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THEORETICAL ARTICLE

# Weak efficiency of higher order for multiobjective fractional variational problem

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**Abstract** The notion of weak efficiency of higher order for multiobjective fractional variational problem has been introduced. Examples are presented to illustrate this new solution concept. $\rho$-invexity is extended to $\rho$-invexity of higher order. Four generalizations of $\rho$-invexity of higher order are given. It is shown with the help of examples that this new class of $\rho$-invex functionals of higher order is larger than the existing class of functionals. Sufficient optimality conditions are proved under newly defined generalized $\rho-$invexity assumptions on the functionals involved. Parametric dual for multiobjective fractional variational problem is proposed for which weak duality theorem is proved. Further, we introduce weighted variational parametric problem to prove strong duality theorem.

**Keywords** Multiobjective fractional variational problem · Weak efficiency of higher order · Vector Kuhn Tucker point · Generalized invexity of higher order · Sufficient optimality conditions · Duality

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## 1 Introduction

Its a matter of enduring interest in the theory of optimization, to define and characterize its solutions. One may come across several variants of solution concepts of vector optimization problems while browsing the literature. However, there has been more concern with the situation where the solution is described in terms of weak efficiency [1, 2] / efficiency / proper efficiency. The concept of local minimizer of higher order was introduced by Auslender [3]. Ward [4] presented the notion of strict local minimizer of order m for scalar minimization problem which was further extended to the notion of strict local minimizer of order m for vector minimization problem by Jimenez [5]. Recently Bhatia [6] extended the notion of Ward [4] to define a global strict minimizer of order m for a multi-objective optimization problem.

All these authors worked for static problems in the sense that time does not enter into consideration. Whereas in practical problems we come across situations where time plays an important role and hence cannot be neglected. Variational problems [7, 8] where optimization is done over an interval of time is more general and applicable. We, in this article extend the notion of weak efficiency of higher order to multi-objective fractional variational problem.

Invexity plays a vital role in many aspects of mathematical programming and hence in calculus of variation. Invex functions were defined by Hanson [9]. To relax invexity assumption imposed on the functions involved, various generalized concepts have been proposed in literature. One of the useful generalizations is $\rho$-invexity [10, 11]. Invexity was extended for variational problems by Mond, Chandra and Husain [12] while Bhatia and Kumar [13] defined B-vexity for variational problems. Bhatia and Sahay [14] introduced higher order strong invexity for multi-objective optimization problem but for the static case.

In this article we define a class of $\rho$-invex functionals of higher order and utilize these functionals to establish sufficient optimality and duality results for multiobjective fractional variational problem. The significance of this new notion of invexity of higher order is two folds. It not only allows us to relax the notion of convexity/invexity associated with optimality and duality results for an optimization problem but also enable us to derive the various requisite results associated with the new solution concept of weak efficiency of higher order.

The paper is organized as follows: In Section 2 some basis definitions and preliminaries are given. In Section 3 sufficient optimality conditions are established for vector fractional variational problem (MFVP) using the concept of weak efficiency of higher order. In Section 4, parametric dual for (MFVP) are considered and duality results are obtained under assumptions of generalized $\rho$-invexity of higher order.

## 2 Definitions and preliminaries

Let $r$, $n$ and $p$ be three positive integers. For a given real interval $I = [a, b]$, let $x : I \to \mathbb{R}^n$ be a piecewise smooth state function with its derivative $\dot{x}$. For notational convenience $x(t)$ will be written as $x$. Let $f^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, k^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to$

$\mathbb{R}$, $i = 1, 2, \ldots, r$ and $g^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, p$ are continuously differentiable functions with respect to each of their argument. Let us denote the partial derivative of $f^i$, $i = 1, 2, \ldots, r$ with respect to $t$, $x$ and $\dot{x}$ by $f_t^i$, $f_x^i$, $f_{\dot{x}}^i$ respectively. Analogously, write the partial derivative of $k^i$, $i = 1, 2, \ldots, r$ and $g^i$, $i = 1, 2, \ldots, p$. Here $\mathbb{R}^n$ denotes an n-dimensional Euclidean space,

$\mathbb{R}_+^n = \{(x^1, x^2, \ldots, x^n)^T \in \mathbb{R}^n | x^i \geq 0,\ i = 1, 2, \ldots, n\}$ and $\operatorname{int}\mathbb{R}_+^n$ denote interior of $\mathbb{R}_+^n$ that is $\operatorname{int}\mathbb{R}_+^n = \{(x^1, x^2, \ldots, x^n)^T \in \mathbb{R}^n | x^i > 0,\ i = 1, 2, \ldots, n\}$. For any $x = (x^1, x^2, \ldots, x^n)^T$, $y = (y^1, y^2, \ldots, y^n)^T \in \mathbb{R}^n$. Define

(i) $x = y \Leftrightarrow x^i = y^i$ for all $i = 1, 2, \ldots, n$.
(ii) $x < y \Leftrightarrow x^i < y^i$ for all $i = 1, 2, \ldots, n$.
(iii) $x \leqq y \Leftrightarrow x^i \leq y^i$ for all $i = 1, 2, \ldots, n$.
(iv) $x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$.

