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RESEARCH ARTICLE

Quantum correlations in a coherently pumped cascade three-level laser in the presence of dephasing

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Abstract This study considers the quantum correlations between two field modes produced by a coherently pumped cascade three-level laser, taking into account the inevitable dephasing processes. The photon-photon quantum correlations are investigated by applying the stationary solutions of the various expectation values of the product of cavity mode operators. In the relatively weak coherent pumping regime, it is found that the stationary entanglement and squeezing between the two field modes increase with the coherent pumping amplitude, while in the relatively strong coherent pumping regime, they decrease. We also show that the quantification of quantum correlations through Gaussian quantum discord, entanglement, and squeezing shows a similar evolution pattern. Moreover, we show that the degree of quantum correlations is increased by the atomic injection rate and decreased by the dephasing processes.

Keywords Dephasing · Squeezing · Entanglement · Discord

Introduction

Quantum correlation, a fundamental aspect of quantum mechanics, has extensive use in quantum information processing [1, 2] and quantum technology [3]. A number of models, including photons [4–11], atoms [12–16], ions [17–20], and phonons [21–24], have been proposed for processing quantum information. Therefore, the scientific community has given a great emphasis on the creation and enhancement of quantum correlations at various levels, microscopic and macroscopic, in recent years.

Quadrature squeezing, a result of the uncertainty relation, is a condition in which one quadrature's noise is suppressed below the shot-noise limit with enhanced noise in the conjugate quadrature without violating the uncertainty relation. This leads to an elliptical uncertainty region in phase space, with the major and minor axes corresponding to the increased and decreased uncertainties, respectively, while, for a coherent state, the uncertainties are equal and visualized in phase space as a circular uncertainty region [25, 26]. A Typical process that results in quadrature squeezing is parametric oscillations [27–29]. It has also been theoretically predicted that various physical systems, such as correlated emission lasers [30–35], anharmonic oscillator [36–38], driven damped harmonic oscillator with time dependent mass and frequency [39, 40], and degenerate hyper Raman processes [41] are reliable sources of squeezed light.

Quantum entanglement is another typical example of non-classical correlation that has been extensively employed in quantum information processing in a range of uses such as quantum cryptography [42], quantum teleportation [43], and telecloning [44]. In an entangled system, the quantum state of the composite system cannot be separately expressed by the independent knowledge of the states of the parts. The generation of entanglement between the cavity modes has been realized in a cascade scheme [33–35], and double Raman scheme [45, 46].

However, entanglement is not the only non-classical correlation that has been used in these fields. Because entanglement cannot measure all quantum correlations between two cavity modes of a bipartite system, Gaussian quantum discord has been introduced to measure total quantum

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correlations beyond entanglement [47]. This suggests that for some separable mixed state, quantum discord may be non-zero, showing the existence of quantum correlations that entanglement measures are unable to detect. The quantum discord between two cavity modes has been studied in a correlated emission laser with coherence induced through the initial coherent superposition of the top and bottom energy states [48]. This study indicates that the atomic injection rate and the initial coherent superposition lead to an increase in the amount of quantum discord between the two field modes.

Decoherence, on the other hand, is essential for the dynamic transition from quantum to classical phenomena. A realistic quantum system is susceptible to environment induced-decoherence, which is a main impediment to the development of devices for quantum information processing. For instance, dephasing, or the gradual loss of quantum coherences, arises from the quantum system's interaction with its surroundings. More recently, the impact of dephasing processes on the stationary entanglement and squeezing of the field modes caused by correlated emission laser has been studied in [33]. This study indicates that dephasing processes reduce the non-classical features of the field modes.

