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THEORETICAL ARTICLE

A revisit to queueing-inventory system with reservation, cancellation and common life time

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Abstract In this paper we consider a single server queueing-inventory system having capacity to store S items which have a common-life time (*CLT*), exponentially distributed with parameter γ. On realization of *CLT* a replenishment order is placed so as to bring the inventory level back to S, the lead time of which follows exponential distribution with parameter β. Items remaining are discarded on realization of *CLT*. Customers waiting in the system stay back on realization of common life time. Reservation of items and cancellation of sold items before its expiry time is permitted. Cancellation takes place according to an exponentially distributed inter-occurrence time with parameter $i\theta$ when there are $(S - i)$ items in the inventory. In this paper we assume that the time required to cancel the reservation is negligible. Customers arrive according to a Poisson process of rate λ and service time follows exponential distribution with parameter μ. The main assumption that no customer joins the system when inventory level is zero leads to a product form solution of the system state distribution. Several system performance measures are obtained.

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1 Introduction

It is common to purchase an item in the inventory and later cancel (return) it. We shall refer purchase of an item from inventory as reservation (for example, reservation of a seat in bus/train/flight for a future journey). Sometimes the purchased item may be returned. We call this as cancellation (for example, canceling a reserved seat). Each item on hand may have an expiry date which in some cases is common to all. Several examples can be cited: batch of medicines that were manufactured together have a common expiry date; once a bus/train/flight departs the vacant seats have no use, those could not board the transport system before departure miss it. In this paper we study a queueing inventory process consisting of S items which have expiry time, called common life time where reservation of items and cancellation of sold items within the expiry time is allowed. The common life time (CLT) of items is exponentially distributed with parameter γ, on realization of which the remaining items are discarded, but the waiting customers stay back in the system. Cancellation and reservation are permitted as long as common life time is not realized. Inter-cancellation time follows exponential distribution with parameter depending on the number of items in reservation list at that moment. Time required to cancel a reservation is assumed to be negligible. Demands for the item form a Poisson process of rate λ; one unit of item is supplied to a customer at the end of the service. The service time follows exponential distribution with parameter μ.

Queueing-inventory models (inventory with positive service time) is introduced by Sigman and Simchi-Levy [11]. It was followed by several researchers. A survey of work in this area is given in Krishnamoorthy et al. [4]. Among these certain significant contributions in stochastic decomposition could be found (see Schwarz et al. [9], Saffari et al. [8], Krishnamoorthy and Viswanath [5]). A new class of stochastic network that exhibit a product form steady state distribution was investigated by Schwarz et al. [10]. The stochastic networks developed there are integrated models for networks of service stations and inventories. They assume that though the server with attached inventory does not accept new customer when the inventory is depleted, lost sales are not lost to the system. A necessary and sufficient condition for a product form steady state distribution of joint queueing environment process which has applications in inventory theory is given by Krenzler and Daduna [2]. A product form steady state distribution for queueing inventory process where service system and random environment interact in both directions is derived in Krenzler and Daduna [1]. Discrete time (s, S) inventory model in which the stored items have a random common life time with a discrete phase type distribution where demands arrive in batches following a discrete phase type renewal process is considered by Lian et al. [6]

Queueing inventory with reservation, cancellation, common life time and retrial is introduced by Krishnamoorthy et al. [3]. They assumed that a customer on arrival to an idle server with at least one item in inventory is immediately taken for service or else he joins the buffer of varying size depending on the number of items in the inventory. If there is no item in the inventory the arriving customer first queue up in a finite waiting space of capacity K. When it overflows an arrival goes to an orbit of infinite capacity with probability $\boldsymbol{p}$ or is lost forever with probability $1-p$. From the orbit he retries for service. However, they fail to produce a product form solution. For the model discussed in the present paper we do away with the buffer, waiting space and orbit; instead a single queue is considered. This is at the expense of loss of some crucial information - the finite waiting list is gone and is replaced by the number in the waiting room at any time. Nevertheless, under the crucial assumption that no customer joins the system when inventory level is zero, we establish the stochastic decomposition property of the system state.

The rest of the paper is arranged as follows. Mathematical formulation is taken up in Sect. 2. Section 3 provides the steady state analysis of the model. Some important performance measures are derived in this section. Numerical examples and optimization problems are discussed in Sect. 4.

