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A batch arrival multi phase queueing system with random feedback in service and single vacation policy

A. Badamchi Zadeh

Abstract We study a batch arrival Poisson input, k phases heterogeneous services with randomly feedback in services. The first phase of service is essential for all customers, but with probability γ_1 a tagged customer chose second phase, with probability α_1 feedback to tail of original queue or with probability $\beta_1 = 1 - \gamma_1 - \alpha_1$ leave the system. Also, after completion of the second phase, with probability γ_2 the customer leads to the third service, or with probability α_2 feedback to tail of original queue, or leaves the system with probability $\beta_2 = 1 - \gamma_2 - \alpha_2$. Like that, in kth phase feedback with probability α_k or leaves the system with probability $1 - \alpha_k$. After completion of each phases, the server either goes for a vacation with probability $\theta(0 \leq \theta \leq 1)$, or may continue to serve the next unit with probability $1 - \theta$, if any. Otherwise, it remains in the system until a customer arrives, which is single vacation policy. We assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. In this paper we derive the steady- state equations, PGF's of the system, and measures of sysytem.

Keywords M/G/1 queue · Multi phases of heterogeneous service · Feedback · Admissibility restricted · Single vacation · Mean queue size · Mean response time

1 Introduction

In modern Qeueing theory it is often assumed that a customer's service may have k phases and also be incredible, so he/she/it must repeated the service. This events was studied for the first time in Shahkar and Badamchizah [17]. Moreover vacation and restricted admissibility are usual events that occur in queueing systems.

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For example, in communication networks a packet which is transmitted from the source is returned to the transmitter until it is successfully received at destination and in production system each defective item is sent back to the server for rework by the quality control desk. Unlike most of the other single server queues in which the server provides only one kind of service, the server provides k types of heterogeneous services and a customer may choose either type of service under single vacation policy. In the analysis of this model, steady state results for the probability generating functions of the queue size and the system size, the expected number of customers and the expected waiting time of customer in the queue and the system are obtained.

In general, queueing theory is an important subject in computers and operations research. Buffers /queues are used to store information that can not be transmitted instantaneously. Classical queueing systems assume that customers are in continuous contact with the server, that is, they can see whether or not the server is busy and thus commence service immediately whenever the service station becomes idle. However, queueing systems with elapse differ in that customers do not know the server state and consequently must verify if the server is idle from time to time. It is natural for telephone callers to break contact when the line is occupied and re-apply for connection later as a vacation. Madan [10], Madan and Choudhury [14] studied the $M/G/1$ queueing system with two phases of heterogeneous services such that the first phase of service follows by the second phase of service, under a Bernoulli vacationr schedule. An $M^X/G/1$ queue system with an additional service channel were analyzed by Choudhury [8]. Furthermore, a similar work can be found in Artalejo [2] and Ke [9].

Madan and Choudhury [13] proposed an $M^X/G/1$ queueing system, assuming batch arrivals with restricted admissibility of arriving batches and Bernoulli schedule server vacation. Earlier, Madan and Abu-Dayyeh [11, 12] dealt with this type of model and studied some aspects of batch arrivals Bernoulli vacation models with restricted admissibility, where all arriving batches are not allowed into the system at all time. Furtheremore in Chaudhry [5] and Alnowibet [1] the $M/G/1$ queueing systems with optional service are analyzed.

Recently in Badamchizadeh and Shahkar [3], Badamchizadeh [4], Salehirad and Badamchizadeh [16] has extended this models.

In many applications such as hospital services, production systems, bank services, computer and communication networks, there is many phases of services such that after completion of services, customers may leave the system or may immediately go for the next phase of service, or the services must be repeated.

In this paper the general case is study such that there are k phases of services, vacation and restricted admissibility. Unlike the usual batch arrival queueing systems, the policy of restricted admissibility of batches in which not all batches are allowed to join the system at all times, has been assumed in this model. In other words, an arriving batch will be allowed to join the system during the server's non-vacation and vacation periods with constant probabilities. Also in this system for overhauling or maintenance purposes of the system, or serving other customers, the server being fatigue or for other reasons not mentioned here, the server may go to vacation.