In this work, we examine the photon-photon quantum correlations caused by a two-photon three-level laser with the addition of coherence through coupling the upper and bottom energy states of the injected atoms by coherent light. The stationary squeezing and entanglement features of the field modes caused by an analogous scheme without taking into account the effect of dephasing processes has been examined in [49, 50]. Moreover, the measure of non-classical features in a correlated emission laser with the atomic coherence added through the coherent superposition of the upper and bottom energy states of the injected atoms, has been studied in the absence of dephasing processes [31]. In this study, we will control and quantify quantum correlations generated by a coherently pumped cascade three-level laser that can be applied to various tasks of quantum information processing and quantum computation. Here, we mainly interested in studying the effect of inevitable dephasing processes on quantum correlations, such as entanglement, squeezing, and quantum discord between the field modes produced by the scheme under consideration. In this regard, we first establish the temporal evolution of the reduced density operator, accounting for the different decay processes. The photon-photon quantum correlations are then studied using the stationary solutions of the field mode operators.

Master equation

Our model comprises a collection of three-level atoms whose upper and bottom energy states are coupled by a coherent light and injected at a constant rate r_a into a cavity as displayed below in Fig. 1.

We symbolize the upper, intermediate, and bottom levels of the atom by the energy eigenstates $|a\rangle, |b\rangle$, and $|c\rangle$ respectively. Moreover, we assume the atomic transition from $|a\rangle$ to $|b\rangle$ and from $|b\rangle$ to $|c\rangle$ to be dipole allowed and the direct transition from $|a\rangle$ to $|c\rangle$ to be dipole forbidden. The transition of a three-level atom from $|a\rangle$ to $|b\rangle$ results in the emission of the first cavity mode, while the transition from $|b\rangle$ to $|c\rangle$ results in the emission of the second cavity mode. The fields modes are assumed to be in resonance with the dipole-allowed atomic transitions. The total Hamiltonian of the three-level atom's interaction with field mode operators and the pumping coherent field can be described as

$$\hat{H} = ig\Big[|a\rangle\langle b|\hat{a} - \hat{a}^{\dagger}|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^{\dagger}|c\rangle\langle b|\Big] + i\frac{\Omega}{2}\big(|c\rangle\langle a| - |a\rangle\langle c|\big), \tag{1}$$

where $\hat{a}^{\dagger}(\hat{a})$ and $\hat{b}^{\dagger}(\hat{b})$ are creation(destruction) field mode operators, g is the atom-field modes coupling strength, and $\Omega = 2\phi\varepsilon_o$. Here ϕ is the coupling strength between the atom and the pumping coherent field, and ε_o is the amplitude of the coherent field. We assume the atoms to be initially in the upper energy levels, and we write the initial state and density operator of a single three-level atom as $|\psi_A(0)\rangle = |a\rangle$ and $\hat{\rho}_A(0) = |a\rangle\langle a|$, respectively.

Moreover, we require $\hat{\rho}_{AR}(t, t_j)$ to be the composite density operator for the field modes and the three-level atom at time t with the atom injected at time t_j. The three-level atoms are introduced into the cavity at a fixed rate r_a, stay within the cavity for a period of τ, and are withdrawn from the cavity after they are relaxed to a level different from the intermediate and bottom levels. It is easy to deduce

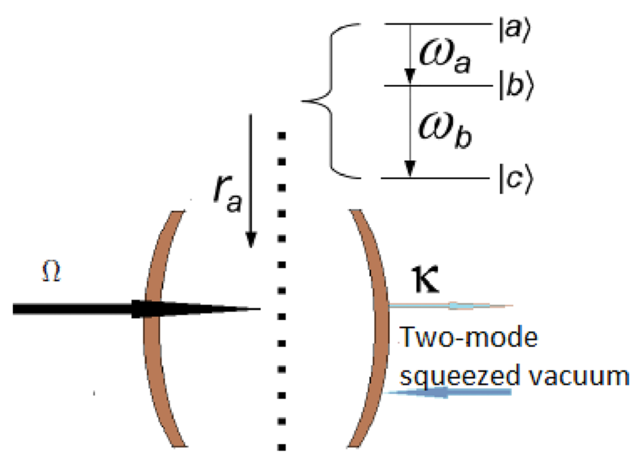

Fig. 1 Schematic representation of a two-photon three-level laser with the upper and bottom levels coupled by coherent field

that $t-\tau \leq t_j \leq t$. The density operator of all the atoms plus the fields at time t is then given by