Let $X$ be the space of piecewise smooth state functions $x : I \to \mathbb{R}^n$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ where the differential operator $D$ is given by $u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s)ds$. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

Now, consider the following multiobjective fractional variational problem:

$$\text{(MFVP) Minimize} \left( \frac{\int_a^b f^1(t, x, \dot{x})dt}{\int_a^b k^1(t, x, \dot{x})dt}, \frac{\int_a^b f^2(t, x, \dot{x})dt}{\int_a^b k^2(t, x, \dot{x})dt}, \ldots, \frac{\int_a^b f^r(t, x, \dot{x})dt}{\int_a^b k^r(t, x, \dot{x})dt} \right)$$

subject to

$$g(t, x, \dot{x}) = (g^1(t, x, \dot{x}), g^2(t, x, \dot{x}), \ldots, g^p(t, x, \dot{x})) \leqq 0,\ t \in I, \tag{1}$$

$$x(a) = \alpha,\ x(b) = \beta, \alpha, \beta \in \mathbb{R}^n. \tag{2}$$

Assume that $\int_a^b f^i(t, x, \dot{x})dt \geq 0$ and $\int_a^b k^i(t, x, \dot{x})dt > 0$ for all $i \in \{1, 2, \ldots, r\}$ and for all $x \in X$. Let $X_0$ be the set of feasible solutions of (MFVP), that is, $X_0 = \{x \in X \mid g(t, x, \dot{x}) \leqq 0,\ t \in I, x(a) = \alpha,\ x(b) = \beta\}$.

Bector et al. [15] defined weak efficient solution for multiobjective variational problem. A class of fractional programming problem, in which objective function is the ratio of two functions, has received considerable importance during past few decades. Because of its ratio structure, it finds its application in various fields like economics, informational theory, engineering, heat exchange networking and numerical analysis. This perhaps was the reason for this class to dominate the research. Hence we are motivated to extend the notion of weak efficiency to a fractional case.

**Definition 1** $\bar{x} \in X_0$ is said to be a weak efficient solution for (MFVP) if there is no other $x \in X_0$ such that

$$\frac{\int_a^b f^i(t, x, \dot{x})dt}{\int_a^b k^i(t, x, \dot{x})dt} < \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt}, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

Motivated by strict efficiency of higher order [14] for multiobjective programing problem, we propose the notion of weak efficiency of higher order for (MFVP). This may be considered as the dynamic generalization of the concept of strict efficiency of higher order.

Let $m \geq 1$ be an integer and $\xi : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise smooth function.

**Definition 2** $\bar{x} \in X_0$ is said to be a weak efficient solution of order $m$ with respect to $\xi$ for (MFVP) if there exist $c = (c^1, \ldots, c^r) \in \operatorname{int} \mathbb{R}^r_+$ and $d = (d^1, \ldots, d^r) \in \operatorname{int} \mathbb{R}^r_+$ such that for no other $x \in X_0$

$$\frac{\int_a^b \{f^i(t, x, \dot{x}) - c^i \|\xi(t, x, \bar{x})\|^m\} dt}{\int_a^b \{k^i(t, x, \dot{x}) + d^i \|\xi(t, x, \bar{x})\|^m\} dt} < \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}}) dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}}) dt}, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

*Remark 1* (a) Let $k \geq m$ be an integer, then a weak efficient solution of order $m$ with respect to $\xi$ for (MFVP) is a weak efficient solution of order $k$ with respect to same $\xi$ for (MFVP), but converse may not be true, as illustrates in the following example.

Consider the following (MFVP):

$$\text{Minimize } \left( \frac{\int_0^1 \{x^2(t) + \frac{\dot{x}^2(t)}{10}\} dt}{\int_0^1 \{\dot{x}^2(t) + 1\} dt}, \frac{\int_0^1 \{2x^2(t)\} dt}{\int_0^1 \{x^2(t) + 1\} dt} \right)$$

subject to

$$x(t) \leq 0,\ t \in I = [0, 1],\ x : I \to \mathbb{R}$$

$$x(0) = 0,\ x(1) = 0.$$

Then $\bar{x}(t) = 0,\ t \in I$ is a weak efficient solution of order 2 with respect to $\xi$, for given (MFVP) where $\xi$ defined by

$$\xi(t, x, \bar{x}) = x(t) + \bar{x}(t).$$

As there exist $c = (c^1, c^2) = (1, 1) \in \operatorname{int} \mathbb{R}^2_+$ and $d = (d^1, d^2) = (1, 1) \in \operatorname{int} \mathbb{R}^2_+$ such that for no other $x \in X_0$

$$\begin{aligned}\frac{\int_0^1 \{x^2(t) + \frac{\dot{x}^2(t)}{10} - c^1 \|\xi(t, x, \bar{x})\|^2\} dt}{\int_0^1 \{\dot{x}^2(t) + 1 + d^1 \|\xi(t, x, \bar{x})\|^2\} dt} &= \frac{\int_0^1 \{\frac{\dot{x}^2(t)}{10}\} dt}{\int_0^1 \{\dot{x}^2(t) + x^2(t) + 1\} dt} < 0 \\ &= \frac{\int_a^b f^1(t, \bar{x}, \dot{\bar{x}}) dt}{\int_a^b k^1(t, \bar{x}, \dot{\bar{x}}) dt},\end{aligned}$$

$$\frac{\int_0^1 \{2x^2(t) - c^2 \|\xi(t, x, \bar{x})\|^2\} dt}{\int_0^1 \{x^2(t) + 1 + d^2 \|\xi(t, x, \bar{x})\|^2\} dt} = \frac{\int_0^1 \{x^2(t)\} dt}{\int_0^1 \{2x^2(t) + 1\} dt} < 0 = \frac{\int_a^b f^2(t, \bar{x}, \dot{\bar{x}}) dt}{\int_a^b k^2(t, \bar{x}, \dot{\bar{x}}) dt}.$$

But not a weak efficient solution of order 1 with respect to same $\xi$, for given (MFVP).

