OPSEARCH (2020) 57:584–602
https://doi.org/10.1007/s12597-019-00431-x

APPLICATION ARTICLE

Modified distance measure on hesitant fuzzy sets and its application in multi-criteria decision making problem

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Accepted: 17 November 2019 / Published online: 27 November 2019

Abstract
The distance measure based on hesitant fuzzy sets is an effective tool in the field of treating similar objects where it distinguishes the difference between two objects. Several distance measures have been proposed so far by different researchers. In this paper, we have proposed modifications in the existing distance measure so that some situations in real life conditions can be handled easily with the proposed distance measure whereas the existing one can not. Finally, the validity and applicability of the proposed distance measure is discussed with some existing examples.

Keywords Hesitant fuzzy sets · Hesitant fuzzy elements · Distance measure · Similarity measure

1 Introduction

Distance and similarity measures are important tools for finding the differences between two objects. Distance and similarity measure can be applied in many areas such as decision making, pattern recognition, image processing, machine learning, market prediction and so on. Initially Wang [1] introduced the concept of fuzzy sets' similarity measure with a computational formula. Since then many researchers started following this topic and extended further. There are many distance and similarity measures proposed for fuzzy set, intuitionistic fuzzy set and fuzzy multiset etc. The Hamming distance, the Euclidean distance and the Housdorff distance are three popular and widely used distance measures. The relationship measure, the similarity measure and the fuzziness of fuzzy set are investigated by Zeng and Li [2]. Szmidt and Kacprzyk [3] studied a new distance between two intuitionistic fuzzy set. Torra and Narukawa [4] introduced weighted distance measure for intuitionistic fuzzy sets based on the Choquet integral with respect to the non-monotonic fuzzy measure. Xia and Xu [5] extended the distance and similarity measure based on hesitant fuzzy set.

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Peng et al. [6] proposed the generalized hesitant fuzzy synergetic weighted distance measure and applied it to multi-criteria decision making problem. Li et al. [7] also studied new distance and similarity measure on hesitant fuzzy set and their application in multi-criteria decision making problem. Wenyi et.al. [8] applied distance and similarity measures in pattern recognition. Harish et al. [9] used a new concept of dual hesitant fuzzy soft set in distance and similarity measure and have shown its application in MCDM problems. Pratiksha et al. [10] studied distance and similarity measures in interval-valued intuitionistic fuzzy set. Wei [11] introduced similarity measures for picture fuzzy sets and its application.

In our real life we come across certain situations where decision making method plays vital role. The decisions given by experts are based on their experience and perception about the conditions of the system and some times there is variation in perceiving the conditions among the experts. So even there is a variation in the variation among the experts in perception all the decisions or the outcomes are reasonable though their impact on the system performance may differ. We have to consider each and every value given by those experts. For example two persons have drawn a picture from a certain given picture. Then the pictures are judged by two experts in terms of degree of their closeness/similarity to the original sample supplied to them. The view of the two experts on the degree of closeness of two pictures with the original is likely to be slightly different. And now if we want to find the degree of closeness of the two pictures to the original one with existing distance and similarity measures, we may come out with equal value of degree of closeness for both the pictures which means that the method fails to solve the problem as it has failed to address the subtle difference in the perception of the experts. To solve these types of problems, we have proposed the modified distance measures in this paper.

The rest of the paper is organized as follows:

Section 2 gives the preliminaries and definition of hesitant fuzzy set and its related distance measures. In Sect. 3, the proposed modified distance measure is detailed. In Sect. 4, the proposed distance measure is applied on some existing examples and validity is studied. Section 5 consists of the conclusion and short note on future work.

2 Preliminaries

Throughout the paper, we use $X = \{x_1, x_2, x_3, \dots, x_n\}$ to denote the universal set, HFS and HFE stand for hesitant fuzzy set and hesitant fuzzy element, respectively. A stands for a HFS and $A(x)$ stands for a HFE, $\tilde{A}$ stands for the set of all hesitant fuzzy sets in X. $l(A(x))$ stands for the total number of elements in $A(x)$.

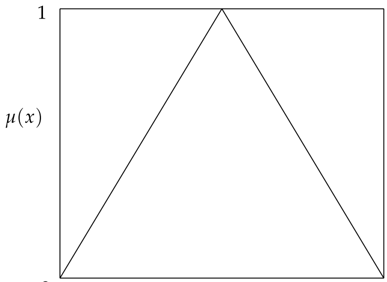
Definition 1 [12] Given a fixed set X, then a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of [0, 1].

For convenience, the HFS is often expressed simply by mathematical symbol

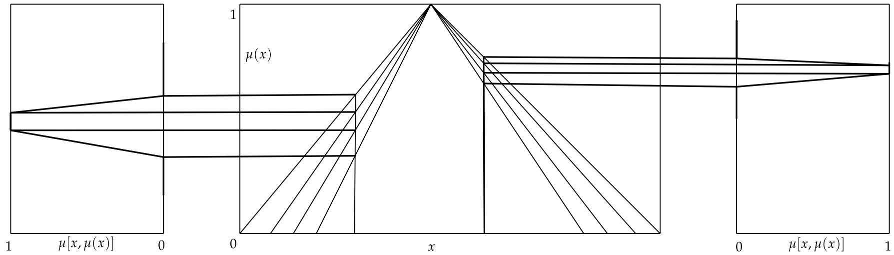
$$A = (< x, A(x) > | x \in X)$$

where $A(x)$ is a set of some values in [0, 1], denoting the possible membership degree of the element $x \in X$ to the set A. $A(x)$ is called a hesitant fuzzy element (HFE).

Graphically we can represent fuzzy set as follows:



Type-1 Fuzzy Set



Type-2 Fuzzy Set

Figure a describes the concept of fuzzy set where $\mu(x)$ is the membership values from 0 to 1 and x is an element of the crisp set. When decision maker gives his own view about the belongingness of a crisp value of x to the perception about the fact represented by x that means he shares his feelings towards the state of that particular problem. If he is fully satisfied and certain about the belongingness of the value of x in the particular state represented by a membership function or if he fully or confidently disagree, he assigns the membership value 1 or 0 respectively. And for these types of un-ambiguous situations there is no hesitancy. However, if the designer is partially satisfied or has reservation in judging the belongingness of the value x in the state represented by the membership function then the membership value lies between 0 and 1. In case of fuzzy set we get only one membership value as shown in triangular membership function in Fig. a.

