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A survey on inventory models with positive service time

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Abstract A detailed review of inventory models involving positive service time is given. These include classical and retrial cases. Also contributions to production inventory with service time is indicated towards the end. In addition directions for future work are indicated.

Keywords Inventory with positive service time · Classical inventory · Retrial inventory with/without production

1 Introduction

Inventory models were studied in detail in *Churchman et al.* [9], *Hadley and Whitin* [13], *Naddor* [35], and more recently in *Sahin* [42]. In the first three a large number of deterministic models are discussed whereas in the book by *Sahin*, stochastic models are highlighted. We call these models and problems as *Classical type*, since in all these the amount of time required to serve the items is negligible. In contrast, most of the real life situations need positive amount of time to serve the inventory. It may appear that there is no difference between a queue and an inventory with positive service time. However this is not the case. In a queue we do not speak about the availability of resources for service—if the customers are available and server is ready to serve then the service starts. Nevertheless, this is not the case in inventory with positive service time. Server may be ready to serve and there may be customers waiting to get service. However, inventory may not be available on stock. Thus a queue

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of customers builds up. Even in the case when lead time is zero, the above problem can very well arise. Needless to say that in the case of positive lead time the server may remain idle even when customers are waiting for want of items in inventory.

This review contains an overview of the work done in inventory with positive service time. We have included all those papers that we felt significant in the following sense: the contributions are novel and/new techniques are employed for the analysis. Further we have also looked at the applicability of the models described in those papers. It may be recalled that literature on production inventory with positive service time is scarce. To the best of our knowledge there are just two papers in this direction. Though in the retrial case again situation is far from satisfactory, yet there are a few contributions. The review of work done so far in inventory with positive service time is given in the next section.

2 Inventory with positive service time—a review

Several problems in inventory with positive service time have been extensively investigated in the classical set up. However, those involving retrial of unsatisfied customers have received much less attention. In the following we provide a survey of the investigations that have taken place in classical case first, followed by those involving retrial of unsatisfied customers.

2.1 Classsical models

Inventory with positive service time is first investigated by *Berman et al.* [2] where demands and service formed two distinct deterministic processes. They consider an inventory management system at a service facility which uses one item of inventory for each service provided. According to the assumptions made queue can be formed only during stock out. They determine optimal order quantity Q that minimizes the total cost rate using dynamic programing technique. Examples of the type of service facilities considered in this paper include installing bumpers at car service stations, hospitals where units of blood are necessary for surgery etc,.

Parthasarathy and Vijayalakshmi [40] are the first to examine the transient behavior of inventory problems. It may appear that this paper deals with inventory involving positive service time. However it is concerned only with production inventory of $(S-1, S)$ type. A queue is formed only when system runs out of stock. Thus when item is available, it is provided immediately on demand; else there is a processing time. In this connection see also *Krishnamoorthy et al.* [21] and *Deepak et al.* [11].

Berman and Kim [3] consider stochastic inventory models with positive service time wherein the average cost is minimized using dynamic programing technique. More specifically, they analyze a problem in a stochastic environment where customers arrive at service facilities according to a Poisson process. The service times are exponentially distributed with mean inter-arrival time assumed to be larger than the mean service time. The main result of their work is that under both the discounted cost case and average cost case, the optimal policy of both the finite and infinite time horizon problem is a threshold ordering policy. They derive the optimal policy under the condition that the order quantity is known.

Berman and Sapna [4] study an inventory control problem at a service facility which requires one item of the inventory. They assume Poisson arrivals of demands, arbitrarily distributed service times and zero lead time. They analyze the system with finite waiting room and come up with an optimal ordering quantity that minimizes the long-run expected cost per unit time under a specified cost structure. Essentially the Markov renewal theoretic approach is employed there. Since the state space is assumed to be finite the resulting Markov renewal equation has unique solution.

Berman and Sapna [5] consider the problem of optimally controlling service rates for an inventory system of service facilities. They analyze a lost sales inventory system at a service facility with positive lead time and Poisson demands for given maximum inventory and reorder levels. Also they discuss an inventory system where speeding up or slowing down the service rate is possible. For a system with finite waiting space, they find the service rates to be employed at each instant of time so that the long-run expected cost of the system is minimized. They identify a semi-Markov decision problem and the optimal solution is obtained by using linear programming method. The main contribution of this paper is the determination of the control policy that yields the specific optimal service rate to be used for every possible state of the system. They conclude that (i) the service rate to be employed is insensitive to changes in inventory level and depends only on the number of customers waiting for service and (ii) the parameters which influence the service rates are the customer arrival rates and waiting time costs. It is not intuitive that the inventory level, replenishment rate and inventory carrying costs have no effect on the service rate.