Since for each $c = (c^1, c^2) \in \operatorname{int}\mathbb{R}^2_+$ and $d = (d^1, d^2) \in \operatorname{int}\mathbb{R}^2_+$, we can take $\hat{x}(t) = q(t^2 - t) \in X_0,\ t \in I$ where $q = min\{c^1, c^2\}$ ( To get the requisite results, it is required to assume $q = min\{c^1, c^2\}$) such that

$$\frac{\int_0^1 \{\hat{x}^2(t) + \frac{\dot{\hat{x}}^2(t)}{10} - c^1\|\xi(t,\hat{x},\bar{x})\|^1\}dt}{\int_0^1 \{\dot{\hat{x}}^2(t) + 1 + d^1\|\xi(t,\hat{x},\bar{x})\|^1\}dt} = \frac{q(2q - 5c^1)}{5(6 + qd^1 + 2q^2)} < 0 = \frac{\int_a^b f^1(t,\bar{x},\dot{\bar{x}})dt}{\int_a^b k^1(t,\bar{x},\dot{\bar{x}})dt},$$

$$\frac{\int_0^1 \{2\hat{x}^2(t) - c^2\|\xi(t,\hat{x},\bar{x})\|^1\}dt}{\int_0^1 \{\hat{x}^2(t) + 1 + d^2\|\xi(t,\hat{x},\bar{x})\|^1\}dt} = \frac{q(2q - 5c^2)}{q^2 + 30 + 5qd^2} < 0 = \frac{\int_a^b f^2(t,\bar{x},\dot{\bar{x}})dt}{\int_a^b k^2(t,\bar{x},\dot{\bar{x}})dt}.$$

(b) It can easily be proved that a weak efficient solution of order $m$ with respect to $\xi$ for (MFVP) is a weak efficient solution for (MFVP), whereas the converse may not be true, as illustrates in the following example.

Consider the following (MFVP):

$$\text{Minimize}\left(\frac{\int_0^1 \{(\dot{x}(t) - 2t)(x(t) - t)\}^2 dt}{\int_0^1 \{x^2(t) + 1\}dt}, \frac{\int_0^1 \{(x(t) - t^2)(\dot{x}(t) - 1)\}^2 dt}{\int_0^1 \{\dot{x}^2(t) + 1\}dt}\right)$$

subject to

$$x(t) \geq 0,\ t \in I = [0, 1],\ x : I \to \mathbb{R}$$

$$x(0) = 0,\ x(1) = 1.$$

Then $\bar{x}(t) = t,\ t \in I$ is a weak efficient solution but not a weak efficient solution of order $m$ with respect to $\xi$, for given (MFVP) where $\xi$ defined by

$$\xi(t, x, \bar{x}) = x(t) - \bar{x}(t).$$

Since for each $c = (c^1, c^2) \in \operatorname{int}\mathbb{R}^2_+$ and $d = (d^1, d^2) \in \operatorname{int}\mathbb{R}^2_+$ and for each positive integer $m$, we can take $\hat{x}(t) = t^2 \in X_0,\ t \in I$ such that

$$\frac{\int_0^1 \{f^1(t,\hat{x},\dot{\hat{x}}) - c^1\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_0^1 \{k^1(t,\hat{x},\dot{\hat{x}}) + d^1\|\xi(t,\hat{x},\bar{x})\|^m\}dt} = \frac{-c^1 K}{\frac{6}{5} + d^1 K} < 0 = \frac{\int_0^1 f^1(t,\bar{x},\dot{\bar{x}})dt}{\int_0^1 k^1(t,\bar{x},\dot{\bar{x}})dt},$$

$$\frac{\int_0^1 \{f^2(t,\hat{x},\dot{\hat{x}}) - c^2\|\xi(t,\hat{x},\bar{x})\|^m\}dt}{\int_0^1 \{k^2(t,\hat{x},\dot{\hat{x}}) + d^2\|\xi(t,\hat{x},\bar{x})\|^m\}dt} = \frac{-c^2 K}{\frac{7}{3} + d^2 K} < 0 = \frac{\int_0^1 f^2(t,\bar{x},\dot{\bar{x}})dt}{\int_0^1 k^2(t,\bar{x},\dot{\bar{x}})dt},$$

$$\text{where } K = \int_0^1 \{\|\xi(t,\hat{x},\bar{x})\|^m\}dt = \int_0^1 \{t^m(1-t)^m\}dt = \frac{(m\,!)^2}{(2m+1)\,!}.$$

For the sake of convenience we write $f^i_x(t)$ for $f^i_x(t, x(t), \dot{x}(t))$ and $f^i_{\dot{x}}(t)$ for $f^i_{\dot{x}}(t, x(t), \dot{x}(t))$ for $i = 1, 2, \ldots, r$.

Following Arana-Jimenez et al. [1, 2], we now define vector Kuhn Tucker point for multiobjective fractional variational problem.

**Definition 3** $\bar{x} \in X_0$ is said to be a vector Kuhn Tucker point for (MFVP) if there exist $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^r) \in \mathbb{R}^r$, $\bar{v} = (\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^r) \in \mathbb{R}^r_+$ and a piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ such that

$$\sum_{i=1}^{r} \bar{\lambda}^i (f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t)) + \bar{y}(t)^T g_{\bar{x}}(t) = \frac{d}{dt} \left[ \sum_{i=1}^{r} \bar{\lambda}^i (f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)) + \bar{y}(t)^T g_{\dot{\bar{x}}}(t) \right], t \in I, \tag{3}$$

$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) = 0, \ t \in I \tag{4}$$

$$\bar{\lambda} \geq 0, \sum_{i=1}^{r} \bar{\lambda}^i = 1, \bar{y}(t) \geqq 0, t \in I, \bar{v}^i = \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}}) dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}}) dt}, \ i = 1, 2, \ldots, r. \tag{5}$$

In order to study weak efficient solutions of higher order for (MFVP), the existing class of functionals is insufficient. Therefore it requires some sort of extension, the notion of $\rho-$invexity of higher order and its generalizations introduced in this paper, solves the purpose.

