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RESEARCH ARTICLE

Solitons in magneto-optic waveguides with generalized Kudryashov's form of self-phase modulation structure

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Abstract The current paper recovers soliton solutions in magneto-optic waveguides having generalized form of Kudryashov's self-phase modulation structure. The retrieval of soliton solutions is achieved by the well-known and primitive G'/G-expansion scheme. The intermediary functions that made this retrieval possible are Jacobi's elliptic functions and Weierstrass' elliptic functions. The parameter constraints for the solitons to exist are also presented.

Keywords Solitons · Waveguides · G'/G-expansion

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Introduction

One of the inherent hurdles for soliton propagation through fibers and other waveguides is the clutter they form [1–5]. Therefore to circumvent this problem is to introduce magneto-optic waveguides [6–10]. Such form of waveguides transforms the solitons state of clutter to a state of separation so that a smooth and laminar flow of solitons is ensured through the fibers for intercontinental distances [11–15]. The current paper is about the transmission of solitons through a magneto-optic waveguide that comes with Kudryashov's form of self-phase modulation structure [16–20]. This is the model that is being considered for the first time in this paper. The model is integrated by the aid of generalized G'/G-expansion scheme [21–25]. This

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is an efficient scheme that yielded a full spectrum of optical solitons through magneto-optic waveguides through a couple of intermediary function. These are Jacobi's elliptic functions [26–30] and Weierstrass' elliptic functions [31–35]. In both cases optical soliton solutions emerged as special cases when the modulus approached unity in its limit or as a special case of the Weierstrass' elliptic functions. These solitons came with their respective parameter constraints that are also presented in the work. The results are exhibited after a detailed introduction to the model. The details are jotted in the rest of the paper.

Governing model

$$iq_t + a_1 q_{xx} + \left(\frac{b_1}{|q|^n} + \frac{c_1}{|q|^{2n}} + d_1|q|^n + e_1|q|^{2n} + \frac{f_1}{|r|^n} + \frac{g_1}{|r|^{2n}} + h_1|r|^n + k_1|r|^{2n} \right) q = Q_1 r + i\left[\beta_1 q_x + \lambda_1 \left(|q|^{2n} q\right)_x + \gamma_1 \left(|q|^{2n}\right)_x q + \theta_1 |q|^{2n} q_x\right], \tag{1}$$

$$ir_t + a_2 r_{xx} + \left(\frac{b_2}{|r|^n} + \frac{c_2}{|r|^{2n}} + d_2|r|^n + e_2|r|^{2n} + \frac{f_2}{|q|^n} + \frac{g_2}{|q|^{2n}} + h_2|q|^n + k_2|q|^{2n} \right) r = Q_2 q + i\left[\beta_2 r_x + \lambda_2 \left(|r|^{2n} r\right)_x + \gamma_2 \left(|r|^{2n}\right)_x r + \theta_2 |r|^{2n} r_x\right], \tag{2}$$

Considering the variables $a_l, b_l, c_l, d_l, e_l, f_l, g_l, h_l, k_l, Q_l, \beta_l, \lambda_l, \gamma_l$, and θ_l for $l = 1, 2$ as real-valued constants and $i = \sqrt{-1}$, this framework encompasses Eqs. (1) and (2) that focus on the complex-valued soliton profiles represented by the dependent variables $q(x, t)$ and $r(x, t)$. These equations, set against the backdrop of spatial and temporal dimensions denoted by x and t, outline the linear temporal evolutions through their first terms. Within this setup, a_l coefficients are chromatic dispersions, managing the spread of light in a spectrum, while b_l, c_l, d_l, and e_l correspond to self-phase modulation, impacting the light beam's phase. The coefficients f_l, g_l, h_l, and k_l are related to cross-phase modulation, indicating interactions among multiple light waves. Additionally, Q_l and β_l stand for magneto-optic parameters and inter-modal dispersions (IMD), critical in light propagation dynamics. The terms λ_l are associated with preventing shock waves through self-steepening (SS), while γ_l and θ_l account for nonlinear dispersion (ND), crucial in the phase velocity's intensity-dependent variation. This study also supplements its results with those obtained through the unified auxiliary equation method and the new mapping method, as discussed in references [24] and [25], enriching the analysis with diverse methodological perspectives.

Mathematical preliminaries

In order to solve Eqs. (1) and (2), we assume that the solutions of Eqs. (1) and (2) have the following forms:

$$q(x, t) = \psi_1(\zeta) e^{i\digamma(x,t)}, \tag{3}$$

$$r(x, t) = \psi_2(\zeta) e^{i\digamma(x,t)}, \tag{4}$$

$$\zeta = x - vt, \ \digamma(x, t) = -kx + \omega t + \theta_0, \tag{5}$$

where the real functions $\psi_j(\zeta)$ for $j = 1, 2$ represent the amplitude portion of the soliton and $\digamma(x, t)$ is the phase component of the pulse, v is the speed of the wave, k is the frequency, ω is the wave number and θ_0 is the phase constant.