Some notations and abbreviations used in the sequel:

- 0^* : Inventory level on common life time realization but before the replenishment.
- $\mathbf{e}$ = Column vector of $1'$s of appropriate order.
- *CTMC* : Continuous Time Markov Chain.
- *LIQBD* : Level Independent Quasi-Birth and Death Process.
- *CLT* : Common Life Time

2 Mathematical formulation

We consider a single server queueing-inventory system consisting of a homogeneous item having a *CLT*. The time duration from the epoch at which we start with maximum inventory level S at a replenishment epoch, to the moment when the *CLT* is realized is called a **cycle**. The *CLT* of items is exponentially distributed with parameter γ. On realization of *CLT* customers waiting in the system stay back in the system. When *CLT* is reached a replenishment order is placed, which is realized on completion of a positive lead time that is exponentially distributed with parameter β. Reservation of items and cancellation of sold items before the *CLT* realization is permitted. Cancellation takes place according to an exponentially distributed inter-occurrence time with parameter $i\theta$, when $(S-i)$ items present in the inventory. Through cancellation of purchased item, inventory gets added to the existing one; nevertheless, inventory level will not go above S (the sum of items in sold list and those in store equal to S). The customers arrive according to a Poisson process of rate λ. Each customer requires a single homogeneous item, having a random duration of service time which follows an exponential distribution with parameter μ. No customer joins the system when inventory level is zero.

The above system is modelled as a continuous time Markov chain $\Gamma = \{(N(t), I(t)), t \geq 0\}$ with state space $\{(n, 0^*), n \geq 0\} \bigcup \{(n, i), n \geq 0, 0 \leq i \leq S\}$ where $N(t)$ = Number of customers at time t and $I(t)$ = Number of items in the inventory at that time. The transitions in the Markov Chain are

- Transitions due to arrival:
  $\boldsymbol{(n, i)} \rightarrow (n+1, i)$ at the rate λ for $n \geq 0$, $1 \leq i \leq S$
- Transitions due to service completions:
  $\boldsymbol{(n, i)} \rightarrow (n-1, i-1)$ at the rate μ for $n \geq 1$, $1 \leq i \leq S$
- Transitions due to common life time realization:
  $\boldsymbol{(n, i)} \rightarrow (n, 0^*)$ at the rate γ for $n \geq 0$, $0 \leq i \leq S$
- First transition that is counted after *CLT* is realized (which is due to replenishment):
  $(n, 0^*) \rightarrow \boldsymbol{(n, S)}$ at the rate β for $n \geq 0$
- Transition due to cancellation:
  $\boldsymbol{(n, i)} \rightarrow (n, i+1)$ at the rate $(S-i)\theta$ for $n \geq 0$, $0 \leq i \leq S-1$

The infinitesimal generator of Γ with entries governed as described above is

$$\mathcal{Q} = \begin{bmatrix} B & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where $\boldsymbol{B}$ contains transitions within level 0; A_0 represents transitions from $\boldsymbol{n}$ to $n+1$ for $n \geq 0$, A_1 represents transitions within $\boldsymbol{n}$ for $n \geq 1$ and A_2 represents transitions from $\boldsymbol{n}$ to $n-1$ for $n \geq 1$. All these are square matrices of order $S+2$.

$$B = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array} \begin{pmatrix} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \\ b_S & S\theta & & & & & \gamma \\ & b_{S-1} & (S-1)\theta & & & & \gamma \\ & & b_{S-2} & (S-2)\theta & & & \gamma \\ & & & \ddots & \ddots & & \vdots \\ & & & & b_1 & \theta & \gamma \\ & & & & & b_0 & \gamma \\ & & & & & \beta & -\beta \end{pmatrix},$$

$$A_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array} \begin{pmatrix} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \\ b_S & S\theta & & & & & \gamma \\ & a_{S-1} & (S-1)\theta & & & & \gamma \\ & & a_{S-2} & (S-2)\theta & & & \gamma \\ & & & \ddots & \ddots & & \vdots \\ & & & & a_1 & \theta & \gamma \\ & & & & & a_0 & \gamma \\ & & & & & \beta & -\beta \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & & & & \\ \mu & 0 & & & \\ & \ddots & \ddots & & \\ & & \mu & 0 & \\ & & & 0 & 0 \end{pmatrix}.$$

with $b_S = -(\gamma + S\theta), b_i = -(\lambda + i\theta + \gamma), a_i = -(\lambda + \mu + i\theta + \gamma)$ for $0 \leq i \leq S-1$.

2.1 Stability condition

To establish the stability condition, we consider the Markov chain $\{\boldsymbol{I(t)}, t \geq 0\}$, where $I(t)$ is as defined earlier with state space given by $\{0, 1, 2, \ldots S, 0^*\}$. Let $\boldsymbol{\phi} = (\phi_0, \phi_1, \ldots, \phi_S, \phi_0^*)$ be the steady-state probability vector of this Markov chain. Its infinitesimal generator is $A = (A_0 + A_1 + A_2) =$

$$\begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array} &
\begin{pmatrix}
b_S & S\theta & & & & & \gamma \\
\mu & b'_{S-1} & (S-1)\theta & & & & \gamma \\
 & \mu & b'_{S-2} & (S-2)\theta & & & \gamma \\
 & & \ddots & \ddots & \ddots & & \vdots \\
 & & & \mu & b'_1 & \theta & \gamma \\
 & & & & \mu & b'_0 & \gamma \\
 & & & & & \beta & -\beta
\end{pmatrix}
\end{array}$$

with $b_S = -(\gamma + S\theta), b'_i = -(\mu + i\theta + \gamma)$ for $0 \leq i \leq S-1$.