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t_j, \tag{2}$$

with $r_a \Delta t_j$ denoting the number of atoms introduced into the cavity in a period of Δt_j. It is possible to replace the summation by integration by assuming that a large number of atoms are introduced into a cavity in a very small time interval Δt_j. It then follows that

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')\, dt', \tag{3}$$

so that differentiating this expression with respect to time, we find

$$\begin{aligned} \frac{d}{dt}\hat{\rho}_{AR}(t) &= r_a\Big[\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t, t-\tau)\Big] \\ &\quad + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')\, dt'. \end{aligned} \tag{4}$$

Because the atomic and field mode variables are uncorrelated at the time of injection and after the atoms are relaxed to levels excluding $|b\rangle$ and $|c\rangle$, we can write

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(0)\hat{\rho}(t) \tag{5}$$

and

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t), \tag{6}$$

where $\hat{\rho}_A(0)$ is the density operator of the atom at the time the atom is being introduced into the cavity, $\hat{\rho}(t)$ is the density operator for the field modes alone, and $\hat{\rho}_A(t-\tau)$ is the density operator of the atom after the atom is withdrawn from the cavity. On the other hand, according to the Heisenberg equation of motion, one can write

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -i\Big[\hat{H}, \hat{\rho}_{AR}(t,t')\Big], \tag{7}$$

so that making use of Eqs. (3)–(7), we find that

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a\Big[\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau)\Big]\hat{\rho}(t) - i\Big[\hat{H}, \hat{\rho}_{AR}(t)\Big]. \tag{8}$$

Here we are concerned with the quantum properties of the cavity mode only. This can be done by tracing Eq. (8) over the atomic variables, using the Hamiltonian specified by Eq. (1), and taking into account the damping of the field modes by two-mode vacuum reservoir. We thus see that

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = {} & g\Big[-\hat{a}^\dagger\hat{\rho}_{ab} + \hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger - \hat{\rho}_{ba}\hat{a} - \hat{b}^\dagger\hat{\rho}_{bc} \\ & + \hat{b}\hat{\rho}_{cb} + \rho_{bc}\hat{b}^\dagger - \hat{\rho}_{cb}\hat{b}\Big] \\ & + \frac{\kappa}{2}\big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\big) + \frac{\kappa}{2}\big(2\hat{b}\hat{\rho}\hat{b}^\dagger \\ & - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\big), \end{aligned} \tag{9}$$

where $\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle$ with α, β=a, b, c; and κ is the cavity dissipation rate.

We proceed to obtain the matrix elements $\hat{\rho}_{ab}$ that appear in the expression of Eq. (9). To this end, on multiplying Eq. (8) from the left by $\langle\alpha|$ and from the right by $|\beta\rangle$, we find

$$\begin{aligned} \frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) &= r_a\Big[\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle\Big]\hat{\rho}(t) \\ &\quad - i\langle\alpha|\Big[\hat{H}, \hat{\rho}_{AR}(t)\Big]|\beta\rangle - \gamma_{\alpha\beta}\hat{\rho}_{\alpha\beta}, \end{aligned} \tag{10}$$

where the last term is added to take into consideration the dephasing rates and the relaxation rates of the atoms by spontaneous emission, with $\gamma_{\alpha\alpha}$ being the spontaneous emission rate and $\gamma_{\alpha\beta}$ for $\alpha \neq \beta$ being the dephasing rates. Assuming the atoms are extracted from the cavity after they have relaxed to levels aside from $|b\rangle$ and $|c\rangle$ and taking into account Eq. (1) and $\hat{\rho}_A(0) = |a\rangle\langle a|$, one can easily establish the following relations:

$$\begin{aligned} \frac{d}{dt}\hat{\rho}_{aa} &= r_a\hat{\rho}(t) + g\big(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger\big) \\ &\quad - \frac{\Omega}{2}\big(\hat{\rho}_{ca} + \hat{\rho}_{ac}\big) - \gamma_{aa}\hat{\rho}_{aa}, \end{aligned} \tag{11}$$