In this paper the aim is the analyze of steady state conditon in a single server queue with a batch arrival Poisson input, k phases of heterogeneous service, restricted admissibility of batches, randomly feedback in services, and Bernoulli vacation for server. In Section 2 we deal with the mathematical model and definitions. Steady-State conditions and generating functions are discussed in Section 3. Mean queue size and mean response time are computed in Section 4, where in Section 5 some special cases are investigated. Finaly wiyh some numerical method the validity of model has been examined.

2 Mathematical model and definitions

We consider a queueing system such that:

i)   Customers arrive at the system in a compound Poisson process with a batch of random size X and mean rate $\lambda > 0$. Size of succesive arriving batches are $X_1, X_2, \ldots$, where i.i.d random variables, distributed with probability mass function(p.m.f) $d_n = Prob[X_i = n]; n \geqslant 1$, probability genrerating function(PGF) $d(z) = E[z_1^X]$. The first and second moments $d^{(1)} = E[X]$ and $d^{(2)} = E[X^2]$; respectively, are assumed to be finite.

ii) The server provides k phases of heterogenous service in succession. The service discipline is assumed to be on the basis of first come, first serve(FCFS). The first phase of service is essential for all customers, but as soon as the essential service is completed, a tagged customer moves for second phase with probability γ_1, feedback to tail of original queue with probability α_1 or leaves the system with probability $\beta_1 = 1 - \alpha_1 - \gamma_1$.

  Similarly after completion of the second phase with probability γ_2 the customer leads to the third service or with probability α_2 feedback to the tail of original queue, or leaves the system with probability $\beta_2 = 1 - \alpha_2 - \gamma_2$. Like that, in k'th phase he/she/it feedback with probability α_k or leaves the system with probability $1 - \alpha_k$.

  The service times for k phases are independent random variables, denoted by $B_i, i = 1, 2, \ldots, k$. Their Laplace-Stieltjes transform (LST)are $B_i^*(s), i = 1, 2, \ldots, k$ where we assume they have finite moments $E(B_i^l)$ for $l \geqq 1$ and $i = 1, 2, \ldots k$.

iii) There is a policy restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let $a(0 \leqq a \leqq 1)$ and $b(0 \leqq b \leqq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of the server's non-vacation period(service time) and vacation period respectively.

iv) As soon as each phase of a customer's services is completed, the server may go for a vacation of random length V with probability $\theta(0 \leq \theta \leq 1)$ or it may continue to serve the next customer, if any, with probability $(1 - \theta)$, otherwise it remains in the system and waits for a new arrival. We denote $V(x), V^*(s)$ and $E(V^l)$ for distribution function (DF), LST and $l'th$ finite moment of V, respectively, where $l \geq 1$.

v)   The random variables B_i and V are all independent variables.

Definition 2.1 The modified service time or the time required by a customer to complete the service cycle is given by:

$$B = \begin{cases} B_1 + \sum_{i=2}^{k} \gamma_1\gamma_2 \ldots \gamma_{i-1} B_i & \text{with probability}(1-\theta) \\ B_1 + \sum_{i=2}^{k} \gamma_1\gamma_2 \ldots \gamma_{i-1} B_i + V & \text{with probability } \theta \end{cases} \quad (2.1)$$

then the LST $B^*(s)$ of B is given by

$$B^*(s) = \theta B_1^*(s) \prod_{i=2}^{k} \gamma_1\gamma_2 \ldots \gamma_{i-1} B_i^*(s) V^*(s) + (1-\theta) B_1^*(s) \prod_{i=2}^{k} \gamma_1\gamma_2 \ldots \gamma_{i-1} B_i^*(s)$$

$$(2.2)$$

Further, for $i = 1, 2, \ldots, k$ we assume that $B_i(0) = 0$, $B_i(\infty) = 1$ and $B_i(x)$ are continuous at $x = 0$, so that

$$\mu_i(x)dx = \frac{dB_i(x)}{1 - B_i(x)} \quad (2.3)$$

is the first order differential equation(hazard rate functions) of B_i.

Also, $V(0) = 0$, $V(\infty) = 1$ and $V(x)$ is continuous at $x = 0$, so that

$$\nu(x)dx = \frac{dV(x)}{1 - V(x)} \quad (2.4)$$

is hazard rate function of V.

