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THEORETICAL ARTICLE

Set-valued fractional programming problems under generalized cone convexity

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Abstract In this paper, we establish the sufficient Karush-Kuhn-Tucker (KKT) optimality conditions for the set-valued fractional programming problem (FP) via contingent epiderivative under ρ-cone convexity. We also study the duality results of parametric (PD), Mond-Weir (MWD), Wolfe (WD) and mixed (MD) types for the problem (FP).

Keywords Convex cone · Set-valued map · Contingent epiderivative · Duality

1 Introduction

In the past few years, the set-valued optimization theory has attracted the attention of many researchers towards this expanding branch of optimization. Many optimization problems in mathematical economics, optimal control, differential inclusions, image processing, viability theory and many more are set-valued optimization problems (SVOP) that involve set-valued maps as objective functions and constraints. Various types of differentiability notions of set-valued maps have been introduced in (SVOP). The notion of contingent epiderivative of set-valued maps, introduced by Jahn and Rauh [8], has a significant role to establish optimality conditions of (SVOP). The notion of cone convexity of set-valued maps, introduced by Borwein

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[6], also plays a vital role in this case. An important class of (SVOP) is set-valued fractional programming problems. In 1997, Bhatia and Mehra [4] introduced the notion of cone preinvexity for set-valued maps and obtained the Lagrangian duality results for the set-valued fractional programming problems. Later, they [5] proved the duality results for Geoffrion efficient solutions of the set-valued fractional programming problems under cone convexity assumptions. In 2013, Gadhi and Jawhar [7] established the necessary optimality conditions of the set-valued fractional programming problems without any convex separation approach. Many authors like Kaul and Lyall [9], Bhatia and Garg [3], Suneja and Gupta [11], Suneja and Lalitha [12] and Lee and Ho [10] established the optimality conditions and proved the duality theorems for vector-valued fractional programming problems under generalized convexity assumptions.

This paper is organized as follows. In Section 2, we recall some definitions and preliminary concepts of the set-valued optimization theory. In Section 3, we establish the sufficient optimality conditions for weak efficiency of the set-valued fractional programming problems under generalized cone convexity assumptions. We also establish the duality results of various types.

2 Definitions and preliminaries

Let Y be a real normed space and K be a nonempty subset of Y. Then K is called a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Further, the cone K is called pointed if $K \cap (-K) = \{\mathbf{0}_Y\}$, solid if $\text{int}(K) \neq \emptyset$, closed if $\overline{K} = K$ and convex if

$$\lambda K + (1-\lambda)K \subseteq K, \forall \lambda \in [0, 1],$$

where int(K) and $\overline{K}$ denote the interior and closure of K, respectively and $\mathbf{0}_Y$ is the zero element of Y.

The positive orthant $\mathbb{R}^m_+$ of $\mathbb{R}^m$, defined by

$$\mathbb{R}^m_+ = \left\{y = (y_1, ..., y_m) \in \mathbb{R}^m : y_i \geq 0, \forall i = 1, ..., m\right\},$$

is a solid pointed closed convex cone of $\mathbb{R}^m$.

Various types of minimal points can be defined with respect to a solid pointed convex cone in a normed space.

Definition 2.1 Let B be a nonempty subset of a normed space Y, K be a solid pointed convex cone in Y and $y' \in B$. Then,

(i) y' is an ideal minimal point of B if $y' - y \in -K$, for all $y \in B$.
(ii) y' is a minimal point of B if there is no $y \in B \setminus \{y'\}$ such that $y - y' \in K$.
(iii) y' is a weakly minimal point of B if there is no $y \in B$ such that $y - y' \in \text{int}(K)$.

The contingent epiderivative of set-valued map is defined with the help of contingent cone. Aubin [1, 2] introduced the contingent cone in normed spaces.

Definition 2.2 [1, 2] Let B be a nonempty subset of a normed space Y and $y' \in \overline{B}$. Then, the contingent cone to B at y', denoted by $T(B, y')$, is defined as:

$y \in T(B, y')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in Y, with $y_n \to y$, such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to y, \text{ as } n \to \infty.$$

Let X and Y be real normed spaces, 2^Y be the set of all subsets of Y and K be a solid pointed convex cone in Y. Let $F : X \to 2^Y$ be a set-valued map from X to Y i.e., $F(x) \subseteq Y$, for all $x \in X$.

The effective domain, graph and epigraph of the set-valued map F are defined by:

$$\text{dom}(F) = \{x \in X : F(x) \neq \emptyset\},$$

$$\text{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$$

and

$$\text{epi}(F) = \{(x, y) \in X \times Y : y \in F(x) + K\}.$$

In 1997, Jahn and Rauh [8] introduced the notion of contingent epiderivative of set-valued maps.

Definition 2.3 [8] Let $F : X \to 2^Y$ be a set-valued map and $(x', y') \in \text{gr}(F)$. Then the single-valued map $D_\uparrow F(x', y') : X \to Y$ is called the contingent epiderivative of F at (x', y') if

$$\text{epi}\left(D_\uparrow F(x', y')\right) = T\left(\text{epi}(F), (x', y')\right).$$

Jahn and Rauh [8] also showed that when $f : X \to \mathbb{R}$ is a real-valued map, being continuous at the point $x' \in X$ and f is convex, then

$$D_\uparrow f\left(x', f(x')\right)(u) = f'(x')(u), \forall u \in X,$$

where $f'(x')(u)$ is the directional derivative of f at x' in the direction u.

Borwein [6] introduced cone convexity of set-valued maps. Later, Jahn and Rauh [8] characterized cone convex set-valued maps in terms of contingent epiderivative.

Definition 2.4 [6] Let A be a nonempty convex subset of X. A set-valued map $F : X \to 2^Y$, with $A \subseteq \text{dom}(F)$, is called K-convex on A if $\forall x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.$$

It is clear that if the set-valued map $F : X \to 2^Y$ is K-convex on A, then $\text{epi}(F)$ is a convex subset of $X \times Y$.