However, the issue is with the partial or reserved view about the belongingness of x to the particular membership state as there are chances of lot of variation among the experts/designers in their perception of the state. The challenge is how to represent the wide variation of views/reservation of designers more reasonably and appropriately in these types of cases? The efforts in the making of more appropriate and accurate representation of belongingness of x to the membership state resulted into the development of higher level of fuzzy sets like type-2 fuzzy set, interval type-2 fuzzy set and intuitionistic fuzzy set to address the conditions with higher level of uncertainty or ambiguity. Why the extension? The answer is the requirement for more accurate and reasonable representation of the variation of degree of belongingness of x to the membership state due to higher uncertainty or ambiguity. Hesitant fuzzy set comes under this category. However, there is a difference. While in other level fuzzy sets efforts are made for representing the varied values of degree of membership among the designers due to variation in their feelings about the belongingness of x to a particular membership state with another mathematical function which requires again another approximation (continuous) of really varied options (discrete) of experts. Whereas in hesitant fuzzy set all the varied opinions (values) about the belongingness of x to the particular membership set are considered instead of mathematical function thereby making the representation more realistic and true to the facts. The representation of type-2 fuzzy set is shown in Fig. b.

Further, there are some cases where the decision maker is confused to assign only one membership value because of higher uncertainty. So instead of giving only one membership value, if a set of values are considered then the representation will be closer to his feeling and more realistic. Moreover, some other cases are also there where more than one experts are engaged to evaluate a particular problem then each and every decision maker may suggested his own opinion as a value for the degree of belongingness. So, we get a number of membership values against a single crisp value and this extension is called hesitant fuzzy set. More variation means more hesitancy associated. In real life very often we can not take decision about a complex system depending upon only one's opinion. If we conduct an interview to select the best candidate where multiple criteria are to be judged but only one expert is engaged then the outcome may not be satisfactory because one expert is not able to judge all the criteria at the same time but if we engage more than one experts then the result will be better than the first case being varied opinions and ultimately wiser. So, in decision making problems hesitant fuzzy works as an important tool.

Definition 2 [13] Let A_1 and A_2 be two hesitant fuzzy sets on $X = \{x_1, x_2, x_3, \dots, x_n\}$, then the distance measure between A_1 and A_2 is denoted as $d(A_1, A_2)$, which satisfies the following properties:

(D1) $0 \le d(A_1, A_2) \le 1$;
(D2) $d(A_1, A_2) = 0$ if and only if $A_1 = A_2$;
(D3) $d(A_1, A_2) = d(A_2, A_1)$.
(D4) For three hesitant fuzzy elements $A_1(x)$, $A_2(x)$ and $A_3(x)$ which have the same length l and $A_k(x) = \{A_k^1(x), A_k^2(x), \dots, A_k^l(x)\}$, $k = 1, 2, 3$ if $A_1^i(x) \le A_2^i(x) \le A_3^i(x), i = \{1, 2, \dots, l\}$ then $d(A_1(x), A_2(x)) \le d(A_1(x), A_3(x)), d(A_2(x), A_3(x)) \le d(A_1(x), A_3(x))$.

Definition 3 [13] Let A_1 and A_2 be two hesitant fuzzy sets on $X = \{x_1, x_2, x_3, \dots, x_n\}$, then the similarity measure between A_1 and A_2 is denoted as $s(A_1, A_2)$, which satisfies the following properties:

(S1) $0 \le s(A_1, A_2) \le 1$;
(S2) $s(A_1, A_2) = 1$ if and only if $A_1 = A_2$;
(S3) $s(A_1, A_2) = s(A_2, A_1)$.

The following property can be obtained as:

Property 1 *If d is the distance measure between hesitant fuzzy sets A_1 and A_2, then $s(A_1, A_2) = 1 - d(A_1, A_2)$ is the similarity measure between hesitant fuzzy sets A_1 and A_2.*

Property 2 *If s is the similarity measure between hesitant fuzzy sets A_1 and A_2, then $d(A_1, A_2) = 1 - s(A_1, A_2)$ is the distance measure between hesitant fuzzy sets A_1 and A_2.*

Sometimes it can be happened that the number of elements in different hesitant fuzzy elements may be different i.e. if A_1 and A_2 be two HFEs then in most of the cases $l(A_1(x)) \neq l(A_2(x))$. According to Xu and Xia [13], the optimist expert can extend the shorter one by adding the maximum value while the pessimist expert add the minimum value to make the lengths equal. This selection of the values mainly depends on the decision makers' risk preferences. In this paper, the shorter one is extended by adding minimum value.

The hesitant normalized Hamming distance, Euclidean distance and generalized hesitant normalized distance given in [13] as follows:

$$d_h(A_1, A_2) = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |A_1^j(x_i) - A_2^j(x_i)| \right] \tag{1}$$

and

$$d_e(A_1, A_2) = \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |A_1^j(x_i) - A_2^j(x_i)|^2 \right) \right]^{\frac{1}{2}} \tag{2}$$

and

$$d_g(A_1, A_2) = \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |A_1^j(x_i) - A_2^j(x_i)|^\lambda \right) \right]^{\frac{1}{\lambda}} \tag{3}$$

where $\lambda > 0, A_1^j(x_i)$ and $A_2^j(x_i)$ are the jth values in $A_1(x_i)$ and $A_2(x_i)$, respectively, and $l_{x_i} = \max\{l(A_1(x_i)), l(A_2(x_i))\}$.