Then $\boldsymbol{\phi}$ satisfies the equations

$$\boldsymbol{\phi} A = 0, \qquad \boldsymbol{\phi}\mathbf{e} = \mathbf{1}. \tag{1}$$

The components of $\boldsymbol{\phi}$ are obtained as

$$\phi_i = \begin{cases} V_i \phi_0 & 1 \leq i \leq S \\ V_0^* \phi_0 & i = 0^* \end{cases}$$

where

$$V_i = \begin{cases} 1 & i = 0 \\ \dfrac{\gamma + S\theta}{\mu} & i = 1 \\ \dfrac{(\gamma + \mu + (S-(i-1))\theta)V_{i-1} - (S-(i-2))\theta)V_{i-2}}{\mu} & 2 \leq i \leq S \\ \dfrac{\gamma}{\beta}\displaystyle\sum_{i=0}^{S} V_i & i = 0^* \end{cases}$$

The unknown probability ϕ_0 can be found from the normalizing condition $\boldsymbol{\phi}\mathbf{e} = \mathbf{1}$ as

$$\phi_0 = \left(\sum_{i=0}^{S} V_i + V_0^* \right)^{-1}. \tag{2}$$

The *LIQBD* description of the model indicates that the queueing-inventory system is stable (see Neuts [7]) if and only if the left drift exceeds that of right drift. That is,

$$\boldsymbol{\phi} A_0 \mathbf{e} < \boldsymbol{\phi} \mathbf{A_2} \mathbf{e} \tag{3}$$

which on simplification gives the stability condition as

$$\lambda < \mu. \tag{4}$$

This leads to

Lemma 2.1 *The process* $\Gamma = \{(N(t), I(t)), t \geq 0\}$ *is stable if and only if* $\lambda < \mu$.

3 Steady-state analysis

For finding the steady state vector of the process Γ, we first consider an inventory system with negligible service time and no backlog of demands. The corresponding Markov chain may be defined as $\tilde{\Gamma} = \{\boldsymbol{I(t)}, t \geq 0\}$ where $I(t)$ has the same definition as described in Sect. 2. Its infinitesimal generator is given by

$$\mathcal{H} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \end{array} \\ \begin{pmatrix} b_S & S\theta & & & & & \gamma \\ \lambda & b_{S-1} & (S-1)\theta & & & & \gamma \\ & \lambda & b_{S-2} & (S-2)\theta & & & \gamma \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & \lambda & b_1 & \theta & \gamma \\ & & & & \lambda & b_0 & \gamma \\ & & & & & \beta & -\beta \end{pmatrix} \end{array}$$

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots, \pi_S, \pi_0^*)$ be the steady state vector of the process $\tilde{\Gamma}$. Then $\boldsymbol{\pi}$ satisfies the equations

$$\boldsymbol{\pi}\mathcal{H} = 0, \qquad \boldsymbol{\pi}\mathbf{e} = \mathbf{1}. \tag{5}$$

Then the components of $\boldsymbol{\pi}$ can be obtained as

$$\pi_i = \begin{cases} U_i \pi_0 & 1 \leq i \leq S \\ U_0^* \pi_0 & i = 0^* \end{cases}$$

where

$$U_i = \begin{cases} 1 & i = 0 \\ \dfrac{\gamma + S\theta}{\lambda} & i = 1 \\ \dfrac{(\gamma + \lambda + (S-(i-1))\theta)U_{i-1} - (S-(i-2))\theta)U_{i-2}}{\lambda} & 2 \leq i \leq S \\ \dfrac{\gamma}{\beta}\sum_{i=0}^{S} U_i & i = 0^* \end{cases}$$

The unknown probability π_0 can be found from the normalizing condition $\pi\mathbf{e} = \mathbf{1}$ as

$$\pi_0 = \left(\sum_{i=0}^{S} U_i + U_0^* \right)^{-1}. \tag{6}$$

Assuming that (4) is satisfied, we compute the steady state probability of the original system. Let $\mathbf{x}$ denote the steady-state probability vector of this system. Then

$$\mathbf{x}\mathcal{Q} = \mathbf{0}, \qquad \mathbf{xe} = \mathbf{1}. \tag{7}$$

Partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots)$ where $\mathbf{x_i} = (\mathbf{x_i}(\mathbf{0}), \mathbf{x_i}(\mathbf{1}), \ldots \mathbf{x_i}(\mathbf{S}), \mathbf{x_i}(\mathbf{0}^*))$, for $i \geq 0$.Then by (7) we get

$$\mathbf{x_0 B} + \mathbf{x_1 A_2} = \mathbf{0}, \tag{8}$$

$$\mathbf{x_i A_0} + \mathbf{x_{i+1} A_1} + \mathbf{x_{i+2} A_2} = \mathbf{0}; \quad \mathbf{i} \geq \mathbf{0}. \tag{9}$$