Lemma 2.1 [8] *Let $F : X \to 2^Y$ be K-convex on a nonempty convex subset A of X. Let $x' \in A$ and $y' \in F(x')$. Assume that F is contingent epiderivable at (x', y'). Then, for all $x \in A$,*

$$F(x) - y' \subseteq D_\uparrow F(x', y')(x - x') + K.$$

Let X be a real normed space and A be a nonempty subset of X. Let $F : X \to 2^{\mathbb{R}^m}$, $G : X \to 2^{\mathbb{R}^m}$ and $H : X \to 2^{\mathbb{R}^k}$ be set-valued maps, with

$$A \subseteq \mathrm{dom}(F) \cap \mathrm{dom}(G) \cap \mathrm{dom}(H).$$

Throughout the paper, denote

$$\mathbf{0}_{\mathbb{R}^m} = (0, ..., 0) \in \mathbb{R}^m$$

and

$$\mathbf{1}_{\mathbb{R}^m} = (1, ..., 1) \in \mathbb{R}^m.$$

Let $F = (F_1, F_2, ..., F_m)$, $G = (G_1, G_2, ..., G_m)$ and $H = (H_1, H_2, ..., H_k)$, where the set-valued maps $F_i : X \to 2^{\mathbb{R}}$, $G_i : X \to 2^{\mathbb{R}}$, $i = 1, 2, ..., m$ and $H_j : X \to 2^{\mathbb{R}}$, $j = 1, 2, ..., k$, are defined by:

$$\mathrm{dom}(F_i) = \mathrm{dom}(F), \mathrm{dom}(G_i) = \mathrm{dom}(G) \text{ and } \mathrm{dom}(H_j) = \mathrm{dom}(H),$$

$$x \in A, y = (y_1, y_2, ..., y_m) \in F(x) \Longleftrightarrow y_i \in F_i(x), \forall i = 1, 2, ..., m,$$

$$z = (z_1, z_2, ..., z_m) \in G(x) \Longleftrightarrow z_i \in G_i(x), \forall i = 1, 2, ..., m$$

and

$$w = (w_1, w_2, ..., w_m) \in H(x) \Longleftrightarrow w_j \in H_j(x), \forall j = 1, 2, ..., k.$$

Assume that $F_i(x) \subseteq \mathbb{R}_+$ and $G_i(x) \subseteq \mathrm{int}(\mathbb{R}_+)$, $\forall i = 1, 2, ..., m$ and $x \in A$.
Let $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_m) \in \mathbb{R}^m_+$.
Define $\frac{y}{z} \in \mathbb{R}^m$ and $\lambda' z \in \mathbb{R}^m$ by:

$$\frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, ..., \frac{y_m}{z_m}\right)$$

and

$$\lambda' z = (\lambda'_1 z_1, \lambda'_1 z_2, ..., \lambda'_m z_m).$$

For $x \in A$, define the subset $\frac{F(x)}{G(x)}$ of $\mathbb{R}^m$ by:

$$\frac{F(x)}{G(x)} = \left\{\frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, ..., \frac{y_m}{z_m}\right) : y_i \in F_i(x), z_i \in G_i(x), i = 1, 2, ..., m\right\}.$$

Consider the set-valued fractional programming problem:

$$\begin{array}{ll} \underset{x \in A}{\text{minimize}} & \frac{F(x)}{G(x)} = \left(\frac{F_1(x)}{G_1(x)}, \frac{F_2(x)}{G_2(x)}, ..., \frac{F_m(x)}{G_m(x)}\right) \\ \text{subject to,} & H(x) \cap \left(-\mathbb{R}^k_+\right) \neq \emptyset. \end{array} \tag{FP}$$

The feasible set of the problem (FP) is

$$S = \left\{x \in A : H(x) \cap \left(-\mathbb{R}^k_+\right) \neq \emptyset\right\}.$$

Definition 2.5 A point $\left(x', \frac{y'}{z'}\right) \in X \times \mathbb{R}^m$, with $x' \in S$, $y' \in F(x')$ and $z' \in G(x')$, is called a minimizer of the problem (FP) if there exist no $x \in S$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} - \frac{y'}{z'} \in -\mathbb{R}^m_+ \setminus \{\mathbf{0}_{\mathbb{R}^m}\}.$$

Definition 2.6 A point $\left(x', \frac{y'}{z'}\right) \in X \times \mathbb{R}^m$, with $x' \in S$, $y' \in F(x')$ and $z' \in G(x')$, is called a weak minimizer of the problem (FP) if there exist no $x \in S$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} - \frac{y'}{z'} \in -\text{int}\left(\mathbb{R}^m_+\right).$$

Consider the parametric problem $(FP_{\lambda'})$ associated with the set-valued fractional programming problem (FP):

$$\begin{aligned} &\underset{x\in A}{\text{minimize}} && F(x) - \lambda' G(x) \\ &\text{subject to,} && H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset. \end{aligned} \qquad (FP_{\lambda'})$$

Definition 2.7 A point $(x', y' - \lambda' z') \in X \times \mathbb{R}^m$, with $x' \in S$, $y' \in F(x')$ and $z' \in G(x')$, is called a minimizer of the problem $(FP_{\lambda'})$, if there exist no $x \in S$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda' z) - (y' - \lambda' z') \in -\mathbb{R}^m_+ \setminus \{\mathbf{0}_{\mathbb{R}^m}\}.$$

Definition 2.8 A point $(x', y' - \lambda' z') \in X \times \mathbb{R}^m$, with $x' \in S$, $y' \in F(x')$ and $z' \in G(x')$, is called a weak minimizer of the problem $(FP_{\lambda'})$, if there exist no $x \in S$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda' z) - (y' - \lambda' z') \in -\text{int}\left(\mathbb{R}^m_+\right).$$

Gadhi and Jawhar [7] proved the relation between the solutions of the problems (FP) and $(FP_{\lambda'})$ in the following theorem.

Lemma 2.2 [7] *A point* $\left(x', \frac{y'}{z'}\right) \in X \times \mathbb{R}^m$ *is a weak minimizer of the problem (FP) if and only if* $(x', \mathbf{0}_{\mathbb{R}^m})$ *is a weak minimizer of the problem* $(FP_{\lambda'})$, *where* $\lambda' = \frac{y'}{z'}$.

Lemma 2.3 [13] *Let* $x_1, x_2 \in \mathbb{R}^n$ *and* $\lambda \in [0, 1]$.
Then,

$$\|\lambda x_1 + (1-\lambda)x_2\|^2 = \lambda\|x_1\|^2 + (1-\lambda)\|x_2\|^2 - \lambda(1-\lambda)\|x_1 - x_2\|^2.$$

3 Main results

We introduce the notion of ρ-cone convexity of set-valued maps. For $\rho = 0$, we have the usual notion of cone convexity of set-valued maps.

Definition 3.1 Let X, Y be real normed spaces, A be a nonempty convex subset of X, K be a solid pointed convex cone in Y, $e \in \text{int}(K)$ and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$.

Then F is called ρ-K-convex with respect to e on A if there exists $\rho \in \mathbb{R}$ such that

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + \rho\lambda(1-\lambda)\|x_1 - x_2\|^2 e + K,$$
$$\forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0,1].$$

In the following theorem, we characterize ρ-cone convexity for contingent epiderivable set-valued maps.