Next, insert (3) with (4) into Eqs. (1) and (2), and then the real parts are:

$$a_1\psi_1'' - [\omega + \beta_1 k + a_1 k^2]\psi_1 - k(\lambda_1 + \theta_1)\psi_1^{2n+1} - Q_1\psi_2 + \frac{b_1}{\psi_1^{n-1}} + \frac{c_1}{\psi_1^{2n-1}} + d_1\psi_1^{n+1} + e_1\psi_1^{2n+1} + \left(\frac{f_1}{\psi_2^n} + \frac{g_1}{\psi_2^{2n}} + h_1\psi_2^n + k_1\psi_2^{2n}\right)\psi_1 = 0, \tag{6}$$

and

$$a_2\psi_2'' - [\omega + \beta_2 k + a_2 k^2]\psi_2 - k\left(\lambda_2 + \theta_2\right)\psi_2^{2n+1} - Q_2\psi_1 + \frac{b_2}{\psi_2^{n-1}} + \frac{c_2}{\psi_2^{2n-1}} + d_2\psi_2^{n+1} + e_2\psi_2^{2n+1} + \left(\frac{f_2}{\psi_1^n} + \frac{g_2}{\psi_1^{2n}} + h_2\psi_1^n + k_2\psi_1^{2n}\right)\psi_2 = 0, \tag{7}$$

where $' = \frac{d}{d\zeta}$. The imaginary parts are:

$$(v + 2a_1 k + \beta_1)\psi_1' + [(2n + 1)\lambda_1 + 2n\gamma_1 + \theta_1]\psi_1^{2n}\psi_1' = 0, \tag{8}$$

$$(v + 2a_2 k + \beta_2)\psi_2' + [(2n + 1)\lambda_2 + 2n\gamma_2 + \theta_2]\psi_2^{2n}\psi_2' = 0. \tag{9}$$

The linearly independent principle is applied on Eqs. (8) and (9) to obtain:

$$v = -(2a_j k + \beta_j), \tag{10}$$

$$(2n+1)\lambda_j + 2n\gamma_j + \theta_j = 0, \tag{11}$$

where $j = 1,\ 2.$

Let us set

$$\psi_2(\zeta) = \mho\psi_1(\zeta), \tag{12}$$

where $\mho$ is a nonzero constant, such that $\mho \neq 1$. As a results Eqs. (6) and (7) reduce to

$$\begin{aligned} &a_1\psi_1^{2n-1}\psi_1'' + (c_1 + g_1\mho^{-2n}) + (b_1 + f_1\mho^{-n})\psi_1^n - [\omega + \beta_1 k + a_1k^2 + Q_1\mho]\psi_1^{2n} \\ &\quad +(d_1 + h_1\mho^n)\psi_1^{3n} + [(e_1 + k_1\mho^{2n}) - k(\lambda_1 + \theta_1)\psi_1^{4n} = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &a_2\mho\psi_1^{2n-1}\psi_1'' + (c_2\mho^{1-2n} + g_2\mho) + (b_2\mho^{1-n} + f_2\mho)\psi_1^n - [\omega + \beta_2 k + a_2k^2)\mho + Q_2]\psi_1^{2n} \\ &\quad +(d_2\mho^{n+1} + h_2\mho)\psi_1^{3n} + [e_2\mho^{2n+1} + k_2\mho - k(\lambda_2 + \theta_2)\mho^{2n+1}]\psi_1^{4n} = 0. \end{aligned} \tag{14}$$

Equations (13) and (14) have the same form under the following constraint conditions:

$$a_1 = a_2\mho, \tag{15}$$

$$c_1 + g_1\mho^{-2n} = c_2\mho^{1-2n} + g_2\mho, \tag{16}$$

$$b_1 + f_1\mho^{-n} = b_2\mho^{1-n} + f_2\mho, \tag{17}$$

$$\omega + \beta_1 k + a_1k^2 + Q_1\mho = [\omega + \beta_2 k + a_2k^2]\mho + Q_2, \tag{18}$$

$$d_1 + h_1\mho^n = d_2\mho^{n+1} + h_2\mho, \tag{19}$$

$$(e_1 + k_1\mho^{2n}) - k(\lambda_1 + \theta_1) = e_2\mho^{2n+1} + k_2\mho - k(\lambda_2 + \theta_2)\mho^{2n+1}. \tag{20}$$

From (18), the wave number ω is given by

$$\omega = \frac{\mho Q_1 - Q_2 + k\left(\beta_1 + ka_1\right) - \mho k\left(\beta_2 + ka_2\right)}{\mho - 1}. \tag{21}$$

Balancing $\psi_1^{2n-1}\psi_1''$ and ψ_1^{4n} in Eq. (13) gives $N = \frac{1}{n}$. Therefore, the new wave transformation

$$\psi_1(\zeta) = [\phi(\zeta)]^{\frac{1}{n}}, \tag{22}$$

changes Eq. (13) to the following new nonlinear ordinary differential equation:

$$na_1\phi\phi'' + (1-n)a_1\phi'^2 + n^2(\Delta_0 + \Delta_1\phi + \Delta_2\phi^2 + \Delta_3\phi^3 + \Delta_4\phi^4) = 0, \tag{23}$$

where $\phi(\zeta)$ is a new function of ζ, such that $\phi(\zeta) > 0$ for $n > 0$, and

$$\begin{cases} \Delta_0 = c_1 + g_1\mho^{-2n}, \\ \Delta_1 = b_1 + f_1\mho^{-n}, \\ \Delta_3 = d_1 + h_1\mho^n, \\ \Delta_4 = e_1 + k_1\mho^{2n} - k(\lambda_1 + \theta_1). \end{cases} \tag{24}$$