If the weight w_i of each element $x_i \in X$ is taken into account, Xu and Xia [13] defined the generalized hesitant weighted distance as follows:

$$d_{wg}(A_1, A_2) = \left[\sum_{i=1}^{n} w_i \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |A_1^j(x_i) - A_2^j(x_i)|^\lambda \right) \right]^{\frac{1}{\lambda}} \tag{4}$$

For two HFEs $A_1(x)$ and $A_2(x)$ on $X = \{x_1, x_2, x_3, \ldots, x_n\}$, Xia and Xu [14] proposed several distance measure between $A_1(x)$ and $A_2(x)$ as follows:

$$d_1(A_1, A_2) = \left[\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |A_1^j(x_i) - A_2^j(x_i)| \right] \tag{5}$$

and

$$d_2(A_1, A_2) = \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |A_1^j(x_i) - A_2^j(x_i)|^2 \right)^{\frac{1}{2}} \tag{6}$$

where $A_1^j(x_i)$ and $A_2^j(x_i)$ are the jth values in $A_1(x_i)$ and $A_2(x_i)$, respectively, and $l_{x_i} = \max\{l(A_1(x_i)), l(A_2(x_i))\}$.

Definition 4 [7] Let A be a hesitant fuzzy set on $X = \{x_1, x_2, x_3, \ldots, x_n\}$, and for any $x_i \in X$, $l(A(x_i))$ be the length of $A(x_i)$. Denote

$$u(A(x_i)) = 1 - \frac{1}{l(A(x_i))}$$

$$u(A) = \frac{1}{n} \sum_{i=1}^{n} u(A(x_i))$$

where $u(A(x_i))$ is the hesitant degree of $(A(x_i))$ and $u(A)$ is the hesitant degree of A.

For two HFSs $A_1(x)$ and $A_2(x)$ on $X = \{x_1, x_2, x_3, \ldots, x_n\}$, Li et al. [7] proposed the normalized Hamming distance, normalized Euclidean distance, normalized generalized distance and generalized hesitant weighted distance, all including hesitant degree as follows:

$$d_{hh}(A_1, A_2) = \frac{1}{2n} \sum_{i=1}^{n} \left[|u(A_1(x_i)) - u(A_2(x_i))| + \frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |A_1^j(x_i) - A_2^j(x_i)| \right] \tag{7}$$

and

$$d_{he}(A_1,A_2)=\left[\frac{1}{2n}\sum_{i=1}^{n}\left(|u(A_1(x_i))-u(A_2(x_i))|^2+\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|A_1^j(x_i)-A_2^j(x_i)|^2\right)\right]^{\frac{1}{2}} \tag{8}$$

and

$$d_{hg}(A_1,A_2)=\left[\frac{1}{2n}\sum_{i=1}^{n}\left(|u(A_1(x_i))-u(A_2(x_i))|^\lambda+\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|A_1^j(x_i)-A_2^j(x_i)|^\lambda\right)\right]^{\frac{1}{\lambda}} \tag{9}$$

and

$$d_{whg}(A_1,A_2)=\left[\frac{1}{2}\sum_{i=1}^{n}w_i\left(|u(A_1(x_i))-u(A_2(x_i))|^\lambda+\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|A_1^j(x_i)-A_2^j(x_i)|^\lambda\right)\right]^{\frac{1}{\lambda}} \tag{10}$$

where $\lambda>0, A_1^j(x_i)$ and $A_2^j(x_i)$ are the jth values in $A_1(x_i)$ and $A_2(x_i)$, respectively, and $l_{x_i}=\max\{l(A_1(x_i)), l(A_2(x_i))\}, w_i$ is the weight of each $x_i\in X$ s.t. $0\le w_i\le 1$ and $\sum_{i=1}^{n}w_i=1$.

In hesitant fuzzy set, when the similarity between two objects is calculated, the divergence of HFSs is taken into account. The divergence of HFSs includes the divergence of HFEs. If we notice more closely, the similarity is measured according to the divergence of HFEs which consists of their lengths and values. Li et al. [7] introduced the hesitant degree where they have considered the length of divergence because the divergence of the values only can not solve some special type of problems. But real life problems are so complicated that the existing distance measures also fail to give any reasonable results for such problems.

Example 1 Let $X=\{x\}$. Let us assume that there exists two patterns which are represented by hesitant fuzzy sets $A_1=\{0.9,0.8,0.7\}$ and $A_2=\{0.7,0.6,0.2\}$. Now there is a sample to be recognized which is represented by a hesitant set $A=\{0.7,0.65,0.6\}$, the principle of minimum distance measure of hesitant fuzzy set is given by:

$$d(A_{i0},A)=\min_{1\le i\le 2}\{d(A_1,A),d(A_2,A)\}$$

which means the sample A belongs to the pattern A_{i0}.

Now applying the above mentioned distance measures Eqs. (1) and (7), we get, $d_h(A_1,A)=0.15,\ d_h(A_2,A)=0.15,$ and $d_{hh}(A_1,A)=0.075,\ d_{hh}(A_2,A)=0.075,$ respectively.

From Eq. (1) we can not get the minimum distance and after applying hesitant degree in Eq. (7), again the problem remains unsolved.

The difference between the membership values of A_2 is very large. This shows that the pattern A_2 is unique. Finally we can conclude that the sample A belongs to the pattern A_1 as having similar small difference between the values. After applying the standard distance measure on the above example, we observe equal distances in all the cases. So the above example increases our hesitancy and for that reason it can not be clearly stated that A belongs to the pattern A_1.

Therefore, to solve these types of problems we need to modify the existing distance measure. In the following section we have proposed a new distance measure for overcoming the above limitations.

3 Proposed methodology

In the following, some new distance measures are proposed.

Definition 5 Let A_1 and A_2 be two HFSs on $X = \{x_1, x_2, x_3, \dots, x_n\}$, then the modified normalized Hamming distance between $A_1(x_i)$ and $A_2(x_i)$ is defined as:

$$d_{mh}(A_1, A_2) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|\right] \tag{11}$$

The modified normalized Euclidean distance between $A_1(x_i)$ and $A_2(x_i)$ is defined as

$$d_{me}(A_1, A_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|^2\right)\right]^{\frac{1}{2}} \tag{12}$$

and the modified normalized generalized distance between $A_1(x_i)$ and $A_2(x_i)$ is defined as

$$d_{mg}(A_1, A_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|^\lambda\right)\right]^{\frac{1}{\lambda}} \tag{13}$$

where $\lambda > 0, A_1^j(x_i)$ and $A_2^j(x_i)$ are the jth values in $A_1(x_i)$ and $A_2(x_i)$, respectively, and $l_{x_i} = \max\{l(A_1(x_i)), l(A_2(x_i))\}$.