$$\frac{d}{dt}\hat{\rho}_{bb} = -g\big(\hat{a}^\dagger\hat{\rho}_{ab} - \hat{b}\hat{\rho}_{cb} + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{b}^\dagger\big) - \gamma_{bb}\hat{\rho}_{bb}, \tag{12}$$

$$\frac{d}{dt}\hat{\rho}_{cc} = -g\big(\hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}\big) + \frac{\Omega}{2}\big(\hat{\rho}_{ac} + \hat{\rho}_{ca}\big) - \gamma_{cc}\hat{\rho}_{cc}, \tag{13}$$

$$\frac{d}{dt}\hat{\rho}_{ab} = g\big(\hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a} + \hat{\rho}_{ac}\hat{b}^\dagger\big) - \frac{\Omega}{2}\hat{\rho}_{cb} - \gamma_{ab}\hat{\rho}_{ab}, \tag{14}$$

$$\frac{d}{dt}\hat{\rho}_{ac} = g\big(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{b}\big) + \frac{\Omega}{2}\big(\hat{\rho}_{aa} - \hat{\rho}_{cc}\big) - \gamma_{ac}\hat{\rho}_{ac}, \tag{15}$$

$$\frac{d}{dt}\hat{\rho}_{bc} = -g\big(\hat{a}^\dagger\hat{\rho}_{ac} - \hat{b}\hat{\rho}_{cc} + \hat{\rho}_{bb}\hat{b}\big) + \frac{\Omega}{2}\hat{\rho}_{ba} - \gamma_{bc}\hat{\rho}_{bc}. \tag{16}$$

To proceed further, we are interested in linear analysis, which corresponds to eliminating the g terms from Eqs. (11), (12), (13), and (15). We also apply the adiabatic approximation scheme, whereas, in comparison to the cavity mode variables, the atomic variables attain steady state quickly. Thus, taking the time derivatives of atomic variables to be zero and applying the linear analysis, we obtain from Eqs. (11), (12), (13), and (15) that

$$\hat{\rho}_{aa} = \frac{r_a}{\gamma}\hat{\rho}\left(\frac{1+\frac{\Omega^2}{2\gamma\Gamma}}{1+\frac{\Omega^2}{\gamma\Gamma}}\right), \qquad \hat{\rho}_{bb} = 0, \tag{17}$$

$$\hat{\rho}_{cc} = \frac{r_a}{\gamma}\hat{\rho}\frac{\Omega^2}{2(\gamma\Gamma+\Omega^2)}, \qquad \hat{\rho}_{ac} = \frac{\Omega}{2\Gamma}\frac{r_a}{\gamma}\frac{\hat{\rho}}{1+\frac{\Omega^2}{\gamma\Gamma}}, \tag{18}$$

where we have considered the case in which all the spontaneous emission rates are equal ($\gamma_{aa} = \gamma_{bb} = \gamma_{cc} = \gamma$). Upon substituting Eqs. (17) and (18) into Eqs. (14) and (16) and taking the time derivative of atomic variables to be zero, we find

$$\hat{\rho}_{ab} = \frac{-gr_a}{\gamma^2\beta}\left[\left(\frac{\gamma}{\Gamma} - \frac{\Omega^2\gamma}{4\Gamma^3} + \frac{\Omega^2}{2\Gamma^2}\right)\hat{\rho}\hat{a} - \frac{\Omega}{2\Gamma}\left(\frac{\gamma}{\Gamma} - \frac{\Omega^2}{2\Gamma^2}\right)\hat{\rho}\hat{b}^\dagger\right], \tag{19}$$

$$\hat{\rho}_{cb} = \frac{-gr_a}{\gamma^2\beta}\left[\frac{\Omega}{\Gamma}\left(\frac{\gamma}{\Gamma} + \frac{\Omega^2}{4\Gamma^2}\right)\hat{\rho}\hat{a} - \left(\frac{\gamma\Omega^2}{4\Gamma^3} + \frac{\Omega^2}{2\Gamma^2}\right)\hat{\rho}\hat{b}^\dagger\right], \tag{20}$$