Generalized $\left(G'/G\right)$-expansion

By balancing $\phi\phi''$ and ϕ^4 in Eq. (23), one obtains $N = 1$.The $\left(G'/G\right)$ expansion method [26–28] assumes the formal solution of Eq. (23) as:

$$\phi(\zeta) = A_0 + A_1\left[\frac{G'(\zeta)}{G(\zeta)}\right], \tag{25}$$

and the function $G(\zeta)$ satisfies the following Jacobi elliptic equation:

$$G'^2(\zeta) = R + QG^2(\zeta) + PG^4(\zeta), \tag{26}$$

where $A_0,\ A_1,\ R,\ Q$ and P are constants, with the condition $A_1 \neq 0$. It is well-known that Eq. (26) gives many explicit solutions in terms of Jacobi elliptic functions and Weierstrass-elliptic functions [26–30] as the following tables:

By differentiating Eq. (25) and successively applying Eq. (26), one can obtain the following derivatives:

$$\begin{cases} \phi'^2(\zeta) = A_1^2Q^2 - 2A_1^2Q\left(\frac{G'(\zeta)}{G(\zeta)}\right)^2 + A_1^2\left(\frac{G'(\zeta)}{G(\zeta)}\right)^4 - 4PA_1^2R \\ \phi''(\zeta) = 2A_1\left(\frac{G'(\zeta)}{G(\zeta)}\right)\left[\left(\frac{G'(\zeta)}{G(\zeta)}\right)^2 - Q\right]. \end{cases} \tag{27}$$

By substituting (25) and (27) into Eq. (23) and collecting all the coefficients of $\left(\frac{G'(\zeta)}{G(\zeta)}\right)^i$, $(i = 0,\ 1,\ 2,\ 3,\ 4)$ and setting them equal to zero, we obtain the following algebraic equations:

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^4 : 2na_1A_1^2 + (1-n)a_1A_1^2 + n^2\Delta_4A_1^4 = 0,$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^3 : 2na_1A_0A_1 + n^2\left(4\Delta_4A_0A_1^3 + \Delta_3A_1^3\right) = 0,$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^2 : -2na_1A_1^2Q - 2(1-n)a_1QA_1^2 + n^2\left(6\Delta_4A_0^2A_1^2 + 3\Delta_3A_0A_1^2 + \Delta_2A_1^2\right) = 0,$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right) : -2na_1A_0QA_1 + n^2\left(4\Delta_4A_0^3A_1 + 3\Delta_3A_0^2A_1 + 2\Delta_2A_0A_1 + \Delta_1A_1\right) = 0,$$

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right)^0 : (1-n)a_1\left(A_1^2Q^2 - 4PA_1^2R\right) + n^2\left(\Delta_4A_0^4 + \Delta_3A_0^3 + \Delta_2A_0^2 + \Delta_1A_0 + \Delta_0\right) = 0.$$

On solving the algebraic equations $\left(\frac{G'(\zeta)}{G(\zeta)}\right)^0 - \left(\frac{G'(\zeta)}{G(\zeta)}\right)^4$ using Maple or Mathematica, we have the following result:

$$A_0 = \sqrt{\frac{(n+1)\left(n^2\Delta_2 - 2Qa_1\right)}{6n^2\Delta_4}},\ A_1 = \sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}, \tag{28}$$

via the restrictions

$$\begin{cases} \Delta_0 = \frac{1}{36}\frac{(n^2-1)\left[16a_1^2\left(9PR-Q^2\right)-n^2\Delta_2\left(n^2\Delta_2+8Qa_1\right)\right]}{\Delta_4n^4}, \\ \Delta_1 = \sqrt{\frac{(n+1)(n-2)^2\left(n^2\Delta_2-2Qa_1\right)\left(n^2\Delta_2+4Qa_1\right)^2}{54n^6\Delta_4}}, \\ \Delta_3 = -\sqrt{\frac{2\Delta_4(n+2)^2\left(n^2\Delta_2-2Qa_1\right)}{3n^2(n+1)}}, \end{cases} \tag{29}$$

provided $a_1\Delta_4 < 0$ and $\Delta_4\left(n^2\Delta_2 - 2Qa_1\right) > 0$.

By substituting (28) into (25), the general exact solution of Eq. (23) obtains as follows:

$$\phi(\zeta) = \sqrt{\frac{(n+1)\left(n^2\Delta_2-2Qa_1\right)}{6n^2\Delta_4}} + \sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left[\frac{G'(\zeta)}{G(\zeta)}\right]. \tag{30}$$

Table 2 The Weierstrass-elliptic function solutions $\wp(\zeta;g_2,g_3)$ for Eq. (26), where $\wp'(\zeta;g_2,g_3) = \frac{d\wp(\zeta;g_2,g_3)}{d\zeta}$ is its derivative with respect to ζ.

Case	g_2	g_3	$G(\zeta)$
19	$\frac{4(Q^2-3PR)}{3}$	$\frac{4Q(-2Q^2+9PR)}{27}$	$\sqrt{\frac{1}{P}\left(\wp(\zeta;g_2,g_3) - \frac{Q}{3}\right)}$
20	$\frac{4(Q^2-3PR)}{3}$	$\frac{4Q(-2Q^2+9PR)}{27}$	$\sqrt{\frac{3R}{3\wp(\zeta;g_2,g_3)-Q}}$
21	$\frac{Q^2}{12} + PR$	$\frac{Q(36PR-Q^2)}{216}$	$\frac{\sqrt{R}[6\wp(\zeta;g_2,g_3)+Q]}{3\wp'(\zeta;g_2,g_3)}$
22	$\frac{Q^2}{12} + PR$	$\frac{Q(36PR-Q^2)}{216}$	$\frac{3\wp'(\zeta;g_2,g_3)}{\sqrt{P}[6\wp(\zeta;g_2,g_3)+Q]}$
23	$\frac{2Q^2}{9}$	$\frac{Q^3}{54}$	$\frac{Q\sqrt{-\frac{15Q}{2P}}\wp(\zeta;g_2,g_3)}{3\wp(\zeta;g_2,g_3)+Q},\ R = \frac{5Q^2}{36P}$