Let $\Phi : X \to \mathbb{R}$ defined by $\Phi(x) = \int_a^b \phi(t, x, \dot{x}) dt$ be Frechet differentiable, where $\phi(t, x, \dot{x})$ is a scalar function with continuous derivatives upto and including second order with respect to each of its arguments. Let there exist a real number $\rho$ and a differentiable vector function $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with

$$\eta(t, x, \bar{x}) = 0 \text{ at } t \text{ if } x(t) = \bar{x}(t) \tag{6}$$

**Definition 4** A functional $\Phi(x)$ is said to be $\rho$-invex of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ if

$$\Phi(x) - \Phi(\bar{x}) \geq \int_a^b \{\eta(t, x, \bar{x}) \phi_{\bar{x}}(t) + [\frac{d\eta(t, x, \bar{x})}{dt}] \phi_{\dot{\bar{x}}}(t) + \rho \|\xi(t, x, \bar{x})\|^m\} dt, \text{for all } x \in X.$$

*Remark 2* (a) If $\xi(t, x, \bar{x}) = 0$, definition 4 reduces to the classical definition of Invexity [16, 17].

(b) For $m = 2$, definition 4 reduces to $\rho - (\eta, \xi)-$ Invexity [11].

**Definition 5** A functional $\Phi(x)$ is said to be $\rho$-pseudoinvex type 1 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ if

$$\int_a^b \{\eta(t, x, \bar{x}) \phi_{\bar{x}}(t) + [\frac{d\eta(t, x, \bar{x})}{dt}] \phi_{\dot{\bar{x}}}(t) + \rho \|\xi(t, x, \bar{x})\|^m\} dt \geq 0 \Rightarrow \Phi(x) \geq \Phi(\bar{x}),$$

for all $x \in X$.

Or equivalently

$$\Phi(x) < \Phi(\bar{x}) \Rightarrow \int_a^b \left\{ \eta(t, x, \bar{x}) \phi_{\bar{x}}(t) + \left[ \frac{d\eta(t, x, \bar{x})}{dt} \right] \phi_{\dot{\bar{x}}}(t) + \rho \|\xi(t, x, \bar{x})\|^m \right\} dt < 0,$$

for all $x \in X$.

**Definition 6** A functional $\Phi(x)$ is said to be $\rho$-pseudoinvex type 2 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ if

$$\int_a^b \left\{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t,x,\bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t)\right\} dt \geq 0 \Rightarrow \Phi(x) \geq \Phi(\bar{x}) + \rho \int_a^b \{\|\xi(t,x,\bar{x})\|^m\} dt,$$

for all $x \in X$.

Or equivalently

$$\Phi(x) < \Phi(\bar{x}) + \rho \int_a^b \{\|\xi(t,x,\bar{x})\|^m\} dt \Rightarrow \int_a^b \left\{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t,x,\bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t)\right\} dt < 0,$$

for all $x \in X$.

**Definition 7** A functional $\Phi(x)$ is said to be $\rho$-quasiinvex type 1 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ if

$$\Phi(x) \leq \Phi(\bar{x}) \Rightarrow \int_a^b \left\{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t,x,\bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t) + \rho\|\xi(t,x,\bar{x})\|^m\right\} dt \leq 0,$$

for all $x \in X$.

Or equivalently

$$\int_a^b \left\{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t,x,\bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t) + \rho\|\xi(t,x,\bar{x})\|^m\right\} dt > 0 \Rightarrow \Phi(x) > \Phi(\bar{x}),$$

for all $x \in X$.

**Definition 8** A functional $\Phi(x)$ is said to be $\rho$-quasiinvex type 2 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ if

$$\Phi(x) \leq \Phi(\bar{x}) + \rho \int_a^b \{\|\xi(t,x,\bar{x})\|^m\} dt \Rightarrow \int_a^b \left\{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t,x,\bar{x})}{dt}\right] \times \phi_{\dot{\bar{x}}}(t)\right\} dt \leq 0,$$

for all $x \in X$.

Or equivalently

$$\int_a^b \left\{\eta(t,x,\bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t,x,\bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t)\right\} dt > 0 \Rightarrow \Phi(x) > \Phi(\bar{x}) + \rho \int_a^b \{\|\xi(t,x,\bar{x})\|^m\} dt,$$

for all $x \in X$.

**Definition 9** A functional $\Phi(x)$ is said to be strictly $\rho$-quasiinvex type 1 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ if

$$\Phi(x) \leq \Phi(\bar{x}) \Rightarrow \int_a^b \left\{ \eta(t, x, \bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t, x, \bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t) + \rho\|\xi(t, x, \bar{x})\|^m \right\} dt < 0,$$

for all $x \in X - \{\bar{x}\}$.

Or equivalently

$$\int_a^b \left\{ \eta(t, x, \bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t, x, \bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t) + \rho\|\xi(t, x, \bar{x})\|^m \right\} dt \geq 0 \Rightarrow \Phi(x) > \Phi(\bar{x}),$$

for all $x \in X - \{\bar{x}\}$.

*Remark 3* Through the following example, it is evident that class of above defined functionals is larger than the class of functionals that already exists in the literature [11, 16].

In the following examples $x : I \to \mathbb{R}$ is assume to be a piecewise smooth function.

