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THEORETICAL ARTICLE

Introduction to neutrosophic soft topological space

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Abstract The primary aim of this paper is to construct a topology on a neutrosophic soft set (NSS). The notion of neutrosophic soft interior, neutrosophic soft closure, neutrosophic soft neighbourhood, neutrosophic soft boundary, regular NSS are introduced and some of their basic properties are studied in this paper. Then the base for neutrosophic soft topology and subspace topology on NSS have been defined with suitable examples. Some related properties have been developed, too. Moreover, the concept of separation axioms on neutrosophic soft topological space have been introduced along with investigation of several structural characteristics.

Keywords Neutrosophic soft topology · Base for neutrosophic soft topology · Subspace topology on neutrosophic soft set · Separation axioms on neutrosophic soft topological space

1 Introduction

The complexity generally arises from uncertainty in the form of ambiguity in real world. Researchers in economics, sociology, medical science and many other several fields deal daily with the complexities of modeling uncertain data. Classical methods do not give fruitful result always because the uncertain appearing in these domains may be of different kinds. The probability theory has been an age old and effective tool to handle uncertainty but it can be applied only on random process.

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After that, theory of evidence, theory of fuzzy set by Zadeh [1], intuitionistic fuzzy set theory by Atanassov [2] were introduced to solve uncertain problems. But each of these theories has it's inherent difficulties as pointed out by Molodtsov [3]. The basic reason for these difficulties is inadequacy of parametrization tool of the theories.

Molodtsov [3] initiated the soft set theory as a new mathematical tool which is free from the parametrization inadequacy syndrome of different theory dealing with uncertainty in 1999. This makes the theory very convenient and easily applicable in practice. Molodtsov [3] successfully applied several directions for the applications of soft set theory, such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration and probability etc. Now, soft set theory and it's applications are progressing rapidly in different fields. Shabir and Naz [4] presented soft topological spaces and defined some concepts of soft sets on this spaces and separation axioms. Moreover, topological structure on fuzzy, fuzzy soft, intuitionistic fuzzy and intuitionistic fuzzy soft set was defined by Chang [5], Tanay and Kandemir [6], Coker [7], Li and Cui [8], Osmanoglu and Tokat [9], Bayramov and Gunduz [10, 11], Neog et al. [12], Varol and Aygun [13].

The concept of Neutrosophic Set (NS) was first introduced by Smarandache [14, 15] which is a generalisation of classical sets, fuzzy set, intuitionistic fuzzy set etc. Later, Maji [16] has introduced a combined concept Neutrosophic soft set (NSS). Using this concept, several mathematicians have produced their research works in different mathematical structures for instance Sahin et al. [17], Broumi [18], Deli and Broumi [19, 20], Maji [21], Broumi and Smarandache [22], Bera and Mahapatra [23–25], Deli [26, 27], Salama and Alblowi [28], Arockiarani et al. [29, 30], Saroja and Kalaichelvi [31] and others.

The primary aim of this paper is to construct a topology on an NSS. The notion of neutrosophic soft interior, neutrosophic soft closure, neutrosophic soft neighbourhood, neutrosophic soft boundary, regular neutrosophic soft set are introduced and some of their basic properties are studied in this paper. The content of the paper is organised as following:

In Sect. 2, some basic definitions and preliminary results are given which will be used in rest of the paper. The notion of neutrosophic soft topological space has been introduced along with some related properties and several structural characteristics in Sect. 3. Section 4 gives the concept of base for neutrosophic soft topology with suitable examples and some related theorems. In Sect. 5, the idea of subspace topology on an NSS set is proposed along with some properties. Then, the concept of separation axioms of neutrosophic soft topological space has been introduced along with investigation of several structural characteristics in Sect. 6. Finally Sect. 7 presents the conclusion of our work.

2 Preliminaries

We recall some necessary definitions related to fuzzy set, soft set, neutrosophic set, neutrosophic soft set for completeness.

2.1 Definitions related to fuzzy set and soft set

This section gives some important definitions related to Fuzzy set, Soft Set [3, 32]:

1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t - norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative.
(ii) $*$ is continuous.
(iii) $a * 1 = 1 * a = a, \ \forall a \in [0,1]$.
(iv) $a * b \le c * d$ if $a \le c,\ b \le d$ with $a,b,c,d \in [0,1]$.

A few examples of continuous t-norm are $a * b = ab$, $a * b = min\{a,b\}$, $a * b = max\{a + b - 1, 0\}$.

2. A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t - conorm (s - norm) if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative.
(ii) $\diamond$ is continuous.
(iii) $a \diamond 0 = 0 \diamond a = a, \ \forall a \in [0,1]$.
(iv) $a \diamond b \le c \diamond d$ if $a \le c,\ b \le d$ with $a,b,c,d \in [0,1]$.

A few examples of continuous s-norm are $a \diamond b = a + b - ab$, $a \diamond b = max\{a, b\}$, $a \diamond b = min\{a + b, 1\}$.

3. Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \to P(U)$ is a mapping.

2.2 Definitions related to neutrosophic set and neutrosophic soft set

Few relevant definitions [14, 16, 20] are given below:

1. A neutrosophic set (NS) on the universe of discourse U is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in U\},$$

where $T, I, F : U \to]^{-}0, 1^{+}[$ and $^{-}0 \le T_A(x) + I_A(x) + F_A(x) \le 3^{+}$.

From philosophical point of view, the neutrosophic set (NS) takes the value from real standard or nonstandard subsets of $]^{-}0, 1^{+}[$. But in real life application in scientific and engineering problems, it is difficult to use NS with value from real standard or nonstandard subset of $]^{-}0, 1^{+}[$. Hence we consider the NS which takes the value from the subset of [0,1].

2. Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NSS over U, where $F : A \to P(U)$ is a mapping.

This concept has been modified by Deli and Broumi as given below:

3. Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the set of all NSs of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to P(U)$ where f_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $P(U)$ and therefore it can be written as a set of ordered pairs,

$$N = \{(e, \{\langle x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)\rangle : x \in U\}) : e \in E\}$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0, 1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

4. The complement of a neutrosophic soft set N is denoted by N^c and is defined by:

$$N^c = \{(e, \{\langle x, F_{f_N(e)}(x), 1 - I_{f_N(e)}(x), T_{f_N(e)}(x)\rangle : x \in U\}) : e \in E\}$$

5. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then N_1 is said to be the neutrosophic soft subset of N_2 if

$$T_{f_{N_1}(e)}(x) \leq T_{f_{N_2}(e)}(x);\ I_{f_{N_1}(e)}(x) \geq I_{f_{N_2}(e)}(x);\ F_{f_{N_1}(e)}(x) \geq F_{f_{N_2}(e)}(x); \quad \forall e \in E \text{ and } x \in U.$$

We write $N_1 \subseteq N_2$ and then N_2 is the neutrosophic soft superset of N_1.

6. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined by:

$$N_3 = \left\{\left(e, \left\{\langle x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x)\rangle : x \in U\right\}\right) : e \in E\right\}$$

where

$$T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x),\ I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x),$$
$$F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x);$$

7. Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their intersection is denoted by $N_1 \cap N_2 = N_3$ and is defined by:

$$N_3 = \left\{\left(e, \left\{\langle x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x)\rangle : x \in U\right\}\right) : e \in E\right\}$$

where

$$T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x),\ I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x), F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x);$$

3 Neutrosophic soft topology

In this section, the concept of neutrosophic soft topology has been introduced. Some related basic properties have been developed in continuation.

Unless otherwise stated, E is treated as the parametric set through out this paper and $e \in E$, an arbitrary parameter.

3.1 Definition

1. A neutrosophic soft set N over (U, E) is said to be null neutrosophic soft set if $T_{f_N(e)}(x) = 0, I_{f_N(e)}(x) = 1, F_{f_N(e)}(x) = 1; \forall e \in E, \forall x \in U$. It is denoted by ϕ_u.
2. A neutrosophic soft set N over (U, E) is said to be absolute neutrosophic soft set if $T_{f_N(e)}(x) = 1, I_{f_N(e)}(x) = 0, F_{f_N(e)}(x) = 0; \forall e \in E, \forall x \in U$. It is denoted by 1_u.

Clearly, $\phi_u^c = 1_u$ and $1_u^c = \phi_u$.

3.2 Definition

Let $NSS(U, E)$ be the family of all neutrosophic soft sets over U via parameters in E and $\tau_u \subset NSS(U, E)$. Then τ_u is called neutrosophic soft topology on (U, E) if the following conditions are satisfied.

(i) $\phi_u, 1_u \in \tau_u$
(ii) the intersection of any finite number of members of τ_u also belongs to τ_u.
(iii) the union of any collection of members of τ_u belongs to τ_u.

Then the triplet (U, E, τ_u) is called a neutrosophic soft topological space. Every member of τ_u is called τ_u-open neutrosophic soft set. An NSS is called τ_u-closed iff it's complement is τ_u-open. There may be a number of topologies on (U, E). If τ_{u^1} and τ_{u^2} are two topologies on (U, E) such that $\tau_{u^1} \subset \tau_{u^2}$, then τ_{u^1} is called neutrosophic soft strictly weaker (coarser) than τ_{u^2} and in that case τ_{u^2} is neutrosophic soft strict finer than τ_{u^1}. Moreover $NSS(U, E)$ is a neutrosophic soft topology on (U, E).

3.2.1 Example

1. Let $U = \{h_1, h_2\}$, $E = \{e_1, e_2\}$ and $\tau_u = \{\phi_u, 1_u, N_1, N_2, N_3, N_4\}$ where N_1, N_2, N_3, N_4 being NSSs are defined as following:

$$
\begin{aligned}
f_{N_1}(e_1) &= \{\langle h_1, (1,0,1)\rangle, \langle h_2, (0,0,1)\rangle\}; \\
f_{N_1}(e_2) &= \{\langle h_1, (0,1,0)\rangle, \langle h_2, (1,0,0)\rangle\}; \\
f_{N_2}(e_1) &= \{\langle h_1, (0,1,0)\rangle, \langle h_2, (1,1,0)\rangle\}; \\
f_{N_2}(e_2) &= \{\langle h_1, (1,0,1)\rangle, \langle h_2, (0,1,1)\rangle\}; \\
f_{N_3}(e_1) &= \{\langle h_1, (1,1,1)\rangle, \langle h_2, (0,1,1)\rangle\}; \\
f_{N_3}(e_2) &= \{\langle h_1, (0,1,0)\rangle, \langle h_2, (0,1,1)\rangle\}; \\
f_{N_4}(e_1) &= \{\langle h_1, (1,1,0)\rangle, \langle h_2, (1,1,0)\rangle\}; \\
f_{N_4}(e_2) &= \{\langle h_1, (1,0,0)\rangle, \langle h_2, (0,1,1)\rangle\};
\end{aligned}
$$

Here $N_1 \cap N_1 = N_1, N_1 \cap N_2 = \phi_u, N_1 \cap N_3 = N_3, N_1 \cap N_4 = N_3, N_2 \cap N_2 = N_2$, $N_2 \cap N_3 = \phi_u, N_2 \cap N_4 = N_2, N_3 \cap N_3 = N_3, N_3 \cap N_4 = N_3, N_4 \cap N_4 = N_4$; and

$N_1 \cup N_1 = N_1, N_1 \cup N_2 = 1_u, N_1 \cup N_3 = N_1, \quad N_1 \cup N_4 = 1_u, N_2 \cup N_2 = N_2, N_2 \cup N_3 = N_4, N_2 \cup N_4 = N_4, N_3 \cup N_3 = N_3, N_3 \cup N_4 = N_4, N_4 \cup N_4 = N_4$;

Corresponding t-norm and s-norm are defined as $a * b = max\{a + b - 1, 0\}$ and $a \diamond b = min\{a + b, 1\}$. Then τ_u is a neutrosophic soft topology on (*U*, *E*) and so (U, E, τ_u) is a neutrosophic soft topological space over (*U*, *E*).