Kalpakam and Shanthi [14] analyze a lost sales $(S-1, S)$ perishable system, under Poisson demands and exponential life times in which the orders are placed at every demand epoch so as to take the inventory position back to its maximum level S. The items are replenished one at a time,with this resupply time having an arbitrary distribution. Various operating characteristics are obtained using Markov renewal techniques. This paper also deals with a base stock policy, with variable ordering quantity and arbitrary unit resupply times. The operating characteristics of this complex model are derived using the techniques of semi-regenerative processes. They use a matrix recursive scheme to obtain the stationary distribution of the underlying Markov chain.

Arivarignan et al. [1] consider a perishable inventory management system at a service facility, with arrival of customers forming a Poisson process. Each customer requires a single item and it is delivered through a service of duration having exponential distribution. The control policy is (s, S) and waiting space is assumed to have finite capacity. They derive the stationary distribution of the system state and also compute a few performance measures of the system.

Optimization of service time of perishable goods is considered in *Berman and Sapna* [6]. They consider a finite capacity system where arrivals form a Poisson process and replenishment is instantaneous. They determine the service rates to be employed at each instant of time so that the long-run expected cost rate is minimized for fixed maximum inventory level and capacity. The problem is modeled as a semi-Markov decision problem. Also they establish the existence of a stationary optimal policy and is derived by reducing it to a linear programing problem. More specifically, in this paper *Berman* and *Sapna* consider a system having perishable items and a non Markovian service times. They assume that customers arrive according to a Poisson process to the facility which has limited waiting space N. The life time of items used for service are exponentially distributed. There is no lead time for orders and customers are served on an FCFS basis. Their main objective is to determine optimal mean service times(or optimal service rates) to be employed each time a customer finishes service according to a general service time distribution. These decisions are based on the number of units in inventory and the number of customers waiting for service. As an example of the model discussed therein, consider surgeries in a hospital that may require blood transfusion. The service provided is the surgery, and the time to complete surgery is the service time. The perishable item is blood, and the problem is how to control the time of the surgery. This can be accomplished by using the appropriate number of surgeons and nurses or by using different types of equipments. For example, when there are many patients waiting and a large amount of blood available, a fast service time is employed and when there are very few patients and the amount of blood available is small, there is no need to speed up the service time.

Dynamic inventory strategies for profit maximization in a service facility requiring exponentially distributed service time and lead time is considered by *Berman and Kim* [7]. Demands arrive according to a Poisson process. The authors model the system as a Markov decision problem to identify a replenishment policy that maximizes the systems profit subject to the costs due to service delay, inventory holding and replenishment. They examine analytically the effect of changes in the system parameters on the optimal profit and optimal replenishment policies. The authors then extend these results to *Erlang* distributed lead time. They conclude with numerical investigation of the optimal (Q, r) replenishment policy.

Krishnamoorthy and Islam [17] analyze an (s, S) inventory system with postponed demands. Here they assume arrival of customers to form a Poisson process. When inventory level reaches zero due to demands, further demands are send to a pool(of postponed demands) which has capacity $M(< \infty)$. Service to pooled customers would be considered only after replenishment against the order placed. Further such customers are served only if the inventory level is at least $s + 1$. They assume that the lead time is exponentially distributed. Queues with postponed work is discussed in *Deepak et al.* [10]. This extended to inventory with positive service time is the theme of *Krishnamoorthy and Islam* [17]. They compute the condition for system stability and derive several performance characteristics of the system.

Sivakumar and Arivarignan [44] analyze a continuous review perishable (s, S) inventory system with a service facility consisting of a waiting line of finite capacity and a single server. They consider two types of customers, ordinary and negative, arriving according to a Markovian Arrival Process(MAP). An ordinary customer joins the queue whereas a negative customer removes some ordinary customers from the queue. A negative customer removes a random number of customers depending on the number in the system. The life time of each item, the service time and the lead time of the reorders are all assumed to have independent exponential distributions. They derive the joint probability distribution of the inventory level and the number of customers in the transient and steady state cases. They also obtain measures of system performance and compute total expected cost rate. Since the state space is finite usual matrix manipulation is handy.