Usually, the weight of the element $x \in X$ should be taken into account. The weighted distance measures for HFSs are presented as follows:

Assume that the weight of $x_i \in X$ is $w_i (i = 1, 2, \dots, n)$, where $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$, then we have the following weighted distance measures:

The modified normalized weighted Hamming distance between $A_1(x_i)$ and $A_2(x_i)$ is defined as:

$$d_{mwh}(A_1, A_2) = \frac{1}{n}\sum_{i=1}^{n} w_i\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|\right] \tag{14}$$

The modified normalized weighted Euclidean distance between $A_1(x_i)$ and $A_2(x_i)$ is defined as

$$d_{mwe}(A_1, A_2) = \left[\frac{1}{n}\sum_{i=1}^{n} w_i\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|^2\right)\right]^{\frac{1}{2}} \tag{15}$$

and the modified normalized weighted generalized distance between $A_1(x_i)$ and $A_2(x_i)$ is defined as

$$d_{mwg}(A_1, A_2) = \left[\frac{1}{n}\sum_{i=1}^{n} w_i\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|^\lambda\right)\right]^{\frac{1}{\lambda}} \tag{16}$$

where $\lambda > 0, A_1^j(x_i)$ and $A_2^j(x_i)$ are the jth values in $A_1(x_i)$ and $A_2(x_i)$ respectively, and $l_{x_i} = \max\{l(A_1(x_i)), l(A_2(x_i))\}$.

4 Application

Applying the proposed distance measure in the above Example 1, we get $d_{mh}(A_1, A) = 0.1017$ and $d_{mh}(A_2, A) = 0.18$. The distance measure between A and A_1 is less than that of A and A_2. So clearly we can say that A belongs to the pattern A_1 which is exactly matched what we have observed initially and also logically we have explained the similarity of A and A_1. But the same example is not solved by the existing distance measures.

To validate the proposed distance measure, the following example is taken into account.

Example 2 [7] Let $X = \{x\}$. Let us assume that there exist two patterns which are represented by hesitant fuzzy sets $h_1 = \{0.97, 0.95, 0.88, 0.86, 0.82, 0.8\}$ and $h_2 = \{0.45\}$. Now there is a sample to be recognized which is represented by a hesitant set $h = \{0.75, 0.73, 0.7, 0.65, 0.6, 0.55\}$.

Firstly, we extend h_2 as $h_2 = \{0.45, 0.45, 0.45, 0.45, 0.45, 0.45\}$, and apply the proposed distance measure, $d_{mh}(h, h_1) = 0.142, d_{mh}(h, h_2) = 0.187$. Since the distance between h and h_1 is less than that of h and h_2. This shows that h belongs to the pattern h_1. According to Li et. al. [7], $d_{hh}(h, h_1) = 0.10835, d_{hh}(h, h_2) = 0.52165$.

That means h belongs to the pattern h_1 which is exactly similar with our result. Therefore, this proves that the proposed distance measure is valid.

The proposed distance measure can solve the following example which is not solved by the existing measures:

Example 3 Suppose that E denotes the set of all equilateral triangle, where $E = \{\alpha, \beta, \gamma | \alpha = \beta = \gamma = 60°\}$. For every triangle T, then T is considered as a fuzzy set in E, thus the membership degree of the fuzzy set T is used to reflect the degree that the triangle (fuzzy set) T is related to the equilateral triangle (Fig. 1).

Let us consider a real life example. Let E_1 be a triangle such that $E_1 = \{\alpha, \beta, \gamma | \alpha = 70°, \beta = 55°, \gamma = 55°\}$ which is similar to the equilateral triangle.

Graphically, it is shown in Fig. 2.

Now consider another two triangles T_1 and T_2 such that $T_1 = \{\alpha, \beta, \gamma | \alpha = 75°, \beta = 55°, \gamma = 50°\}$ and $T_2 = \{\alpha, \beta, \gamma | \alpha = 80°, \beta = 50°, \gamma = 50°\}$ represented by Fig. 3 and Fig. 4 respectively.

Suppose the membership values of E_1, T_1 and T_2 given by the decision makers are as follows:

$A = \{(E_1, h), (T_1, h_1), (T_2, h_2)\}$ i.e. $A = \{(E_1, \{0.6, 0.5\}), (T_1, \{0.7, 0.65\}), (T_2, \{0.7, 0.35\})\}$.

According to the expert's decision, none of the triangles belongs to the set of equilateral triangles. Now, if we compare the similarity among these three triangles E_1, T_1 and T_2, we have to decide a sample image among these three. Initially let us consider E_1 as the sample image. Using distance measures, the minimum distance between (E_1, T_1) and (E_1, T_2) can be calculated.

From the above membership values, we get $h_1 = \{0.7, 0.65\}$ and $h_2 = \{0.7, 0.35\}$, and $h = \{0.6, 0.5\}$. If we think logically again we get that