Table 1 The Jacobi elliptic function solutions for Eq. (26)

Case	P	Q	R	$G(\zeta)$
1	m^2	$-(\tau^2+1)$	1	$\mathrm{sn}(\zeta,\tau)$
2	τ^2	$-\left(1+\tau^2\right)$	1	$\mathrm{cd}(\zeta,\tau) = \frac{\mathrm{cn}(\zeta,\tau)}{\mathrm{dn}(\zeta,\tau)}$
3	$-\tau^2$	$2\tau^2-1$	$1-\tau^2$	$\mathrm{cn}(\zeta,\tau)$
4	-1	$2-\tau^2$	τ^2-1	$\mathrm{dn}(\zeta,\tau)$
5	$1-\tau^2$	$2-\tau^2$	1	$\mathrm{sc}(\zeta,\tau) = \frac{\mathrm{sn}(\zeta,\tau)}{\mathrm{cn}(\zeta,\tau)}$
6	$-\tau^2\left(1-\tau^2\right)$	$2\tau^2-1$	1	$\mathrm{sd}(\zeta,\tau) = \frac{\mathrm{sn}(\zeta,\tau)}{\mathrm{dn}(\zeta,\tau)}$
7	$\frac{1}{4}$	$\frac{1-2\tau^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}(\zeta,\tau) \pm \mathrm{cs}(\zeta,\tau)$
8	$\frac{1-\tau^2}{4}$	$\frac{1+\tau^2}{2}$	$\frac{1-\tau^2}{4}$	$\mathrm{nc}(\zeta,\tau) \pm \mathrm{sc}(\zeta,\tau)$
9	$\frac{\tau^2}{4}$	$\frac{\tau^2-2}{2}$	$\frac{\tau^2}{4}$	$\mathrm{ns}(\zeta,\tau) \pm \mathrm{ds}(\zeta,\tau)$
10	$\frac{\tau^2}{4}$	$\frac{\tau^2-2}{2}$	$\frac{\tau^2}{4}$	$\sqrt{\tau^2-1}\mathrm{sd}(\zeta,\tau) \pm \mathrm{cd}(\zeta,\tau)$
11	$\frac{\tau^2-1}{4}$	$\frac{1+\tau^2}{2}$	$\frac{\tau^2-1}{4}$	$\tau\mathrm{sd}(\zeta,\tau) \pm \mathrm{nd}(\zeta,\tau)$
12	$\frac{\tau^2}{4}$	$\frac{\tau^2-2}{2}$	$\frac{\tau^2}{4}$	$\mathrm{dc}(\zeta,\tau) \pm \sqrt{1-\tau^2}\mathrm{nc}(\zeta,\tau)$
13	1	$2\left(1-2\tau^2\right)$	1	$\mathrm{sc}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)$
14	τ^4	$2\left(\tau^2-2\right)$	1	$\mathrm{sd}(\zeta,\tau)\mathrm{cn}(\zeta,\tau)$
15	1	$2\left(\tau^2+1\right)$	$1-2\tau^2+\tau^4$	$\mathrm{cs}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)$
16	$P>0$	$Q<0$	$\frac{\tau^2Q^2}{\left(\tau^2+1\right)^2P}$	$\sqrt{-\frac{\tau^2Q}{\left(\tau^2+1\right)^2P}}\mathrm{sn}\left(\sqrt{-\frac{Q}{(\tau^2+1)}}\zeta,\tau\right)$
17	$P<0$	$Q>0$	$\frac{(1-\tau^2)Q^2}{\left(\tau^2-2\right)^2P}$	$\sqrt{-\frac{Q}{(2-\tau^2)P}}\mathrm{dn}\left(\sqrt{\frac{Q}{(2-\tau^2)}}\zeta,\tau\right)$
18	$P<0$	$Q>0$	$\frac{\tau^2(\tau^2-1)Q^2}{\left(2\tau^2-1\right)^2P}$	$\sqrt{-\frac{\tau^2Q}{(2\tau^2-1)P}}\mathrm{cn}\left(\sqrt{\frac{Q}{(2\tau^2-1)}}\zeta,\tau\right)$

Now, according to Tables 1 and 2. and the general formal solution (30), we deduce the following set of optical solitons of Eqs. (1) and (2):

Case-1: If $P=\tau^2,\ Q=-(\tau^2+1),\ R=1,\ 0<\tau<1$ and $G(\zeta)=\ \mathrm{sn}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2+2a_1(\tau^2+1)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\mathrm{cn}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)}{\mathrm{sn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{31}$$

$$r(x,t)=\mho q(x,t),\tag{32}$$

provided $\frac{n^2\Delta_2+2a_1(\tau^2+1)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the straddled solitons

$$q(x,t)=\left\{\sqrt{\frac{(n+1)(n^2\Delta_2+4a_1)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\mathrm{sech}(\zeta)\mathrm{csch}(\zeta)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{33}$$

$$r(x,t)=\mho q(x,t).\tag{34}$$

Case-2: If $P=\tau^2,\ Q=-\left(1+\tau^2\right),\ R=1,\ 0<\tau<1$ and $G(\zeta)=\mathrm{cd}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2+2a_1(1+\tau^2)\right)}{6n^2\Delta_4}}-\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{(1-\tau^2)\mathrm{sn}(\zeta,\tau)}{\mathrm{dn}(\zeta,\tau)^2\mathrm{cd}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{35}$$

$$r(x,t)=\mho q(x,t),\tag{36}$$

provided $\frac{n^2\Delta_2+2a_1(1+\tau^2)}{\Delta_4}>0$.