Control policies like N, D, T and their combinations are extensively studied in queuing systems. *Krishnamoorthy et al.* [20] consider an (s, S) inventory system, where customers require a random amount(positive)of service time. With all underlying distributions independent exponentials they analyze the classical N-policy for inventory with positive service time. Lead time for replenishment of orders is assumed to be zero. Using matrix geometric method and a bit of heuristics the authors obtain the joint distribution of the system state in product form; this essentially means that the system state has the stochastic decomposition property. They further prove that the optimal order quantity agrees with the classical EOQ formula. In addition they prove that the cost function that was constructed is convex in N and hence a global optimam value for N exists.

Can we always get a product form solution when the lead time is zero and the probability distributions involved are all exponentials? The answer is, surprisingly, 'NO'. *Krishnamoorthy et al.* [21] considered an (s, S) inventory system with service time in which it is assumed that when the server is idle he continues to process the items. In case a processed item is available at a customer arrival epoch, then it is instantaneously served resulting in negligible service time. However, in the absence of processed item at the epoch of arrival of a customer, he has to wait until the item is processed. Of course he has to wait until all ahead of him, if any, are served. Unlike in *Krishnamoorthy et al.* [20], here the authors are not able to produce closed form solution. Instead they obtain a matrix geometric solution. Numerically they compute the optimal values of s and S. Unlike its predecessor, in the present case optimal s is not zero. Whereas *Krishnamoorthy et al.* [21] failed to get closed form solution for the model where the purpose was to improve the utilization of server idle time and decrease waiting time of customers, *Deepak et al.* [11] (see also *Krishnamoorthy et al.* [25]) consider another variation of *Krishnamoorthy et al.* [20] where a customer demands a processed item or an unprocessed with probability p and $1 - p$ respectively at the time when he enters one for service. If unprocessed item is demanded, then service time is negligible whereas if processed item is needed then there is a positive service time involved which we assume to be exponential. They assume that customers arrive according to a Poisson process. Lead time is assumed to be zero as in the last two problems discussed. Surprisingly here the authors succeeded in producing closed form solution for the system state probability, which further turned out to be in product form. It is important to enquire as to why *Krishnamoorthy et al.* [21] failed to get analytical solution for the system state probability. Following can be attributed to this failure: in their model the state $(0, 0)$ can be reached from $(1, 0, 1)$ or from $(1, 1, 0)$. Transitions among the intermediate state also takes place in a similar way. These result in nonexistence of analytical solution. However, in the case of *Deepak et al.* [11] this complex situation does not arise. Suppose that item is available at a service completion epoch, the server processes it (even in the absence of a customers); this is done as long as item(equivalent material) is available in the inventory for processing, keeping the sum total of processed and unprocessed items (inclusive of the one being processed, if any) at most at S (the maximum that can be held in the inventory). This model leads to drastic reduction in waiting time of customers and hence the system running cost.

Schwarz et al. [43] examine an $M/M/1$ queuing system with attached inventory, exponentially distributed lead time having the additional constraint that customers do not join the queue when inventory level is zero. However, those who are already in the system do not renege. This assumption has lead to a product form expression for the joint distribution of the number of customers in the system and inventory level. This is remarkable since there is a strong correlation between the lead time and the number of customers joining during the lead time. It may be noted that the condion for system stability does not involve the parameter of the lead time exponential random variable. This can be explained away as a consequence of the assumption that no customer joins the system when the inventory level is zero. Had reneging of customers been incorporated when inventory level is zero, then the lead time parameter would have appeared in the stability condition. (These remarks are applicable to a very recent paper by Krishnamoorthy and Viswanath C. Narayanan [32]. In the product form one factor is the number of customers in classical $M/M/1$ queue. Various ordering policies such as the (r, Q), (s, S), general randomized policies are examined. An important fact is that the state dependent service rates can be incorporated into the different models without any difficulty. The authors provide expressions for optimal order quantity and optimal reorder level for all models considered. Very recently this has been generalised to the case of production inventory with positive service time, which we elaborate towards the end.