2. Let $U = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau_u = \{\phi_u, 1_u, N_1, N_2, N_3\}$ where N_1, N_2, N_3 being NSSs over (*U*, *E*) are defined as follow:

$$\begin{aligned}
f_{N_1}(e_1) &= \{\langle x_1, (1.0, 0.5, 0.4)\rangle, \langle x_2, (0.6, 0.6, 0.6)\rangle, \langle x_3, (0.5, 0.6, 0.4)\rangle\};\\
f_{N_1}(e_2) &= \{\langle x_1, (0.8, 0.4, 0.5)\rangle, \langle x_2, (0.7, 0.7, 0.3)\rangle, \langle x_3, (0.7, 0.5, 0.6)\rangle\};\\
f_{N_2}(e_1) &= \{\langle x_1, (0.8, 0.5, 0.6)\rangle, \langle x_2, (0.5, 0.7, 0.6)\rangle, \langle x_3, (0.4, 0.7, 0.5)\rangle\};\\
f_{N_2}(e_2) &= \{\langle x_1, (0.7, 0.6, 0.5)\rangle, \langle x_2, (0.6, 0.8, 0.4)\rangle, \langle x_3, (0.5, 0.8, 0.6)\rangle\};\\
f_{N_3}(e_1) &= \{\langle x_1, (0.6, 0.6, 0.7)\rangle, \langle x_2, (0.4, 0.8, 0.8)\rangle, \langle x_3, (0.3, 0.8, 0.6)\rangle\};\\
f_{N_3}(e_2) &= \{\langle x_1, (0.5, 0.8, 0.6)\rangle, \langle x_2, (0.5, 0.9, 0.5)\rangle, \langle x_3, (0.2, 0.9, 0.7)\rangle\};
\end{aligned}$$

The t-norm and s-norm are defined as $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$. Here $N_1 \cap N_1 = N_1, N_1 \cap N_2 = N_2, N_1 \cap N_3 = N_3, N_2 \cap N_2 = N_2, N_2 \cap N_3 = N_3, N_3 \cap N_3 = N_3$ and $N_1 \cup N_1 = N_1, N_1 \cup N_2 = N_1, N_1 \cup N_3 = N_1, \quad N_2 \cup N_2 = N_2, N_2 \cup N_3 = N_2, N_3 \cup N_3 = N_3$. Then τ_u is a neutrosophic soft topology on (*U*, *E*) and so (U, E, τ_u) is a neutrosophic soft topological space over (*U*, *E*).

3. Let $NSS(U, E)$ be the family of all neutrosophic soft sets over (*U*, *E*). Then $\{\phi_u, 1_u\}$ and $NSS(U, E)$ are two examples of the neutrosophic soft topology over (*U*, *E*). They are called, respectively, indiscrete (trivial) and discrete neutrosophic soft topology. Clearly, they are the smallest and largest neutrosophic soft topology on (*U*, *E*), respectively.

3.3 Proposition

Let (U, E, τ_{u^1}) and (U, E, τ_{u^2}) be two neutrosophic soft topological spaces over (*U*, *E*). Suppose, $\tau_{u^1} \cap \tau_{u^2} = \{M \in NSS(U, E) : M \in \tau_{u^1} \cap \tau_{u^2}\}$; Then $\tau_{u^1} \cap \tau_{u^2}$ is also a neutrosophic soft topology on (*U*, *E*).

Proof

(i) Clearly $\phi_u, 1_u \in \tau_{u^1} \cap \tau_{u^2}$;

(ii) Let $M_1, M_2 \in \tau_{u^1} \cap \tau_{u^2}$

$$\begin{aligned}
&\Rightarrow M_1, M_2 \in \tau_{u^1} \text{ and } M_1, M_2 \in \tau_{u^2};\\
&\Rightarrow M_1 \cap M_2 \in \tau_{u^1} \text{ and } M_1 \cap M_2 \in \tau_{u^2};\\
&\Rightarrow M_1 \cap M_2 \in \tau_{u^1} \cap \tau_{u^2}
\end{aligned}$$

(iii) Let $\{M_i : i \in \Gamma\} \in \tau_{u^1} \cap \tau_{u^2}$

$$\begin{aligned}
&\Rightarrow \{M_i\} \in \tau_{u^1} \text{ and } \{M_i\} \in \tau_{u^2}\\
&\Rightarrow \cup_i M_i \in \tau_{u^1} \text{ and } \cup_i M_i \in \tau_{u^2}\\
&\Rightarrow \cup_i M_i \in \tau_{u^1} \cap \tau_{u^2}
\end{aligned}$$

Thus $\tau_{u^1} \cap \tau_{u^2}$ is a neutrosophic soft topology on (*U*, *E*).

□

3.3.1 Remark

The union of two neutrosophic soft topologies may not be so.

We consider $U = \{x_1, x_2, x_3\},\ E = \{e_1, e_2\}$ and $\tau_{u^1} = \{\phi_u, 1_u, N_1, N_2, N_3\}$, $\tau_{u^2} = \{\phi_u, 1_u, N_4, N_5\}$ be two neutrosophic soft topologies on (*U*, *E*). Let τ_{u^1}, t-norm and s-norm be as in example (2) of subsection [3.2.1]. We define $N_4 = N_2$ and N_5 as:

$$f_{N_5}(e_1) = \{\langle x_1, (0.7, 0.5, 0.8)\rangle, \langle x_2, (0.4, 0.8, 0.6)\rangle, \langle x_3, (0.4, 0.9, 0.7)\rangle\};$$
$$f_{N_5}(e_2) = \{\langle x_1, (0.6, 0.7, 0.8)\rangle, \langle x_2, (0.5, 0.9, 0.6)\rangle, \langle x_3, (0.3, 0.8, 0.8)\rangle\};$$

Let $\tau_{u^3} = \tau_{u^1} \cup \tau_{u^2} = \{\phi_u, 1_u, N_1, N_2, N_3, N_5\}$. But τ_{u^3} is not a topology on (*U*, *E*), as $N_3 \cap N_5 \notin \tau_{u^3}$. Here $N_3 \cap N_5 = M$, say, is given as following:

$$f_M(e_1) = \{\langle x_1, (0.6, 0.6, 0.8)\rangle, \langle x_2, (0.4, 0.8, 0.8)\rangle, \langle x_3, (0.3, 0.9, 0.7)\rangle\};$$
$$f_M(e_2) = \{\langle x_1, (0.5, 0.8, 0.8)\rangle, \langle x_2, (0.5, 0.9, 0.6)\rangle, \langle x_3, (0.2, 0.9, 0.8)\rangle\};$$

3.4 Proposition (De-Morgan's law)

Let N_1, N_2 be two neutrosophic soft sets over (*U*, *E*). Then,

(i) $(N_1 \cup N_2)^c = {N_1}^c \cap {N_2}^c$
(ii) $(N_1 \cap N_2)^c = {N_1}^c \cup {N_2}^c$

Proof (i) For all $e \in E, x \in U$, we have,

$$N_1 \cup N_2 = \left\{\left\langle x, \left[T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x), I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x), F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x)\right]\right\rangle\right\}$$

Then,

$$(N_1 \cup N_2)^c = \left\{\left\langle x, \left[F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x), 1 - \left(I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x)\right), T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x)\right]\right\rangle\right\}$$

Now,

$$N_1^c = \left\{\left\langle x, \left[F_{f_{N_1}(e)}(x), 1 - I_{f_{N_1}(e)}(x), T_{f_{N_1}(e)}(x)\right]\right\rangle\right\}$$

and

$$N_2^c = \left\{\left\langle x, \left[F_{f_{N_2}(e)}(x), 1 - I_{f_{N_2}(e)}(x), T_{f_{N_2}(e)}(x)\right]\right\rangle\right\}.$$

Then,

$$N_1^c \cap N_2^c = \left\{\left\langle x, \left[F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x), \left(1 - I_{f_{N_1}(e)}(x)\right) \diamond (1 - I_{f_{N_2}(e)}(x)), T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x)\right]\right\rangle\right\}$$
$$= \left\{\left\langle x, \left[F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x), 1 - \left(I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x)\right), T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x)\right]\right\rangle\right\}$$

Hence, $(N_1 \cup N_2)^c = {N_1}^c \cap {N_2}^c$. □

Note:- Here, $(1 - I_{f_{N_1}(e)}(x)) \diamond (1 - I_{f_{N_2}(e)}(x)) = 1 - (I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x))$ holds for dual pairs of non-parameterized t-norms and s-norms e.g., $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$, $a * b = max\{a + b - 1, 0\}$ and $a \diamond b = min\{a + b, 1\}$ etc.

In a similar fashion, 2nd part can be established.

The theorems can be extended as: (i) $\{\cup_i N_i\}^c = \cap_i N_i^c$ (ii) $\{\cap_i N_i\}^c = \cup_i N_i^c$ for a family of NSSs $\{N_i\}_{i \in \Gamma}$ over (U, E).

3.5 Proposition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $\tau_u = \{N_i : N_i \in NSS(U, E)\} = \{[e, f_{N_i}(e)]_{e \in E} : N_i \in NSS(U, E)\}$ where $f_{N_i}(e) = \{\langle x, T_{f_{N_i}(e)}(x), I_{f_{N_i}(e)}(x), F_{f_{N_i}(e)}(x)\rangle : x \in U\}$; Then the collection $\tau_1 = \{[T_{f_{N_i}(e)}(U)]_{e \in E}\}$, $\tau_2 = \{[I_{f_{N_i}(e)}(U)]^c{}_{e \in E}\}$ and $\tau_3 = \{[F_{f_{N_i}(e)}(U)]^c{}_{e \in E}\}$ define fuzzy soft topologies on (U, E).