Case-3: If $P=-\tau^2,\ Q=2\tau^2-1,\ R=1-\tau^2,\ 0<\tau<1$ and $G(\zeta)=\mathrm{cn}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a_1(2\tau^2-1)\right)}{6n^2\Delta_4}}-\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\mathrm{dn}(\zeta,\tau)\mathrm{sn}(\zeta,\tau)}{\mathrm{cn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{37}$$

$$r(x,t)=\mho q(x,t),\tag{38}$$

provided $\frac{n^2\Delta_2-2a_1(2\tau^2-1)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the dark solitons

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a_1\right)}{6n^2\Delta_4}}-\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\tanh(\zeta)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{39}$$

$$r(x,t)=\mho q(x,t),\tag{40}$$

provided $\frac{n^2\Delta_2-2a_1}{\Delta_4}>0$.

Case-4: If $P=-1,\ Q=2-\tau^2,\ R=\tau^2-1,\ 0<\tau<1$ and $G(\zeta)=\mathrm{dn}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a_1(2-\tau^2)\right)}{6n^2\Delta_4}}-\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\tau^2\mathrm{cn}(\zeta,\tau)\mathrm{sn}(\zeta,\tau)}{\mathrm{dn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{41}$$

$$r(x,t)=\mho q(x,t),\tag{42}$$

provided $\frac{n^2\Delta_2-2a_1(2-\tau^2)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the same dark solitons (39) and (40).

Case-5: If $P=1-\tau^2,\ Q=2-\tau^2,\ R=1,\ 0<\tau<1$ and $G(\zeta)=\ \mathrm{sc}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a_1(2-\tau^2)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\mathrm{dn}(\zeta,\tau)}{\mathrm{cn}(\zeta,\tau)\mathrm{sn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{43}$$

$$r(x,t)=\mho q(x,t),\tag{44}$$

provided $\frac{n^2\Delta_2-2a_1(2-\tau^2)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the singular solitons

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\coth(\zeta)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{45}$$

$$r(x,t)=\mho q(x,t),\tag{46}$$

provided $\frac{n^2\Delta_2-2a_1}{\Delta_4}>0$.

Case-6: If $P=-\tau^2(1-\tau^2)$, $Q=2\tau^2-1$, $R=1$, $0<\tau<1$ and $G(\zeta)=\mathrm{sd}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a_1(2\tau^2-1)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\mathrm{cn}(\zeta,\tau)}{\mathrm{dn}(\zeta,\tau)\mathrm{sn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{47}$$

$$r(x,t)=\mho q(x,t),\tag{48}$$

provided $\frac{n^2\Delta_2-2a_1(2\tau^2-1)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the same singular solitons (45) and (46).

Case-7: If $P=\frac{1}{4}$, $Q=\frac{1-2\tau^2}{2}$, $R=\frac{1}{4}$, $0<\tau<1$ and $G(\zeta)=\mathrm{ns}(\zeta,\tau)\pm\mathrm{cs}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-a_1(1-2\tau^2)\right)}{6n^2\Delta_4}}\mp\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\mathrm{ds}(\zeta,\tau)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{49}$$

$$r(x,t)=\mho q(x,t),\tag{50}$$

provided $\frac{n^2\Delta_2-a_1(1-2\tau^2)}{6n^2\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the singular solitons

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2+a_1\right)}{6n^2\Delta_4}}\mp\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\mathrm{csch}(\zeta)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{51}$$

$$r(x,t)=\mho q(x,t),\tag{52}$$

provided $\frac{n^2\Delta_2+a_1}{\Delta_4}>0$.

Case-8: If $P=\frac{1-\tau^2}{4}$, $Q=\frac{1+\tau^2}{2}$, $R=\frac{1-\tau^2}{4}$, $0<\tau<1$ and $G(\zeta)=\mathrm{nc}(\zeta,\tau)\pm\mathrm{sc}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-a_1(1+\tau^2)\right)}{6n^2\Delta_4}}\pm\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\mathrm{dc}(\zeta,\tau)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{53}$$

$$r(x,t)=\mho q(x,t),\tag{54}$$

provided $\frac{n^2\Delta_2-a_1(1+\tau^2)}{\Delta_4}>0$.

Case-9: If $P=\frac{\tau^2}{4}$, $Q=\frac{\tau^2-2}{2}$, $R=\frac{\tau^2}{4}$, $0<\tau<1$ and $G(\zeta)=\mathrm{ns}(\zeta,\tau)\pm\mathrm{ds}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-a_1(\tau^2-2)\right)}{6n^2\Delta_4}}\mp\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\mathrm{cs}(\zeta,\tau)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{55}$$

$$r(x,t)=\mho q(x,t),\tag{56}$$

provided $\frac{n^2\Delta_2-a_1(\tau^2-2)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the same singular solitons (51) and (52).