in which $\beta = \left(1+\frac{\Omega^2}{\gamma\Gamma}\right)\left(1+\frac{\Omega^2}{4\Gamma^2}\right)$ and all the dephasing rates to be equal $\gamma_{ab} = \gamma_{ac} = \gamma_{bc} = \Gamma$. Now insertion of Eqs. (19) and (20) along with their adjoint into Eq. (9), the master equation for our quantum optical system can be written as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \eta(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) + \lambda(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho})\\ &+ \chi(-\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger - \hat{b}\hat{\rho}\hat{a})\\ &+ \xi(\hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a}\hat{b})\\ &+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}),\end{aligned} \tag{21}$$

where

$$\eta = \frac{A}{2\beta}\left(\frac{\gamma}{\Gamma} - \frac{\Omega^2\gamma}{4\Gamma^3} + \frac{\Omega^2}{2\Gamma^2}\right), \qquad \lambda = \frac{A}{2\beta}\left(\frac{\Omega^2\gamma}{4\Gamma^3} + \frac{\Omega^2}{2\Gamma^2}\right), \tag{22}$$

$$\chi = \frac{A}{2\beta}\left[\frac{\Omega}{2\Gamma}\left(\frac{\gamma}{\Gamma} - \frac{\Omega^2}{2\Gamma^2}\right)\right], \qquad \xi = \frac{A}{2\beta}\left[\frac{\Omega}{\Gamma}\left(\frac{\gamma}{\Gamma} + \frac{\Omega^2}{4\Gamma^2}\right)\right], \tag{23}$$

and

$$A = \frac{2g^2 r_a}{\gamma^2}, \tag{24}$$

is the linear gain coefficient. We can establish the following temporal evolution equations for the field mode operators by employing the identity $\frac{d}{dt}\langle\hat{O}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{O}\right)$ along with Eq. (21) as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle &= -\left[\frac{\kappa}{2} - \eta\right]\langle\hat{a}\rangle - \chi\langle\hat{b}^\dagger(t)\rangle,\\ \frac{d}{dt}\langle\hat{b}\rangle &= -\left[\frac{\kappa}{2} + \lambda\right]\langle\hat{b}\rangle + \xi\langle\hat{a}^\dagger(t)\rangle,\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2\rangle &= -\left[\kappa - 2\eta\right]\langle\hat{a}^2\rangle - 2\chi\langle\hat{b}^\dagger\hat{a}\rangle,\\ \frac{d}{dt}\langle\hat{b}^2\rangle &= -\left[\kappa + 2\lambda\right]\langle\hat{b}^2\rangle + 2\xi\langle\hat{b}\hat{a}^\dagger\rangle,\end{aligned} \tag{26}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = 2\eta(\langle\hat{a}^\dagger\hat{a}\rangle + 1) - \chi(\langle\hat{b}^\dagger\hat{a}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle) - \kappa\langle\hat{a}^\dagger\hat{a}\rangle, \tag{27}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -2\lambda\langle\hat{b}^\dagger\hat{b}\rangle + \xi(\langle\hat{b}^\dagger\hat{a}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle) - \kappa\langle\hat{b}^\dagger\hat{b}\rangle, \tag{28}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -(\kappa + \lambda - \eta)\langle\hat{a}\hat{b}\rangle + \xi(\langle\hat{a}^\dagger\hat{a}\rangle + 1) - \chi\langle\hat{b}^\dagger\hat{b}\rangle, \tag{29}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -(\kappa + \lambda - \eta)\langle\hat{a}^\dagger\hat{b}\rangle - \chi\langle\hat{b}^2\rangle + \xi\langle\hat{a}^{\dagger 2}\rangle. \tag{30}$$