Definition 2.2 Let $N_Q(t)$ be the queue size(including one being served, if any) at time 't' and for $i = 1, 2, \ldots, k$ the supplementary variables are defined as:

$$B_i^0(t) \equiv \text{ the elapsed } kth \text{ phase of service at time 't'}$$
$$V^0(t) \equiv \text{ the elapsed vacation time at time 't'}$$

Now let us introduce the following random variables:

$$Y(t) = \begin{cases} 0 & \text{if the server is idle at time 't'}, \\ i & \text{if the server is busy with } ith \text{ phase of service at time 't'}, \quad i = 1, 2, \ldots, k \\ k+1 & \text{if the server is on vacation at time 't'}. \end{cases}$$

$$(2.5)$$

From this we have a bivariate Markov process $\{N_Q(t), L(t)\}$ where $L(t) = 0$ if $Y(t) = 0$; $L(t) = B_i^0(t)$ if $Y(t) = i$ for $i = 1, 2, \ldots, k$ and $L(t) = V^0(t)$ if $Y(t) = k+1$. Now for $i = 1, 2, \ldots, k$ the following probabilities are define as:

$$Q_n(x, t) = Prob[N_Q(t) = n, L(t) = V^0(t); x < V^0(t) \leqslant x + dx] \quad x > 0, \quad n \geqslant 0$$
$$P_{i,n}(x, t) = Prob[N_Q(t) = n, L(t) = B_i^0(t); x < B_i^0(t) \leqslant x + dx] \quad x > 0, \quad n \geqslant 1$$

and

$$R_0(t) = Prob[N_Q(t) = 0, L(t) = 0]$$

Now the analysis of the limiting behaviour of this queueing process at a random epoch can be performed with the help of Kolmogorov forward equations, provided the following limits exist and are independent of initial state :

$$R_0 = \lim_{t\to\infty} R_0(t)$$
$$P_{i,n}(x)dx = \lim_{t\to\infty} P_{i,n}(x,t)dx \quad i = 1, 2, \ldots, k \quad x > 0, \, n \geqslant 1$$
$$Q_n(x)dx = \lim_{t\longrightarrow\infty} Q_n(x,t)dx \quad x > 0, \quad n \geqslant 0$$

Definition 2.3 For $i = 1, 2, \ldots, k$ the PGF of these probabilities are defined as follow:

$$P_i(x, z) = \sum_{n=1}^{\infty} z^n P_{i,n}(x) \quad |z| \leqslant 1, \quad x > 0 \tag{2.6}$$

$$P_i(0, z) = \sum_{n=1}^{\infty} z^n P_{i,n}(0) \quad |z| \leqslant 1 \tag{2.7}$$

Also

$$Q(x, z) = \sum_{n=0}^{\infty} z^n Q_n(x) \quad |z| \leqslant 1, \quad x > 0 \tag{2.8}$$

$$Q(0, z) = \sum_{n=0}^{\infty} z^n Q_n(0) \tag{2.9}$$

3 Steady-state probability generating function

From Kolmogorov forward equations , for $i = 1, 2, \ldots, k$ the steady-state conditions can be written as follow

$$\frac{d}{dx}P_{i,n}(x) + [\lambda + \mu_i(x)]P_{i,n}(x) = \lambda(1-a)P_{i,n}(x) + \lambda a \sum_{k=1}^{n} d_k P_{i,n-k}(x) \quad n \geq 2, \quad x > 0 \tag{3.10}$$

and for $n = 1$

$$\frac{d}{dx}P_{i,1}(x) + [\lambda + \mu_i(x)]P_{i,1}(x) = \lambda(1 - a)P_{i,1}(x), \quad x > 0 \tag{3.11}$$

$$\frac{d}{dx}Q_n(x) + [\lambda + \nu(x)]Q_n(x) = \lambda(1-b)Q_n(x) + \lambda b \sum_{k=1}^{n} d_k Q_{n-k}(x) \quad n \geq 1, \quad x > 0 \tag{3.12}$$

$$\frac{d}{dx}Q_0(x) + [\lambda + \nu(x)]Q_0(x) = \lambda(1 - b)Q_0(x) \tag{3.13}$$

also

$$\lambda R_0 = \lambda(1 - a)R_0 + (1 - \theta)\sum_{i=1}^{k} \beta_i \int_0^{+\infty} \mu_i(x)P_{i,1}(x)dx + (1 - \theta)(1 - \alpha_k)$$

$$\times \int_0^{+\infty} \mu_k(x)P_{k,1}(x)dx + \int_0^{+\infty} \nu(x)Q_0(x)dx \tag{3.14}$$