We produce a solution of the form

$$\mathbf{x_i} = \mathbf{K}\left(\frac{\lambda}{\mu}\right)^{\mathbf{i}} \boldsymbol{\pi}; \quad \mathbf{i} \geq \mathbf{0} \tag{10}$$

where $\boldsymbol{K}$ is a constant to be determined. With these $\mathbf{x_i}$ substituted in $\mathbf{x}\mathcal{Q} = \mathbf{0}$ we get

$$\mathbf{x_0 B} + \mathbf{x_1 A_2} = \mathbf{K}\boldsymbol{\pi}\left(\mathbf{B} + \frac{\lambda}{\mu}\mathbf{A_2}\right) = \mathbf{K}\boldsymbol{\pi}\mathcal{H} = \mathbf{0},$$

$$\mathbf{x_i A_0} + \mathbf{x_{i+1} A_1} + \mathbf{x_{i+2} A_2} = \mathbf{K}\left(\frac{\lambda}{\mu}\right)^{\mathbf{i+1}} \boldsymbol{\pi}\left(\mathbf{B} + \frac{\lambda}{\mu}\mathbf{A_2}\right) = \mathbf{K}\left(\frac{\lambda}{\mu}\right)^{\mathbf{i+1}} \boldsymbol{\pi}\mathcal{H} = \mathbf{0}.$$

Thus we can see that (10) satisfy the Eqs. (8) and (9). Now applying the normalizing condition $\mathbf{xe} = 1$ we get

$$K\left(1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \ldots\right) = 1.$$

Hence under the condition $\lambda < \mu$ we have $K = 1 - \frac{\lambda}{\mu}$.

Thus under the condition that $\lambda < \mu$, the steady state probability vector of the process Γ with generator matrix $\mathcal{Q}$ is given by $\mathbf{x} = (\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots)$, where

$$\mathbf{x_i} = \mathbf{K}\left(\frac{\lambda}{\mu}\right)^{\mathbf{i}} \boldsymbol{\pi}; \quad \mathbf{i} \geq \mathbf{0} \tag{11}$$

where

$$K = 1 - \frac{\lambda}{\mu}, \tag{12}$$

and conversely. Thus, the system state distribution under the stability condition is the product of marginal distributions of the number of customers in an M / M / 1 system and the number of items in the inventory.

Now we look at a few of the system characteristics that throw light on the performance of the system.

3.1 Performance measures

1. Expected number of customers in the system, $E_C = \dfrac{\lambda}{\mu - \lambda}$
2. Expected number of item in the inventory, $E_I = \sum_{i=1}^{S} i\pi_i$
3. Expected cancellation rate, $E_{CR} = \sum\limits_{i=0}^{S}(S-i)\theta\pi_i$
4. Expected number of cancellation, $E_{CN} = \dfrac{\sum_{i=0}^{S}(S-i)\theta\pi_i}{\gamma}$
5. Expected inventory purchase rate by customers, $E_{PR} = \lambda\sum_{i=1}^{S}\pi_i$
6. Expected number of inventory purchased by customers in a cycle, $E_{PN} = \dfrac{\lambda\sum_{i=1}^{S}\pi_i}{\gamma}$
7. Expected loss rate of customers, $E_L = \lambda\pi_0$
8. Probability that all items are in sold list before *CLT* realization, $Pvacant = \pi_0$
9. Probability that all items are in the system before *CLT* realization, $Pfull = \pi_S$

3.2 Expected sojourn time in zero inventory level in a cycle before realization of *CLT*

This is the expected time during which the system stays with no inventory. We derive this for a finite capacity system. For that consider the Markov Chain $\{(N(t), I(t)), t \geq 0\}$. The state space is $\{(n,0), 0 \leq n \leq K\}\bigcup\{\Delta\}$ where $\{\Delta\}$ denotes the absorbing state of the Markov chain which is realization of *CLT* or cancellation and K is the maximum number of customers accommodated in the system. Its infinitesimal generator is of the form

$$\mathcal{H}_1 = \begin{bmatrix} T & T^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where $T =$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ K-1 \\ K \\ \Delta \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & \cdots & K-1 & K & \Delta \end{array} \\
\left(\begin{array}{ccccccc}
-(S\theta+\gamma) & & & & & & (S\theta+\gamma) \\
 & -(S\theta+\gamma) & & & & & (S\theta+\gamma) \\
 & & -(S\theta+\gamma) & & & & (S\theta+\gamma) \\
 & & & & & & \vdots \\
 & & & & -(S\theta+\gamma) & & (S\theta+\gamma) \\
 & & & & & -(S\theta+\gamma) & (S\theta+\gamma) \\
0 & & & & 0 & 0 & 0
\end{array}\right)
\end{array}$$

Thus expected sojourn time in zero inventory level, $E_T^0 = -\alpha_K T^{-1}\mathbf{e}$ where $\alpha_K = (x_0(0), x_1(0), \ldots, x_K(0))$. Expected number of visits $= \frac{\mu\rho}{\gamma}\pi_1$. Thus the expected sojourn time in zero inventory level in a cycle $= \frac{\mu\rho}{\gamma}\pi_1(-\alpha_K T^{-1}\mathbf{e})$.