Proof

(i) $\phi_u, 1_u \in \tau_u$

$$\Rightarrow 0, 1 \in \tau_1;\ 1, 0 \in \tau_2;\ 1, 0 \in \tau_3;$$

(ii) $N_1, N_2 \in \tau_u$ implies $N_1 \cap N_2 \in \tau_u$ i.e.,

$$\left\{\left[T_{f_{N_1}(e)}(U) * T_{f_{N_2}(e)}(U),\ I_{f_{N_1}(e)}(U) \diamond I_{f_{N_2}(e)}(U),\ F_{f_{N_1}(e)}(U) \diamond F_{f_{N_2}(e)}(U)\right]_{e \in E}\right\} \in \tau_u$$
$$\Rightarrow \left\{(T_{f_{N_1}(e)}(U) * T_{f_{N_2}(e)}(U))_{e \in E}\right\} \in \tau_1,$$
$$\left\{\left(\left[I_{f_{N_1}(e)}(U)\right]^c * \left[I_{f_{N_2}(e)}(U)\right]^c\right)_{e \in E}\right\} \in \tau_2,$$
$$\left\{\left(\left[F_{f_{N_1}(e)}(U)\right]^c * \left[F_{f_{N_2}(e)}(U)\right]^c\right)_{e \in E}\right\} \in \tau_3;$$

(iii) $\{N_i : i \in \Gamma\} \in \tau_u$ implies $\cup_i N_i \in \tau_u$

$$\Rightarrow \left\{\left[\diamond_i T_{f_{N_i}(e)}(U),\ *_i I_{f_{N_i}(e)}(U),\ *_i F_{f_{N_i}(e)}(U)\right]_{e \in E}\right\} \in \tau_u$$
$$\Rightarrow \left\{\left[\diamond_i T_{f_{N_i}(e)}(U)\right]_{e \in E}\right\} \in \tau_1,\quad \left\{\diamond_i \left[I_{f_{N_i}(e)}(x)\right]^c_{e \in E}\right\} \in \tau_2,\quad \left\{\diamond_i \left[F_{f_{N_i}(e)}(U)\right]^c_{e \in E}\right\} \in \tau_3;$$

This ends the proposition.

It can be verified by example (2) in the subsection [3.2.1]. □

3.5.1 Remark

Converse of the above proposition is not true as shown by the following example.

Let $U = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and $\tau_u = \{\phi_u, 1_u, N_1, N_2, N_3\}$ where N_1, N_2, N_3 being NSSs in $NSS(U, E)$ are defined as following:

$$\begin{aligned}
f_{N_1}(e_1) &= \{\langle x_1, (0.8, 0.4, 0.3)\rangle, \langle x_2, (0.6, 0.5, 0.7)\rangle, \langle x_3, (0.5, 0.4, 0.2)\rangle\};\\
f_{N_1}(e_2) &= \{\langle x_1, (0.5, 0.6, 0.4)\rangle, \langle x_2, (0.3, 0.4, 0.6)\rangle, \langle x_3, (0.1, 0.5, 0.6)\rangle\};\\
f_{N_2}(e_1) &= \{\langle x_1, (0.7, 0.6, 0.4)\rangle, \langle x_2, (0.5, 0.7, 0.7)\rangle, \langle x_3, (0.4, 0.6, 0.7)\rangle\};\\
f_{N_2}(e_2) &= \{\langle x_1, (0.4, 0.7, 0.6)\rangle, \langle x_2, (0.2, 0.8, 0.7)\rangle, \langle x_3, (0.1, 0.7, 0.6)\rangle\};\\
f_{N_3}(e_1) &= \{\langle x_1, (0.9, 0.5, 0.6)\rangle, \langle x_2, (0.6, 0.6, 0.9)\rangle, \langle x_3, (0.6, 0.5, 0.8)\rangle\};\\
f_{N_3}(e_2) &= \{\langle x_1, (0.6, 0.6, 0.8)\rangle, \langle x_2, (0.5, 0.7, 0.8)\rangle, \langle x_3, (0.7, 0.6, 1.0)\rangle\};
\end{aligned}$$

The t-norm and s-norm are defined as $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$. Then,

$$\begin{aligned}
\tau_1 &= \left\{\left\langle T_{f_{\phi_u}(e)}(U), T_{f_{1_u}(e)}(U), T_{f_{N_1}(e)}(U), T_{f_{N_2}(e)}(U), T_{f_{N_3}(e)}(U)\right\rangle_{e\in E}\right\}\\
\tau_2 &= \left\{\left\langle I_{f_{\phi_u}(e)}(U), I_{f_{1_u}(e)}(U), I_{f_{N_1}(e)}(U), I_{f_{N_2}(e)}(U), I_{f_{N_3}(e)}(U)\right\rangle_{e\in E}\right\}\\
\tau_3 &= \left\{\left\langle F_{f_{\phi_u}(e)}(U), F_{f_{1_u}(e)}(U), F_{f_{N_1}(e)}(U), F_{f_{N_2}(e)}(U), F_{f_{N_3}(e)}(U)\right\rangle_{e\in E}\right\}
\end{aligned}$$

are fuzzy soft topologies on (*U*, *E*). Elaborately,

$$\begin{aligned}
\tau_1 = \{&\langle (0,0,0), (1,1,1), (0.8, 0.6, 0.5), (0.7, 0.5, 0.4), (0.9, 0.6, 0.6)\rangle_{e_1},\\
&\langle (0,0,0), (1,1,1), (0.5, 0.3, 0.1), (0.4, 0.2, 0.1), (0.6, 0.5, 0.7)\rangle_{e_2}\}
\end{aligned}$$

and so on.

Here, $\tau_u = \{\phi_u, 1_u, N_1, N_2, N_3\}$ is not a neutrosophic soft topology on (*U*, *E*), because $N_2 \cap N_3 \notin \tau_u$, where $N_2 \cap N_3 = N_4$, say, is given as following:

$$\begin{aligned}
f_{N_4}(e_1) &= \{\langle x_1, (0.7, 0.6, 0.6)\rangle, \langle x_2, (0.5, 0.7, 0.9)\rangle, \langle x_3, (0.4, 0.6, 0.8)\rangle\};\\
f_{N_4}(e_2) &= \{\langle x_1, (0.4, 0.7, 0.8)\rangle, \langle x_2, (0.2, 0.8, 0.8)\rangle, \langle x_3, (0.1, 0.7, 1.0)\rangle\};
\end{aligned}$$

3.5.2 Proposition

Let (U, E, τ_u) be a neutrosophic soft topological space defined over (*U*, *E*). Then $\tau_{1e} = \{[T_{f_M(e)}(U)] : M \in \tau_u\}$, $\tau_{2e} = \{[I_{f_M(e)}(U)]^c : M \in \tau_u\}$, $\tau_{3e} = \{[F_{f_M(e)}(U)]^c : M \in \tau_u\}$ for each $e \in E$, define fuzzy topologies on (*U*, *E*).

Proof By proposition (3.5), $\tau_1 = \{[T_{f_{N_i}(e)}(U)]_{e\in E}\}$, $\tau_2 = \{[I_{f_{N_i}(e)}(U)]^c{}_{e\in E}\}$ and $\tau_3 = \{[F_{f_{N_i}(e)}(U)]^c{}_{e\in E}\}$ are three fuzzy soft topologies on (*U*, *E*).

So for each $e \in E$, $\tau_{1e}, \tau_{2e}, \tau_{3e}$ are fuzzy topologies on (*U*, *E*). Let $\tau_e = \{\tau_{1e}, \tau_{2e}, \tau_{3e}\}$; It is called as fuzzy tritopology on (*U*, *E*). Thus corresponding to each parameter $e \in E$, we have a fuzzy tritopology τ_e on (*U*, *E*). Hence, a neutrosophic soft topology on (*U*, *E*) gives a parameterized family of fuzzy tritopologies on (*U*, *E*).

The reverse of that proposition may not be true. Following example shows the fact. □

3.5.3 Example

We consider the example in remark [3.5.1]. Then,

$$\tau_{1e_1} = \left\{T_{f_{\phi_u}(e_1)}(U), T_{f_{1_u}(e_1)}(U), T_{f_{N_1}(e_1)}(U), T_{f_{N_2}(e_1)}(U), T_{f_{N_3}(e_1)}(U)\right\}$$
$$\tau_{2e_1} = \left\{I_{f_{\phi_u}(e_1)}(U), I_{f_{1_u}(e_1)}(U), I_{f_{N_1}(e_1)}(U), I_{f_{N_2}(e_1)}(U), I_{f_{N_3}(e_1)}(U)\right\}$$
$$\tau_{3e_1} = \left\{F_{f_{\phi_u}(e_1)}(U), F_{f_{1_u}(e_1)}(U), F_{f_{N_1}(e_1)}(U), F_{f_{N_2}(e_1)}(U), F_{f_{N_3}(e_1)}(U)\right\}$$

are fuzzy topologies on (*U*, *E*), where,

$$\tau_{1e_1} = \{(0,0,0), (1,1,1), (0.8,0.6,0.5), (0.7,0.5,0.4), (0.9,0.6,0.6)\} \text{ and so on.}$$

Here, $\{\tau_{1e_1}, \tau_{2e_1}, \tau_{3e_1}\}$ is a fuzzy tritopology on (*U*, *E*). Also,

$$\tau_{1e_2} = \left\{T_{f_{\phi_N}(e_2)}(U), T_{f_{1_N}(e_2)}(U), T_{f_{N_1}(e_2)}(U), T_{f_{N_2}(e_2)}(U), T_{f_{N_3}(e_2)}(U)\right\}$$
$$\tau_{2e_2} = \left\{I_{f_{\phi_N}(e_2)}(U), I_{f_{1_N}(e_2)}(U), I_{f_{N_1}(e_2)}(U), I_{f_{N_2}(e_2)}(U), I_{f_{N_3}(e_2)}(U)\right\}$$
$$\tau_{3e_2} = \left\{F_{f_{\phi_N}(e_2)}(U), F_{f_{1_N}(e_2)}(U), F_{f_{N_1}(e_2)}(U), F_{f_{N_2}(e_2)}(U), F_{f_{N_3}(e_2)}(U)\right\}$$

is another set of fuzzy topologies on (*U*, *E*) and consequently $\{\tau_{1e_2}, \tau_{2e_2}, \tau_{3e_2}\}$ is also a fuzzy tritopology on (*U*, *E*). But $\tau_u = \{\phi_u, 1_u, N_1, N_2, N_3\}$ is not a neutrosophic soft topology on (*U*, *E*).

3.6 Definition

Let τ_u be a neutrosophic soft topology on (*U*, *E*) and $N_1, N_2 \in NSS(U,E)$. Then N_2 is called a neighbourhood of N_1 if there exists an open neutrosophic soft set *M* (i.e., $M \in \tau_u$) such that $N_1 \subset M \subset N_2$.

In the example (2) of subsection [3.2.1], $N_3 \subset N_2 \subset N_1$ and so N_1 is a neighbourhood of N_3.

3.6.1 Theorem

An NSS $M \in NSS(U,E)$ is an open NSS iff *M* is a neighbourhood of each NSS N_1 contained in *M*.

Proof Let *M* be an open NSS and N_1 be any NSS contained in *M*. Since we have $N_1 \subset M \subset M$, it follows that *M* is a neighbourhood of N_1.