Theorem 3.1 *Let A be a nonempty convex subset of X, $e \in \text{int}(K)$ and $F : X \to 2^Y$ be ρ-K-convex with respect to e on A. Let $x' \in A$ and $y' \in F(x')$. Assume that F is contingent epiderivable at (x', y').*
Then,

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(x - x') + \rho\|x - x'\|^2 e + K, \forall x \in A.$$

Proof Let $x \in A$ and $y \in F(x)$.
As F is ρ-K-convex with respect to e on A,

$$\lambda F(x) + (1-\lambda)F(x') \subseteq F(x' + \lambda(x - x')) + \rho\lambda(1-\lambda)\|x - x'\|^2 e + K,$$
$$\forall \lambda \in [0,1].$$

Let $\{\lambda_n\}$ be a sequence in $\mathbb{R}$ such that $\lambda_n \in (0,1)$ and $\lambda_n \to 0$, as $n \to \infty$.
Consider two sequences $\{x_n\}$ in X and $\{y_n\}$ in Y, defined by:

$$x_n = x' + \lambda_n(x - x')$$

and

$$y_n = \lambda_n y + (1-\lambda_n)y' - \rho\lambda_n(1-\lambda_n)\|x - x'\|^2 e.$$

Therefore,

$$y_n \in F(x_n) + K.$$

It is clear that

$$x_n \to x', y_n \to y', \frac{x_n - x'}{\lambda_n} \to x - x', \text{ when } n \to \infty$$

and

$$\frac{y_n - y'}{\lambda_n} = y - y' - \rho(1-\lambda_n)\|x - x'\|^2 e \to y - y' - \rho\|x - x'\|^2 e, \text{ when } n \to \infty.$$

Therefore,

$$(x - x', y - y' - \rho\|x - x'\|^2 e) \in T\left(\text{epi}(F), (x', y')\right) = \text{epi}\left(D_{\uparrow}F(x', y')\right).$$

Consequently,

$$y - y' - \rho\|x - x'\|^2 e \in D_{\uparrow}F(x', y')(x - x') + K,$$

which is true for all $y \in F(x)$.
Therefore,

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(x - x') + \rho\|x - x'\|^2 e + K.$$

□

We also establish a relation between the notions of cone convexity and ρ-cone convexity of set-valued maps, when $X = \mathbb{R}^n$.

Theorem 3.2 *Let A be a nonempty convex subset of $\mathbb{R}^n$, $e \in \text{int}(K)$ and $F : \mathbb{R}^n \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then $F : \mathbb{R}^n \to 2^Y$ is ρ-K-convex with respect to e on A if and only if there exists a K-convex set-valued map $\widetilde{F} : \mathbb{R}^n \to 2^Y$ on A, such that*

$$F(x) = \widetilde{F}(x) + \rho\|x\|^2 e, \forall x \in \text{dom}(F). \tag{3.1}$$

Proof Suppose that there exists a K-convex set-valued map $\widetilde{F} : \mathbb{R}^n \to 2^Y$ on A such that Eq. 3.1 holds.
We show that $F : \mathbb{R}^n \to 2^Y$ is ρ-K-convex with respect to e on A, i.e., for all $x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + \rho\lambda(1-\lambda)\|x_1 - x_2\|^2 e + K.$$

Let $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$.
Then,

$$y_1 = z_1 + \rho\|x_1\|^2 e$$

and

$$y_2 = z_2 + \rho\|x_2\|^2 e,$$

for some $z_1 \in \widetilde{F}(x_1)$ and $z_2 \in \widetilde{F}(x_2)$.
As $\widetilde{F} : \mathbb{R}^n \to 2^Y$ is K-convex on A, we have

$$\lambda\widetilde{F}(x_1) + (1-\lambda)\widetilde{F}(x_2) \subseteq \widetilde{F}(\lambda x_1 + (1-\lambda)x_2) + K.$$

Therefore,

$$\lambda z_1 + (1-\lambda)z_2 \in \widetilde{F}(\lambda x_1 + (1-\lambda)x_2) + K.$$

Again, from Lemma 2.3, we have

$$\|\lambda x_1 + (1-\lambda)x_2\|^2 = \lambda\|x_1\|^2 + (1-\lambda)\|x_2\|^2 - \lambda(1-\lambda)\|x_1 - x_2\|^2.$$

Hence,

$$\begin{aligned}&\lambda(z_1 + \rho\|x_1\|^2 e) + (1-\lambda)(z_2 + \rho\|x_2\|^2 e)\\ &\in \widetilde{F}(\lambda x_1 + (1-\lambda)x_2) + \rho\|\lambda x_1 + (1-\lambda)x_2\|^2 e + \rho\lambda(1-\lambda)\|x_1 - x_2\|^2 e + K.\end{aligned}$$

It follows that

$$\lambda y_1 + (1-\lambda)y_2 \in F(\lambda x_1 + (1-\lambda)x_2) + \rho\lambda(1-\lambda)\|x_1 - x_2\|^2 e + K.$$

Therefore, $F : \mathbb{R}^n \to 2^Y$ is ρ-K-convex with respect to e on A.

Conversely, let $F : \mathbb{R}^n \to 2^Y$ be ρ-K-convex with respect to e on A.
Consider the set-valued map $\widetilde{F} : \mathbb{R}^n \to 2^Y$ defined by

$$\widetilde{F}(x) = F(x) - \rho\|x\|^2 e, \forall x \in \text{dom}(F). \tag{3.2}$$

To show that $\widetilde{F}$ is K-convex on A, let $x_1', x_2' \in A$, $z_1' \in \widetilde{F}(x_1')$ and $z_2' \in \widetilde{F}\left(x_2'\right)$.
Then,

$$z_1' = y_1' - \rho\|x_1'\|^2 e$$

and

$$z_2' = y_2' - \rho\|x_2'\|^2 e,$$

for some $y_1' \in F\left(x_1'\right)$ and $y_2' \in F\left(x_2'\right)$.
Therefore,

$$\lambda y_1' + (1-\lambda) y_2' \in F\left(\lambda x_1' + (1-\lambda) x_2'\right) + \rho\lambda(1-\lambda)\|x_1' - x_2'\|^2 e + K.$$

Hence,

$$\begin{aligned}
&\lambda z_1' + (1-\lambda) z_2' \\
&= \lambda\left(y_1' - \rho\|x_1'\|^2 e\right) + (1-\lambda)\left(y_2' - \rho\|x_2'\|^2 e\right) \\
&\in F\left(\lambda x_1' + (1-\lambda) x_2'\right) - \rho\left(\lambda\|x_1'\|^2 + (1-\lambda)\|x_2'\|^2\right) e + \rho\lambda(1-\lambda)\|x_1' - x_2'\|^2 e + K \\
&= F\left(\lambda x_1' + (1-\lambda) x_2'\right) - \rho\left\|\lambda x_1' + (1-\lambda) x_2'\right\|^2 e + K \text{ (From Lemma 2.3)} \\
&= \widetilde{F}\left(\lambda x_1' + (1-\lambda) x_2'\right) + K.
\end{aligned}$$

Consequently, $\widetilde{F}$ is a K-convex set-valued map on A. □

When $\rho = 0$, ρ-cone convex set-valued map becomes cone convex. We construct an example of ρ-cone convex set-valued map, which is not cone convex.