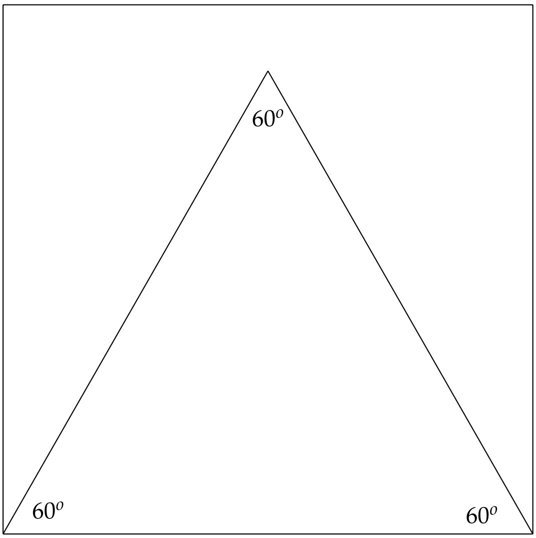


Fig. 1 Graphical representation of the equilateral triangle E

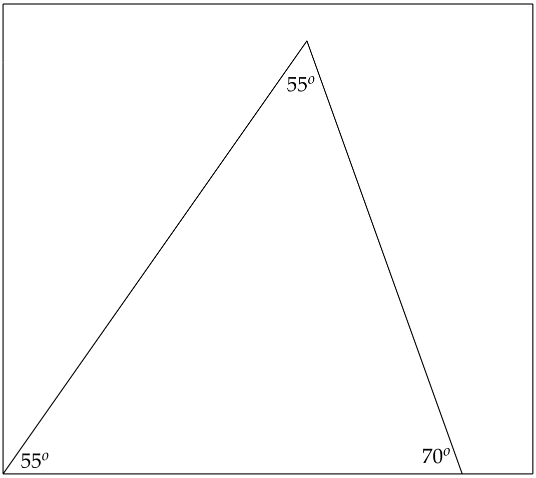


Fig. 2 Graphical representation of the triangle E_1

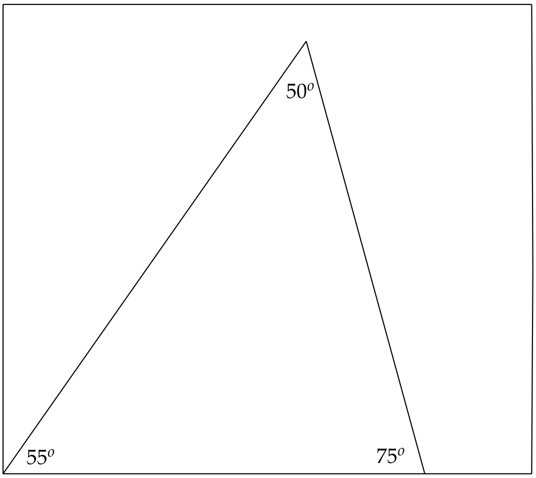


Fig. 3 Graphical representation of the triangle T_1

the difference between the values of h_2 is very large, so the shape of the triangle h_2 is unique and never similar to h. This means h belongs to h_1. When we apply the Hamming distance measure equation (1), then we have $d_h(h, h_1) = 0.125, d_h(h, h_2) = 0.125$. If we apply the Euclidean distance measure equation (2), then $d_e(h, h_1) = 0.127, d_e(h, h_2) = 0.127$. If we apply the Hamming distance measure including hesitant degree equation (7), then $d_{hh}(h, h_1) = 0.0625, d_{hh}(h, h_2) = 0.0625$. That means none of the existing distance measures can solve this problem. Now applying the proposed distance measure equation (11) and (12), then we have $d_{mh}(h, h_1) = 0.1037, d_{mh}(h, h_2) = 0.1267$, $d_{me}(h, h_1) = 0.1070, d_{me}(h, h_2) = 0.1361$. That means the sample h belongs to the pattern h_1, which is exactly matched with our initial assumption. It indicates that d_{mh} and d_{me} are reasonable.

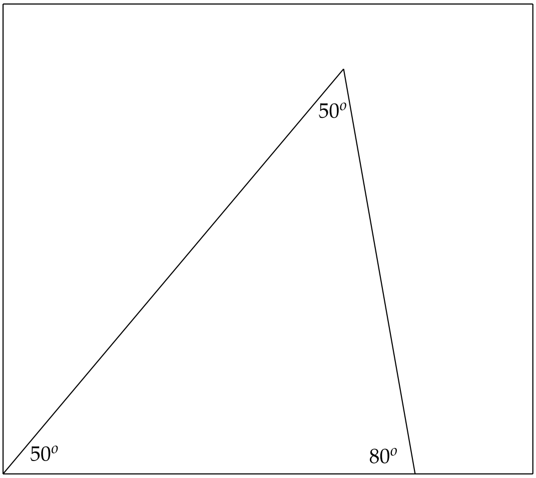


Fig. 4 Graphical representation of the triangle T_2

Also the following example is a special type of example which can not be solved by the distance measure including hesitant degree but it is solved by the standard distance measure as well as the proposed distance measure.

Example 4 Let us consider another real life example where E_1 is almost similar to equilateral triangle such that $E_1 = \{\alpha, \beta, \gamma | \alpha = 62°, \beta = 59°, \gamma = 59°\}$ graphically shown in Fig. 5.

Also let us consider another two triangles T_1 and T_2 such that $T_1 = \{\alpha, \beta, \gamma | \alpha = 64°, \beta = 58°, \gamma = 58°\}$ and $T_2 = \{\alpha, \beta, \gamma | \alpha = 82°, \beta = 50°, \gamma = 48°\}$ (Fig. 6 and Fig. 7 respectively).

Suppose the membership values of E_1, T_1 and T_2 given by the decision makers are as follows:

$A = \{(E_1, h), (T_1, h_1), (T_2, h_2)\}$ i.e. $A = \{(E_1, \{0.9\}), (T_1, \{0.95, 0.9\}), (T_2, \{0.4\})\}$.