(a) The functional $\Phi(x) = \int_0^1 \{x^6(t) - x^4(t)\}dt$ is $\rho$-invex of order 4 for $\rho = -1$, at $\bar{x}(t) = 0, t \in I = [0, 1]$, with respect to $\eta$ and $\xi$, defined as

$$\eta(t, x, \bar{x}) = x^2(t) - \bar{x}^2(t),$$

$$\xi(t, x, \bar{x}) = x(t) - \bar{x}(t).$$

But it is not invex at $\bar{x}(t) = 0, t \in I = [0, 1]$ since

$$\Phi(\hat{x}) - \Phi(\bar{x}) < 0 = \int_0^1 \left\{ \eta(t, \hat{x}, \bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t, \hat{x}, \bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t) \right\} dt, \text{ for } \hat{x}(t) = t, t \in I.$$

(b) The functional $\Phi(x) = \int_0^1 \frac{x^4(t)}{2}dt$ is $\rho$-invex of order 4 for $\rho = \frac{1}{2}$, at $\bar{x}(t) = 0, t \in I$, with respect to $\eta$ and $\xi$, defined as

$$\eta(t, x, \bar{x}) = x^2(t) - \bar{x}^2(t),$$

$$\xi(t, x, \bar{x}) = x(t) - \bar{x}(t).$$

But it is not $\rho - (\eta, \xi)-$ Invex at $\bar{x}(t) = 0, t \in I = [0, 1]$, since

$$\Phi(\hat{x}) - \Phi(\bar{x}) < \int_0^1 \left\{ \eta(t, \hat{x}, \bar{x})\phi_{\bar{x}}(t) + \left[\frac{d\eta(t, \hat{x}, \bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t) + \frac{1}{2}\|\xi(t, \hat{x}, \bar{x})\|^2 \right\} dt, \text{ for } \hat{x}(t) = t, t \in I.$$

(c) The functional $\Phi(x) = \int_0^1 x(t)dt$ is $\rho$-pseudoinvex type 1 of order 3 for $\rho = 27$, at $\bar{x}(t) = -2t, t \in I = [0, 1]$ with respect to $\eta$ and $\xi$, defined as

$$\eta(t, x, \bar{x}) = x(t) + \bar{x}(t)$$

$$\xi(t, x, \bar{x}) = (\bar{x}(t) + 2t)x(t) + \frac{1}{3}.$$

But it is not $\rho-(\eta,\xi)-$ pseudoinvex at $\bar{x}(t)=-2t,\ t\in I=[0,1]$, as

$$\int_0^1\left\{\eta(t,\hat{x},\bar{x})\phi_{\bar{x}}(t)+\left[\frac{d\eta(t,\hat{x},\bar{x})}{dt}\right]\phi_{\dot{\bar{x}}}(t)+\rho\|\xi(t,\hat{x},\bar{x})\|^2\right\}dt\geq 0,$$

but $\Phi(\hat{x})<\Phi(\bar{x})$, for $\hat{x}(t)=-3t, t\in I$.

(d) The functional $\Phi(x)=\int_0^1\{3x(t)+1\}dt$ is $\rho$-quasiinvex type 1 of order 3 for $\rho=16$, at $\bar{x}(t)=-t^2, t\in I=[0,1]$ with respect to $\eta$ and $\xi$, defined as

$$\eta(t,x,\bar{x})=x(t)+\bar{x}(t)$$

$$\xi(t,x,\bar{x})=\bar{x}(t)+t^2+\frac{1}{2}.$$

But it is not $\rho-(\eta,\xi)-$ quasiinvex at $\bar{x}(t)=-t^2, t\in I=[0,1]$, as $\Phi(\hat{x})\leq\Phi(\bar{x})$, but $\int_0^1\{\eta(t,\hat{x},\bar{x})\phi_{\bar{x}}(t)+[\frac{d\eta(t,\hat{x},\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)+\rho\|\xi(t,\hat{x},\bar{x})\|^2\}dt>0$, for $\hat{x}(t)=\frac{-1}{2}, t\in I$.

(e) The functional $\Phi(x)=\int_0^1\{-x(t)\}dt$ is $\rho$-pseudoinvex type 2 of order 3 for $\rho=1$, at $\bar{x}(t)=0, t\in I=[0,1]$ with respect to $\eta$ and $\xi$, defined as

$$\eta(t,x,\bar{x})=\bar{x}(t)+1$$

$$\xi(t,x,\bar{x})=x(t)-\bar{x}(t).$$

But it is not $\rho-(\eta,\xi)-$ pseudoinvex at $\bar{x}(t)=0, t\in I=[0,1]$ as

$$\int_0^1\{\eta(t,\hat{x},\bar{x})\phi_{\bar{x}}(t)+[\frac{d\eta(t,\hat{x},\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)+\rho\|\xi(t,\hat{x},\bar{x})\|^2\}dt\geq 0,$$

but $\Phi(\hat{x})<\Phi(\bar{x})$, for $\hat{x}(t)=2t, t\in I$.

(f) The functional $\Phi(x)=\int_0^1\{-x^2(t)\}dt$ is $\rho$-quasiinvex type 2 of order 3 for $\rho=1$, at $\bar{x}(t)=0, t\in I=[0,1]$ with respect to $\eta$ and $\xi$, defined as

$$\eta(t,x,\bar{x})=x(t)-\bar{x}(t)$$

$$\xi(t,x,\bar{x})=x(t)-\bar{x}(t).$$

But it is not $\rho-(\eta,\xi)-$ quasiinvex at $\bar{x}(t)=0, t\in I=[0,1]$, as $\Phi(\hat{x})\leq\Phi(\bar{x})$, but $\int_0^1\{\eta(t,\hat{x},\bar{x})\phi_{\bar{x}}(t)+[\frac{d\eta(t,\hat{x},\bar{x})}{dt}]\phi_{\dot{\bar{x}}}(t)+\rho\|\xi(t,\hat{x},\bar{x})\|^2\}dt>0$, for $\hat{x}(t)=t, t\in I$.