Example 3.1 Consider the set-valued map $F : [-1, 1] \subseteq \mathbb{R} \to 2^{\mathbb{R}^2}$ defined by:

$$F(t) = \begin{cases} \left\{\left(x - 2t^2, x^2 - 2t^2\right) : x \geq 0\right\}, & if\ 0 \leq t \leq 1, \\ \left\{\left(x - 2t^2, x - 2t^2\right) : x \leq 0\right\}, & if\ -1 \leq t < 0. \end{cases}$$

Let $t_1 = -1$, $t_2 = 1$ and $\lambda = \frac{1}{2}$.
So, $\lambda t_1 + (1-\lambda) t_2 = 0$.
Therefore,

$$F(\lambda t_1 + (1-\lambda) t_2) + \mathbb{R}_+^m = \mathbb{R}_+^m.$$

It is clear that

$$(-2, -2) \in F(t_1) \cap F(t_2) \text{ but, } (-2, -2) \notin \mathbb{R}_+^m.$$

So,

$$\lambda(-2, -2) + (1-\lambda)(-2, -2) = (-2, -2) \notin F(\lambda t_1 + (1-\lambda) t_2) + \mathbb{R}_+^m.$$

Hence, F is not $\mathbb{R}_+^2$-convex on $[-1, 1]$.
Assume that $\rho = -2$.
Then the set-valued map $\widetilde{F} : [-1, 1] \subseteq \mathbb{R} \to 2^{\mathbb{R}^2}$, defined by Eq. 3.2, is given by

$$\widetilde{F}(t) = \begin{cases} \{(x, x^2) : x \geq 0\}, & if\ 0 \leq t \leq 1, \\ \{(x, x) : x \leq 0\}, & if\ -1 \leq t < 0. \end{cases}$$

We can easily prove that $\widetilde{F}$ is $\mathbb{R}_+^2$-convex on $[-1, 1]$.
Hence, F is (-2)-$\mathbb{R}_+^2$-convex with respect to $e = (1, 1)$ on $[-1, 1]$.

3.1 Optimality conditions

We establish the sufficient optimality conditions for the problem (FP), assuming that the objective and constraint set-valued maps are ρ-cone convex as well as contingent epiderivable.

Theorem 3.3 ***(Sufficient optimality conditions)*** *Let A be a nonempty convex subset of X, x' be an element of the feasible set S of the problem (FP), $y' \in F(x')$, $z' \in G(x')$, $\lambda' = \frac{y'}{z'}$ and $w' \in H(x') \cap (-L)$. Assume that F is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-\lambda' G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A. Let F be contingent epiderivable at (x', y'), $-\lambda' G$ be contingent epiderivable at $(x', -\lambda' z')$ and H be contingent epiderivable at (x', w'). Suppose that there exists $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^* \neq \mathbf{0}_{\mathbb{R}^m}$, and*

$$(\rho_1 + \rho_2)\langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle + \rho_2\langle z^*, \mathbf{1}_{\mathbb{R}^k}\rangle \geq 0, \tag{3.3}$$

such that

$$\begin{aligned}&\left\langle y^*, D_\uparrow F(x', y')(x - x') + D_\uparrow(-\lambda' G)(x', -\lambda' z')(x - x')\right\rangle \\ &+\left\langle z^*, D_\uparrow H(x', w')(x - x')\right\rangle \geq 0, \forall x \in A,\end{aligned} \tag{3.4}$$

$$y' - \lambda' z' = 0 \tag{3.5}$$

and

$$\langle z^*, w'\rangle = 0. \tag{3.6}$$

Then $\left(x', \frac{y'}{z'}\right)$ is a weak minimizer of the problem (FP).

Proof We prove the theorem by the method of contradiction.
Let $\left(x', \frac{y'}{z'}\right)$ be not a weak minimzer of the problem (FP).
Then there exist $x \in S$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} < \frac{y'}{z'}.$$

As $y' - \lambda' z' = 0$, we have

$$\frac{y}{z} < \lambda'.$$

So,

$$y - \lambda' z < 0.$$

Hence,

$$\langle y^*, y - \lambda' z\rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

Again, as $y' - \lambda' z' = 0$, we have

$$\langle y^*, y' - \lambda' z'\rangle = 0.$$

Since $x \in S$, there exists an element $w \in H(x) \cap \left(-\mathbb{R}^k_+\right)$.
Therefore,

$$\langle z^*, w\rangle \leq 0.$$

So,

$$\langle z^*, w - w'\rangle \leq 0, \text{ as } \langle z^*, w'\rangle = 0.$$

Hence,

$$\langle y^*, y - \lambda' z - (y' - \lambda' z')\rangle + \langle z^*, w - w'\rangle < 0. \tag{3.7}$$

As F is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-\lambda' G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$F(x) - y' \subseteq D_{\uparrow} F(x', y')(x - x') + \rho_1 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$(-\lambda' G)(x) + \lambda' z \subseteq D_{\uparrow}(-\lambda' G)(x', -\lambda' z')(x - x') + \rho_2 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$H(x) - w' \subseteq D_{\uparrow} H(x', w')(x - x') + \rho_3 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$y - y' \in D_{\uparrow} F(x', y')(x - x') + \rho_1 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$-\lambda' z + \lambda' z' \in D_{\uparrow}(-\lambda' G)(x', -\lambda' z')(x - x') + \rho_2 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$w - w' \in D_{\uparrow} H(x', w')(x - x') + \rho_3 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence, from Eqs. 3.3 and 3.4, we have

$$\langle y^*, y - \lambda' z - (y' - \lambda' z') \rangle + \langle z^*, w - w' \rangle \geq 0,$$

which contradicts Eq. 3.7.
Consequently, (x', y') is a weak minimizer of the problem (FP). □

We can also prove the following theorem by the above approach.

Theorem 3.4 *(**Sufficient optimality conditions**) Let A be a nonempty convex subset of X, x' be an element of the feasible set S of the problem (FP), $y' \in F(x')$, $z' \in G(x')$ and $w' \in H(x') \cap (-L)$. Assume that $z' F$ is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-y' G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A. Suppose that there exists $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^* \neq \mathbf{0}_{\mathbb{R}^m}$, and Eqs. 3.3 and 3.6 are satisfied, with*

$$\left\langle y^*, D_{\uparrow}(z' F)(x', y' z')(x - x') + D_{\uparrow}(-y' G)(x', -y' z')(x - x') \right\rangle$$
$$+ \left\langle z^*, D_{\uparrow} H(x', w')(x - x') \right\rangle \geq 0, \forall x \in A. \tag{3.8}$$

Then $\left(x', \frac{y'}{z'}\right)$ is a weak minimizer of the problem (FP).

Now, we formulate the duals of parametric (PD), Mond-Weir (MWD), Wolfe (WD) and mixed (MD) types for the problem (FP) and study the corresponding duality theorems. We give the proofs of the duality theorems of parametric (PD) and Mond-Weir (MWD) types. We state the duality theorems of Wolfe (WD) and mixed (MD) types whose proofs are very similar to the former ones, hence omitted.