Paul et al. [37] analyze a continuous review (s, S) inventory system at a service facility, wherein an item is demanded by a customer which is issued after performing service on the item. The waiting hall size is assumed to be finite. The arrival of customers forms a Poisson process. The service time, life of item in the stock and the lead time of orders are all assumed to be independently distributed exponential random variables. The joint probability distribution of the number of customers in the system and the inventory level is obtained in both transient and steady state cases.Measures of system performance in the steady state are derived.They also obtain long run total expected cost rate. Here again the state space being finite, matrix manipulations simplify analysis of the system.

Sivakumar and Arivarignan [45] consider a perishable inventory management system at a service facility with infinite waiting hall for customers. In addition they introduce negative customers who take away one pending demand. A real life situation for this includes distribution centers/sales centers of specialized or sophisticated or expensive items such as cars, electrical devices, consumers product(television sets,refrigerator) wherein the waiting customers may be wooed or taken away new arriving customers(usually touts of the competitive sales organizations). In this article they assume that the arrival of customers from a Poisson process with parameter $\lambda(> 0)$. The probability for an arriving customer to be ordinary is p and that for being negative customer is $q(= 1 - p)$. The customer demands one item at a time and it is delivered to the demanded customer after a random time of service. The service times are assumed to be distributed as negative exponential with parameter μ. The maximum capacity of the inventory is fixed as S. The life time of each item is exponentially distributed and it is assumed that the lead time is also distributed as exponential. They have obtained joint probability distribution of inventory level and queue length in the steady state. Also they analyze expected total cost rate under certain cost structures. The authors could have called the arrival process of regular and negative customers togethor as marked poisson.

Sivakumar et al. [46] examine a continous review (s, S) inventory system at a service facility in which the waiting hall for customers is of size N. Also they assume that customers arrive according to a compound Poisson process with parameter and that the number of customers arriving at an arrival instant is a random variable Y with probability function $P_k = P_r\{Y = k\}$, k = 1,2,3...Individual customer demands single item and it is delivered after completing service. The service times are assumed to be distributed as negative exponential with parameter depending on the number of customers in the system. The life time of each item is assumed to have negative exponential distribution. The lead time is distributed as exponential. They derive the joint probability distribution of the inventory level and the number of customers, both in the transient and steady state. Finally they provide a numerical example to illustrate the convexity of the total expected cost rate in steady state.

Krishnamoorthy and Jose [24] analyze an (s, S) inventory system with reneging of customers and finite shortage. Queuing systems with reneging have received wide attention. However inventory with service time and customer reneging has not been studied extensively and their contribution seems to be the first in that direction. In this model they consider an (s, S) inventory system with service and reneging of customers and finite shortage of items. Arrival of customers form a Poisson process. Service times are exponentially distributed with parameter, linearly depending on the number of customers present in the system. The lead time is assumed to be zero. The inter reneging times of customer from the system have exponential distribution with parameter depending linearly on the number of waiting customers in the queue. At a time maximum of $k(> 0)$ shortages is allowed. Using Matrix-analytic method they perform the steady state analysis of the inventory model. Several measures of the system performance in the steady state are derived. One can determine the range for the costs that will yield nice analytical properties for the objective function, which can be exploited for arriving at an optimal solution. The cost function is numerically shown to have the following properties: (i) as service rate increases (keeping all other parameter fixed), the cost function is convex; (ii) with either the reneging rate or the arrival rate λ increasing, the cost function increases monotonically and (iii) with either the number of shortages k or the maximum inventory level S increasing, the cost function is convex.

Paul et al. [38] consider a continuous review perishable (s, S) inventory system with service facility consisting of a waiting hall of finite capacity and a single server. They assume two types of customers, ordinary and negative, arriving according to a Markovian arrival process(MAP) as observed earlier, the authors could have considered a Marked Markovian arrival process (MMAP). An ordinary customer joins the queue and a negative customer, instead of joining the queue removes one ordinary customer from the queue. The removal rule adopted in that paper is RCE (removal of a customer from the end). The individual customers demand for one unit of the item is satisfied after a random time of service which is assumed to have a phase type distribution. The life duration of each item and the lead time of the reorders have been assumed to be independent exponential distributions. Joint probability distribution of the number of customers in the system and the inventory level are computed algorithmically. Further, performance measures are computed and the total expected cost rate is evaluated. The results are illustrated numerically. Here again finiteness of the state space becomes handy for analysis.