3.3 Expected sojourn time in maximum inventory level S in a cycle before realization of *CLT*

This is the expected time system stays with maximum inventory. Here also derivation is done in case of finite number of customers. For that consider the Markov Chain $\{(N(t), I(t)), t \geq 0\}$. The state space is $\{(n, S), 0 \leq n \leq K\} \bigcup \{\Delta\}$, where $\{\Delta\}$ denotes the absorbing state of the Markov chain which represents realization of *CLT* or service completion. Its infinitesimal generator is of the form

$$\mathcal{H}_2 = \begin{bmatrix} T_1 & T_1^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$T_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ K-1 \\ K \\ \Delta \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & K-1 & K & \Delta \end{array} \\ \begin{pmatrix} -(\lambda+\mu+\gamma) & \lambda & & & & & (\mu+\gamma) \\ & -(\lambda+\mu+\gamma) & \lambda & & & & (\mu+\gamma) \\ & & -(\lambda+\mu+\gamma) & \lambda & & & (\mu+\gamma) \\ & & & & & & \vdots \\ & & & & -(\lambda+\mu+\gamma) & & (\mu+\gamma) \\ & & & & & -(\lambda+\mu+\gamma) & (\mu+\gamma) \\ 0 & & & & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Thus the expected sojourn time in maximum inventory level, $E_{T_1}^S = -\alpha_K T_1^{-1}\mathbf{e}$ where $\alpha_K = (x_0(S), x_1(S), \ldots, x_K(S))$ and expected number of visits to $S = \frac{\theta}{\gamma}\pi_{S-1}$. Thus, expected sojourn time in maximum inventory level in a cycle $= (-\alpha_K T_1^{-1}\mathbf{e})\frac{\theta}{\gamma}\pi_{\mathbf{S-1}}$.

4 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

4.1 Effect of λ on various performance measures

Table 1 indicates that increase in λ value makes increase in expected number of customers in the system, expected loss rate, expected purchase rate, expected cancellation rate. As λ increases there is a decrease in the expected number of items in the inventory. Also, as λ increases probability that all items are in the sold list prior to realization of *CLT* increases and probability that all items are in the system just prior to realization of *CLT* decreases. These are all natural consequences as arrival rate increases.

Table 1 Effect of λ: fix $S = 20, \theta = 3, \mu = 15, \gamma = 0.1, \beta = 2$

λ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
9	1.4992	16.2817	1.3246×10^{-6}	8.2979	8.5697	1.4717×10^{-7}	0.0602
10	1.9946	15.9726	6.1139×10^{-6}	9.2258	9.5154	6.1139×10^{-7}	0.0459
11	2.7185	15.6629	2.1047×10^{-5}	10.1563	10.4442	1.9133×10^{-6}	0.0355
12	3.8342	15.3587	5.5471×10^{-5}	11.0717	11.3313	4.6226×10^{-6}	0.0280
13	5.6433	15.0756	1.1340×10^{-4}	11.9242	12.1367	8.7234×10^{-6}	0.0229

Table 2 Effect of μ: $S = 20, \theta = 3, \lambda = 11, \gamma = 0.1, \beta = 2$

μ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
12	7.7781	15.7020	8.3553×10^{-5}	10.0471	10.1282	7.5957×10^{-6}	0.0380
13	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
14	3.5499	15.6618	3.7134×10^{-5}	10.1616	10.4034	3.3759×10^{-6}	0.0357
15	2.7185	15.6629	2.1047×10^{-5}	10.1563	10.4442	1.9133×10^{-6}	0.0355
16	2.1905	15.6665	1.1363×10^{-5}	10.1490	10.4622	1.0330×10^{-6}	0.0354
17	1.8302	15.6665	5.9801×10^{-6}	10.1440	10.4700	5.4364×10^{-7}	0.0353

4.1.1 Effect of the service rate μ

Table 2 indicates that increase in μ value decreases the expected number of customers and expected loss rate of customers in the system. As service rate increases it is natural that loss rate of customers and expected number of customers in the system decreases. As μ increases expected number of items in the inventory shows a decreasing tendency first and then it increases. This could be attributed to the increase in cancellation of purchased items. Expected purchase rate increases, which is on expected lines. However expected cancellation rate increases first and then decreases as μ value increases. The initial increase in cancellation rate is due to large number of purchases taking place consequent to increasing value of μ; however with further increase in value of μ, the traffic intensity decreases and so the number of actual purchase decreases, which in turn results in the decrease of the rate of cancellations. Probability for all items in sold list prior to *CLT* realization decreases and probability for all items in system also decreases.