For $n \geq 1$ these sets of equations are to be solved under the following boundary conditions at $x = 0$

$$P_{1,n}(0) = \lambda a d_n R_0 + \sum_{i=1}^{k} \alpha_i \int_0^{+\infty} \mu_i(x) P_{i,n}(x) dx + (1-\theta) \sum_{i=1}^{k-1} \beta_i \int_0^{+\infty} \mu_i(x) P_{i,n+1}(x) dx$$

$$+ (1-\theta)(1-\alpha_k) \int_0^{+\infty} \mu_k(x) P_{k,n+1}(x) dx + \int_0^{+\infty} \nu(x) Q_n(x) dx \qquad (3.15)$$

and

$$P_{i,n}(0) = \gamma_{i-1} \int_0^{+\infty} \mu_i(x) P_{i-1,n}(x) dx, \quad n \geqslant 1 \qquad (3.16)$$

also for $n \geq 0$

$$Q_n(0) = \theta \sum_{i=1}^{k-1} \beta_i \int_0^{+\infty} \mu_i(x) P_{i,n+1}(x) dx + \theta(1-\alpha_k) \int_0^{+\infty} \mu_k(x) P_{k,n+1}(x) dx$$

$$(3.17)$$

Finally the normalizing condition is

$$R_0 + \sum_{i=1}^{k} \sum_{n=1}^{\infty} \int_o^{+\infty} P_{i,n}(x) dx + \sum_{n=0}^{\infty} \int_0^{+\infty} Q_n(x) dx = 1 \qquad (3.18)$$

Theorem 3.1 *For $i = 1, 2, \ldots, k$ from Eqs. 3.10 and 3.11 we have*

$$P_i(x, z) = P_i(0, z)[1 - B_i(x)] e^{-\lambda a(1-d(z))x} \quad x > 0 \qquad (3.19)$$

and from Eqs. 3.12, 3.13

$$Q(x, z) = Q(0, z)[1 - V(x)] e^{-\lambda b(1-d(z))x} \quad x > 0 \qquad (3.20)$$

Lemma 3.2 *For $i = 1, 2, \ldots, k$ let*

$$B_i^*(\lambda a(1 - d(z))) = \int_0^{+\infty} e^{-\lambda a(1-d(z))x} dB_i(x) \qquad (3.21)$$

$$V^*(\lambda b(1 - d(z))) = \int_0^{+\infty} e^{-\lambda b(1-d(z))x} dV(x) \qquad (3.22)$$

be the z-transform of B_i and V respectively, then

$$I) z P_1(0, z) = \lambda a d(z) R_0 z + z P_1(0, z) \sum_{i=1}^{k} \alpha_i B_i^*(\lambda a(1-d(z))) + (1-\theta) P_1(0, z) \sum_{i=1}^{k} \beta_i B_i^*(\lambda a(1-d(z)))$$

$$+ (1-\theta)(1-\alpha_k) B_k^*(\lambda a(1 - d(z))) + z Q(0, z) V^*(\lambda b(1 - d(z))) - \lambda a R_0 z \qquad (3.23)$$

and for $i = 1, 2, \ldots, k$

$$II) P_i(0, z) = \Gamma_{i-1} A_{i-1}^*(\lambda a(1 - d(z))) P_1(0, z) \qquad (3.24)$$

where

$$\Gamma_j = \gamma_1 \gamma_2 \cdots \gamma_j$$

and

$$A_j^*(\lambda a(1 - d(z))) = \prod_{i=1}^{j} B_i^* (\lambda a(1 - d(z))) \tag{3.25}$$

$$III) zQ(0, z) = \theta \sum_{i=1}^{k-1} \beta_i B_i^* (\lambda a(1 - d(z))) \, P_i(0, z) + \theta(1 - \alpha_k) B_k^* (\lambda a(1 - d(z))) \, P_k(0, z) \tag{3.26}$$

Proof   I):   By multiplying Eq. 3.15 in z^n and summation from $n = 1$ to $+\infty$, adding Eq. 3.14 to result and using Eqs. 3.21 and 3.22.

II):   By multiplying Eq. 3.16 in z^n and summation from $n = 1$ to $+\infty$.

III):   By multiplying Eq. 3.17 in z^n and summation from $n = 1$ to $+\infty$.

□

In the rest of this section for simplifying the formulas we omit $(\lambda a(1 - d(z)))$ from B_i^* and $(\lambda b(1 - d(z)))$ from V^*.