Yadavalli et al. [48] consider a continuous review perishable inventory management system at a service facility with a finite waiting room. They assume that the customers arrive according to a Markovian arrival process(MAP) and that each customer demands a single item which is delivered after completing the service. The service time duration is assumed to be distributed as negative exponential. The maximum capacity of the inventory is fixed as S and the maximum number of customers at any time is N, including the one at the service point. They consider a set of reorder levels with a specified probability of placing an order at a particular reorder level. The ordering quantity depends upon the reorder level at which an order is triggered and the lead time is distributed exponentially with its parameter depending upon the reorder level. They derive the joint probability distribution of the inventory level and the number of customers in the system in the steady state. They also obtain various measures of system performance in the steady state. Needless to mention that the system state being finite, matrix manipulations help in the analysis of the system.

Krishnamoorthy et al. [25] consider the inter-arrival time distribution of customers to be phase type distributed. With service time also *iid* phase type distributed random variables the authors, investigate the system numerically when reneging of customers and shortage of items are permitted. The lead time is assumed to be zero. Customers join the system and tend to leave from it with positive probability without getting service. They compute the long run behavior of the system in the steady sate. A number of descriptors of the system are provided. A cost function associated with the model is numerically investigated. When the shortage cost is finite and relatively less than the holding cost of the inventoried items, the system orders for replenishment of the item only on accumulation of a certain number of customers. The shortage cost can be measured in terms the waiting cost of customers in the absence of inventory. When the number of customers in the system accumulates to a positive number K, an order is placed for replenishment and received instantly since the lead time is zero. However, customers tend to renege from the system. Here they assume the reneging rate to be a constant, irrespective of the number of customers waiting (i.e., excluding one in service). An objective of this paper is to compute the optimal value of K. Apart from this they also investigate the optimal number S of items to be purchased at each replenishment epoch. Since the model is quite complex one can not expect analytical tractability of the problem. Hence they approach the problem algorithmically and the optimal shortage level is numerically evaluated. Some measures of the system performance in the steady state are also derived.

He and Neuts [41], in order to minimize waiting time of customers in a queue, investigated the effect of transfer of customers between two parallel queues. Specifically, they assume that as long as the difference between number of customers in the two queues remain below a preassigned number K, customers stay at their respective queues. However, the moment this difference reaches K, a certain number of $L(< K)$ of customers is transfered form the longer to the shorter queue. The L customers chosen for transfer are from the rear end of the longer queue and they are placed behind the waiting customers of the shorter queue. They prove that the combined system is stable if and only if the combined arrival rate is less than the combined service rate. Further they observe that one of the queue can be unstable; however the combined system can be stable. Optimal value of K and L are investigated when all underlying distributions are exponential. Note that the transfer time of customers is assumed to be negligible. This model has great practical utility. Now what happens if each queue in the above model is attached with inventory? In both, the same control policy (s, S) is used. Lead times for replenishment in them are independent but nonidentical exponentially distributed random variables. *Deepak et al.* [12] discuss this inventory model. In addition to transfer of customers as in *He and Neuts* [41], the authors introduce transfer of inventory subject to availability. Even when customers are not transferred, inventory could be transfered to facilitate reduction in waiting time of customers. For example, one of the waiting lines may have no inventory on stock, but has at least one customer waiting. Then as many items as possible could be transferred to restore service in the waiting line where customers are waiting without any service being given. This model is pretty complex. Nevertheless the authors succeed in providing a detailed analysis of the system which as such is level dependent; it is made a level independent quasi birth death processes(LIQBD) by suitably defining the level and phases.

Paul et al. [39] consider a continuous review perishable (s, S) inventory system with service facility consisting of finite waiting room and a single server. The customers arrive according to a Markovian arrival process(MAP). The individual customer's unit demand is satisfied after a random time of service which is assumed to have phase type distribution. The life time of each item and the lead time of reorders are assumed to have independent exponential distributions. Any arriving customer, who finds the waiting room full, enters an orbit of infinite capacity. These orbital customers compete for service by sending out signals, the duration between two successive attempts being exponentially distributed. The joint probability distribution of the number of customers in the waiting room, number of customers in the orbit and the number of inventoried items is obtained in the steady state case. Various system performance measures are computed and total expected cost rate is calculated in the case when system is stable. As repeated in their earlier work, here again the authors restrict to finite state space model.

