503.94
A	200	0.0512	0.0803	101,605.80
	225	0.0548	0.0855	101,134.70
	250	0.0581	0.0903	100,692.40
	275	0.0613	0.0948	100,274.10
M	0.03	0.0543	0.0820	101,083.00
	0.04	0.0536	0.0818	101,224.20
	0.05	0.0525	0.0812	101,398.00
	0.06	0.0512	0.0803	101,605.80
m	0.4	0.0372	0.0606	99,046.40
	0.5	0.0421	0.0675	100,139.10
	0.6	0.0468	0.0741	100,958.30
	0.7	0.0512	0.0803	101,605.80

exhibit a longer expiration date m. That is, since deteriorating items can be kept for a longer time, the quantity and quality losses can slightly slowdown, which stimulates the market demand to some degree and ultimately enhances the retailer's performance. For the given expiration date m, along with the increases of c (or decrease of p and K), the retailer's performance can be gradually weakened

Table 3 Sensitivity analysis on parameters of Example 3

Changing parameters	Change	T^*	N^*	TP_2^*
K	2250	0.0555	0.9790	74,643.42
	2500	0.0526	0.9832	83,347.70
	2750	0.0502	0.9868	92,071.11
	3000	0.0480	0.9900	100,811.10
a	0.245	0.0484	0.5413	100,326.40
	0.247	0.0483	0.6986	100,456.20
	0.249	0.0482	0.8479	100,618.20
	0.251	0.0480	0.9900	100,811.10
b	0.145	0.0480	0.9900	100,811.10
	0.147	0.0482	0.8204	100,592.50
	0.149	0.0483	0.6464	100,416.90
	0.151	0.0484	0.4672	100,284.80
p	34	0.0527	0.3881	82,891.96
	36	0.0510	0.5992	88,802.64
	38	0.0495	0.7995	94,776.61
	40	0.0480	0.9900	100,811.10
c	4	0.0480	0.9900	100,811.10
	4.2	0.0482	0.8119	100,029.10
	4.4	0.0484	0.6425	99,292.52
	4.6	0.0486	0.4812	98,597.00
A	200	0.0480	0.9900	100,811.10
	225	0.0510	0.9857	100,306.50
	250	0.0537	0.9816	99,829.41
	275	0.0564	0.9777	99,375.73
M	0.15	0.0481	0.9531	100,589.30
	0.20	0.0481	0.9654	100,663.00
	0.25	0.0481	0.9776	100,736.90
	0.30	0.0480	0.9900	100,811.10
m	0.55	0.0430	0.9945	99,830.85
	0.60	0.0447	0.9928	100,198.60
	0.65	0.0464	0.9913	100,522.70
	0.70	0.0480	0.9900	100,811.10

(improved). Further, it is apparent that the retailer's order incentive would be slightly impaired if more holding cost should be paid off. And the retailer would decrease order cycles when confronted with higher selling price or demand rate.

6 Conclusions

In this paper, we have developed an inventory model for deteriorating items with expiration date which are expected to experience both quantity loss and quality loss during the retail stage that can only be maintained within expiration date to investigate the optimal retailer's replenishment decisions within the EOQ framework to reflect the realistic business situations. In addition, two-level trade credit has been incorporated to hedge against negative impacts of expiration date towards the retailer's order incentive. The supplier provides the retailer a delay period and the retailer also adopts the trade credit policy to stimulate his/her customer demand to reflect realistic business situations. The quantity loss decreases inventory level sharply, while the quality loss impairs customers' willingness to consume items, which ultimately influence the retailer's operation performance. This paper also considered a multiplicative demand function assumed to be function of the product freshness and credit period granted by the retailer, and also granting credit period increases not only demand but also default risk. First, we investigate the retailer's inventory system for deteriorating items as a profit maximization problem to determine the retailer's optimal inventory policies. From the view point of the profits, decision rules to find the optimal cycle time T^* and optimal credit period N^* contains two cases: (i) $N \leq M$ and (ii) $N \geq M$. In order to obtain the optimal ordering policies, we propose some lemmas to help the retailer in accurately and quickly determine the optimal replenishment decisions under maximizing the annual total profit. Finally, we have used software MATLAB to study the sensitivity analysis on the optimal solution with respect to each parameter to illustrate the model and provide some managerial insights.

From the numerical analysis of Example 1–3, it is clear that the sub-case 1.1 ($T + N \leq M$) is more profitable for all entities. Because in this case, the retailer receives sales revenue of all items at time $T + N$ and is able to pay off total purchasing cost by M. Therefore, there is no interest charged. On the other hand, during the period $[N, T + N]$ retailer can earned interest on the sale revenues received from customers and on full sales revenue during the period $[T + N, M]$. Thus, the retailer will follow sub-case 1.1.

This work can be extended in the following dimensions. One immediate possible extension could be allowable shortages, cash discounts, etc. Additionally, in traditional marketing and economic theory, price is a major factor on the demand rate. As a result, one could take pricing strategy into consideration in the future research.

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