Next, suppose that *M* is a neighbourhood of each NSS contained in *M*. Since $M \subset M$, there exists an open NSS N_2 such that $M \subset N_2 \subset M$. Hence, $M = N_2$ and *M* is open. □

3.6.2 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space on (U, E) and $M \in NSS(U, E)$. The family of all neighbourhoods of M is called the neighbourhood system or neighbourhood filter of M up to topology τ_u and is denoted by $Nbd(M)$.

3.6.3 Theorem

Let $Nbd(M)$ be the neighbourhood system of the NSS M. Then,

(i) finite intersections of the members of $Nbd(M)$ belongs to $Nbd(M)$.
(ii) each neutrosophic soft set containing a member of $Nbd(M)$ belongs to $Nbd(M)$.

Proof (i) Let $N_1, N_2 \in Nbd(M)$. Then there exists $N_1', N_2' \in \tau_u$ such that $M \subset N_1' \subset N_1$ and $M \subset N_2' \subset N_2$. Since, $N_1' \cap N_2' \in \tau_u$, we have, $M \subset N_1' \cap N_2' \subset N_1 \cap N_2$. Thus, $N_1 \cap N_2 \in Nbd(M)$.

(ii) If $N_1 \in Nbd(M)$ and N_2 be a neutrosophic soft set containing N_1, then there exists $N_1' \in \tau_u$ such that $M \subset N_1' \subset N_1 \subset N_2$. This shows that $N_2 \in Nbd(M)$. □

3.7 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$ be arbitrary. Then the interior of M is denoted by M^o and is defined by

$$M^o = \cup\{N_1 : N_1 \text{ is neutrosophic soft open and } N_1 \subset M\}$$

i.e., it is the union of all open neutrosophic soft subsets of M.

3.7.1 Example

We consider the example (2) in subsection [3.2.1]. Let an arbitrary $M \in NSS(U, E)$ be defined as following:

$$f_M(e_1) = \{\langle x_1, (0.9, 0.4, 0.5)\rangle, \langle x_2, (0.5, 0.6, 0.6)\rangle, \langle x_2, (0.7, 0.6, 0.5)\rangle\};$$
$$f_M(e_2) = \{\langle x_1, (0.7, 0.5, 0.4)\rangle, \langle x_2, (0.8, 0.3, 0.4)\rangle, \langle x_2, (0.6, 0.7, 0.5)\rangle\};$$

Then $\phi_u, N_2, N_3 \subset M$ and so, $M^o = \phi_u \cup N_2 \cup N_3 = N_2$.

3.7.2 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M, P \in NSS(U, E)$. Then,

(i) $M^o \subset M$ and M^o is the largest open set.
(ii) $M \subset P \Rightarrow M^o \subset P^o$.
(iii) M^o is an open neutrosophic soft set i.e., $M^o \in \tau_u$.

(iv) M is neutrosophic soft open set iff $M^o = M$.
(v) $(M^o)^o = M^o$.
(vi) $(\phi_u)^o = \phi_u$ and $1_u^o = 1_u$.
(vii) $(M \cap P)^o = M^o \cap P^o$.
(viii) $M^o \cup P^o \subset (M \cup P)^o$.

Proof

(i) It is obvious from the definition.
(ii) Here $M^o \subset M \subset P$. So $M^o \subset P$ and also $P^o \subset P$. But P^o is the largest open set contained in P, hence $M^o \subset P^o$.
(iii) By definition of τ_u and M^o, it is obvious.
(iv) Here, $M^o \subset M$ and let M be neutrosophic soft open. Now,

$$M \subset M \Rightarrow M \subset \cap\{P \in \tau_u : P \subset M\} = M^o \Rightarrow M \subset M^o$$

Hence, $M = M^o$
Conversely, let $M = M^o$. Then, $M = M^o \in \tau_u \Rightarrow M$ is an open NSS.
(v) If N is an open NSS then $N^o = N$. Clearly, M^o is an open NSS. Replacing N by M^o, we have the result.
(vi) Since, $\phi_u, 1_u \in \tau_u$ so they are open NSSs. Hence, the result follows from (iv).
(vii) $M \cap P \subset M$ and $M \cap P \subset P \Rightarrow (M \cap P)^o \subset M^o$ and $(M \cap P)^o \subset P^o \Rightarrow (M \cap P)^o \subset M^o \cap P^o$
Further, $M^o \subset M$ and $P^o \subset P$. Then $M^o \cap P^o \subset M \cap P$. But, $(M \cap P)^o \subset M \cap P$ and it is largest open NSS. So, $M^o \cap P^o \subset (M \cap P)^o$. Thus, $(M \cap P)^o = M^o \cap P^o$.
(viii) $M \subset M \cup P$ and $P \subset M \cup P \Rightarrow M^o \subset (M \cup P)^o$ and $P^o \subset (M \cup P)^o \Rightarrow M^o \cup P^o \subset (M \cup P)^o$. □

3.7.3 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$. The the associated interior of M is an NSS over (U, E), denoted by $\{(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})_{e \in E}\}$ and is defined as $\{(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})_{e \in E}\} = \{(\diamond_i T_{f_{N_i}(e)}, *_i I_{f_{N_i}(e)}, *_i F_{f_{N_i}(e)})_{e \in E}\}$, where $N_i \in \tau_u$ and $f_{N_i}(e) \subset f_M(e)$.

3.7.4 Proposition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$. Then, $M^o \subset \{(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})_{e \in E}\}$ where $M^o = \{(f_M(e))_{e \in E}\}^o$.

Proof For each $e \in E$, $(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})$ is the largest open neutrosophic set contained in $f_M(e)$. Also $M^o \subset M$. Hence, $M^o \subset \{(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})_{e \in E}\}$. □

3.7.5 Example

We consider the example (2) of subsection [3.2.1]. Let $M \in NSS(U, E)$ be defined as following:

$$f_M(e_1) = \{\langle x_1, (0.8, 0.5, 0.4)\rangle, \langle x_2, (0.7, 0.7, 0.6)\rangle, \langle x_3, (0.4, 0.6, 0.4)\rangle\};$$
$$f_M(e_2) = \{\langle x_1, (0.9, 0.4, 0.4)\rangle, \langle x_2, (0.8, 0.7, 0.2)\rangle, \langle x_3, (0.8, 0.5, 0.5)\rangle\};$$

Then, $N_2 \subset M$ and $N_3 \subset M$. So, $M^o = N_2 \cup N_3 = N_2$.

Further, $f_{N_2}(e_1)$,$f_{N_3}(e_1) \subset f_M(e_1)$ and $f_{N_1}(e_2)$,$f_{N_2}(e_2)$,$f_{N_3}(e_2) \subset f_M(e_2)$.

The tabular representation of $\{(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})_{e \in E}\}$ is as (Table 1):

Hence, $M^o \neq \{(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})_{e \in E}\}$

3.8 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$ be arbitrary. Then the closure of M is denoted by $\overline{M}$ and is defined by:

$$\overline{M} = \cap\{N_1 : N_1 \text{ is neutrosophic soft closed and } N_1 \supset M\}$$

i.e., it is the intersection of all closed neutrosophic soft supersets of M.

3.8.1 Example

Consider the example (2) in subsection [3.2.1]. Let an arbitrary $M \in NSS(U, E)$ be defined as:

$$f_M(e_1) = \{\langle x_1, (0.6, 0.7, 0.8)\rangle, \langle x_2, (0.5, 0.3, 0.7)\rangle, \langle x_3, (0.4, 0.4, 0.5)\rangle\};$$
$$f_M(e_2) = \{\langle x_1, (0.4, 0.5, 0.7)\rangle, \langle x_2, (0.3, 0.4, 0.8)\rangle, \langle x_3, (0.6, 0.4, 0.6)\rangle\};$$

Clearly, ϕ_u, $1_u, N_1^c, N_2^c, N_3^c, N_4^c$ are all closed NSSs over (U, E). They are given as:

$$f_{N_1^c}(e_1) = \{\langle x_1, (0.4, 0.5, 1.0)\rangle, \langle x_2, (0.6, 0.4, 0.6)\rangle, \langle x_3, (0.4, 0.4, 0.5)\rangle\};$$
$$f_{N_1^c}(e_2) = \{\langle x_1, (0.5, 0.6, 0.8)\rangle, \langle x_2, (0.3, 0.3, 0.7)\rangle, \langle x_3, (0.6, 0.5, 0.7)\rangle\};$$
$$f_{N_2^c}(e_1) = \{\langle x_1, (0.6, 0.5, 0.8)\rangle, \langle x_2, (0.6, 0.3, 0.5)\rangle, \langle x_3, (0.5, 0.3, 0.4)\rangle\};$$
$$f_{N_2^c}(e_2) = \{\langle x_1, (0.5, 0.4, 0.7)\rangle, \langle x_2, (0.4, 0.2, 0.6)\rangle, \langle x_3, (0.6, 0.2, 0.5)\rangle\};$$
$$f_{N_3^c}(e_1) = \{\langle x_1, (0.7, 0.4, 0.6)\rangle, \langle x_2, (0.8, 0.2, 0.4)\rangle, \langle x_3, (0.6, 0.2, 0.3)\rangle\};$$
$$f_{N_3^c}(e_2) = \{\langle x_1, (0.6, 0.2, 0.5)\rangle, \langle x_2, (0.5, 0.1, 0.5)\rangle, \langle x_3, (0.7, 0.1, 0.2)\rangle\};$$

Then $1_u, N_2^c, N_3^c \supset M$ and so, $\overline{M} = 1_u \cap N_2^c \cap N_3^c = N_2^c$.

Table 1 Tabular form of $\{(T^o_{f_M(e)}, I^o_{f_M(e)}, F^o_{f_M(e)})_{e \in E}\}$

	$(T^o_{f_M(e_1)}, I^o_{f_M(e_1)}, F^o_{f_M(e_1)})$	$(T^o_{f_M(e_2)}, I^o_{f_M(e_2)}, F^o_{f_M(e_2)})$
x_1	(0.8,0.5,0.6)	(0.8,0.4,0.5)
x_2	(0.5,0.7,0.6)	(0.7,0.7,0.3)
x_3	(0.4,0.7,0.5)	(0.7,0.5,0.6)

3.8.2 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (*U*, *E*) and $M, P \in NSS(U, E)$. Then,

(i) $M \subset \overline{M}$ and $\overline{M}$ is the smallest closed set.
(ii) $M \subset P \Rightarrow \overline{M} \subset \overline{P}$.
(iii) $\overline{M}$ is closed neutrosophic soft set i.e., $\overline{M} \in \tau_u^c$.
(iv) M is neutrosophic soft closed set iff $\overline{M} = M$.
(v) $\overline{\overline{M}} = \overline{M}$.
(vi) $\overline{\phi_u} = \phi_u$ and $\overline{1_u} = 1_u$.
(vii) $\overline{M \cup P} = \overline{M} \cup \overline{P}$.
(viii) $\overline{M \cap P} \subset \overline{M} \cap \overline{P}$.