Case-10: If $P=\frac{\tau^2}{4}$, $Q=\frac{\tau^2-2}{2}$, $R=\frac{\tau^2}{4}$, $0<\tau<1$ and $G(\zeta)=\sqrt{\tau^2-1}\mathrm{sd}(\zeta,\tau)\pm\mathrm{cd}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-a_1(\tau^2-2)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\sqrt{\tau^2-1}\mathrm{cn}(\zeta,\tau)\pm(\tau^2-1)\mathrm{sn}(\zeta,\tau)}{\mathrm{dn}(\zeta,\tau)(\sqrt{\tau^2-1}\mathrm{sn}(\zeta,\tau)\pm\mathrm{cn}(\zeta,\tau))}\right)\right\}^{\frac{1}{n}}\times\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{57}$$

$$r(x,t)=\mho q(x,t),\tag{58}$$

provided $\frac{n^2\Delta_2-a_1(\tau^2-2)}{\Delta_4}>0$.

Case-11: If $P=\frac{\tau^2-1}{4}$, $Q=\frac{1+\tau^2}{2}$, $R=\frac{\tau^2-1}{4}$, $0<\tau<1$ and $G(\zeta)=\tau\mathrm{sd}(\zeta,\tau)\pm\mathrm{nd}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-a_1(1+\tau^2)\right)}{6n^2\Delta_4}}\pm\tau\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\mathrm{cd}(\zeta,\tau)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\tag{59}$$

$$r(x,t)=\mho q(x,t),\tag{60}$$

provided $\frac{n^2\Delta_2-a_1(1+\tau^2)}{\Delta_4}>0$.

Case-12: If $P=\frac{\tau^2}{4}$, $Q=\frac{\tau^2-2}{2}$, $R=\frac{\tau^2}{4}$, $0<\tau<1$ and $G(\zeta)=\mathrm{dc}(\zeta,\tau)\pm\sqrt{1-\tau^2}\mathrm{nc}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-a_1\left(\tau^2-2\right)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\mathrm{sc}(\zeta,\tau)\left(\left(\tau^2-1\right)\mp\sqrt{1-\tau^2}\,\mathrm{dn}(\zeta,\tau)\right)}{\sqrt{1-\tau^2}\pm\mathrm{dn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\times e^{i(-kx+\omega t+\theta_0)}, \tag{61}$$

$$r(x,t)=\mho q(x,t), \tag{62}$$

provided $\frac{n^2\Delta_2-a_1(\tau^2-2)}{\Delta_4}>0$.

Case-13: If $P=1,\ Q=2\left(1-2\tau^2\right),\ R=1,\ 0<\tau<1$ and $G(\zeta)=\mathrm{sc}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-4a_1\left(1-2\tau^2\right)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\tau^2\mathrm{sn}(\zeta,\tau)^2\left(\mathrm{sn}(\zeta,\tau)^2-2\right)+1}{\mathrm{sn}(\zeta,\tau)\mathrm{cn}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\times e^{i(-kx+\omega t+\theta_0)}, \tag{63}$$

$$r(x,t)=\mho q(x,t), \tag{64}$$

provided $\frac{n^2\Delta_2-4a_1(1-2\tau^2)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the same straddled solitons (33) and (34).

Case-14: If $P=\tau^4,\ Q=2\left(\tau^2-2\right),\ R=1,\ 0<\tau<1$ and $G(\zeta)=\mathrm{sd}(\zeta,\tau)\mathrm{cn}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-4a_1\left(\tau^2-2\right)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\mathrm{sn}(\zeta,\tau)^2\left(\tau^2\mathrm{sn}(\zeta,\tau)^2-2\right)+1}{\mathrm{sn}(\zeta,\tau)\mathrm{cn}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}\times e^{i(-kx+\omega t+\theta_0)}, \tag{65}$$

$$r(x,t)=\mho q(x,t), \tag{66}$$

provided $\frac{n^2\Delta_2-4a_1(\tau^2-2)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the same straddled solitons (33) and (34).

Case-15: If $P=1,\ Q=2\left(\tau^2+1\right),\ R=1-2\tau^2+\tau^4,\ 0<\tau<1$ and $G(\zeta)=\mathrm{cs}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-4a_1\left(\tau^2+1\right)\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\tau^2\mathrm{sn}(\zeta,\tau)^4-1}{\mathrm{sn}(\zeta,\tau)\mathrm{cn}(\zeta,\tau)\mathrm{dn}(\zeta,\tau)}\right)\right\}^{\frac{1}{n}}e^{i(-kx+\omega t+\theta_0)}, \tag{67}$$

$$r(x,t)=\mho q(x,t), \tag{68}$$

provided $\frac{n^2\Delta_2-4a_1(\tau^2+1)}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the straddled solitons

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-8a_1\right)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\mathrm{sech}(\zeta)\mathrm{csch}(\zeta)-2\coth(\zeta)\right)\right\}^{\frac{1}{n}}e^{i(-kx+\omega t+\theta_0)}, \tag{69}$$

$$r(x,t)=\mho q(x,t), \tag{70}$$

provided $\frac{n^2\Delta_2-8a_1}{\Delta_4}>0$.