Corollary 3.3 *From* Eq. 3.23

$$P_1(0, z) = \frac{z\lambda a R_0(d(z) - 1)}{\left[\beta_1 B_1^* - \sum_{i=2}^{k-1} \beta_i \Gamma_{i-1} A_i^* + (1 - \alpha_k)\Gamma_{k-1} A_k^* \right] [1 - \theta(1 - V^*)] - z \left(1 - \alpha_1 B_1^* - \sum_{i=1}^{k} \alpha_i \Gamma_{i-1} A_i^* \right)} \tag{3.27}$$

Remark 3.4 We set $G(z) = \left[1 - \theta \left(1 - V^*(\lambda b(1 - d(z))) \right) \right]$. In the systems with vacation, this function has a main role. Also if $H(z) = [\beta_1 B_1^* + \sum_{i=2}^{k-1} \beta_i \Gamma_{i-1} A_i^* + (1 - \alpha_k)\Gamma_{k-1} A_k^*]$ and $K(z) = 1 - \alpha_1 B_1^* - \sum_{i=2}^{k} \alpha_i \Gamma_{i-1} A_i^*$, then

$$P_1(0, z) = \frac{z\lambda a R_0 \, (d(z) - 1))}{z K(z) - H(z)G(z)} \tag{3.28}$$

and from Eqs. 3.24 and 3.25 we have respectively

$$P_i(0, z) = \frac{z\lambda a R_0 \, (d(z) - 1)) \, \Gamma_{i-1} A_i^*}{z K(z) - H(z)G(z)} \tag{3.29}$$

and

$$Q(0, z) = \frac{z\lambda a R_0 \, (d(z) - 1)) \, H(z)}{z K(z) - H(z)G(z)} \tag{3.30}$$

Corollary 3.5 *Since*

$$P_1(z) = \int_0^{+\infty} P_1(x, z)dx$$

hence from Eq. 3.19 *for* $i = 1$, *using* Eq. 3.28 *and integration by part we have*

$$P_1(z) = \frac{z R_0 (1 - B_1^*)}{H(z)G(z) - z K(z)} \tag{3.31}$$

Similarly from Eqs. 3.19 *for* $i = 2, \ldots, k$, 3.29 *and* 3.30 *we have*

$$P_i(z) = \int_0^{+\infty} P_i(x, z)dx = \int_0^{+\infty} P_i(0, z)[1 - B_i(x)]e^{-\lambda\alpha(1-d(z))x}dx$$

$$= \frac{zR_0(1 - B_i^*)\Gamma_{i-1}A_{i-1}^*}{H(z)G(z) - zK(z)} \tag{3.32}$$

and

$$Q(z) = \frac{\theta R_0(1 - V^*)H(z)}{H(z)G(z) - zK(z)} \tag{3.33}$$

Remark 3.6 The unknown constant R_0 can be determined by using normalizing condition (3.18) which is

$$R_0 + P_1(1) + \sum_{i=2}^{k} P_2(1) + Q(1) = 1 \tag{3.34}$$

From Eqs. 3.31, 3.32 and 3.33 we have

$$R_0[1 + P_1(z) + \sum_{i=2}^{k} P_i(z) + Q(z)] = R_0\left[\frac{(1 - z)H(z)}{H(z)G(z) - zK(z)}\right] \tag{3.35}$$

R_0 is the steady-state probability that the server is idle but available in the system, hence $1 - R_0 = \rho < 1$ is the utilization factor of this system and the stability condition that under which the steady state solution exists.

The values of H(z), G(z), K(z) and their derivatives have main roles in rest of article, which are in next lemma.

Lemma 3.7 *According to lemma (3.8) from* [17] *we have:*

$$\frac{d}{dz} A_i^* (\lambda a(1 - d(z))) \mid_{z=1} = \lambda a E(X) \sum_{l=1}^{i} E(B_l) \tag{3.36}$$

$$\frac{d^2}{dz^2} A_i^* (\lambda a(1 - d(z))) \mid_{z=1} = [\lambda a E(X)]^2 \left\{\left[\sum_{l=1}^{i} E(B_l)\right]^2 + \sum_{l=1}^{i} var(B_l)\right\} + \lambda a E(X(X-1)) \sum_{l=1}^{j} E(B_l) \tag{3.37}$$