Proof

(i) It follows from definition directly.
(ii) $M \subset \overline{M}$ and $P \subset \overline{P} \Rightarrow M \subset P \subset \overline{P} \Rightarrow M \subset \overline{P}$
But $\overline{M}$ is the smallest closed set containing M i.e., $M \subset \overline{M} \subset \overline{P}$. Hence, $\overline{M} \subset \overline{P}$.
(iii) It is obvious from the definition of τ_u and $\overline{M}$.
(iv) Here, $M \subset \overline{M}$ and let M be closed. Then, $M \in \tau_u^c$ and $M \subset M$.
Now, $\overline{M} = \cap\left\{P \in \tau_u^c : P \supset M\right\} \subset \left\{M \in \tau_u^c : M \supset M\right\} = M$
$\Rightarrow \overline{M} \subset M$ i.e., $M = \overline{M}$
Next, let $M = \overline{M}$. Then $(\overline{M})^c \in \tau_u \Rightarrow M^c \in \tau_u \Rightarrow M^c$ is open $\Rightarrow M$ is closed.
(v) If N is closed then $N = \overline{N}$. But $\overline{N}$ is closed by construction of $\overline{N}$. Replacing N by $\overline{M}$, we get $\overline{\overline{M}} = \overline{M}$.
(vi) $\phi_u, 1_u$ are open as well as closed. So by (iv), the result follows.
(vii) $M \subset M \cup P$ and $P \subset M \cup P \Rightarrow \overline{M} \subset \overline{M \cup P}$ and $\overline{P} \subset \overline{M \cup P}$
$\Rightarrow \overline{M} \cup \overline{P} \subset \overline{M \cup P}$
Also, $M \subset \overline{M}$ and $P \subset \overline{P} \Rightarrow M \cup P \subset \overline{M} \cup \overline{P}$. But we have, $M \cup P \subset \overline{M \cup P} \subset \overline{M} \cup \overline{P}$
Thus, $\overline{M \cup P} = \overline{M} \cup \overline{P}$.
(viii) $M \cup P \subset M$ and $M \cup P \subset P \Rightarrow \overline{M \cup P} \subset \overline{M}$ and $\overline{M \cup P} \subset \overline{P}$
$\Rightarrow \overline{M \cup P} \subset \overline{M} \cup \overline{P}$. □

3.8.3 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (*U*, *E*) and $M \in NSS(U, E)$. The the associated closure of M is an NSS over (*U*, *E*), denoted by $\{(\overline{T_{f_M(e)}}, \overline{I_{f_M(e)}}, \overline{F_{f_M(e)}})_{e\in E}\}$ and is defined as $\{(\overline{T_{f_M(e)}}, \overline{I_{f_M(e)}}, \overline{F_{f_M(e)}})_{e\in E}\} = \{(*_i F_{f_{N_i}(e)}, \diamond_i(1 - I_{f_{N_i}(e)}), \diamond_i T_{f_{N_i}(e)})_{e\in E}\}$, where $N_i \in \tau_u$ and $f_M(e) \subset f_{N_i^c}(e)$.

3.8.4 Proposition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$. Then, $\{(\overline{T_{f_M(e)}}, \overline{I_{f_M(e)}}, \overline{F_{f_M(e)}})_{e\in E}\} \subset \overline{M}$ where $\overline{M} = \overline{\{(f_M(e))_{e\in E}\}}$.

Proof For each $e \in E$, $(\overline{T_{f_M(e)}}, \overline{I_{f_M(e)}}, \overline{F_{f_M(e)}})$ is the smallest closed neutrosophic set containing $f_M(e)$. Also $M \subset \overline{M}$. Hence, $\{(\overline{T_{f_M(e)}}, \overline{I_{f_M(e)}}, \overline{F_{f_M(e)}})_{e\in E}\} \subset \overline{M}$. □

3.8.5 Example

Consider the example (2) of subsection [3.2.1]. Clearly N_1^c, N_2^c, N_3^c are all neutrosophic soft closed. Define an arbitrary NSS $M \in NSS(U, E)$ as:

$$f_M(e_1) = \{\langle x_1, (0.5, 0.6, 1.0)\rangle, \langle x_2, (0.5, 0.5, 0.6)\rangle, \langle x_3, (0.3, 0.5, 0.7)\rangle\};$$
$$f_M(e_2) = \{\langle x_1, (0.5, 0.7, 0.8)\rangle, \langle x_2, (0.2, 0.6, 0.8)\rangle, \langle x_3, (0.4, 0.7, 0.9)\rangle\};$$

Then, $M \subset N_2^c$ and $M \subset N_3^c$. So, $\overline{M} = N_2^c \cap N_3^c = N_2^c$ which is as following:

$$f_{\overline{M}}(e_1) = \{\langle x_1, (0.6, 0.5, 0.8)\rangle, \langle x_2, (0.6, 0.3, 0.5)\rangle, \langle x_3, (0.5, 0.3, 0.4)\rangle\};$$
$$f_{\overline{M}}(e_2) = \{\langle x_1, (0.5, 0.4, 0.7)\rangle, \langle x_2, (0.4, 0.2, 0.6)\rangle, \langle x_3, (0.6, 0.2, 0.5)\rangle\};$$

Further, $f_M(e_1) \subset f_{N_2^c}(e_1), f_{N_3^c}(e_1)$ and $f_M(e_2) \subset f_{N_1^c}(e_2), f_{N_2^c}(e_2), f_{N_3^c}(e_2)$;

The tabular representation of $\{(\overline{T_{f_M(e)}}, \overline{I_{f_M(e)}}, \overline{F_{f_M(e)}})_{e\in E}\}$ is as (Table 2):

Hence, $\overline{M} \neq \{(\overline{T_{f_M(e_1)}}, \overline{I_{f_M(e_1)}}, \overline{F_{f_M(e_1)}})_{e\in E}\}$.

3.8.6 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$. Then, (i) $(\overline{M})^c = (M^c)^o$ (ii)$(M^o)^c = \overline{(M^c)}$

Proof

(i) $\overline{M} = \cap\{Q \in \tau_u^c : Q \supset M\}$

$\Rightarrow (\overline{M})^c = (\cap\{Q \in \tau_u^c : Q \supset M\})^c = \cup\{Q^c \in \tau_u : Q^c \subset M^c\} = (M^c)^o$

Table 2 Tabular form of $\{(\overline{T_{f_M(e)}}, \overline{I_{f_M(e)}}, \overline{F_{f_M(e)}})_{e\in E}\}$

	$(\overline{T_{f_M(e_1)}}, \overline{I_{f_M(e_1)}}, \overline{F_{f_M(e_1)}})$	$(\overline{T_{f_M(e_2)}}, \overline{I_{f_M(e_2)}}, \overline{F_{f_M(e_2)}})$
x_1	(0.6,0.5,0.8)	(0.5,0.6,0.8)
x_2	(0.6,0.3,0.5)	(0.3,0.3,0.7)
x_3	(0.5,0.3,0.4)	(0.6,0.5,0.7)

(ii) $M^o = \cup\{P \in \tau_u : P \subset M\}$
$\Rightarrow (M^o)^c = (\cup\{P \in \tau_u : P \subset M\})^c = \cap\{P^c \in \tau_u^c : P^c \supset M^c\} = \overline{(M^c)}.$

□

3.9 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$. Then boundary of M is denoted by $Bd(M)$ and is defined by $Bd(M) = \overline{M} \cap \overline{M^c}$.

3.9.1 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $N \in NSS(U, E)$. Then,

(i) $N^o \cap Bd(N) = \phi_u$.
(ii) $\overline{N} = N^o \cup Bd(N)$.
(iii) $Bd(N) = \phi_u$ iff N is closed and open.
(iv) $Bd(N) = \overline{N} \cap (N^o)^c$.

Proof

(i) $$N^o \cap Bd(N) = N^o \cap (\overline{N} \cap \overline{N^c}) = N^o \cap (\overline{N} \cap (N^o)^c) = N^o \cap (N^o)^c \cap \overline{N} = \phi_u \cap \overline{N} = \phi_u$$

(ii) $$N^o \cup Bd(N) = N^o \cup (\overline{N} \cap \overline{N^c}) = N^o \cup (\overline{N} \cap (N^o)^c) = (N^o \cup \overline{N}) \cap (N^o \cup (N^o)^c) = (N^o \cup \overline{N}) \cap 1_u = (N^o \cup \overline{N}) = \overline{N}, \text{ as } N^o \subset N \subset \overline{N}.$$

(iii) $$Bd(N) = \overline{N} \cup \overline{N^c} = \phi_u$$
$$\Rightarrow \overline{N} \cap (N^o)^c = \phi_u \Rightarrow \overline{N} \cap ((N^o)^c)^c \neq \phi_u \Rightarrow \overline{N} \cap N^o \neq \phi_u$$
$$\Rightarrow \overline{N} \subset N^o \text{ i.e., } N \subset \overline{N} \subset N^o \Rightarrow N \subset N^o;$$

Also we have, $N^o \subset N$. Thus, $N = N^o$ i.e., N is open.
Further, $\overline{N} \subset N^o \subset N \Rightarrow \overline{N} \subset N$; But we have, $N \subset \overline{N}$. Thus, $N = \overline{N}$ i.e., N is closed.
Conversely, if N is closed and open then $N = N^o$ and $N = \overline{N}$.
Now, $Bd(N) = \overline{N} \cap \overline{N^c} = \overline{N} \cap (N^o)^c = N \cap N^c = \phi_u$.

(iv) $$Bd(N) = \overline{N} \cap \overline{N^c} = \overline{N} \cap (N^o)^c;$$

3.10 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M \in NSS(U, E)$. Then,

(i) an open neutrosophic soft set M is called regular if $M = (\overline{M})^o$ and

(ii) a closed neutrosophic soft set M is called regular if $M = \overline{(M^o)}$.

3.10.1 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M, P \in NSS(U, E)$. Then,

(i) if M be closed NSS then M^o is a regular open NSS.

(ii) if M be open NSS then $\overline{M}$ is a regular closed NSS.

(iii) if M, P are regular open NSSs then $M \subset P$ iff $\overline{M} \subset \overline{P}$.

(iv) if M, P are regular closed NSSs then $M \subset P$ iff $M^o \subset P^o$.

(v) complement of regular closed (open) NSS is a regular open (closed) NSS.

Proof

(i) Since, M is closed then $M = \overline{M}$. Also, $(M^o)^o = M^o$, as M^o is open.
Now we have, $M^o \subset M \Rightarrow \overline{M^o} \subset \overline{M} = M \Rightarrow \overline{M^o} \subset M \Rightarrow (\overline{M^o})^o \subset M^o$
Further, $M^o \subset \overline{M^o} \Rightarrow (M^o)^o \subset (\overline{M^o})^o \Rightarrow M^o \subset (\overline{M^o})^o$;
Hence, M^o is regular.