Case-16: If $R=\frac{\tau^2Q^2}{\left(\tau^2+1\right)^2P},\ Q<0,\ P>0,\ 0<\tau<1$ and $G(\zeta)=\sqrt{-\frac{\tau^2Q}{\left(\tau^2+1\right)^2P}}\mathrm{sn}\left(\sqrt{-\frac{Q}{\left(\tau^2+1\right)}}\zeta,\tau\right)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a_1Q\right)}{6n^2\Delta_4}}+\sqrt{\frac{a_1(n+1)Q}{n^2\Delta_4\left(\tau^2+1\right)}}\left(\frac{\mathrm{cn}\left(\sqrt{-\frac{Q}{\tau^2+1}}\zeta,\tau\right)\mathrm{dn}\left(\sqrt{-\frac{Q}{\tau^2+1}}\zeta,\tau\right)}{\mathrm{sn}\left(\sqrt{-\frac{Q}{\tau^2+1}}\zeta,\tau\right)}\right)\right\}^{\frac{1}{n}}\times e^{i(-kx+\omega t+\theta_0)}, \tag{71}$$

$$r(x,t)=\mho q(x,t), \tag{72}$$

provided $\frac{n^2\Delta_2-2a_1Q}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the straddled solitons

$$q(x,t)=\left\{\sqrt{\frac{(n+1)\left(n^2\Delta_2-2a_1Q\right)}{6n^2\Delta_4}}+\sqrt{\frac{a_1(n+1)Q}{2n^2\Delta_4}}\left(\mathrm{sech}\left(\sqrt{-\frac{Q}{2}}\zeta\right)\mathrm{csch}\left(\sqrt{-\frac{Q}{2}}\zeta\right)\right)\right\}^{\frac{1}{n}}\times e^{i(-kx+\omega t+\theta_0)}, \tag{73}$$

$$r(x,t)=\mho q(x,t). \tag{74}$$

Case-17: If $R=\frac{(1-\tau^2)Q^2}{(\tau^2-2)^2P}$, $Q>0$, $P<0$, $0<\tau<1$ and $G(\zeta)=\sqrt{-\frac{Q}{(2-\tau^2)P}}\mathrm{dn}\left(\sqrt{\frac{Q}{(2-\tau^2)}}\zeta,\tau\right)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)(n^2\Delta_2-2a_1Q)}{6n^2\Delta_4}}-\sqrt{-\frac{a_1\tau^4(n+1)Q}{n^2\Delta_4(2-\tau^2)}}\left(\frac{\mathrm{cn}\left(\sqrt{\frac{Q}{2-\tau^2}}\zeta,\tau\right)\mathrm{sn}\left(\sqrt{\frac{Q}{2-\tau^2}}\zeta,\tau\right)}{\mathrm{dn}\left(\sqrt{\frac{Q}{2-\tau^2}}\zeta,\tau\right)}\right)\right\}^{\frac{1}{n}}\times\mathrm{e}^{i(-kx+\omega t+\theta_0)}, \tag{75}$$

$$r(x,t)=\mho q(x,t), \tag{76}$$

provided $\frac{n^2\Delta_2-2a_1Q}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the dark solitons

$$q(x,t)=\left\{\sqrt{\frac{(n+1)(n^2\Delta_2-2a_1Q)}{6n^2\Delta_4}}-\sqrt{-\frac{a_1(n+1)Q}{n^2\Delta_4}}\tanh\left(\sqrt{Q}\zeta\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)}, \tag{77}$$

$$r(x,t)=\mho q(x,t). \tag{78}$$

Case-18: If $R=\frac{\tau^2(\tau^2-1)Q^2}{(2\tau^2-1)^2P}$, $Q>0$, $P<0$, $0<\tau<1$ and $G(\zeta)=\sqrt{-\frac{\tau^2Q}{(2\tau^2-1)P}}\mathrm{cn}\left(\sqrt{\frac{Q}{(2\tau^2-1)}}\zeta,\tau\right)$, we have the Jacobi elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)(n^2\Delta_2-2a_1Q)}{6n^2\Delta_4}}-\sqrt{-\frac{a_1(n+1)Q}{n^2\Delta_4(2\tau^2-1)}}\left(\frac{\mathrm{dn}\left(\sqrt{\frac{Q}{2\tau^2-1}}\zeta,\tau\right)\mathrm{sn}\left(\sqrt{\frac{Q}{2\tau^2-1}}\zeta,\tau\right)}{\mathrm{cn}\left(\sqrt{\frac{Q}{2\tau^2-1}}\zeta,\tau\right)}\right)\right\}^{\frac{1}{n}}\times\mathrm{e}^{i(-kx+\omega t+\theta_0)}, \tag{79}$$

$$r(x,t)=\mho q(x,t), \tag{80}$$

provided $\frac{n^2\Delta_2-2a_1Q}{\Delta_4}>0$.

In particular, if $\tau\to1$, then Eqs. (1) and (2) have the same dark solitons (77) and (78).

Case-19: If $g_2=\frac{4(Q^2-3PR)}{3}$, $g_3=\frac{4Q(-2Q^2+9PR)}{27}$ and $G(\zeta)=\sqrt{\frac{1}{P}\left(\wp(\zeta;g_2,g_3)-\frac{Q}{3}\right)}$, we have the Weierstrass-elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)(n^2\Delta_2-2a_1Q)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{\wp'(\zeta;g_2,g_3)}{2\left(\wp(\zeta;g_2,g_3)-\frac{Q}{3}\right)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)}, \tag{81}$$

$$r(x,t)=\mho q(x,t), \tag{82}$$

provided $\frac{n^2\Delta_2-2a_1Q}{\Delta_4}>0$.