## 3 Optimality conditions

**Theorem 1** *(Sufficient optimality conditions) Let $\bar{x}\in X_0$ be a vector Kuhn Tucker point for (MFVP). Let us write $\theta^i(x)=\int_a^b\{f^i(t,x,\dot{x})-\bar{v}^ik^i(t,x,\dot{x})\}dt, i=$*

*$1, 2, \ldots, r$ and $G(x) = \int_a^b \{\bar{y}(t)^T g(t, x, \dot{x})\}dt$. Further suppose that any one of the following holds:*

(a) *$\theta^i(x)$, for $i = 1, 2, \ldots, r$ are $\rho^i$-pseudoinvex type 2 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ and $G(x)$ is $\rho'$-quasiinvex type 1 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$, where $\rho'$, $\rho^i \in int\, \mathbb{R}_+$, for $i = 1, 2, \ldots, r$.*

(b) *$\theta^i(x)$, for $i = 1, 2, \ldots, r$ are $\rho^i$-quasiinvex type 2 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ and $G(x)$ is strict $\rho'$-quasiinvex type 1 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$, where $\rho'$, $\rho^i \in int\, \mathbb{R}_+$, for $i = 1, 2, \ldots, r$.*

*Then $\bar{x}$ is a weak efficient solution of order $m$ with respect to $\xi$ for (MFVP).*

*Proof* (a) If possible, suppose $\bar{x}$ is not a weak efficient solution of order $m$ with respect to $\xi$ for (MFVP). Then for any $c = (c^1, c^2, \ldots, c^r) \in \operatorname{int} \mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in \operatorname{int} \mathbb{R}^r_+$, there exist $\hat{x} \in X_0$ such that

$$\frac{\int_a^b \{f^i(t, \hat{x}, \dot{\hat{x}}) - c^i \|\xi(t, \hat{x}, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, \hat{x}, \dot{\hat{x}}) + d^i \|\xi(t, \hat{x}, \bar{x})\|^m\}dt} < \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt} = \bar{v}^i, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

That is,

$$\int_a^b \{f^i(t, \hat{x}, \dot{\hat{x}}) - \bar{v}^i k^i(t, \hat{x}, \dot{\hat{x}})\}dt < (c^i + d^i \bar{v}^i) \int_a^b \{\|\xi(t, \hat{x}, \bar{x})\|^m\}dt,$$

for all $i \in \{1, 2, \ldots, r\}$.

Choose $\rho^i = c^i + d^i \bar{v}^i$, for $i = 1, 2, \ldots, r$ and using Eq. (5), we get

$$\int_a^b \{f^i(t, \hat{x}, \dot{\hat{x}}) - \bar{v}^i k^i(t, \hat{x}, \dot{\hat{x}})\}dt < \int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - \bar{v}^i k^i(t, \bar{x}, \dot{\bar{x}}) + \rho^i \|\xi(t, \hat{x}, \bar{x})\|^m\}dt,$$

for all $i \in \{1, 2, \ldots, r\}$.

Since $\theta^i(x)$, for $i = 1, 2, \ldots, r$ are $\rho^i$-pseudoinvex type 2 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$, we obtain

$$\int_a^b \{\eta(t, \hat{x}, \bar{x})[f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t)] + \frac{d\eta(t, \hat{x}, \bar{x})}{dt}[f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)]\}dt < 0, \quad (7)$$

for all $i \in \{1, 2, \ldots, r\}$.

Multiplying each inequality (7) by $\bar{\lambda}^i$, $i = 1, 2, \ldots, r$ and then adding, we get

$$\int_a^b \left\{\eta(t, \hat{x}, \bar{x}) \left[\sum_{i=1}^r \bar{\lambda}^i (f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t))\right] + \frac{d\eta(t, \hat{x}, \bar{x})}{dt} \times \left[\sum_{i=1}^r \bar{\lambda}^i (f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t))\right]\right\} dt < 0. \quad (8)$$

Since $\bar{y}(t) \geqq 0$, $t \in I$, $\hat{x} \in X_0$ and using Eq. (4), we obtain

$$\int_a^b \bar{y}(t)^T g(t, \hat{x}, \dot{\hat{x}})dt \leq 0 = \int_a^b \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}})dt. \quad (9)$$

Since $G(x)$ is $\rho'$-quasiinvex type 1 of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$, Eq. (9) yields

$$\int_a^b \left\{\eta(t,\hat{x},\bar{x})[\bar{y}(t)^T g_{\bar{x}}(t)] + \frac{d\eta(t,\hat{x},\bar{x})}{dt}[\bar{y}(t)^T g_{\dot{\bar{x}}}(t)] + \rho'\|\xi(t,\hat{x},\bar{x})\|^m\right\} dt \leq 0. \tag{10}$$

Adding both sides of the inequalities (8), (10) and using (3), we get

$$\int_a^b \left\{\frac{d}{dt}\left\{\eta(t,\hat{x},\bar{x})\left[\sum_{i=1}^r \bar{\lambda}^i(f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)) + \bar{y}(t)^T g_{\dot{\bar{x}}}(t)\right]\right\}\right\} \times dt + \int_a^b \{\rho'\|\xi(t,\hat{x},\bar{x})\|^m\} dt < 0,$$

$$\Rightarrow \left\{\eta(t,\hat{x},\bar{x})\left[\sum_{i=1}^r \bar{\lambda}^i(f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)) + \bar{y}(t)^T g_{\dot{\bar{x}}}(t)\right]\right\}\Big|_a^b + \int_a^b \{\rho'\|\xi(t,\hat{x},\bar{x})\|^m\} dt < 0.$$

Since $\hat{x},\ \bar{x} \in X_0$ and (6), we get

$$\int_a^b \{\rho'\|\xi(t,\hat{x},\bar{x})\|^m\} dt < 0.$$

This is a contradiction as $\rho' > 0$, $\|\xi(t,\hat{x},\bar{x})\|^m \geq 0$, for all positive integer $m$. This completes the proof.
(b) Proof runs on the same lines as the proof of part (a) and is hence omitted. □

**Corollary 1** *Let $\bar{x} \in X_0$ be a vector Kuhn Tucker point for (MFVP). Let $\theta^i(x)$, for $i = 1, 2, \ldots, r$ are $\rho^i$-invex of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$ and $G(x)$ is $\rho'$-invex of order $m$ at $\bar{x}$ with respect to $\eta$ and $\xi$, where $\rho', \rho^i \in int\,\mathbb{R}_+$, for $i = 1, 2, \ldots, r$. Then $\bar{x}$ is a weak efficient solution of order $m$ with respect to $\xi$ for (MFVP).*