3.2 Parametric type dual

We consider the parametric type dual (PD) associated the problem (FP).

$$\begin{array}{ll} \text{maximize} & \lambda', \\ \text{subject to,} & \\ & \langle y^*, D_{\uparrow}F(x', y')(x - x') + D_{\uparrow}(-\lambda' G)(x', -\lambda' z')(x - x')\rangle \\ & +\langle z^*, D_{\uparrow}H(x', w')(x - x')\rangle \geq 0, \forall x \in A, \\ & y'_i - \lambda'_i z'_i \geq 0, \forall i = 1, ..., m, \\ & x' \in A, y' \in F(x'), z' \in G(x'), \lambda' \in \frac{F(x)}{G(x)}, w' \in H(x), \\ & y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+, \langle z^*, w'\rangle \geq 0 \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1. \end{array} \tag{PD}$$

A point $(x', y', z', \lambda', w', y^*, z^*)$ satisfying all the constraints of the problem (PD) is called a feasible point of (PD).

Definition 3.2 A feasible point $(x', y', z', \lambda', w', y^*, z^*)$ of the problem (PD) is called a weak maximizer of (PD) if there exists no feasible point $\left(x, y, z, \lambda, w, y^*_1, z^*_1\right)$ of (PD) such that

$$\lambda - \lambda' \in \text{int}\left(\mathbb{R}^m_+\right).$$

Theorem 3.5 *(**Weak Duality**) Let A be a nonempty convex subset of X, $\overline{x}$ be an element of the feasible set S of the problem (FP) and $(x', y', z', \lambda', y^*, z^*)$ be a feasible point of the problem (PD). Assume that F is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-\lambda' G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, such that*

$$(\rho_1 + \rho_2) + \rho_3 \left\langle z^*, \mathbf{1}_{\mathbb{R}^k}\right\rangle \geq 0. \tag{3.9}$$

Then,

$$\frac{F(\overline{x})}{G(\overline{x})} - \lambda' \subseteq \mathbb{R}^m \setminus -\text{int}\left(\mathbb{R}^m_+\right).$$

Proof We prove the theorem by the method of contradiction.
Suppose that for some $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$,

$$\frac{\overline{y}}{\overline{z}} - \lambda' \in -\text{int}\left(\mathbb{R}^m_+\right).$$

Therefore,

$$\frac{\overline{y}}{\overline{z}} < \lambda'.$$

Hence,

$$\frac{\overline{y}_i}{\overline{z}_i} < \lambda'_i, \forall i = 1, ..., m.$$

So,

$$\overline{y}_i - \lambda'_i \overline{z}_i < 0, \forall i = 1, ..., m.$$

Therefore,

$$\langle y^*, \overline{y} - \lambda' \overline{z}\rangle < 0, \;\text{since } \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

Again, from the constraints of (PD),

$$y'_i - \lambda'_i z'_i \geq 0, \forall i = 1, ..., m.$$

Therefore,

$$\langle y^*, y' - \lambda' z' \rangle \geq 0.$$

Again, since $\overline{x} \in S$, we have

$$H(\overline{x}) \cap \left(-\mathbb{R}^k_+\right) \neq \emptyset.$$

We choose $\overline{w} \in H(\overline{x}) \cap \left(-\mathbb{R}^k_+\right)$.
So,

$$\langle z^*, \overline{w} \rangle \leq 0.$$

Again, from the constraints of (PD), we have

$$\langle z^*, w' \rangle \geq 0.$$

So,

$$\langle z^*, \overline{w} - w' \rangle = \langle z^*, \overline{w} \rangle - \langle z^*, w' \rangle \leq 0.$$

Hence,

$$\langle y^*, \overline{y} - \lambda' \overline{z} - (y' - \lambda' z') \rangle + \langle z^*, \overline{w} - w' \rangle < 0. \tag{3.10}$$

As F is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-\lambda' G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$F(\overline{x}) - y' \subseteq D_\uparrow F(x', y')(\overline{x} - x') + \rho_1 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$(-\lambda' G)(\overline{x}) + \lambda' z' \subseteq D_\uparrow (-\lambda' G)(x', -\lambda' z')(\overline{x} - x') + \rho_2 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$H(\overline{x}) - w' \subseteq D_\uparrow H(x', w')(\overline{x} - x') + \rho_3 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$\overline{y} - y' \in D_\uparrow F(x', y')(\overline{x} - x') + \rho_1 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$-\lambda' \overline{z} + \lambda' z' \in D_\uparrow (-\lambda' G)(x', -\lambda' z')(\overline{x} - x') + \rho_2 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$\overline{w} - w' \in D_\uparrow H(x', w')(\overline{x} - x') + \rho_3 \|\overline{x} - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence, from the constraints of (PD) and Eq. 3.9, we have

$$\langle y^*, \overline{y} - \lambda' \overline{z} - (y' - \lambda' z') \rangle + \langle z^*, \overline{w} - w' \rangle \geq 0,$$

which contradicts Eq. 3.10.
Therefore,

$$\frac{\overline{y}}{\overline{z}} - \lambda' \notin -\text{int}\left(\mathbb{R}^m_+\right).$$

Since $\overline{y} \in F(\overline{x})$ is arbitrary, we have

$$\frac{F(\overline{x})}{G(\overline{x})} - \lambda' \subseteq \mathbb{R}^m \setminus -\text{int}\left(\mathbb{R}^m_+\right).$$

□

Theorem 3.6 *(**Strong Duality**) Let $(x', \frac{y'}{z'})$ be a weak minimizer of the problem (FP) and $w' \in H(x') \cap \left(-\mathbb{R}^k_+\right)$. Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $\langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1$ and $\lambda' \in \mathbb{R}^m$, Eqs. 3.4, 3.5 and 3.6 are satisfied at $(x', y', z', \lambda', y^*, z^*)$. Then $(x', y', z', \lambda', y^*, z^*)$ is a feasible solution of the problem (PD). Further, if the weak duality Theorem 3.5 between (FP) and (PD) holds, then $(x', y', z', \lambda', y^*, z^*)$ is a weak maximizer of (PD).*

Proof As the Eqs. 3.4, 3.5 and 3.6 are satisfied at $(x', y', z', \lambda', y^*, z^*)$, we have

$$\langle y^*, D_\uparrow F(x', y')(x - x') + D_\uparrow(-\lambda' G)(x', -\lambda' z')(x - x')\rangle$$
$$+\langle z^*, D_\uparrow H(x', w')(x - x')\rangle \geq 0, \forall x \in A,$$
$$y' - \lambda' z' = 0$$

and

$$\langle z^*, w'\rangle = 0.$$

Hence $(x', y', z', \lambda', y^*, z^*)$ is a feasible solution of (PD).
Suppose that the weak duality Theorem 3.5 between (FP) and (PD) holds and $(x', y', z', \lambda', y^*, z^*)$ is not a weak maximizer of (PD).
Then there exits a feasible point $(x, y, z, \lambda, y_1^*, z_1^*)$ of (PD) such that

$$\lambda - \lambda' \in \text{int}\left(\mathbb{R}^m_+\right).$$

As $y' - \lambda' z' = 0$,

$$\lambda - \frac{y'}{z'} \in \text{int}\left(\mathbb{R}^m_+\right).$$

which contradicts the weak duality Theorem 3.5 between (FP) and (PD).
Consequently, $(x', y', z', \lambda', y^*, z^*)$ is a weak maximizer of (PD). □