Now from Eq. 3.35

$$\rho = 1 - R_0 = \frac{z[H(z) - K(z)] + H(z)[G(z) - 1]}{(z - 1)H(z)} \mid_{z=1}$$

From remark (3.4)

$$H(1) - K(1) = 0, \qquad G(1) = 1 \tag{3.38}$$

hence ρ is $\dfrac{0}{0}$, therfore by L'Hopital rule we have

$$\rho = \frac{H'(1) - K'(1) + H(1)G'(1)}{H(1)} = \frac{\left[(\alpha_1 + \beta_1)(B_1^*)' + \sum\limits_{i=2}^{k-1} (\alpha_i + \beta_i)\Gamma_{i-1} \left(A_i^* \right)' + \left(A_k^* \right)' \right]}{\beta_1 + \sum\limits_{i=2}^{k-1} \beta_i \Gamma_{i-1} + (1 - \alpha_k)\Gamma_{k-1}} + \theta(V^*)'$$

$$(3.39)$$

4 Mean queue size and other measures of system

From Eqs. 3.31, 3.32, 3.33 and 3.35 the PGF of the queue size distribution at a random epoch is

$$
\begin{aligned}
P_Q(z) &= R_0 + P_1(z) + P_2(z) + zQ(z) \\
&= (1 - \rho)\frac{(z - 1)H(z)}{zK(z) - H(z)G(z)}
\end{aligned}
$$

$$(4.40)$$

Let L_Q be the mean number of customers in the queue (i.e mean queue size), then we have

$$L_Q = \frac{dP_Q(z)}{dz} \Big|_{z=1}$$

$$(4.41)$$

Let $P_Q(z) = (1-\rho)\dfrac{f(z)}{g(z)}$ where $f(z) = (z-1)H(z)$ and $g(z) = zK(z) - H(z)G(z)$.

$f(z)$ and $g(z)$ are zero at $z = 1$, hence L_Q is the form $\dfrac{0}{0}$. So using L'Hopital rule we have

$$L_Q = (1 - \rho)\frac{f''(1)g'(1) - f'(1)g''(1)}{2[g'(1)]^2}$$

$$(4.42)$$

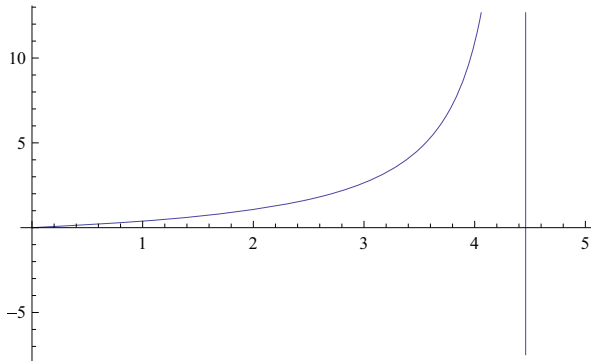By computing $f'(1)$, $f''(1)$, $g'(1)$ and $g''(1)$ we have

$$
\begin{aligned}
L_Q &= \frac{g'(1)}{f'(1)} \frac{f''(1)g'(1) - g''(1)f'(1)}{2[g'(1)]^2} = \frac{f''(1)}{2f'(1)} - \frac{g''(1)}{2g'(1)} \\
&= \frac{H'(1)}{H(1)} - \frac{2K'(1) + K''(1) - H''(1) - 2H'(1)G'(1) - H(1)G''(1)}{2[K(1) + K'(1) - H'(1) - H(1)G'(1)]}
\end{aligned}
$$

$$(4.43)$$

Now for computing the mean waiting-time of a test customer in this model, by using Little's formula, this measure of system is equal

$$W_Q = \frac{L_Q}{\lambda_X}$$

$$(4.44)$$

Table 1 Values of parameters

α_1	β_1	α_2	β_2	γ_1	a	b	θ	μ_1	μ_2	p_1	p_2	v	q	d
.5	.2	.7	.3	.3	.2	.3	.1	5	4	1	2	2	3	.9

Fig. 1 L_Q vis-a-vis λ

where, following the admissibility assumption of our model, λ_X the actual arrival rate of batches is given by

$\lambda_X = \lambda\alpha$ (proportion of non-vacation time) $+ \lambda\beta$ (proportion of vacation time)