(ii) Since, M is open then $M = M^o$;
Now, $(\overline{M})^o \subset \overline{M} \Rightarrow \overline{(\overline{M})^o} \subset \overline{\overline{M}} \Rightarrow \overline{(\overline{M})^o} \subset \overline{M}$
Again, $M \subset \overline{M} \Rightarrow M^o \subset (\overline{M})^o \Rightarrow M \subset (\overline{M})^o \Rightarrow \overline{M} \subset \overline{(\overline{M})^o}$;
This shows that $\overline{M} = \overline{(\overline{M})^o}$ i.e., $\overline{M}$ is regular closed NSS.

(iii) Here, $M = (\overline{M})^o$ and $P = (\overline{P})^o$
If $M \subset P$, then $\overline{M} \subset \overline{P}$.
Next, $\overline{M} \subset \overline{P} \Rightarrow (\overline{M})^o \subset (\overline{P})^o \Rightarrow M \subset N$.

(iv) Here, $M = \overline{M^o}$ and $P = \overline{P^o}$;
Now, $M \subset P \Rightarrow \overline{M^o} \subset \overline{N^o} \Rightarrow M^o \subset N^o$;
Next, $M^o \subset N^o \Rightarrow \overline{M^o} \subset \overline{N^o} \Rightarrow M \subset N$;

(v) Let M be a regular closed NSS. We shall show $M^c = (\overline{M^c})^o$
Here, $M = \overline{M^o}$. Now $(\overline{M^c})^o = ((M^o)^c)^o = (\overline{M^o})^c = M^c$. Next, let M be regular open NSS i.e., $M = (\overline{M})^o$. . Then $\overline{(M^c)^o} = \overline{(\overline{M})^c} = ((\overline{M})^o)^c = M^c$ i.e., M^c is regular closed NSS. □

4 Base for neutrosophic soft topology

4.1 Definition

1. A neutrosophic soft point in an NSS N is defined as an element $(e, f_N(e))$ of N, for $e \in E$ and is denoted by e_N, if $f_N(e) \notin \phi_u$ and $f_N(e') \in \phi_u, \forall e' \in E - \{e\}$.
2. The complement of a neutrosophic soft point e_N is another neutrosophic soft point e_N^c such that $f_N^c(e) = (f_N(e))^c$.
3. A neutrosophic soft point $e_N \in M, M$ being an NSS if for the element $e \in E, f_N(e) \leq f_M(e)$.

4.1.1 Example

Let $U = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Then,

$$e_{1N} = \{\langle x_1, (0.6, 0.4, 0.8)\rangle, \langle x_2, (0.8, 0.3, 0.5)\rangle, \langle x_3, (0.3, 0.7, 0.6)\rangle\}$$

is a neutrosophic soft point whose complement is

$$e_{1N}^c = \{\langle x_1, (0.8, 0.6, 0.6)\rangle, \langle x_2, (0.5, 0.7, 0.8)\rangle, \langle x_3, (0.6, 0.3, 0.3)\rangle\}.$$

For another NSS M defined on same (U, E), let,

$$f_M(e_1) = \{\langle x_1, (0.7, 0.4, 0.7)\rangle, \langle x_2, (0.8, 0.2, 0.4)\rangle, \langle x_3, (0.5, 0.6, 0.5)\rangle\}.$$

Then, $f_N(e_1) \leq f_M(e_1)$ i.e., $e_{1N} \in M$.

4.2 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E). Then a collection $ß_u \subset \tau_u$ is a base for τ_u if arbitrary $M \in \tau_u$ can be expressed as the union of the members of $ß_u$ i.e., if $M = \cup_i B_i$ for $B_i \in ß_u$. Members of $ß_u$ are called basic open sets.

4.2.1 Example

In the example (1) in [3.2.1], $ß_u = \{\phi_u, N_1, N_2, N_3\}$ is a base for neutrosophic soft topological space (U, E, τ_u) over (U, E). Because,

$$\phi_u = \phi_u \cup \phi_u, 1_u = N_1 \cup N_2, N_1 = N_1 \cup N_3, N_2 = N_2 \cup N_2, N_3 = N_3 \cup N_3, N_4 = N_2 \cup N_3;$$

4.2.2 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $ß_u \subset \tau_u$. Then the followings are equivalent.

(i) $ß_u$ is a basis of τ_u.
(ii) For every $M \in \tau_u$ and $e_M \in M$ there exists $B_i \in ß_u$ such that $e_M \in B_k \subset M$.

Proof (i) → (ii) Let $ß_u$ be a basis of τ_N and $M \in \tau_u$. Then, $M = \cup_i B_i$ for $B_i \in ß_u$. Now, $e_M \in M \Rightarrow e_M \in \cup_i B_i \Rightarrow e_M \in B_k$, for some k.

Thus, $e_M \in B_k \subset \cup_i B_i = M$.

(ii) → (i) Here M is open NSS and $e_M \in B_i \subset M$ for $B_i \in ß_u$.

Then, $M = \cup_i B_i \Rightarrow ß_u$ is a basis for τ_u. □

4.2.3 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $ß_u \subset NSS(U, E)$. Then $ß_u$ is a basis for topology τ_u iff followings hold for arbitrary $N \in \tau_u$.

(i) for every $e_N \in N$ there exists $B_i \in ß_u$ such that $e_N \in B_i$.
(ii) for every $B_1, B_2 \in ß_u$ and $e_N \in B_1 \cap B_2$, there exists $B_3 \in ß_u$ such that $e_N \in B_3 \subset B_1 \cap B_2$.

Proof First suppose that $ß_u$ is a basis for topology τ_u on (U, E). Then $N = \cup_i B_i$ for $B_i \in ß_N$. Now $e_N \in N \Rightarrow e_N \in B_i$ for some i. Thus condition (i) is satisfied.

Next suppose that $B_1, B_2 \in ß_u$. Then B_1, B_2 are basic open NSS and $B_1 \cap B_2$ is also an open NSS. So for $e_N \in B_1 \cap B_2$, there exists another $B_3 \in ß_u$ such that $e_N \in B_3 \subset B_1 \cap B_2$.

Conversely, suppose that the given conditions hold. We are to prove that $ß_u$ is a basis for topology τ_u on (U, E).

Let τ_u be a family of union of sets of $ß_u$. Since by condition (i), $N = \cup_i B_i$ for $B_i \in ß_u$ then $N \in \tau_u$. Now if $N_1, N_2 \in \tau_u$ and $e_N \in N_1 \cap N_2$ there exists $B_1, B_2 \in ß_u$ such that $e_N \in B_1 \subset N_1$ and $e_N \in B_2 \subset N_2$. This implies $e_N \in B_1 \cap B_2 \subset N_1 \cap N_2$.

By condition (ii), there exists $B_3 \in ß_u$ such that $e_N \in B_3 \subset B_1 \cap B_2 \subset N_1 \cap N_2$. This shows that $N_1 \cap N_2$ can be expressed as the union of the members in $ß_u$ i.e., $N_1 \cap N_2 \in \tau_u$. Finally, $N_1 \cap N_2 = \phi_u \in \tau_u$ if N_1, N_2 are disjoint. Hence, $ß_u$ is a base for the topology τ_u on (U, E). □

4.2.4 Theorem

If $ß_u$ is a base for a topology τ_u, then τ_u is the smallest topology containing $ß_u$.

Proof Let τ_{u^1} be another topology containing $ß_u$ and $M \in \tau_u$. Then for $e_M \in M$, there exists $B_{e_M} \in ß_u$ such that $e_M \in B_{e_M} \subset M$ i.e., $M = \cup_{e_M} B_{e_M}$ for $e_M \in M$. Now since τ_{u^1} contains $ß_u$ and $B_{e_M} \in ß_u$ then $B_{e_M} \in \tau_{u^1}$. Also τ_{u^1} is a topology, so $\cup_{e_M} B_{e_M} \in \tau_{u^1}$ i.e., $M \in \tau_{u^1}$. This shows that $\tau_u \subset \tau_{u^1}$. □

4.2.5 Theorem

Let N be an NSS over (*U*, *E*). Suppose τ_{u^1} and τ_{u^2} be two topologies on (*U*, *E*) generated by the bases $ß_{u^1}$ and $ß_{u^2}$, respectively. Then $\tau_{u^1} \subset \tau_{u^2}$ iff for each $e_N \in N$ and for each $B_1 \subset ß_{u^1}$ containing e_N, there exists $B_2 \in ß_{u^2}$ such that $e_N \in B_2 \subset B_1$.

Proof First suppose, $\tau_{u^1} \subset \tau_{u^2}$ and $e_N \in N, B_1 \in ß_{u^1}$ such that $e_N \in B_1$. Since $ß_{u^1}$ is a basis for neutrosophic soft topology τ_{u^1} on (*U*, *E*), then $ß_{u^1} \subset \tau_{u^1} \Rightarrow e_N \in B_1 \in ß_{u^1} \subset \tau_{u^1} \subset \tau_{u^2}$ i.e., $e_N \in B_1 \in \tau_{u^2}$. Since, $ß_{u^2}$ is the base for the topology τ_{u^2}, so for $B_2 \in ß_{u^2}$ we have, $e_N \in B_2 \subset B_1$.

Conversely suppose that the hypothesis holds. we show that $\tau_{u^1} \subset \tau_{u^2}$.

Let $M \in \tau_{u^1}$. Since $ß_{u^1}$ is a basis for the topology τ_{u^1}, then for $e_N \in M$ there exists $B_1 \in ß_{u^1}$ such that $e_N \in B_1 \subset M$. Now by hypothesis, there exists $B_2 \in ß_{u^2}$ such that $B_2 \subset B_1 \Rightarrow B_2 \subset B_1 \subset M \Rightarrow B_2 \subset M \Rightarrow M \in \tau_{u^2}$. This shows that $\tau_{u^1} \subset \tau_{u^2}$. □

5 Subspace topology

5.1 Definition

Let (U, E, τ_u) be a neutrosophic soft topological space over (*U*, *E*) where τ_u is a topology on (*U*, *E*) and $M \in NSS(U, E)$ an arbitrary NSS. Suppose $\tau_M = \{M \cap N_i : N_i \in \tau_u\}$. Then τ_M forms also a topology on (*U*, *E*). Thus (U, E, τ_M) is a neutrosophic soft subspace topology of (U, E, τ_u).