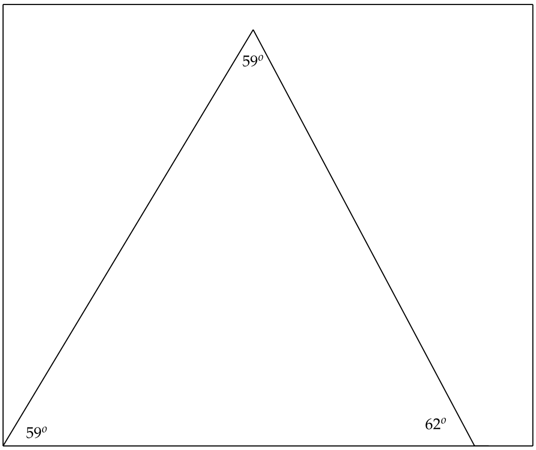


Fig. 5 Graphical representation of the triangle E_1

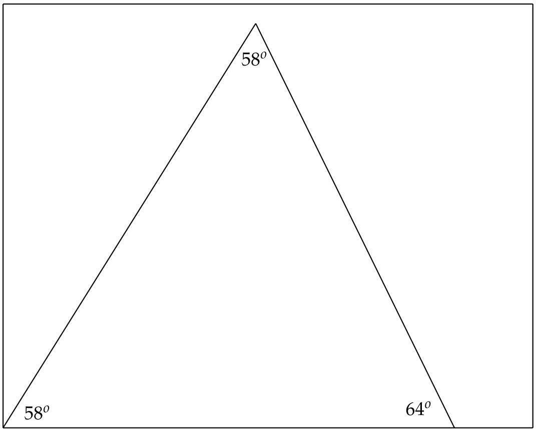


Fig. 6 Graphical representation of the triangle T_1

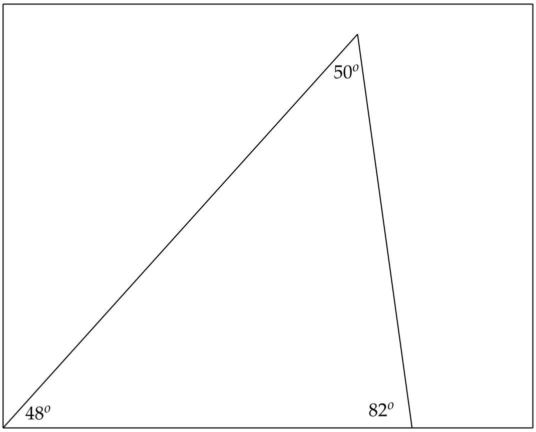


Fig. 7 Graphical representation of the triangle T_2

According to the expert's decision, the triangles E_1 and T_1 are almost similar to the equilateral triangle. Using distance measures the similarity among these triangles E_1, T_1 and T_2 can be compared. Initially let the sample image be E_1. The minimum distance between (E_1, T_1) and (E_1, T_2) can be calculated as follows.

For calculation we have $h_1 = \{0.95, 0.9\}$ and $h_2 = \{0.4\}$, and a triangle $h = \{0.9\}$. If we think logically again we get that the difference between the values of h_1 is very small and this indicates that the shape of the triangle is almost equivalent to the equilateral triangle, so the shape of the triangle h_1 may be similar to h. This means h belongs to h_1. But the membership value of h_2 is 0.4, which indicates that the shape is not exactly similar to the equilateral triangle because the decision maker is fully confident that the shape is particular and with this confident he has assiged the membership value of h_2. So the shape of h_2 is not matched with h.

For solving this problem existing distance measure including hesitant degree is not sufficient. If we apply the Hamming distance measure including hesitant degree equation (7), then $d_{hh}(h, h_1) = 0.2625, d_{hh}(h, h_2) = 0.25$. That means if we add hesitant degree to the distance measure the result becomes different which is beyond our intuition. So, hesitant degree can not provide the actual result of the solution. When we apply the Hamming distance measure equation (1), then we have $d_h(h, h_1) = 0.025, d_h(h, h_2) = 0.5$. If we apply the Euclidean distance measure equation (2), then $d_e(h, h_1) = 0.035, d_e(h, h_2) = 0.5$. Now applying the proposed distance measure equation (11) and (12), then we have $d_{mh}(h, h_1) = 0.014, d_{mh}(h, h_2) = 0.385$, $d_{me}(h, h_1) = 0.019, d_{me}(h, h_2) = 0.385$. That means the sample h belongs to the pattern h_1, which is exactly matched with our initial assumption. This indicates that the proposed measure reasonable and acceptable.

The proposed distance measure is applied in multi criteria decision making problem with some special assumption:

Example 5 [13], (*Alternative selection*) Energy is an indispensable factor for the socio-economic development of societies. Suppose that there are five alternatives (energy projects) $P_i (i = 1, 2, 3, 4, 5)$ to be invested, and four attributes to be considered: c_1: technological; c_2: environmental; c_3: socio-political; c_4: economic. The attribute weight vector is $w = (0.15, 0.3, 0.2, 0.35)$. Several decision makers are invited to evaluate the performance of the five alternatives. To get a more reasonable result, it is better that the decision makers give their evaluations anonymously. Thus, each value provided only means that it is possible value, but its importance is unknown. So, it is reasonable to allow these values repeated many times appear only once, and all possible evaluations for an alternative under the attributes can be considered as HFS. The result evaluated by the decision makers are contained in a hesitant fuzzy decision matrix, shown in Table 1.

Suppose that the ideal alternative is $P^* = \{1\}$ seen as a special HFS, we can select the best alternative by calculating the distance between each alternative and the ideal alternative.

Now the generalized hesitant weighted distance d_{wg} proposed by Xu and Xia [13] Eq. (4) and the modified generalized weighted distance d_{mwg} proposed in this paper Eq. (16) are shown in Tables 2 and 3, respectively to calculate the deviations between each alternative and the ideal alternative.

Table 1 Hesitant fuzzy decision matrix

	c_1	c_2	c_3	c_4
P_1	$\{0.5, 0.4, 0.3\}$	$\{0.9, 0.8, 0.7, 0.1\}$	$\{0.5, 0.4, 0.2\}$	$\{0.9, 0.6, 0.5, 0.3\}$
P_2	$\{0.5, 0.3\}$	$\{0.9, 0.7, 0.6, 0.5, 0.2\}$	$\{0.8, 0.6, 0.5, 0.1\}$	$\{0.7, 0.4, 0.3\}$
P_3	$\{0.7, 0.6\}$	$\{0.9, 0.6\}$	$\{0.7, 0.5, 0.3\}$	$\{0.6, 0.4\}$
P_4	$\{0.8, 0.7, 0.4, 0.3\}$	$\{0.7, 0.4, 0.2\}$	$\{0.8, 0.1\}$	$\{0.9, 0.8, 0.6\}$
P_5	$\{0.9, 0.7, 0.6, 0.3, 0.1\}$	$\{0.8, 0.7, 0.6, 0.4\}$	$\{0.9, 0.8, 0.7\}$	$\{0.9, 0.7, 0.6, 0.3\}$

The result obtained by using the proposed distance measure d_{mwg} is exactly similar to that of d_{wg}. To investigate this, we consider $\lambda = 1$ in the following. For calculating d_{wg}, the divergence of the values are considered, so P_5 is the best alternative. The proposed distance measure is also depends on the divergence of the values of HFEs. The results obtained from the proposed measure is much smaller than that of the existing one, which indicates that the measure of the divergence of the values gives the clear picture of the actual distance of the given alternatives to the ideal alternative. Since the ranking of the alternative has not changed, so we can say that the proposed distance measure is correct and reasonable.