But from Remark 3.2 we have

$$\text{the proportion of vacation time } = Q(1) = \theta\lambda b E(X)E(V)$$

Hence the proportion of non-vacation time including the first and second service times and idle time, is equal $1 - \theta\lambda b E(X)E(V)$. Consequently

$$\lambda_X = \lambda a + (b - a)\theta\lambda^2 b E(X)E(V) \tag{4.45}$$

4.1 Particular case

If $\alpha_i \to 0$ for each $i = 1, 2, \ldots, k$, then $K(z) = 1$, and also $H(1) = 1$, therefore $\rho = H'(1) - G'(1)$ and

$$L_Q = H'(1) + \frac{H''(1) + 2H'(1)G'(1) + G''(1)}{2(1 - H'(1) - G'(1))} \tag{4.46}$$

5 Special cases and numerical results

Analyzing a queueing system via actual cases are very important and useful way to confirm the models. In this section we chose known distributions for service times and vacation time, so with this, and by some numerical approches the validity of

Table 2 Values of L_Q with respect λ

λ	1	2	3	4
L_Q	.38	1.7	2.64	10.9

Fig. 2 L_Q vis-a-vis θ for $\lambda = 3$
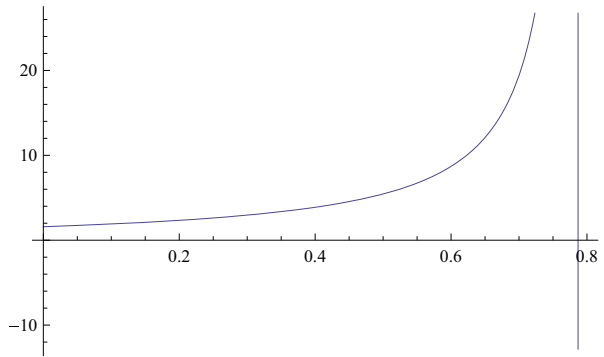
the system are examained. Also this approch explains that our model can function reasonably well for certain practical problems. For simple computations we assume $k = 2$.

Case 1 For each $i = 1, 2$, let the distribution of services time be p_i-Erlang as follows:

$$dB_i(x) = \frac{(p_i\mu_i)^{\mu_i} x^{p_i-1} e^{-p_i\mu_i x}}{(p_i - 1)!} dx \quad x > 0, \, p_i \geq 1$$

hence

$$B_i^*(\lambda - \lambda d(z)) = \frac{(p_i\mu_i)^{p_i}}{[\lambda \, (d(z) - 1) + p_i\mu_i)]^{p_i}}$$

so $E(B_i) = \frac{1}{\mu_i}$ and $E\left(B_i^2\right) = \frac{p_i+1}{p_i\mu_i^2}$.

Also we assume the distribution of vacation time be q-Erlang

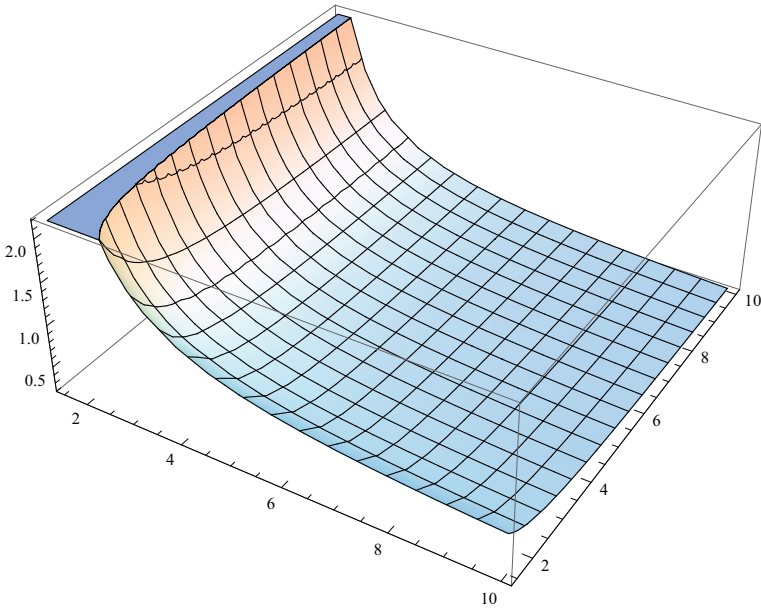$$dV(x) = \frac{(qv)^v x^{q-1} e^{-qvx}}{(q - 1)!} dx \quad x > 0, q \geq 1$$

hence

$$V^*(\lambda - \lambda d(z)) = \frac{(qv)^q}{[\lambda(d(z) - 1) + qv]^q}$$

so $E(V) = \frac{1}{v}$ and $E(V^2) = \frac{q+1}{qv^2}$. If we chose geometric distribution for batch size, i.e $d_n = d(1 - d)^{n-1}, 0 < d < 1$, then $E(X) = \frac{1}{d}$ and $E(X(X - 1)) = \frac{2(1-d)}{d^2}$.