Yadavalli et al. [49] analyze a continuous review inventory system at a service facility, wherein an item demanded by a customer is issued only after performing service of random duration on the item. The service facility is assumed to have waiting hall of infinite capacity. The arrival time points of customers form a renewal process. The service times are assumed to be distributed as negative exponential. The operating policy is (s, S) with instantaneous supply of ordered items. They consider two models which differ in the way that ordered items, when received, is brought into the stock. In their first model, the ordered items are brought into the stock immediately. In the second model, the supplied items are brought into the stock only at the next demand epoch. The stationary distribution of the underlying Markov chain is obtained. The joint probability distribution of the number of customers in the system and the inventory level is obtained in the steady state case.

Vineetha [50] considers an inventory problem with service times arbitrary distributed, arrival of demands constituting a Poisson process and zero lead time with no shortage. Embedding the process at service completion epochs, she analyzes the system as an $M/G/1$ type queuing problem and obtains product form solution for the system state distribution.

Viswanath et al. [47] have studied an inventory model with positive service time introducing the notion of vacation to server. They consider quite general distributions for inter-arrival times(MAP), service time durations (PH distributed), duration of a vacation(also PH distributed); server goes on vacation whenever there is either no customer left behind in the system at a departure epoch or when the inventory level drops to zero or when both occur simultaneously. Since lead time is assumed to be positive, the chance of inventory level dropping to zero is positive. Further the continuous time Markov chain is that analyzed turn out to be LDQBD since reneging of customers when service is not provided(as a consequence of no stock) takes place. To the best of our knowledge this is the first paper on inventory with positive service time involving server vacation. The LDQBD structure of the Markovian chain produces an infinitesimal generator which is not even asymptotically quasi-Toeplitz. As a result an appropriate truncation procedure at least an asymptotically quasi-Toeplitz structure is called for. The authors employ a method due to *Bright and Taylor* [8] that calls for the construction of a dominating process of the LDQBD at hand. This results in the identification of a level(number of customers in the system) beyond which a repeating structure for the infinitesimal generator of the underlying Markov chain could materialize. Then the *Neuts and Rao* [36] truncation procedure is employed to analyze the system. The authors obtain several useful system state characteristics.

Krishnamoorthy and Anbazhagan [27] analyze a perishable stochastic inventory system under continuous review at a service facility in which the waiting hall for customers is of finite capacity M. The service starts only when the customer level reaches $N(< M)$, once the server has become idle for want of customers. They have assumed Poisson arrival of demands for the commodity that are perishable and are issued to the customer after a random duration of service, exponentially distributed is performed on it. Here they consider an (s, S) ordering policy with random lead time for replenishment. It is assumed that demand for the commodity is of unit size. The arrivals of customers to the service station form a Poisson process. The inventoried items have exponential life times. It is also assumed that lead time for the reorders is distributed as exponential and is independent of the service time distribution. The demands that occur during stock out periods are lost. The joint probability distribution of the number of customers in the system and the inventory level is obtained in steady state case. Some measures of system performance in the steady state are derived.The results are numerically illustrated. Matrix manipulation turns out to be handy here again due to finiteness of state space.

An (s, S) inventory problem involving service interruption (according to a Poisson process) is considered by *Krishnamoorthy et al.* [29]. Interruption duration has exponential distribution; service time is also exponentially distributed (provided there is no interruption). However, the effective service time now has a distribution different from the exponential. Lead time for replenishment is zero. Stability of this system is investigated and under such regime, expected number of customers in the system, expected inventory level and expected interruption rate are computed.

2.1.1 Production inventory with positive service time

Krishnamoorthy and Narayanan [30] provide another new direction of enquiry. It is concerned with the introduction of positive service time to a production inventory process. The customer arrival process is governed by MAP (Markovian arrival process); production of items again has this characteristic MPP (Markovian production process); thus these two independent processes have built in intra-correlations. Service time duration has PH-distribution. Under this very general set up, the authors analyze the system for stability.

And then for the stable system they compute several performance measures that has great impact for system design.In order to reduce waiting time of customers due to nonavailability of items during lead time, *Krishnamoorthy and Raju* [15] have introduced through a sequence of papers the notion of local purchase. However they assume that the service time is negligible. *Lalitha* [34] discusses various types of inventory problems involving positive service time, perishablility of items, local purchase in the absence of items on stock. In a few cases she has obtained product form solution. Since service time is positive in the models that she investigates, a queue of customers is formed even when inventory level is positive.