4.1.2 Effect of common life time parameter γ

In Table 3, there are few surprises. These are in the behaviour of E_I, E_{CR} and E_{PR} with increase in value of γ. Increase in γ means the *CLT* realization is faster. We observe that as γ increases there is a decrease in expected number of items in the inventory, expected loss rate of customers. Shorter the *CLT*, lesser will be the purchase rate, so cancellation rate also decreases. Also, we observe that as *CLT*

Table 3 Effect of γ: $S = 20, \theta = 3, \lambda = 11, \mu = 13, \beta = 2$

γ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
0.1	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
0.2	4.9865	15.0569	5.6983×10^{-5}	9.3877	9.8460	5.1803×10^{-6}	0.0449
0.3	4.9871	14.4928	5.4177×10^{-5}	8.7134	9.4188	4.9251×10^{-6}	0.0524
0.4	4.9875	13.9706	5.1615×10^{-5}	8.1101	9.0272	4.6923×10^{-6}	0.0590
0.5	4.9880	13.4857	4.9268×10^{-5}	7.5680	8.6669	4.4789×10^{-6}	0.0648
0.6	4.9884	13.0342	4.7109×10^{-5}	7.0791	8.3342	4.2827×10^{-6}	0.0699

Table 4 Effect of θ: $S = 20, \gamma = 0.1, \lambda = 11, \mu = 13, \beta = 2$

θ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
1	4.9854	9.7793	0.0278	9.2706	10.2863	0.0025	0.0099
2	4.9859	14.0978	1.5007×10^{-4}	9.9043	10.3136	1.3643×10^{-5}	0.0153
3	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
4	4.9860	16.4797	3.4360×10^{-5}	10.2809	10.3137	3.1236×10^{-6}	0.0726
5	4.9860	16.9753	2.3040×10^{-5}	10.3736	10.3137	2.0946×10^{-6}	0.1165
6	4.9860	17.3093	1.7012×10^{-5}	10.4440	10.3137	1.5466×10^{-6}	0.1624

realization decreases probability that all items are in sold list just prior to *CLT* realization decreases and probability that all items are in system prior to *CLT* realization increases.

4.1.3 Effect of cancellation rate θ

Table 4, shows that as cancellation rate increases expected number of customers in the system initially show a slight increase and then it remains constant. Expected number of items in the inventory and expected cancellation rate show an upward trend, which is a consequence of increasing value of θ. Expected purchase rate increases first and then remains constant and expected loss rate of customers decrease with respect to increase in θ. Also, we observe that as cancellation rate increases probability that all items are in sold list just prior to *CLT* realization decreases and probability that all items are in system just prior to *CLT* realization increases.This tendency is a consequence of higher cancellation rate for the same *CLT* parameter value.

4.1.4 Effect of replenishment rate β

From Table 5, we observe that as replenishment rate increases expected number of customers in the system show a slight decreasing tendency and expected loss rate of customers increase. There is an increase in expected number of items in the inventory, expected cancellation rate, expected purchase rate. Also, we observe that

Table 5 Effect of β: $S = 20, \theta = 3, \lambda = 11, \mu = 13, \gamma = 0.1$

β	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
1	4.9865	14.9579	5.720×10^{-5}	9.6846	9.8460	5.2001×10^{-6}	0.0347
2	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
3	4.9858	15.9205	6.1094×10^{-5}	10.3079	10.4796	5.5540×10^{-6}	0.0370
4	4.9857	16.0496	6.1618×10^{-5}	10.3915	10.5646	5.6017×10^{-6}	0.0373
5	4.9856	16.1281	6.1937×10^{-5}	10.4423	10.6162	5.6306×10^{-6}	0.0374
6	4.9856	16.1808	6.2151×10^{-5}	10.4764	10.6509	5.6501×10^{-6}	0.0376

as replenishment rate increases probability that all items are in sold list just prior to *CLT* realization and probability that all items are in system just prior to *CLT* realization increases.

4.2 Optimization problem

Based on the above performance measures we construct a cost function to check the maximality of profit function.

We define a revenue function as $\mathcal{RF}$ as

$$\begin{aligned}\mathcal{RF} &= C_1 E_{PR} + C_2 E_{CR} - h_I E_I - h_C E_C \\ &= \pi_0 \left\{ C_1 \lambda \sum_{1=1}^{S} U_i + C_2 \sum_{i=0}^{S-1} (S-i)\theta U_i - h_I \sum_{i=1}^{S} i U_i \right\} - h_C \frac{\lambda}{\mu - \lambda}\end{aligned}$$

where

- C_1 = revenue to the system due to per unit purchase of item in the inventory
- C_2 = revenue to the system due to per unit cancellation of inventory purchased
- h_I = holding cost per unit time per item in the inventory
- h_C = holding cost of customer per unit per unit time

In order to study the variation in different parameters on profit function we first fix the costs $C_1 = \$150, C_2 = \$50, h_I = \$20, h_C = \$5$.