Now for numerical result we assume the following values for parameters such that the steady state condition($\rho < 1$) obtained

Table 3 values of L_Q with respect θ

θ	0	.1	.2	.3	.4	.5	.6	.7	.75
L_Q	1.58	1.9	2.34	2.95	3.87	5.44	8.69	19.4	46.46

Fig. 3 *L vis-a-vis μ_1 and μ_2*

Now by using above values and Eq. 3.39, the steady state condition is $\rho = .227\lambda < 1$, so $\lambda < 4.4$. By using Eq. 4.43

$$L_Q = .06\lambda + \frac{.144\lambda + .004\lambda^2}{.58 - .13\lambda}$$

The graph of model is in Fig. 1.

Some values of L_Q against λ are in Table 2.

Now in this case we analyze L_Q with respect to θ. Using values of Table 1, and $\lambda = 3$ the steady-state condition is $\rho = .18 + .498\theta < 1$, hence $\theta < .76$. Also

$$L_Q = .18 + \frac{.4782 + .35\theta}{.34 - .445\theta}$$

Figure 2 shows the graph of model. Also in Table 2 some values of L_Q against θ are computed. Untile $\theta = .6$, the system is tolerable, but after $\theta = .7$ the system blows up.

Table 4 values of parameters

α_1	β_1	α_2	β_2	γ_1	a	b	θ	v	d
.3	.4	.6	.4	.3	.3	.2	.3	3	.6

Table 5 values of L_Q with respect μ_i

$\mu_1 = \mu_2$	1.5	2	3	5	10
L_Q	9.54	2.7	1.4	.54	.23

Case 2 In this case we assume for $i = 1, 2$ the distribution of service times and vacation time are exponential as follow

$$dB_i(x) = \mu_i e^{-\mu_i x} dx, \quad E(B_i) = \frac{1}{\mu_i} \quad E\left(B_i^2\right) = \frac{2}{\mu_i^2}$$

and

$$dV(x) = v e^{-vx} dx, \quad E(V) = \frac{1}{v} \quad E\left(V^2\right) = \frac{2}{v^2}$$

With geometric distribution for batch size according to Case 1 and following values for parameters in Table 3 the steady state condition with respect μ_1 and μ_2 is $\rho = \frac{.96}{\mu_1} + \frac{.29}{\mu_2} + .03 < 1$. Also

$$L_Q = \frac{.5}{\mu_1} + \frac{.11}{\mu_2} + \frac{\frac{1.12}{\mu_1} + \frac{.36}{\mu_2} + \frac{.32}{\mu_1^2} + .06\left[\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)^2 + \left(\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\right)\right]}{1 - \frac{1}{\mu_1} + \frac{.3}{\mu_2}}$$

and the graph of model is in Fig. 3. According to this curve and values of Table 4, L decresses with respet μ_i (Table 5), and according to values of tables after $\mu_1 = \mu_2 = 2$ the system is stable. Also since $L_Q = -6.19$ when $\mu_1 = \mu_2 = 1$, then steady state condition begins after $\mu_1 = \mu_2 = 1.28$, but after $\mu_1 = \mu_2 = 2$, the system is stable.

6 Concluding remarks

In this paper we have studied a batch arrival k phases queueing system with randomly feedback, admissibility restricted and server's vacation which generalized classical $M/G/1$ queue. An application of this model can be found in mobile network where the messages are in batch form, the service may have many phases such that services may be unaccepted and customer may repeat the services. Also, because of admissibility restriction in service or system, all batches don't enter in service. Our investigations are concerned with not only queue size distribution but also waiting time distribution. This model extends the systems for example in Artalejo [2], Badamchizadeh and Shahkar [3], Badamchizadeh [4], Choudhury [7], Madan and Choudhury [13, 14]. A practical generalization for this system is to consider optional services and optional vacation.

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