Theorem 3.7 *(**Converse Duality**) Let A be a nonempty convex subset of X and $(x', y', z', \lambda', y^*, z^*)$ be a feasible point of the problem (PD), where $\lambda' = \frac{y'}{z'}$. Assume that F is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-\lambda' G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, satisfying Eq. 3.9. If x' is an element of the feasible set S of the problem (FP), then $\left(x', \frac{y'}{z'}\right)$ is a weak minimizer of the problem (FP).*

Proof We prove the theorem by the method of contradiction.
Suppose $\left(x', \frac{y'}{z'}\right)$ is not a weak minimzer of the problem (FP).
Therefore there exist $x \in S$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} < \frac{y'}{z'}.$$

As $\lambda' = \frac{y'}{z'}$, we have

$$\frac{y}{z} < \lambda'.$$

So,

$$y - \lambda' z < 0.$$

Hence,

$$\langle y^*, y - \lambda' z\rangle < 0,\ \text{since}\ \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

Again, from the constraints of (PD),

$$y_i' - \lambda_i' z_i' \geq 0, \forall i = 1, ..., m.$$

Therefore,

$$\langle y^*, y' - \lambda' z'\rangle \geq 0.$$

Since $x \in S$, there exists an element

$$w \in H(x) \cap \left(-\mathbb{R}^k_+\right).$$

Therefore,

$$\langle z^*, w\rangle \leq 0.$$

We have

$$\langle z^*, w - w'\rangle \leq 0,\ \text{as}\ \langle z^*, w'\rangle = 0.$$

Hence,

$$\langle y^*, y - \lambda' z - (y' - \lambda' z')\rangle + \langle z^*, w - w'\rangle < 0. \tag{3.11}$$

As F is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-\lambda' G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$F(x) - y' \subseteq D_{\uparrow} F(x', y')(x - x') + \rho_1 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$(-\lambda' G)(x) + \lambda' z \subseteq D_{\uparrow}(-\lambda' G)(x', -\lambda' z')(x - x') + \rho_2 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$H(x) - w' \subseteq D_{\uparrow} H(x', w')(x - x') + \rho_3 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$y - y' \in D_{\uparrow} F(x', y')(x - x') + \rho_1 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$-\lambda' z + \lambda' z' \in D_{\uparrow}(-\lambda' G)(x', -\lambda' z')(x - x') + \rho_2 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$w - w' \in D_{\uparrow} H(x', w')(x - x') + \rho_3 \|x - x'\|^2 \mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence, from the constraints of (PD) and Eq. 3.9, we have

$$\langle y^*, y - \lambda' z - (y' - \lambda' z')\rangle + \langle z^*, w - w'\rangle \geq 0,$$

which contradicts Eq. 3.11.

Consequently, $\left(x', \frac{y'}{z'}\right)$ is a weak minimizer of the problem (FP). □

3.3 Mond-Weir type dual

We consider the Mond-Weir type dual (MWD) associated the problem (FP).

$$\begin{aligned} &\text{maximize} \quad \tfrac{y'}{z'},\\ &\text{subject to,}\\ &\qquad \left\langle y^*, D_\uparrow(z'F)(x', y'z')(x-x') + D_\uparrow(-y'G)(x', -y'z')(x-x')\right\rangle\\ &\qquad + \left\langle z^*, D_\uparrow H(x', w')(x-x')\right\rangle \geq 0, \forall x \in A,\\ &\qquad \langle z^*, w'\rangle \geq 0,\\ &\qquad x' \in A, y' \in F(x'), z' \in G(x'), y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+ \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1. \end{aligned} \tag{MWD}$$

A point $(x', y', z', w', y^*, z^*)$ which satisfies all the constraints of the problem (MWD) is called a feasible point of (MWD).

Definition 3.3 A feasible point $(x', y', z', w', y^*, z^*)$ of the problem (MWD) is called a weak maximizer of (MWD) if there exists no feasible point $(x, y, z, w, y_1^*, z_1^*)$ of (MWD) such that

$$\frac{y}{z} - \frac{y'}{z'} \in \text{int}\left(\mathbb{R}^m_+\right).$$

Theorem 3.8 *(**Weak Duality**) Let A be a nonempty convex subset of X, $\overline{x}$ be an element of the feasible set S of the problem (FP) and $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (MWD). Assume that $z'F$ is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-y'G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, satisfying Eq. 3.9.*
Then,

$$\frac{F(\overline{x})}{G(\overline{x})} - \frac{y'}{z'} \subseteq \mathbb{R}^m \setminus -int\left(\mathbb{R}^m_+\right).$$

Proof We prove the theorem by the method of contradiction.
Suppose that for some $\overline{y} \in F(\overline{x})$ and $\overline{z} \in G(\overline{x})$,

$$\frac{\overline{y}}{\overline{z}} - \frac{y'}{z'} \in -\text{int}\left(\mathbb{R}^m_+\right).$$

Therefore,

$$\frac{\overline{y}}{\overline{z}} < \frac{y'}{z'}.$$

So,

$$\overline{y}z' - y'\overline{z} < 0.$$

Hence,

$$\langle y^*, \overline{y}z' - y'\overline{z}\rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

As $\overline{x} \in S$, we have

$$H(\overline{x}) \cap \left(-\mathbb{R}^k_+\right) \neq \emptyset.$$

We choose $\overline{w} \in H(\overline{x}) \cap \left(-\mathbb{R}^k_+\right)$.
So,

$$\langle z^*, \overline{w}\rangle \leq 0.$$

Again, from the constraints of (MWD), we have

$$\langle z^*, w'\rangle \geq 0.$$

So,

$$\langle z^*, \overline{w} - w'\rangle = \langle z^*, \overline{w}\rangle - \langle z^*, w'\rangle \leq 0.$$

Hence,

$$\langle y^*, \overline{y}z' - y'\overline{z}\rangle + \langle z^*, \overline{w} - w'\rangle < 0. \tag{3.12}$$