## 4 Duality results

In this section a parametric dual (MFVDP) to (MFVP) is proposed following the approach of Bector et al. [18]. Weak and strong duality results are established to relate weak efficient solutions of higher order for primal and dual problem.

$$\text{(MFVDP) Maximize } \bar{v} = (\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^r)$$

subject to

$$\sum_{i=1}^r \bar{\lambda}^i(f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t)) + \bar{y}(t)^T g_{\bar{x}}(t) = \frac{d}{dt}\left[\sum_{i=1}^r \bar{\lambda}^i(f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)) + \bar{y}(t)^T g_{\dot{\bar{x}}}(t)\right], t \in I, \tag{11}$$

$$\int_a^b \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}})dt \geq 0, \tag{12}$$

$$\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - \bar{v}^i k^i(t, \bar{x}, \dot{\bar{x}})\}dt \geq 0, i = 1, 2, \ldots, r \tag{13}$$

$$\bar{x} \in X, \bar{\lambda} \geq 0, \bar{\lambda}^T e = 1, \bar{y}(t) \geqq 0, t \in I, \bar{v}^i \geq 0, i = 1, 2, \ldots, r \text{ where } e = (1, 1, \ldots, 1). \tag{14}$$

$$\bar{x}(a) = \alpha, \bar{x}(b) = \beta. \tag{15}$$

Let $Y_1$ be the set of feasible solutions of (MFVDP).

**Theorem 2** *(Weak duality) Let* $x \in X_0$, $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y}) \in Y_1$. *Let us write* $\theta^i(x) = \int_a^b \{f^i(t, x, \dot{x}) - \bar{v}^i k^i(t, x, \dot{x})\}dt$, $i = 1, 2, \ldots, r$ *and* $G(x) = \int_a^b \{\bar{y}(t)^T g(t, x, \dot{x})\}dt$. *Further suppose that any one of the following holds:*

(a) $\theta^i(x)$, *for* $i = 1, 2, \ldots, r$ *are* $\rho^i$*-pseudoinvex type 2 of order* $m$ *at* $\bar{x}$ *with respect to* $\eta$ *and* $\xi$ *and* $G(x)$ *is* $\rho'$*-quasiinvex type 1 of order* $m$ *at* $\bar{x}$ *with respect to* $\eta$ *and* $\xi$, *where* $\rho'$, $\rho^i \in int\, \mathbb{R}_+$, *for* $i = 1, 2, \ldots, r$.
(b) $\theta^i(x)$, *for* $i = 1, 2, \ldots, r$ *are* $\rho^i$*-quasiinvex type 2 of order* $m$ *at* $\bar{x}$ *with respect to* $\eta$ *and* $\xi$ *and* $G(x)$ *is strict* $\rho'$*-quasiinvex type 1 of order* $m$ *at* $\bar{x}$ *with respect to* $\eta$ *and* $\xi$, *where* $\rho'$, $\rho^i \in int\, \mathbb{R}_+$, *for* $i = 1, 2, \ldots, r$.

*Then there exist* $c = (c^1, c^2, \ldots, c^r) \in int\, \mathbb{R}^r_+$ *and* $d = (d^1, d^2, \ldots, d^r) \in int\, \mathbb{R}^r_+$ *such that following cannot hold:*

$$\frac{\int_a^b \{f^i(t, x, \dot{x}) - c^i \|\xi(t, x, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, x, \dot{x}) + d^i \|\xi(t, x, \bar{x})\|^m\}dt} < \bar{v}^i, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

*Proof* (a) Contrary to the results, assume that for any $c = (c^1, c^2, \ldots, c^r) \in \text{int}\, \mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in \text{int}\, \mathbb{R}^r_+$

$$\frac{\int_a^b \{f^i(t, x, \dot{x}) - c^i \|\xi(t, x, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, x, \dot{x}) + d^i \|\xi(t, x, \bar{x})\|^m\}dt} < \bar{v}^i, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

That is,

$$\int_a^b \{f^i(t, x, \dot{x}) - \bar{v}^i k^i(t, x, \dot{x})\}dt < (c^i + d^i \bar{v}^i) \int_a^b \{\|\xi(t, x, \bar{x})\|^m\}dt, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

Using the hypotheses and proceeding as in the proof of Theorem 1, we arrive at

$$\int_a^b \{\frac{d}{dt}\{\eta(t, x, \bar{x}) \left[\sum_{i=1}^r \bar{\lambda}^i (f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)) + \bar{y}(t)^T g_{\dot{\bar{x}}}(t)\right]\}\}dt + \int_a^b \{\rho' \|\xi(t, x, \bar{x})\|^m\}dt < 0,$$

$$\Rightarrow \{\eta(t, x, \bar{x}) \left[\sum_{i=1}^r \bar{\lambda}^i (f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)) + \bar{y}(t)^T g_{\dot{\bar{x}}}(t)\right]\}\Big|_a^b + \int_a^b \{\rho' \|\xi(t, x, \bar{x})\|^m\}dt < 0.$$

This leads to a contradiction on using Eqs. (2), (6) and (15). Hence result follows.
(b) Proof runs on the same lines as the proof of part (a) and is hence omitted. □

Adopting the approach of Arana-Jimenez et al. [2], consider the following weighting variational problem (MF$P_{\bar{v}\lambda}$), for each $\bar{v} = (\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^r) \in \mathbb{R}^r_+$. This facilitates us to establish strong duality theorem.

$$(\text{MF}P_{\bar{v}\lambda}) \text{ Minimize } \left(\sum_{i=1}^{r} \lambda^i \int_a^b \{f^i(t, x, \dot{x}) - \bar{v}^i k^i(t, x, \dot{x})\} dt\right)$$

subject to

$$g(t, x, \dot{x}) \leqq 0, t \in I, x(a) = \alpha, x(b) = \beta.$$

where $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^r) \in \mathbb{R}^r_+ - \{0\}$ is a fixed vector.