Case-20: If $g_2=\frac{4(Q^2-3PR)}{3}$, $g_3=\frac{4Q(-2Q^2+9PR)}{27}$ and $G(\zeta)=\sqrt{\frac{3R}{3\wp(\zeta;g_2,g_3)-Q}}$, we have the Weierstrass-elliptic function solutions

$$q(x,t)=\left\{\sqrt{\frac{(n+1)(n^2\Delta_2-2a_1Q)}{6n^2\Delta_4}}+\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}\left(\frac{-3\wp'(\zeta;g_2,g_3)}{2(3\wp(\zeta;g_2,g_3)-Q)}\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)}, \tag{83}$$

$$r(x,t)=\mho q(x,t), \tag{84}$$

provided $\frac{n^2\Delta_2-2a_1Q}{\Delta_4}>0$.

Case-21: If $g_2=\frac{Q^2}{12}+PR$, $g_3=\frac{Q(36PR-Q^2)}{216}$ and $G(\zeta)=\frac{\sqrt{R}[6\wp(\zeta;g_2,g_3)+Q]}{3\wp'(\zeta;g_2,g_3)}$, we have the Weierstrass-elliptic function solutions

$$\begin{aligned}q(x,t)=&\left\{\sqrt{\frac{(n+1)(n^2\Delta_2-2a_1Q)}{6n^2\Delta_4}}-\frac{\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}}{24\wp'(\zeta;g_2,g_3)(6\wp(\zeta;g_2,g_3)+Q)}\left(864\wp(\zeta;g_2,g_3)^3\right.\right.\\&+144Q\wp(\zeta;g_2,g_3)^2-6(12PR+Q^2)\wp(\zeta;g_2,g_3)\\&\left.\left.-144\wp'(\zeta;g_2,g_3)^2-12PQR-Q^3\right)\right\}^{\frac{1}{n}}\mathrm{e}^{i(-kx+\omega t+\theta_0)},\end{aligned} \tag{85}$$

$$r(x,t)=\mho q(x,t), \tag{86}$$

provided $\frac{n^2\Delta_2-2a_1Q}{6n^2\Delta_4}>0$.

Case-22: If $g_2=\frac{Q^2}{12}+PR$, $g_3=\frac{Q(36PR-Q^2)}{216}$ and $G(\zeta)=\frac{3\wp'(\zeta;g_2,g_3)}{\sqrt{P}[6\wp(\zeta;g_2,g_3)+Q]}$, we have the Weierstrass-elliptic function solutions

$$q(x,t) = \Bigg\{ \sqrt{\frac{(n+1)\left(n^2\Delta_2 - 2a_1 Q\right)}{6n^2\Delta_4}} + \frac{\sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}}}{24\wp'(\zeta;g_2,g_3)(6\wp(\zeta;g_2,g_3)+Q)} \Big(864\wp\left(\zeta;g_2,g_3\right)^3 + 144Q\wp\left(\zeta;g_2,g_3\right)^2 - 6\left(12PR+Q^2\right)\wp\left(\zeta;g_2,g_3\right) - 144\wp'\left(\zeta;g_2,g_3\right)^2 - 12PQR - Q^3\Big)\Bigg\}^{\frac{1}{n}} e^{i(-kx+\omega t+\theta_0)}, \tag{87}$$

$$r(x,t) = \mho q(x,t), \tag{88}$$

provided $\frac{n^2\Delta_2 - 2a_1 Q}{\Delta_4} > 0$.

Case-23: If $g_2 = \frac{2Q^2}{9}$, $g_3 = \frac{Q^3}{54}$, $R = \frac{5Q^2}{36P}$ and $G(\zeta) = \frac{Q\sqrt{-\frac{15Q}{2P}}\wp(\zeta;g_2,g_3)}{3\wp(\zeta;g_2,g_3)+Q}$, we have the Weierstrass-elliptic function solutions

$$q(x,t) = \left\{ \sqrt{\frac{(n+1)\left(n^2\Delta_2 - 2a_1 Q\right)}{6n^2\Delta_4}} + \sqrt{-\frac{a_1(n+1)}{n^2\Delta_4}} \left(\frac{Q\wp'(\zeta;g_2,g_3)}{\wp(\zeta;g_2,g_3)(3\wp(\zeta;g_2,g_3)+Q)} \right) \right\}^{\frac{1}{n}} e^{i(-kx+\omega t+\theta_0)}, \tag{89}$$

$$r(x,t) = \mho q(x,t), \tag{90}$$

provided $\frac{n^2\Delta_2 - 2a_1 Q}{6n^2\Delta_4} > 0$.

Remark-1: All the results obtained in this article satisfy Eq. (23) under the given constraints outlined in (29).

Remark-2: The above solutions (31)–(90) are new and not found in [24, 25] or elsewhere.

Conclusions

This paper recovered a complete spectrum of optical solitons propagating through magneto-optic waveguides that comes with Kudryashov's form of self-phase modulation structure. The integration scheme is the age-old version of the then-popular, although not robust, G'/G-expansion scheme. The recovered results are a complete spectrum of optical solitons that emerged through a couple of intermediary functions. These are the Jacobi's elliptic functions and the Weierstrass' elliptic functions. The soliton solutions came with parameter constraints that are also presented in the entire paper. It is not out of place to point out that the adopted integration scheme fails to recover soliton radiation. Thus the integration algorithm carries a gigantic loophole. Therefore, the results of this manuscript are indeed technically incorrect with no soliton radiation components in them. In future, this will be rectified with the implementation of the inverse scattering transform [36–39] or so. This work is therefore an eye-opener !!!

Declarations

Conflict of interest The authors claim there is no Conflict of interest.

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