As $z'F$ is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-y'G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$z'F(\overline{x}) - y'z' \subseteq D_\uparrow F(x', y')(\overline{x} - x') + \rho_1\|\overline{x} - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$(-y'G)(\overline{x}) + y'z' \subseteq D_\uparrow(-y'G)(x', -y'z')(\overline{x} - x') + \rho_2\|\overline{x} - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$H(\overline{x}) - w' \subseteq D_\uparrow H(x', w')(\overline{x} - x') + \rho_3\|\overline{x} - x'\|^2\mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$\overline{y}z' - y'z' \in D_\uparrow(z'F)(x', y'z')(\overline{x} - x') + \rho_1\|\overline{x} - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$-z'\overline{z} + y'z' \in D_\uparrow(-y'G)(x', -y'z')(\overline{x} - x') + \rho_2\|\overline{x} - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$\overline{w} - w' \in D_\uparrow H(x', w')(\overline{x} - x') + \rho_3\|\overline{x} - x'\|^2\mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence, from the constraints of (MWD) and Eq. 3.9, we have

$$\langle y^*, \overline{y}z' - y'\overline{z}\rangle + \langle z^*, \overline{w} - w'\rangle \geq 0,$$

which contradicts Eq. 3.12.
Therefore,

$$\frac{\overline{y}}{\overline{z}} - \frac{y'}{z'} \notin -\text{int}\left(\mathbb{R}^m_+\right).$$

Since $\overline{y} \in F(\overline{x})$ is arbitrary, we have

$$\frac{F(\overline{x})}{G(\overline{x})} - \frac{y'}{z'} \subseteq \mathbb{R}^m \setminus -\text{int}\left(\mathbb{R}^m_+\right).$$

□

Theorem 3.9 *(**Strong Duality**) Let* $\left(x', \frac{y'}{z'}\right)$ *be a weak minimizer of the problem (FP) and* $w' \in H(x') \cap \left(-\mathbb{R}^k_+\right)$*. Assume that for some* $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$*, with* $\langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1$*, Eqs. 3.6 and 3.8 are satisfied at* $(x', y', z', w', y^*, z^*)$*. Then* $(x', y', z', w', y^*, z^*)$ *is a feasible solution of the problem (MWD). Further, if the weak duality Theorem 3.8 between (FP) and (MWD) holds, then* $(x', y', z', w', y^*, z^*)$ *is a weak maximizer of (MWD).*

Proof As Eqs. 3.6 and 3.8 are satisfied at $(x', y', z', w', y^*, z^*)$, we have

$$\left\langle y^*, D_{\uparrow}(z'F)(x', y'z')(x-x') + D_{\uparrow}(-y'G)(x', -y'z')(x-x')\right\rangle + \left\langle z^*, D_{\uparrow}H(x', w')(x-x')\right\rangle \geq 0, \forall x \in A,$$

and

$$\langle z^*, w'\rangle = 0.$$

So, $(x', y', z', w', y^*, z^*)$ is a feasible solution of (MWD).
Suppose that the weak duality Theorem 3.8 between (FP) and (MWD) holds and $(x', y', z', w', y^*, z^*)$ is not a weak maximizer of (MWD).
Then there exists a feasible point $(x, y, z, w, y_1^*, z_1^*)$ of (MWD) such that

$$\frac{y'}{z'} < \frac{y}{z},$$

which contradicts the weak duality Theorem 3.8 between (FP) and (MWD).
Consequently, $(x', y', z', \lambda', y^*, z^*)$ is a weak maximizer of (MWD). □

Theorem 3.10 *(**Converse Duality**) Let A be a nonempty convex subset of X and $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (MWD). Assume that $z'F$ is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-y'G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, satisfying Eq. 3.9. If x' is an element of the feasible set S of the problem (FP), then $\left(x', \frac{y'}{z'}\right)$ is a weak minimizer of the problem (FP).*

Proof We prove the theorem by the method of contradiction.
Suppose $\left(x', \frac{y'}{z'}\right)$ is not a weak minimzer of the problem (FP).
Therefore there exist $x \in S$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} < \frac{y'}{z'}.$$

So,

$$yz' - y'z < 0.$$

Therefore,

$$\langle y^*, yz' - y'z\rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+.$$

Again, since $x \in S$, we have

$$H(x) \cap \left(-\mathbb{R}^k_+\right) \neq \emptyset.$$

We choose $w \in H(x) \cap \left(-\mathbb{R}^k_+\right)$.
So,

$$\langle z^*, w\rangle \leq 0.$$

From the constraints of (WD), we have

$$\langle z^*, w'\rangle \geq 0.$$

So,

$$\langle z^*, w - w'\rangle = \langle z^*, w\rangle - \langle z^*, w'\rangle \leq 0.$$

Hence,

$$\langle y^*, yz' - y'z\rangle + \langle z^*, w - w'\rangle < 0. \tag{3.13}$$

As $z'F$ is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-y'G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, we have

$$z'F(x) - y'z' \subseteq D_\uparrow F(x', y')(x - x') + \rho_1\|x - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$(-y'G)(x) + y'z' \subseteq D_\uparrow(-y'G)(x', -y'z')(x - x') + \rho_2\|x - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$H(x) - w' \subseteq D_\uparrow H(x', w')(x - x') + \rho_3\|x - x'\|^2\mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

Hence,

$$yz' - y'z' \in D_\uparrow(z'F)(x', y'z')(x - x') + \rho_1\|x - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+,$$
$$-z'z + y'z' \in D_\uparrow(-y'G)(x', -y'z')(x - x') + \rho_2\|x - x'\|^2\mathbf{1}_{\mathbb{R}^m} + \mathbb{R}^m_+$$

and

$$w - w' \in D_\uparrow H(x', w')(x - x') + \rho_3\|x - x'\|^2\mathbf{1}_{\mathbb{R}^k} + \mathbb{R}^k_+.$$

So, from the constraints of (WD) and Eq. 3.9, we have

$$\langle y^*, yz' - y'z\rangle + \langle z^*, w - w'\rangle \geq 0,$$

which contradicts Eq. 3.13.

Therefore $\left(x', \frac{y'}{z'}\right)$ is a weak minimizer of (FP). □

3.4 Wolfe type dual

We consider the Wolfe type dual (WD) associated the problem (FP).

$$\begin{aligned}
&\text{maximize} \quad \frac{y' + \langle z^*, w'\rangle\mathbf{1}_{\mathbb{R}^m}}{z'},\\
&\text{subject to,}\\
&\qquad \left\langle y^*, D_\uparrow(z'F)(x', y'z')(x - x') + D_\uparrow(-y'G)(x', -y'z')(x - x')\right\rangle\\
&\qquad + \left\langle z^*, D_\uparrow H(x', w')(x - x')\right\rangle \geq 0, \forall x \in A,\\
&\qquad x' \in A, y' \in F(x'), z' \in G(x'), y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+ \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1.
\end{aligned} \tag{WD}$$

A point $(x', y', z', w', y^*, z^*)$ which satisfies all the constraints of the problem (WD) is called a feasible point of (WD).