**Lemma 1** *Let $\bar{x} \in X_0$ be an optimal solution of order $m$ with respect to $\xi$ for (MF$P_{\bar{v}\bar{\lambda}}$) which is normal, for some $\bar{\lambda} \in \mathbb{R}^r, \bar{\lambda} \geq 0, \bar{\lambda}^T e = 1, t \in I, \bar{v}^i = \frac{\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt}{\int_a^b k^i(t,\bar{x},\dot{\bar{x}})dt}$. Then $\bar{x}$ is a vector Kuhn Tucker point for (MFVP).*

*Proof* Let $\bar{x} \in X_0$ be an optimal solution of order $m$ with respect to $\xi$ for (MF$P_{\bar{v}\bar{\lambda}}$) which is normal, for some $\bar{\lambda} \in \mathbb{R}^r, \bar{\lambda} \geq 0, \bar{\lambda}^T e = 1, t \in I, \bar{v}^i = \frac{\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt}{\int_a^b k^i(t,\bar{x},\dot{\bar{x}})dt}$. It is trivial that an optimal solution of order $m$ with respect to $\xi$ for (MF$P_{\bar{v}\bar{\lambda}}$) is also an optimal solution of (MF$P_{\bar{v}\bar{\lambda}}$). Hence $\bar{x}$ is a normal optimal solution for (MF$P_{\bar{v}\bar{\lambda}}$). Now result can be easily proved by using necessary optimality conditions [19, Theorem 3.1] □

**Theorem 3** *(Strong duality) Let $\bar{x}$ be a weak efficient solution of order $m$ with respect to $\xi$ for (MFVP). Furthermore, if it is an optimal solution of order $m$ with respect to $\xi$ for (MF$P_{\bar{v}\bar{\lambda}}$) which is normal, for some $\bar{\lambda} \in \mathbb{R}^r, \bar{\lambda} \geq 0, \bar{\lambda}^T e = 1, t \in I, \bar{v}^i = \frac{\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt}{\int_a^b k^i(t,\bar{x},\dot{\bar{x}})dt}$ . Then there exist a piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ such that $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y}) \in Y_1$. Further if weak duality theorem holds and $\xi(t, x, \bar{x}) = \xi(t, \bar{x}, x)$ for all $x \in X$. Then $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y})$ is a weak efficient solution of order $m$ with respect to $\xi$ for (MFVDP).*

*Proof* Since $\bar{x}$ is an optimal solution of order $m$ with respect to $\xi$ for (MF$P_{\bar{v}\bar{\lambda}}$) which is normal. By lemma 1, it is a vector Kuhn Tucker point for (MFVP), that is, there exist a piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ such that $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y}) \in Y_1$. Suppose weak duality theorem holds and $\xi(t, x, \bar{x}) = \xi(t, \bar{x}, x)$ for all $x \in X$.

Let if possible, $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y})$ is not a weak efficient solution of order $m$ with respect to $\xi$ for (MFVDP). Then for any $\rho = (\rho^1, \rho^2, \ldots, \rho^r) \in \text{int}\mathbb{R}^r_+$, there exist $(\hat{x}, \hat{\lambda}, \hat{v}, \hat{y}) \in Y_1$ such that

$$\hat{v}^i + \rho^i \int_a^b \{\|\xi(t, \hat{x}, \bar{x})\|^m\} dt > \bar{v}^i = \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}}) dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}}) dt}, \text{ for all } i \in \{1, 2, \ldots, r\}. \tag{16}$$

Case(i) If $\|\xi(t, \hat{x}, \bar{x})\|^m = 0$. Then for any $c = (c^1, \ldots, c^r) \in \text{int}\,\mathbb{R}^r_+$ and $d = (d^1, \ldots, d^r) \in \text{int}\,\mathbb{R}^r_+$,

$$\frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - c^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt} < \hat{v}^i \text{ for all } i \in \{1, 2, \ldots, r\},$$

which is contradiction to weak duality theorem.
Case(ii) If $\|\xi(t, \hat{x}, \bar{x})\|^m \neq 0$.

Assertion 1: for any $c = (c^1, c^2, ..., c^r) \in \text{int}\,\mathbb{R}^r_+$ and $d = (d^1, d^2, ..., d^r) \in \text{int}\,\mathbb{R}^r_+$, we have

$$\frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - c^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt} < \hat{v}^i, \text{ for all } i \in \{1, 2, ..., r\}.$$

If possible assertion 1 is not true then there exist $c = (c^1, c^2, \ldots, c^r) \in \text{int}\,\mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in \text{int}\,\mathbb{R}^r_+$, we have

$$\frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - c^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt} \not< \hat{v}^i, \text{ for some } i \in \{1, 2, \ldots, r\}, \tag{17}$$

Taking $\rho^i = \frac{c^i \int_a^b k^i(t, \bar{x}, \dot{\bar{x}}) dt + d^i \int_a^b f^i(t, \bar{x}, \dot{\bar{x}}) dt}{\{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt\} \int_a^b k^i(t, \bar{x}, \dot{\bar{x}}) dt}$, $i = 1, 2, \ldots, r$ in Eq. (16)

$$\frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - c^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i \|\xi(t, \hat{x}, \bar{x})\|^m\} dt} < \hat{v}^i, \text{ for all } i \in \{1, 2, \ldots, r\},$$

which is a contradiction to Eq. (17). Therefore assertion 1 holds which deny weak duality theorem. Thus $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y})$ is a weak efficient solution of order $m$ with respect to $\xi$ for (MFVDP). □

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