4.2.1 Effect of variation in S, γ and θ on $\mathcal{RF}$

Table 6 shows that the change in revenue function with respect to S and θ (see Fig. 1). The revenue function increases first with θ and then keep going down. It may be noted that cancellation to some extent prior to common life realization results in higher profit to the system since there is a cancellation penalty imposed on the customer. As common life time realization decreases profit becomes less. This is due to lower cancellation rate. Table 7 shows that the change in revenue function

Table 6 Effect of S and θ. Fix $\lambda = 11, \mu = 13, \gamma = 0.1, \beta = 2$

$S \downarrow \theta \rightarrow$	1	1.5	2	2.5	3	3.5	4
10	1511.2	1789.9	1882.4	1904.6	1906.8	1904.3	1901.2
11	1602	1835.4	1890.2	1895.8	1892.1	1887.6	1883.7
12	1677	1859.5	1885.2	1881.5	1875.2	1869.9	1865.8
13	1735.7	1867.3	1873.1	1864.7	1857.4	1851.9	1847.8
14	1778.6	1863.9	1857.5	1847.1	1839.4	1833.9	1829.8
15	1806.7	1853.5	1840.4	1829.1	1821.4	1815.8	1811.7
16	1882	1839.1	1822.6	1811.1	1803.3	1797.8	1793.7

Table 7 Effect of γ and S. Fix $\lambda = 11, \mu = 13, \beta = 2, \theta = 2$

$S \downarrow \gamma \rightarrow$	0.1	0.15	0.2	0.25	0.3	0.35	0.4
10	1882.4	1828.1	1776.7	1728	1681.6	1637.6	1595.6
11	1890.2	1834.5	1781.7	1731.7	1684.2	1639.2	1596.4
12	1885.2	1828.5	1775.5	1725	1677.1	1631.7	1588.6
13	1873.1	1816.6	1763.2	1712.7	1664.8	1619.4	1576.3
14	1857.5	1801.2	1748	1697.7	1650	1604.8	1561.8
15	1840.4	1784.4	1731.5	1681.4	1634.1	1589.1	1546.5
16	1822.6	1767	1714.4	1664.7	1617.7	1573.1	1530.7

Table 8 Effect of θ and γ. Fix $S = 15$ $\lambda = 11, \mu = 13, \beta = 2$

$\gamma \downarrow \theta \rightarrow$	1	1.5	2	2.5	3
0.1	1806.7	1747.2	1691.3	1638.8	1589.4
0.15	1853.5	1793.3	1736.7	1683.5	1633.3
0.2	1840.4	1784.4	1731.5	1681.4	1634.1
0.25	1829.1	1776	1725.7	1677.9	1632.6
0.3	1821.4	1770.2	1721.6	1675.4	1631.4
0.35	1815.8	1766.0	1718.7	1673.6	1630.7
0.4	1811.7	1763	1716.6	1672.4	1630.3

with respect to S and γ (see Fig. 2) keeping rate of cancellation a constant. Table 8 shows the change in revenue function with respect to γ and θ (Fig. 3).

5 Conclusion

We analyzed an inventory system with reservation and *CLT* for inventory. Purchased items could be returned before expiry of *CLT*. The *CLT* of items is exponentially distributed. On realization of *CLT* customers waiting in the system