Krishnamoorthy et al. [31] deal with production inventory with positive service time. The demand process constitutes a Poisson process. Service times has exponential distribution. The production process is switched on when inventory level drops to s and is switched off when it goes back to S. Items are called to the stock one at a time on account of production. With the assumption that customers do not join when inventory level is zero, the authors come up with a stochastic decompsition of the system state. The results in [43] is shown to follow from this paper. They obtain analytically the optimal values of s and S.

2.2 Problems involving retrial of customers

2.2.1 Retrial inventory without production

Krishnamoorthy and Jose [19] discuss an inventory system with positive service time and retrial of customers. They assume arrival of customers to form a Poisson process and lead time is exponentially distributed. An arriving customer encountering the inventory dry proceeds to an orbit with probability γ and is lost forever with probability, $(1-\gamma)$. A retrial customer in the orbit, finding the inventory dry, returns to the orbit with probability $(1-\delta)$. The inter retrial time is exponentially distributed with linear rate depending on number of customers in the orbit. Also they calculate the expected number of departures after receiving service, the expected number of customers lost without getting service and the expected total cost of the system.

Retrial of unsatisfied customers is also discussed in *Krishnamoorthy et al.* [22]. They assume the arrival process of customers to constitute a Batch Markovian Arrival Process(BMAP), service times are *iid* exponential distributed random variables. The authors discuss an (s, S) retrial inventory with service time. Demands enter the buffer of capacity equal to the number of items held in the inventory at that time. When buffer is full(equal to the number of inventoried items), further demands proceed to an orbit of infinite capacity. The orbital customers try their luck after the elapse of a random time which is exponentially distributed. These customers keep on trying until they succeed in finding a berth in the buffer. Inventory level decreases by one unit for providing service to a customer in the buffer. When inventory level reaches s an order for replenishment is placed. The authors assume that lead time is exponentially distributed. They establish the system stability condition. Under conditions of system stability, several measures of performances are evaluated.

Krishnamoorthy and Jose [23] analyze and compare three (s, S) inventory systems with positive service time and retrial of customers. In all these systems, arrivals of customers form a Poisson process and service times are exponentially distributed. When the inventory level depletes to s due to service, an order for replenishment is placed. The lead time follows an exponential distribution. In the first model, an arriving customer, who finds the inventory dry or server busy, proceeds to an orbit with probability γ and is lost forever with probability $(1 - \gamma)$. A retrial customer in the orbit, who finds the inventory dry or server busy, returns to the orbit with probability δ and is lost forever with probability $(1 - \delta)$. In second and third models it is assumed that an arriving customer who finds the buffer full proceeds to an orbit with probability γ and is lost forever with probability $(1 - \gamma)$. A retrial customer from the orbit who finds buffer full returns to the orbit with probability δ and is lost forever with probability $(1 - \delta)$. In all these cases, the inter retrial times follows an exponential distribution with parameter $i\theta$ when there are i customers in the orbit. That is, the problem considered is LDQBD. They investigate these systems to obtain performance measures and construct suitable cost functions for the three cases. Numerical illustrations are provided.

Service interruptions are a common phenomena. This may be due to the server break down, server taking a break or due to arrival of a higher priority customer. An extensive survey on queues with interruption could be found in [33]. *Krishnamoorthy et al.* [31] examine an inventory model at a service facility which can fail while providing service. It enquires an exponentially distributed amount of time to fix the server. Interruption process forms a Poisson process. At any time the server will not be effected by more than one interruption. Customers on their arrival, encountering a busy server goes to an orbit to retry for service. No customer joins the orbit when the inventory level is zero. Further when server is an interruption a primary arrival joins the orbit with probability p $(0 < p < 1)$. Also at such instances a retrial customer tend to leave the system with probability q $(0 < q < 1)$. The authors show that the resulting LDQBD is always stable. They device a novel approach to compute the expected waiting time of an orbital customer. Several other important system characteristics are also investigated.