If we assume that the given decision matrix where the values of P_3 and P_5 are slightly changed, which is represented in Table 4 as follows:

Now the generalized hesitant weighted distance d_{wg} proposed by Xu and Xia [13] Eq. (4) and the modified generalized weighted distance d_{mwg} proposed in this paper Eq. (16) are shown in Tables 5 and 6, respectively to calculate the deviations between each alternative and the ideal alternative, considering $\lambda = 1$.

According to the problem we have considered, the existing distance measure d_{wg} fails to arrange the ranking because the alternative P_3 and P_5 have the same distance with the ideal alternative. But when we apply the proposed distance measure d_{mwg}, the ranking comes out appropriately. With this assumed multi

Table 2 Results obtain by distance measure d_{wg}

	P_1	P_2	P_3	P_4	P_5	Rankings
$\lambda = 1$	0.4779	0.5027	0.4025	0.4292	0.3558	$P_5 > P_3 > P_4 > P_1 > P_2$
$\lambda = 2$	0.5378	0.5451	0.4366	0.5052	0.4129	$P_5 > P_3 > P_4 > P_1 > P_2$
$\lambda = 6$	0.6599	0.6476	0.5156	0.6704	0.5699	$P_3 > P_5 > P_2 > P_1 > P_4$
$\lambda = 10$	0.7213	0.7046	0.5607	0.7373	0.6537	$P_3 > P_5 > P_2 > P_1 > P_4$

Table 3 Results obtain by distance measure d_{mwg}

	P_1	P_2	P_3	P_4	P_5	Rankings
$\lambda = 1$	0.3499	0.3630	0.2661	0.3155	0.2393	$P_5 > P_3 > P_4 > P_1 > P_2$
$\lambda = 2$	0.4159	0.3708	0.2978	0.3179	0.3005	$P_3 > P_5 > P_4 > P_2 > P_1$
$\lambda = 6$	0.5658	0.4714	0.3778	0.5859	0.4807	$P_3 > P_2 > P_5 > P_1 > P_4$
$\lambda = 10$	0.6400	0.5207	0.4210	0.6583	0.5796	$P_3 > P_2 > P_5 > P_1 > P_4$

Table 4 Assumed hesitant fuzzy decision matrix

	c_1	c_2	c_3	c_4
P_1	{0.5, 0.4, 0.3}	{0.9, 0.8, 0.7, 0.1}	{0.5, 0.4, 0.2}	{0.9, 0.6, 0.5, 0.3}
P_2	{0.5, 0.3}	{0.9, 0.7, 0.6, 0.5, 0.2}	{0.8, 0.6, 0.5, 0.1}	{0.7, 0.4, 0.3}
P_3	{0.9, 0.7, 0.6}	{0.9, 0.8, 0.6}	{0.9, 0.5, 0.3}	{0.6, 0.4}
P_4	{0.8, 0.7, 0.4, 0.3}	{0.7, 0.4, 0.2}	{0.8, 0.1}	{0.9, 0.8, 0.6}
P_5	{0.9, 0.7, 0.6, 0.3, 0.1}	{0.8, 0.7, 0.5, 0.3}	{0.9, 0.8, 0.7}	{0.9, 0.7, 0.6, 0.3}

Table 5 Results obtain by distance measure d_{wg}

	P_1	P_2	P_3	P_4	P_5	Rankings
$\lambda = 1$	0.4779	0.5027	0.372	0.4292	0.372	$P_5 = P_3 > P_4 > P_1 > P_2$

Table 6 Results obtain by distance measure d_{mwg}

	P_1	P_2	P_3	P_4	P_5	Rankings
$\lambda = 1$	0.3499	0.3249	0.2457	0.3155	0.2537	$P_3 > P_5 > P_4 > P_2 > P_1$

criteria decision making problem, we can conclude that the proposed distance measure is better and reasonable then the existing ones.

5 Results and discussion

Suppose a sample object (i.e. equilateral triangle) is given and two more objects (i.e. similar to equilateral triangle) are to be judged by experts that which one of the given objects belongs to the sample object. Then the experts give their own feelings in terms of membership value according to the best of their knowledge. After getting their views, we can easily reach to the final solution. All these events are based on our perception that which object belongs to the given sample. To make the method of decision making based on the perceptions more scientific some mathematical indices like distance measures are proposed. That is why many researchers have proposed distance measure based on some logic. But in real life all the problems we face are not always similar. In this paper we have modified the formula of distance measure to overcome the limitation that we have faced in some examples. All the formulae of distance measure depend upon the differences of the membership values given by experts. But in some cases it may happen that after measuring all the distances, the value of the outcome is either equal or incorrect. We have proposed the Example 3 where the distance measure given by all the traditional methods failed to give the final solution because the distance measure given by them is equal for all the cases. Again in Example 4 we have seen that the outcome given by the existing distance measure [7] is incorrect. In these cases our proposed distance measure works properly and has given an accurate result. Examples 2, 3, 4, 5 are enough to show that our proposed distance measure is better than the existing measures.

We can apply our proposed distance measure successfully in the field of image classification, pattern recognition, image processing, machine learning, market prediction, power plant site selection, many areas in engineering and medical science etc. where decision making method plays an important role. In this paper, the proposed method is applied in Example 5 which is a multi criteria decision making problem and the results what we have discussed are fully logical and more appropriate.