5.1.1 Example

Let us consider the example (2) in subsection [3.2.1]. We define $M \in NSS(U, E)$ as following:

$$f_M(e_1) = \{\langle x_1, (0.4, 0.6, 0.8)\rangle, \langle x_2, (0.7, 0.3, 0.2)\rangle, \langle x_3, (0.5, 0.5, 0.7)\rangle\};$$
$$f_M(e_2) = \{\langle x_1, (0.6, 0.3, 0.5)\rangle, \langle x_2, (0.4, 0.7, 0.6)\rangle, \langle x_3, (0.8, 0.3, 0.5)\rangle\};$$

We denote $M \cap \phi_u = \phi_M, M \cap 1_u = 1_M, M \cap N_1 = M_1, M \cap N_2 = M_2, M \cap N_3 = M_3$; Then M_1, M_2, M_3 are given as following:

$$f_{M_1}(e_1) = \{\langle x_1, (0.4, 0.6, 0.8)\rangle, \langle x_2, (0.6, 0.6, 0.6)\rangle, \langle x_3, (0.5, 0.6, 0.7)\rangle\};$$
$$f_{M_1}(e_2) = \{\langle x_1, (0.6, 0.4, 0.5)\rangle, \langle x_2, (0.4, 0.7, 0.6)\rangle, \langle x_3, (0.7, 0.5, 0.6)\rangle\};$$
$$f_{M_2}(e_1) = \{\langle x_1, (0.4, 0.6, 0.8)\rangle, \langle x_2, (0.5, 0.7, 0.6)\rangle, \langle x_3, (0.4, 0.7, 0.7)\rangle\};$$
$$f_{M_2}(e_2) = \{\langle x_1, (0.6, 0.6, 0.5)\rangle, \langle x_2, (0.4, 0.8, 0.6)\rangle, \langle x_3, (0.5, 0.8, 0.6)\rangle\};$$
$$f_{M_3}(e_1) = \{\langle x_1, (0.4, 0.6, 0.8)\rangle, \langle x_2, (0.4, 0.8, 0.8)\rangle, \langle x_3, (0.3, 0.8, 0.7)\rangle\};$$
$$f_{M_3}(e_2) = \{\langle x_1, (0.5, 0.8, 0.6)\rangle, \langle x_2, (0.4, 0.9, 0.6)\rangle, \langle x_3, (0.2, 0.9, 0.7)\rangle\};$$

Here $M_1 \cap M_2 = M_2, M_1 \cap M_3 = M_3, M_2 \cap M_3 = M_3$ and $M_1 \cup M_2 = M_2, M_1 \cup M_3 = M_3, M_2 \cup M_3 = M_3$. Then $\tau_M = \{\phi_M, 1_M, M_1, M_2, M_3\}$ is neutrosophic soft subspace topology on (*U*, *E*).

5.2 Definition

Let M, N be two NSSs over (U, E). Then $M - N$ may be defined as:

$$M - N = \left\{\langle x, T_{f_M(e)(x)} * F_{f_N(e)(x)}, I_{f_M(e)(x)} \diamond (1 - I_{f_N(e)}(x)), F_{f_M(e)}(x) \diamond T_{f_N(e)}(x)\rangle\right\}$$

for all $x \in U$, $e \in E$.

5.3 Theorem

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E) and $M, N \in NSS(U, E)$. Then,

(i) If β_u is a base of τ_u then $\beta_M = \{B \cap M : B \in \beta_u\}$ is a base for the topology τ_M.
(ii) If Q is closed NSS in M and M is closed NSS in N, then Q is closed in N.
(iii) Let $Q \subset M$. If $\overline{Q}$ is the closure of Q then $\overline{Q} \cap M$ is the closure of Q in M.

Proof

(i) Since β_u is a base for τ_u so for arbitrary $N_K \in \tau_u$, we have $N_K = \cup_{B \in \beta_u} B$. This implies $N_K \cap M = (\cup_{B \in \beta_u} B) \cap M = \cup_{B \in \beta_u}(B \cap M)$ for $N_K \cap M \in \tau_M$. Since arbitrary member of τ_M can be expressed as the union of members of β_M, hence the theorem is completed.

(ii) To prove this part, we first show that if Q is closed in M then there exists a closed set $V \subset N$ i.e., $V \notin \tau_u$ such that $Q = V \cap M$.
Let Q be closed in M. Then Q^c is open in M i.e., Q^c can be put as $Q^c = P \cap M$ for $P \in \tau_u \Rightarrow (Q^c)^c = M \cap (P \cap M)^c = P^c \cap M$. Here $P^c \notin \tau_u$ i.e., P^c is closed NSS. So P^c here acts as $V \subset N$.
Conversely, suppose that $Q = V \cap M$ where $M \subset N$ and V is closed in N. Clearly, $V^c \in \tau_u$ so that $V^c \cap M \in \tau_M$. Now, $V^c \cap M = (N - V) \cap M = (N \cap M) - (V \cap M) = M - Q$. This implies $M - Q$ is open in M i.e., Q is closed in M.
We now prove the main theorem.
Since Q is closed NSS in M, then $Q = V \cap M$, for V being a closed NSS in N. By hypothesis, M is closed in N also. Thus Q is closed in N.

(iii) $\overline{Q} = \cap\{Q_i : Q_i \text{ is closed and } Q_i \supset Q\}$ is the closure of Q and so $\overline{Q}$ is closed.
Now, $\overline{Q} \cap M = \cap\{Q_i : Q_i \text{ is closed and } Q_i \supset Q\} \cap M = \cap(Q_i \cap M)$. Since each Q_i is closed then each $Q_i \cap M$ is closed in M by theorem (ii). Now, $Q \subset Q_i$ and $Q \subset M$. So $Q \cap M \subset Q_i \cap M \Rightarrow Q \subset Q_i \cap M$.
Thus, $\overline{Q} \cap M = \cap\{(Q_i \cap M) : (Q_i \cap M) \text{ is closed and } (Q_i \cap M) \supset Q\}$
Hence $\overline{Q} \cap M$ is a closure of Q in M. □

5.4 Theorem

Let (U, E, τ_M) be a subspace topology of a topological space (U, E, τ_u) over (U, E). If M is open NSS in (U, E, τ_u), then an NSS $M_1 \subset M$ is open in (U, E, τ_M) iff M_1 is open in (U, E, τ_u). The interior of a neutrosophic soft subset P of the open NSS M in (U, E, τ_M) is the same as the interior of P in (U, E, τ_u).

Proof First suppose that M is open NSS in (U, E, τ_u) such that a neutrosophic soft subset M_1 of M is open in (U, E, τ_M). Then $M_1 \in \tau_M$ and so $M_1 = N_1 \cap M$ for $N_1 \in \tau_u$. But M_1 is open NSS in (U, E, τ_u) as N_1, M both are open NSSs in (U, E, τ_u).

Conversely, assume that M_1 is open NSS in (U, E, τ_u) when M is open NSS in (U, E, τ_u) and $M_1 \subset M$. Then $M_1 \in \tau_u$. But $M_1 \cap M = M_1$ and so M_1 is open NSS in (U, E, τ_M). Hence the first part is proved. Now,

$$\begin{aligned} P^o \text{ in } (U, E, \tau_M) &= \cup\{M_1 : M_1 \subset P \text{ and } M_1 \in \tau_M\} \\ &= \cup\{(N_1 \cap M) : (N_1 \cap M) \subset P \text{ and } N_1 \in \tau_u\} \\ &= \cup\{(N_1 \cap M) : (N_1 \cap M) \subset P \text{ and } (N_1 \cap M) \in \tau_u\} \\ &\quad [\text{ as } M \text{ is open NSS in } (U, E, \tau_u)] \\ &= \cup\{M_1 : M_1 \subset P \text{ and } M_1 \in \tau_u\} \\ &= P^o \text{ in } (U, E, \tau_u) \end{aligned}$$

Thus the theorem is proved. □

5.5 Theorem

Let (U, E, τ_Q) be a subspace topology of a topological space (U, E, τ_u) over (U, E). If Q is closed NSS in (U, E, τ_u), then an NSS $Q_1 \subset Q$ is closed in (U, E, τ_Q) iff Q_1 is closed in (U, E, τ_u). The closure of a neutrosophic soft subset S of the closed NSS Q in (U, E, τ_Q) is the same as the closure of S in (U, E, τ_u).

Proof First suppose that Q is closed NSS in (U, E, τ_u) such that a neutrosophic soft subset Q_1 of Q is closed in (U, E, τ_Q). Since Q_1 is closed in (U, E, τ_Q) and so $Q_1 = N_2 \cap Q$ for N_2 being closed in (U, E, τ_u). But Q_1 is closed NSS in (U, E, τ_u) as N_2, Q both are closed NSSs in (U, E, τ_u).

Conversely, assume that Q_1 is closed NSS in (U, E, τ_u) when Q is closed NSS in (U, E, τ_u) and $Q_1 \subset Q$. Then $Q_1 \cap Q = Q_1$ and so Q_1 is closed NSS in (U, E, τ_Q). Hence the first part is proved. Now,

$$\begin{aligned} \overline{S} \text{ in } (U, E, \tau_Q) &= \cap\left\{Q_1 : Q_1 \supset S \text{ and } Q_1^c \in \tau_Q\right\} \\ &= \cap\left\{(N_2 \cap Q) : (N_2 \cap Q) \supset S \text{ and } N_2^c \in \tau_u\right\} \\ &= \cap\{(N_2 \cap Q) : (N_2 \cap Q) \supset S \text{ and } (N_2 \cap Q)^c \in \tau_u\} \\ &\quad [\text{ as } Q \text{ is closed NSS in } (U, E, \tau_u)] \\ &= \cap\left\{Q_1 : Q_1 \supset S \text{ and } Q_1^c \in \tau_u\right\} \\ &= \overline{S} \text{ in } (U, E, \tau_u) \end{aligned}$$

Thus the theorem is proved. □

6 Separation axioms

6.1 Definition

T_0 : A neutrosophic soft topological space (U, E, τ_u) over (U, E) is said to be a neutrosophic soft T_0 space if for every pair of disjoint neutrosophic soft points e_{N_1} and e_{N_2} there exists a neutrosophic soft open set containing one but not the other.

A discrete neutrosophic soft topological space is a neutrosophic soft T_0-space since every neutrosophic soft point is a neutrosophic soft open set in the discrete space.

T_1 : A neutrosophic soft topological space (U, E, τ_u) over (U, E) is said to be a neutrosophic soft T_1 space if for every pair of disjoint neutrosophic soft points e_K and e_S, there exists neutrosophic soft open sets M and P such that $e_K \in M, e_K \notin P$ and $e_S \in P, e_S \notin M$.

Let $U = \{h_1, h_2\},\ E = \{e_1, e_2\}$ and $\tau_u = \{\phi_u, 1_u, M, P\}$ where $M, P \in NSS(U, E)$ are defined as following:

$$
\begin{aligned}
f_M(e_1) &= \{\langle h_1, (1, 0, 1)\rangle, \langle h_2, (0, 0, 1)\rangle\};\\
f_M(e_2) &= \{\langle h_1, (0, 1, 0)\rangle, \langle h_2, (1, 0, 0)\rangle\};\\
f_P(e_1) &= \{\langle h_1, (0, 1, 0)\rangle, \langle h_2, (1, 1, 0)\rangle\};\\
f_P(e_2) &= \{\langle h_1, (1, 0, 1)\rangle, \langle h_2, (0, 1, 1)\rangle\};
\end{aligned}
$$

Then τ_u is a neutrosophic soft topology on (U, E) with respect to the t-norm and s-norm defined as $a * b = max\{a + b - 1, 0\}$ and $a \diamond b = min\{a + b, 1\}$. Here $e_{1M} \in M, e_{1M} \notin P$ and $e_{2P} \in P, e_{2P} \notin M$ though e_{1M}, e_{2P} are distinct.