2.2.2 Retrial inventory with production

Krishnamoorthy and Islam [16] consider an (s, S) production inventory system with retrial of customers. In that paper the authors attempt at investigating retrial of unsuccessful customers in accessing the service station in an (s, S) production inventory system. Also they assume arrivals of customers from outside the system. Whenever the inventory level drops to s the production mechanism is immediately converted to ON mode from the Off mode i.e., production starts. The inter-production times are exponentially distributed. When inventory level reaches zero, further arriving demands are sent to the orbit which has capacity $M(< \infty)$. Demands arrive according to Poisson process. Customers who find the orbit full and inventory level at zero, are lost to the system. Services to the orbital customers or external demands is provided if at least one item is available in the inventory. Demands arising from the orbital customers are exponentially distributed with parameter $K\gamma$, when there are K customers in the orbit. The long run joint probability distribution of the number of customers in the orbit and inventory level is obtained. Also some performance measures are evaluated and an appropriate expected cost function analyzed. Though the model involves level dependence, the system being finite, can be handled easily.

A production inventory with PH-distributed production time and retrial of unsatisfied customers, is considered in *Krishnamoorthy et al.* [18]. They assume that the arrival of customers form a Markovian arrival process, thereby introducing correlation into arrivals. Each customer requires an exponentially distributed amount of time to get the item served. In addition to the orbit of infinite capacity a finite buffer is provided for customers to wait. When the latter is full, customers proceed to orbit. The authors discuss the issue of stability of the system and then proceed to compute its long-run state distribution. When production time to produce one unit of item has exponential distribution they compute a number of optimal design oriented performance measures. The work of *Krishnamoorthy and Viswanath C. Narayanan* [32], which was mentioned earlier, considers a more general production process distribution.

Krishnamoorthy and Jose [26] analyze and compare three production inventory systems with positive service time and retrial of customers. In all these systems, arrivals of customers form a Poisson process and service times are exponentially distributed. When the inventory level depletes to s due to services provided to the arriving customers, production starts, that is, the production is switched to ON mode. The production continues until the number of items in the inventory reaches the level S(the maximum that can be held in the inventory). The time between additions of two successive items (by production) to the inventory is exponentially distributed. In model I, an arriving customer who finds the inventory level zero or sever busy, proceeds to an orbit with a positive probability and is lost forever with complementary probability. A retrial customer in the orbit who finds the inventory level zero or server busy, returns to the orbit with probability δ and is lost forever with probability $(1 - \delta)$. In models II and III, an arriving customer who finds the buffer full is assumed to proceeds to an orbit with probability γ and is lost forever with probability $(1 - \gamma)$. A retrial customer from the orbit who finds the buffer full, returns to the orbit with probability δ and is lost forever with probability $(1 - \delta)$. In all these systems, inter-retrial times follow an exponential distribution with linear parameter $i\theta$ when there are i customers in the orbit. In classical inventory with service time, customers have to wait until their turn of service arrives. In retrial set up, customers will not wait in the system (in the absence of a waiting space) nor are they lost to the system. Instead they can go back home, go for a coffee/library or attend to some other work and then come to the inventory system where they require the service. This especially happens when the item is out of stock or the finite buffer is full. Thus the main distinction between classical and retrial inventory system is that, whereas in the former a customer may be lost in the absence of inventoried items, in the latter customers need not be lost; instead they go to orbit (own house/ library/ shop/attend some other work) and try to access the server at a future time. Using the Matrix geometric method to solve the above mentioned models, some measures of the performance in the steady state are derived. A suitable cost function is defined for all these cases and it is analyzed numerically.

3 Future work

Discrete time inventory models with positive service time have so far received least attention. These cannot be handled that smoothly as discrete time queues since even the simple case will be a two dimensional problem just as in the continuous time case and involve more than two partial generating functions that could not be easily connected. Another aspect of inventory with positive service time can be implemented, are those appearing in supply chain. Such problems are now being attended to. Some of the interesting problems in queues, namely queues with interruption/ self-generation of priorities/ orbital search of customers in retrial set up can all be extended to those in inventory involving positive service time. Of these orbital search for unsatisfied customers, wherein inventory level is positive, is discussed in *Krishnamoorthy et al.* [28]. With all underlying distributions exponential(parameter of the inter retrial time distribution is level dependent and linear), they assume that primary customers do not join orbit when inventory level is zero. The resulting Markov chain turns out to be a level dependent quasi birth and death process(LDQBD). The authors investigate the condition for system stability and further they produce several performance measures that are useful for system design by reducing this LDQBD to LIQBD through truncation procedures and by construction of a dominating process. Search of customers to inventory with positive service time can be employed in several real-life inventory problems.

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