6 Conclusion

Using hesitant fuzzy set many distance and similarity measures have been proposed so far. In this paper, we have proposed modifications of some existing distance measures to overcome the limitations of the existing measures to deal with real life situations and give appropriate results. In our day to day life we come across different situations like pattern recognition i.e. if doctors want to diagnose a patient by drawing similarity of symptoms of two patients based on their perceptions/views on different parameters, some situations may arise due to higher hesitancy leading to failure of the decision making process to come out with clear verdict about the disease if the existing distance measures are used. But if the proposed measures are used then they can handle not only the normal situations which can be handled with the existing measures but also the intricate situations with higher hesitancy wherein the existing measures failed. In these situations our method comes out with a satisfactory solution better than the existing ones because after mathematical calculation the existing formulas give equal value in some cases and that is why we can not conclude the solution of the problem properly but the proposed method gives clear distinctive value so we can easily reach to a clear verdict or solution.

In our future work, we will further modify the hesitant degree so that new distance and similarity measure including hesitant degree can solve more intricate and complex problems where the existing distance measures including hesitant degree can not solve.

Appendix

Theorem 1 *$d_{mh}(A_1, A_2), d_{me}(A_1, A_2)$ and $d_{mg}(A_1, A_2)$ satisfy the properties (D1)–(D4).*

Proof We have,

$$d_{mh}(A_1, A_2) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|\right] \tag{17}$$

If $A_1^j(x_i) = 1$ and $A_2^j(x_i) = 0$ then $d_{mh}(A_1, A_2) = 1$.

If $A_1^j(x_i) = A_2^j(x_i)$ then $d_{mh}(A_1, A_2) = 0$.

And when $A_1^j(x_i) \neq A_2^j(x_i)$ then $0 < d_{mh}(A_1, A_2) < 1$.

Therefore, properties (D1) and (D2) are satisfied.

If we interchange A_1 and A_2, the result will not change i.e. $d_{mh}(A_1, A_2) = d_{mh}(A_2, A_1)$. Therefore property (D3) is satisfied.

Now if $A_1^j(x_i) \leq A_2^j(x_i)$ and $A_1^j(x_i) \leq A_3^j(x_i)$, then we can get easily $d_{mh}(A_1, A_2) \leq d_{mh}(A_1, A_3)$. And if $A_2^j(x_i) \leq A_3^j(x_i)$ and $A_1^j(x_i) \leq A_3^j(x_i)$, then we have $d_{mh}(A_2, A_3) \leq d_{mh}(A_1, A_3)$. This shows that (D4) is satisfied.

Similarly we can proof that $d_{me}(A_1, A_2)$ and $d_{mg}(A_1, A_2)$ satisfy all the properties. □

Theorem 2 *When $A_1^j(x_i) + A_2^j(x_i) = 1$, then $d_h(A_1, A_2), d_e(A_1, A_2)$ and $d_g(A_1, A_2)$ becomes the special case of $d_{mh}(A_1, A_2), d_{me}(A_1, A_2)$ and $d_{mg}(A_1, A_2)$, where $A_1^j(x_i)$ and $A_2^j(x_i)$ are the j th values in $A_1(x_i)$ and $A_2(x_i)$.*

Proof We have,

$$d_{mh}(A_1, A_2) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|\frac{A_1^j(x_i) - A_2^j(x_i)}{A_1^j(x_i) + A_2^j(x_i)}\right|\right] \tag{18}$$

If $A_1^j(x_i) + A_2^j(x_i) = 1$ then the above distance measure becomes,

$$d_h(A_1, A_2) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|A_1^j(x_i) - A_2^j(x_i)|\right] \tag{19}$$

Similarly $d_e(A_1, A_2)$ and $d_g(A_1, A_2)$ are the special cases of $d_{me}(A_1, A_2)$ and $d_{mg}(A_1, A_2)$ respectively. □

References

1. Wang, W.J.: New similarity measures on fuzzy sets and on elements. Fuzzy Sets Syst. **85**, 305–309 (1997)
2. Zeng, W.Y., Li, H.X.: Inclusion measure, similarity measure and the fuzziness of fuzzy sets and their relations. Int. J. Intell. Syst. **21**, 639–653 (2006)
3. Szmidt, E., Kacprzyk, J.: Distance between intuitionistic fuzzy sets. Fuzzy Sets Syst. **114**, 505–518 (2000)
4. Torra, V., Narukawa, Y.: On hesitant fuzzy sets and decision. In: The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, pp. 1378–1382 (2009)
5. Xia, M., Xu, Z.: Hesitant fuzzy information aggregation in decision making. Int. J. Approx. Reason. **52**, 395–407 (2011)
6. Peng, D.H., Gao, C.Y., Gao, Z.F.: Generalized hesitant fuzzy synergetic weighted distance measures and their application to multi-criteria decision making. Appl. Math. Model. **37**, 5837–5850 (2013)
7. Li, D., Zeng, W., Li, J.: New distance and similarity measures on hesitant fuzzy sets and their applications in multi-criteria decision making. Eng. Appl. Artif. Intell. **40**, 11–16 (2015)
8. Zeng, W., Li, D.: Distance and similarity measures between hesitant fuzzy sets and their application in pattern recognition. Pattern Recognit. Lett. **84**, 267–271 (2016)

9. Garg, H., Arora, R.: Distance and similarity measures for dual hesitant fuzzy soft set and their application in MCDM problem. Int. J. Uncertain. Quantif. **7**, 229–248 (2017)
10. Tiwari, P., Gupta, P.: Entropy, distance and Similarity measures under Interval-valued Intuitionistic fuzzy environment. Informatica **42**, 617–627 (2018)
11. Wei, G.W.: Some similarity measures for picture fuzzy sets and their application. Iran. J. Fuzzy Syst. **15**(1), 77–89 (2018)
12. Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**, 529–539 (2010)
13. Xu, Z.S., Xia, M.M.: Distance and similarity measures for hesitant fuzzy sets. Inf. Sci. **181**, 2128–2138 (2011a)
14. Xu, Z.S., Xia, M.M.: On distance and correlation measures of hesitant fuzzy information. Int. J. Intell. Syst. **26**, 410–425 (2011b)