T_2 : Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E). For two distinct neutrosophic soft points e_K, e_S if there exists disjoint neutrosophic soft open sets M, P such that $e_K \in M$ and $e_S \in P$ then (U, E, τ_u) is called T_2 space or Hausdorff space.

Let $U = \{h_1, h_2\},\ E = \{e\}$ and $\tau_u = \{\phi_u, 1_u, M, P\}$ where $M, P \in NSS(U, E)$ are defined as following:

$$
\begin{aligned}
f_M(e) &= \{\langle h_1, (1, 0, 1)\rangle, \langle h_2, (0, 0, 1)\rangle\};\\
f_P(e) &= \{\langle h_1, (0, 1, 0)\rangle, \langle h_2, (1, 1, 0)\rangle\};
\end{aligned}
$$

Then τ_u is a neutrosophic soft topology on (U, E) with respect to the t-norm and s-norm defined as $a * b = max\{a + b - 1, 0\}$ and $a \diamond b = min\{a + b, 1\}$. Here $e_M \in M$ and $e_P \in P$ where $e_M \neq e_P$ and $M \cap P = \phi_u$.

Regular: A neutrosophic soft topological space (U, E, τ_u) over (U, E) is said to be a neutrosophic soft regular space if for every neutrosophic soft point e_K and neutrosophic soft closed set M not containing e_K, there exists disjoint neutrosophic soft open sets P_1, P_2 such that $e_K \in P_1$ and $M \subset P_2$.

Let $U = \{h_1, h_2\},\ E = \{e\}$ and $\tau_u = \{\phi_u, 1_u, P_1, P_2\}$ where $P_1, P_2 \in NSS(U, E)$ are defined as following:

$$f_{P_1}(e) = \{\langle h_1, (0,1,0)\rangle, \langle h_2, (0,1,1)\rangle\};$$
$$f_{P_2}(e) = \{\langle h_1, (1,0,1)\rangle, \langle h_2, (1,0,0)\rangle\};$$

Then τ_u is a neutrosophic soft topology on (U, E) with respect to the t-norm and s-norm defined as $a * b = max\{a + b - 1, 0\}$ and $a \diamond b = min\{a + b, 1\}$ where $P_1 \cap P_2 = \phi_u$. Here, $e_{P_1} \in P_1$ and $e_{P_1} \notin M \subset P_2$ where $M = P_2^c$ is a closed NSS.

T_3 : A neutrosophic soft topological space (U, E, τ_u) over (U, E) is said to be a neutrosophic soft T_3-space if it is neutrosophic soft regular and neutrosophic soft T_1-space.

Normal: A neutrosophic soft topological space (U, E, τ_u) over (U, E) is said to be a neutrosophic soft normal space if for every pair of disjoint neutrosophic soft closed sets M_1, M_2 there exists disjoint neutrosophic soft open sets P_1, P_2 such that $M_1 \subseteq P_1$ and $M_2 \subseteq P_2$.

Let $U = \{x_1, x_2\},\ E = \{e\}$ and $\tau_u = \{\phi_u, 1_u, P_1, P_2\}$ where $P_1, P_2 \in NSS(U, E)$ are defined as following:

$$f_{P_1}(e) = \{\langle x_1, (1,1,0)\rangle, \langle x_2, (0,1,1)\rangle\};$$
$$f_{P_2}(e) = \{\langle x_1, (0,0,1)\rangle, \langle x_2, (1,0,0)\rangle\};$$

Then τ_u is a neutrosophic soft topology on (U, E) with respect to the t-norm and s-norm, $a * b = max\{a + b - 1, 0\}$ and $a \diamond b = min\{a + b, 1\}$. Here P_1, P_2 are disjoint neutrosophic soft open sets. Then P_1^c, P_2^c are disjoint neutrosophic soft closed sets in (U, E, τ_u) with $P_1^c = P_2, P_2^c = P_1$

T_4 : A neutrosophic soft topological space (U, E, τ_u) over (U, E) is said to be a neutrosophic soft T_4-space if it is neutrosophic soft normal and neutrosophic soft T_1-space.

6.2 Proposition

Let (U, E, τ_M) be a neutrosophic soft subspace topology of a neutrosophic soft topological space (U, E, τ_u) over (U, E). Then,

(i) if (U, E, τ_u) is a neutrosophic soft T_0-space then (U, E, τ_M) is also so.
(ii) if (U, E, τ_u) is a neutrosophic soft T_1-space then (U, E, τ_M) is also so.
(iii) if (U, E, τ_u) is a neutrosophic soft T_2-space then (U, E, τ_M) is also so.
(iv) if (U, E, τ_u) is a neutrosophic soft regular space then (U, E, τ_M) is also so.
(v) if (U, E, τ_u) is a neutrosophic soft T_3-space then (U, E, τ_M) is also so.

Proof (i) Let $e_K, e_S \in M$ with $e_K \neq e_S$. Since (U, E, τ_u) is a neutrosophic soft T_0-space then $N_1, N_2 \in \tau_u$ such that $e_K \in N_1, e_S \notin N_1$ or $e_S \in N_2, e_K \notin N_2$. Hence, $e_K \in M \cap N_1, e_S \notin M \cap N_1$ or $e_S \in M \cap N_2, e_K \notin M \cap N_2$. Thus (U, E, τ_M) is a neutrosophic soft T_0-space.

The others can be proved in the similar manner. □

6.3 Proposition

Neutrosophic soft T_4- space $\Rightarrow$ neutrosophic soft T_3- space $\Rightarrow$ neutrosophic soft T_2-space $\Rightarrow$ neutrosophic soft T_1- space $\Rightarrow$ neutrosophic soft T_0- space.

Proof We here give the proof of last $(\Rightarrow)$. The proof of others are in similar way.

Let (U, E, τ_u) be a neutrosophic soft topological space over (U, E). We consider two distinct neutrosophic soft points e_k, e_S. Since (U, E, τ_u) is a neutrosophic soft T_1-space then there exists $M, P \in \tau_u$ such that $e_K \in M, e_K \notin P$ and $e_S \notin M, e_S \in P$. This shows $e_K \in M, e_S \notin M$ or $e_K \notin P, e_S \in P$. Hence, (U, E, τ_u) is a T_0-space over (U, E). □

6.4 Theorem

A neutrosophic soft topological space (U, E, τ_u) is a neutrosophic soft T_2-space iff for distinct neutrosophic soft points e_K, e_S there exists a neutrosophic soft open set M containing e_K but not e_S such that $e_S \notin \overline{M}$.

Proof Let e_K, e_S be two distinct neutrosophic soft points in a neutrosophic soft T_2-space (U, E, τ_u). Then there exists disjoint neutrosophic soft open sets N_1, N_2 such that $e_K \in N_1, e_S \in N_2$. Since $e_K \neq e_S$ and N_1, N_2 are disjoint NSSs, then $e_S \notin N_1$. It implies $e_S \notin \overline{N_1}$ as $N_1 \subset \overline{N_1}$. By similar argument $e_K \notin N_2$ and so $e_K \notin \overline{N_2}$.

Next suppose, for distinct neutrosophic soft points e_K, e_S there exists a neutrosophic soft open set M containing e_K but not e_S such that $e_S \notin \overline{M}$. Then $e_S \in (\overline{M})^c$ i.e., M and $(\overline{M})^c$ are disjoint neutrosophic soft open set containing e_K and e_S, respectively. □

6.5 Theorem

A neutrosophic soft topological space (U, E, τ_u) in which every neutrosophic soft point is neutrosophic soft closed, is neutrosophic soft regular space iff for a neutrosophic soft open set M containing a neutrosophic soft point e_K, there exists a neutrosophic soft open set P containing e_S, such that $\overline{P} \subset M$.

Proof We take a neutrosophic soft open set M containing e_K in a regular neutrosophic soft topological space (U, E, τ_u). Then M^c is neutrosophic soft closed and $e_K \notin M^c$. By hypothesis, there exists disjoint neutrosophic soft open sets N_1, N_2 such that $e_K \in N_1$ and $M^c \subset N_2$. Now since $N_1 \cap N_2 = \phi_u$, so $N_1 \subset N_2^c \Rightarrow \overline{N_1} \subset \overline{N_2^c} = N_2^c$ as N_2^c is closed. But $(M^c)^c \supset N_2^c \Rightarrow M \supset N_2^c$. Thus $\overline{N_1} \subset M$.

Conversely, assume that the hypothesis hold. Take a neutrosophic soft closed set Q not containing a neutrosophic soft point e_K. Then $e_K \in Q^c$ and Q^c is a neutrosophic soft open set. This implies there exists a neutrosophic soft open set P containing e_K such that $\overline{P} \subset Q^c \Rightarrow Q \subset (\overline{P})^c \Rightarrow (\overline{P})^c$ is a neutrosophic soft open set containing Q and $P \cap (\overline{P})^c = \phi_u$ [as $P \subset \overline{P}$]. □

6.6 Theorem

A neutrosophic soft topological space (U, E, τ_u) is neutrosophic soft normal space iff for any neutrosophic soft closed set Q and neutrosophic soft open set P containing Q, there exists a neutrosophic soft open set M such that $Q \subset M$ and $\overline{M} \subset P$.

Proof Let (U, E, τ_u) be a neutrosophic soft normal space and P be a neutrosophic soft open set containing Q where Q be a neutrosophic soft closed set i.e., P^c, Q be two disjoint neutrosophic soft closed sets. Then there exists disjoint neutrosophic soft open sets N_1, N_2 such that $Q \subset N_1$ and $P^c \subset N_2$. Now $N_1 \subset N_2^c \Rightarrow \overline{N_1} \subset \overline{N_2^c} = N_2^c$. Also, $P^c \subset N_2 \Rightarrow N_2^c \subset P \Rightarrow \overline{N_1} \subset P$.

Conversely, let Q_1, Q_2 be two disjoint pair of neutrosophic soft closed sets. Then $Q_1 \subset Q_2^c$. By hypothesis there exists a neutrosophic soft open set M such that $Q_1 \subset M$ and $\overline{M} \subset Q_2^c \Rightarrow Q_2 \subset (\overline{M})^c \Rightarrow M$ and $(\overline{M})^c$ are disjoint neutrosophic soft open sets such that $Q_1 \subset M$ and $Q_2 \subset (\overline{M})^c$. □

7 Conclusion

In the present paper, the topological structure on neutrosophic soft set has been introduced. We propose some properties of neutrosophic soft interior and neutrosophic soft closure. We can say that a neutrosophic soft topological space gives a parameterized family of fuzzy tritopologies on the initial universe but the reverse is not true. Besides, here we also have defined the base for neutrosophic soft topological space, subspace on neutrosophic soft set, separation axioms with suitable examples. Several related properties and structural characteristics in each case have been investigated. This concept will bring a new opportunity in future research and development of NSS theory.

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