Definition 3.4 A feasible point $(x', y', z', w', y^*, z^*)$ of the problem (WD) is called a weak maximizer of (WD) if there exists no feasible point $(x, y, z, w, y_1^*, z_1^*)$ of (WD) such that

$$\frac{y + \langle z_1^*, w\rangle\mathbf{1}_{\mathbb{R}^m}}{z} - \frac{y' + \langle z^*, w'\rangle\mathbf{1}_{\mathbb{R}^m}}{z'} \in \operatorname{int}\left(\mathbb{R}^m_+\right).$$

Theorem 3.11 *(**Weak Duality**) Let A be a nonempty convex subset of X, $\overline{x}$ be an element of the feasible set S of the problem (FP) and $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (WD). Assume that $z'F$ is ρ_1-$\mathbb{R}^m_+$-convex with respect to*

$\mathbf{1}_{\mathbb{R}^m}$, $-y'G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, satisfying Eq. 3.9.
Then,

$$\frac{F(\overline{x})}{G(\overline{x})} - \frac{y' + \langle z^*, w'\rangle \mathbf{1}_{\mathbb{R}^m}}{z'} \subseteq \mathbb{R}^m \setminus -int\left(\mathbb{R}^m_+\right).$$

Theorem 3.12 *(**Strong Duality**) Let $(x', \frac{y'}{z'})$ be a weak minimizer of the problem (FP) and $w' \in H(x') \cap \left(-\mathbb{R}^k_+\right)$. Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $\langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1$, Eqs. 3.6 and 3.8 are satisfied at $(x', y', z', w', y^*, z^*)$. Then $(x', y', z', w', y^*, z^*)$ is a feasible solution of the problem (WD). Further, if the weak duality Theorem 3.11 between (FP) and (WD) holds, then $(x', y', z', w', y^*, z^*)$ is a weak maximizer of (WD).*

Theorem 3.13 *(**Converse Duality**) Let A be a nonempty convex subset of X, $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (WD) and $\langle z^*, w'\rangle \geq 0$. Assume that $z'F$ is ρ_1-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$, $-y'G$ is ρ_2-$\mathbb{R}^m_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^m}$ and H is ρ_3-$\mathbb{R}^k_+$-convex with respect to $\mathbf{1}_{\mathbb{R}^k}$, on A, satisfying Eq. 3.9. If x' is an element of the feasible set S of the problem (FP), then $(x', \frac{y'}{z'})$ is a weak minimizer of the problem (FP).*

3.5 Mixed type dual

We consider the mixed type dual (MD) associated the problem (FP).

$$\begin{aligned}
&\text{maximize} \quad \frac{y' + \langle z^*, w'\rangle \mathbf{1}_{\mathbb{R}^m}}{z'},\\
&\text{subject to,}\\
&\qquad \left\langle y^*, D_{\uparrow}(z'F)(x', y'z')(x - x') + D_{\uparrow}(-y'G)(x', -y'z')(x - x')\right\rangle\\
&\qquad + \left\langle z^*, D_{\uparrow}H(x', w')(x - x')\right\rangle \geq 0, \forall x \in A,\\
&\qquad \langle z^*, w'\rangle \geq 0,\\
&\qquad x' \in A, y' \in F(x'), z' \in G(x'), y^* \in \mathbb{R}^m_+, z^* \in \mathbb{R}^k_+ \text{ and } \langle y^*, \mathbf{1}_{\mathbb{R}^m}\rangle = 1.
\end{aligned} \tag{MD}$$

A point $(x', y', z', w', y^*, z^*)$ satisfying all the constraints of the problem (MD) is called a feasible point of (MD).

Definition 3.5 A feasible point $(x', y', z', w', y^*, z^*)$ of the problem (MD) is called a weak maximizer of (MD) if there exists no feasible point $(x, y, z, w, y_1^*, z_1^*)$ of (MD) such that

$$\frac{y + \langle z_1^*, w\rangle \mathbf{1}_{\mathbb{R}^m}}{z} - \frac{y' + \langle z^*, w'\rangle \mathbf{1}_{\mathbb{R}^m}}{z'} \in \text{int}\left(\mathbb{R}^m_+\right).$$

Theorem 3.14 *(**Weak Duality**) Let A be a nonempty convex subset of X, $\overline{x}$ be an element of the feasible set S of the problem (FP) and $(x', y', z', w', y^*, z^*)$ be a feasible point of the problem (MD). Assume that $z'F$ is $\rho_1 - \mathbb{R}^m_+$-convex with respect*

to $\mathbf{1}_{\mathbb{R}^m}$, $-y'G$ *is* $\rho_2 - \mathbb{R}^m_+$*-convex with respect to* $\mathbf{1}_{\mathbb{R}^m}$ *and* H *is* ρ_3-$\mathbb{R}^k_+$*-convex with respect to* $\mathbf{1}_{\mathbb{R}^k}$*, on* A*, satisfying Eq. 3.9.*
Then,

$$\frac{F(\overline{x})}{G(\overline{x})} - \frac{y' + \langle z^*, w' \rangle \mathbf{1}_{\mathbb{R}^m}}{z'} \subseteq \mathbb{R}^m \setminus -int\left(\mathbb{R}^m_+\right).$$

Theorem 3.15 *(**Strong Duality**) Let* $\left(x', \frac{y'}{z'}\right)$ *be a weak minimizer of the problem (FP) and* $w' \in H(x') \cap \left(-\mathbb{R}^k_+\right)$*. Assume that for some* $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$*, with* $\langle y^*, \mathbf{1}_{\mathbb{R}^m} \rangle = 1$*, Eqs. 3.6 and 3.8 are satisfied at* $(x', y', z', w', y^*, z^*)$*. Then* $(x', y', z', w', y^*, z^*)$ *is a feasible solution of the problem (MD). Further, if the weak duality Theorem 3.14 between (FP) and (MD) holds, then* $(x', y', z', w', y^*, z^*)$ *is a weak maximizer of (MD).*

Theorem 3.16 *(**Converse Duality**) Let* A *be a nonempty convex subset of* X *and* $(x', y', z', w', y^*, z^*)$ *be a feasible point of the problem (MD). Assume that* $z'F$ *is* ρ_1-$\mathbb{R}^m_+$*-convex with respect to* $\mathbf{1}_{\mathbb{R}^m}$*,* $-y'G$ *is* ρ_2-$\mathbb{R}^m_+$*-convex with respect to* $\mathbf{1}_{\mathbb{R}^m}$ *and* H *is* ρ_3-$\mathbb{R}^k_+$*-convex with respect to* $\mathbf{1}_{\mathbb{R}^k}$*, on* A*, satisfying Eq. 3.9. If* x' *is an element of the feasible set* S *of the problem (FP), then* $\left(x', \frac{y'}{z'}\right)$ *is a weak minimizer of the problem (FP).*

4 Concluding remarks

In this paper, we establish the sufficient KKT conditions for the set-valued fractional programming problem (FP) via the contingent epiderivative. We assume generalized cone convexity assumptions on the objective and constraint set-valued maps. We also introduce the duals of parametric (PD), Mond-Weir (MWD), Wolfe (WD) and mixed (MD) types and prove the corresponding weak, strong and converse duality theorems under cone convexity assumptions.

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