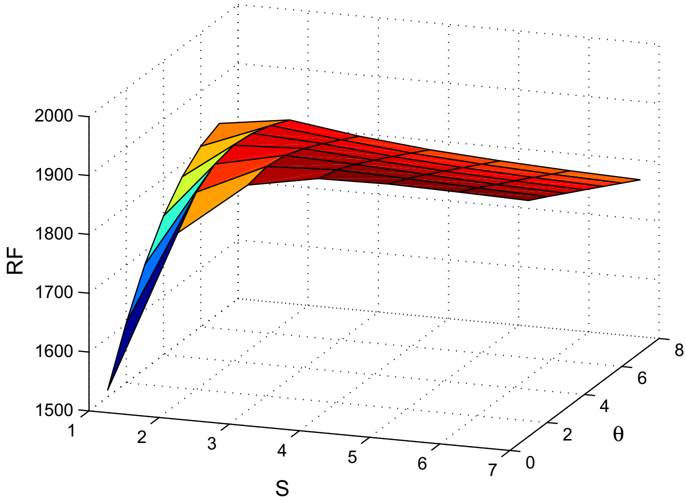

Fig. 1 Effect of S and θ

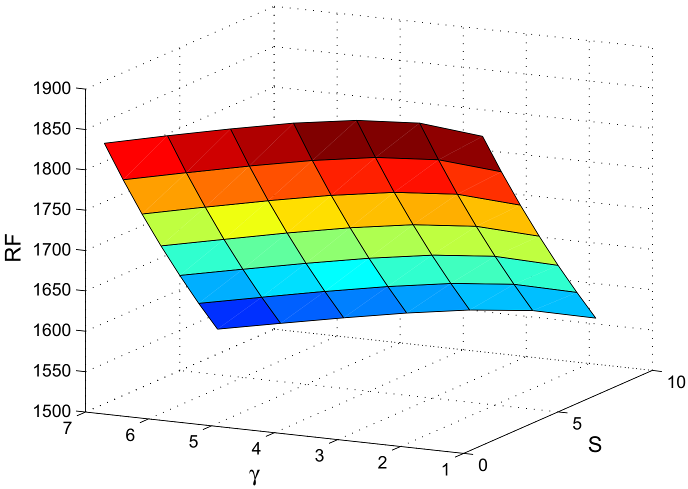

Fig. 2 Effect of S and γ

stay back. When *CLT* is reached a replenishment order is placed, lead time of which follows exponential distribution. No new arrival joins when inventory level is zero. This leads to a product form solution. Under stability condition we computed the long run system state distribution. These are in turn used for computing several system performance measures. Expected sojourn time in

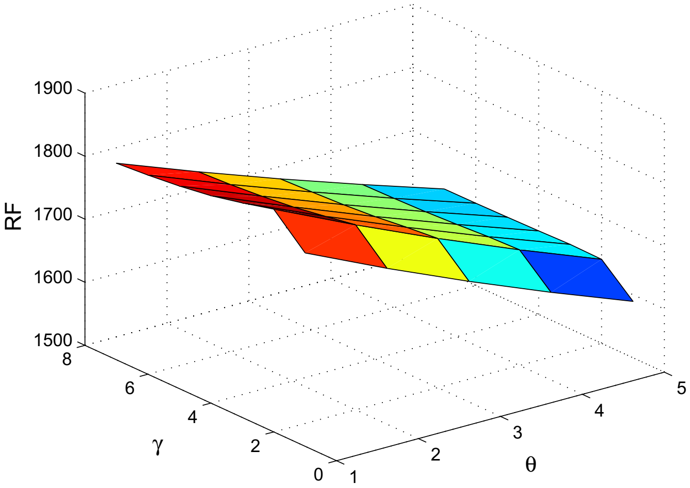

Fig. 3 Effect of γ and θ

maximum inventory level and zero inventory level in a cycle are derived. An optimization of a revenue function is also done numerically.

We propose to examine whether a product form solution exist for a queueing-inventory system with finite capacity.

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References

1. Krenzler, R., Daduna, H.: Loss systems in a random environment steady-state analysis. Queueing Syst. (2014). doi:10.1007/s11134-014-9426-6
2. Krenzler, R., Daduna, H.: Loss systems in a random environment-embedded Markov chains analysis, 1–54. http://preprint.math.unihamburg.de/public/papers/prst/prst2013-02 (2013)
3. Krishnamoorthy, A., Shajin, D., Lakshmy, B.: On a queueing-inventory with reservation, cancellation, common life time and retrial. Ann. Oper. Res. (2015). doi:10.1007/s10479-015-1849-x
4. Krishnamoorthy, A., Lakshmy, B., Manikandan, R.: A survey on inventory models with positive service time. Opsearch **48**(2), 153–169 (2011)
5. Krishnamoorthy, A., Viswanath, N.C.: Stochastic decomposition in production inventory with service time. EJOR (2013). doi:10.1016/j.ejor.2013.01.041
6. Lian, Z., Liu, L., Neuts, M.F.: A discrete-time model for common lifetime inventory systems. Math. Oper. Res. **30**(3), 718–732 (2005)
7. Neuts, M.F.: Matrix-geometric solutions in stochastic models: an algorithmic approach. The Johns Hopkins University Press, Baltimore (1981). [1994 version is Dover Edition]
8. Saffari, M., Asmussen, S., Haji, R.: The M/M/1 queue with inventory, lost sale and general lead times. Queueing Syst. (2013). doi:10.1007/s11134-012-9337-3
9. Schwarz, M., Sauer, C., Daduna, H., Kulik, R., Szekli, R.: M/M/1 queueing systems with inventory. Queueing Syst. **54**, 55–78 (2006)
10. Schwarz, M., Wichelhaus, C., Daduna, H.: Product form models for queueing networks with an inventory. Stoch. Models **23**(4), 627–663 (2007)
11. Sigman, K., Simchi-Levi, D.: Light traffic heuristic for an M/G/1 queue with limited inventory. Ann. OR **40**, 371–380 (1992)