The stationary solutions of these equations have the form

$$\begin{aligned}&\langle\hat{a}\rangle = 0, \qquad \langle\hat{b}\rangle = 0,\\ &\langle\hat{a}\hat{b}^\dagger\rangle = 0, \qquad \langle\hat{a}^2\rangle = 0, \qquad \langle\hat{b}^2\rangle = 0,\end{aligned} \tag{31}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle = \frac{-2\chi(\kappa+2\lambda)}{(\kappa+\lambda-\eta)(\kappa+2\lambda)(\kappa-2\eta) + 4\xi\chi(\kappa+\lambda-\eta)} \left[\frac{2\xi\eta}{\kappa-2\eta} + \xi\right] + \frac{2\eta}{\kappa-2\eta}, \tag{32}$$

$$\langle\hat{b}^\dagger\hat{b}\rangle = \frac{2\xi(\kappa-2\eta)}{(\kappa+\lambda-\eta)(\kappa+2\lambda)(\kappa-2\eta) + 4\xi\chi(\kappa+\lambda-\eta)} \left[\frac{2\xi\eta}{\kappa-2\eta} + \xi\right], \tag{33}$$

$$\langle\hat{a}\hat{b}\rangle = \frac{(\kappa-2\eta)(\kappa+2\lambda)}{(\kappa+\lambda-\eta)(\kappa+2\lambda)(\kappa-2\eta) + 4\xi\chi(\kappa+\lambda-\eta)} \left[\frac{2\xi\eta}{\kappa-2\eta} + \xi\right]. \tag{34}$$

Quadrature squeezing

Here, we study the squeezing features of the field modes generated from a coherently pumped cascade three-level laser. To this end, we introduce two quadrature operators: $\hat{c}_+ = \hat{c} + \hat{c}^\dagger$ and $\hat{c}_- = i(\hat{c}^\dagger - \hat{c})$, representing the position and momentum operators respectively, with $\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b})$. The variances of these quadrature operators are defined by [51]

$$\left(\Delta\hat{c}_\pm\right)^2 = \left\langle \hat{c}_\pm^2 \right\rangle - \left\langle \hat{c}_\pm \right\rangle^2, \tag{35}$$

from which follows

$$\left(\Delta\hat{c}_\pm\right)^2 = 1 + \left\langle \hat{a}^\dagger\hat{a} \right\rangle + \left\langle \hat{b}^\dagger\hat{b} \right\rangle + \left\langle \hat{a}^\dagger\hat{b} \right\rangle + \left\langle \hat{a}\hat{b}^\dagger \right\rangle \pm \frac{1}{2}\left(\left\langle \hat{a}^2 \right\rangle + \left\langle \hat{b}^2 \right\rangle + \left\langle \hat{a}^{\dagger 2} \right\rangle + \left\langle \hat{b}^{\dagger 2} \right\rangle + 2\left\langle \hat{a}\hat{b} \right\rangle + 2\left\langle \hat{a}^\dagger\hat{b}^\dagger \right\rangle \right). \tag{36}$$

Now substitution of Eqs. (31)–(34) into Eq. (36) results in

$$\left(\Delta\hat{c}_-\right)^2 = 1 + \frac{2\eta}{\kappa - 2\eta} + \frac{2\kappa\xi\left(\kappa + 2\lambda\right)}{\left(\kappa + \lambda - \eta\right)\left(\kappa + 2\lambda\right)\left(\kappa - 2\eta\right) + 4\xi\chi\left(\kappa + \lambda - \eta\right)} \times \left[\frac{\xi}{\kappa + 2\lambda} - \frac{\chi}{\kappa - 2\eta} - 1\right]. \tag{37}$$

The quadrature squeezing S of the cavity radiation with respect to the quadrature variance of the vacuum state is defined by [51]

$$S = \frac{\left(\Delta c_-\right)^2_{vac} - \left(\Delta c_-\right)^2}{\left(\Delta c_-\right)^2_{vac}} \tag{38}$$

where $(\Delta c_-)^2_{vac}$ is the quadrature variance of the vacuum state. As demonstrated in Fig. 2, it is possible to generate a robust squeezed light even in the presence of dephasing. For a given value of A, we see that the amount of squeezing increases with the increase of the amplitude of the coherent pumping light until it attains its maximum value in the weak pumping regime. With a further increase in the amplitude of the coherent pumping light, the degree of squeezing decays slowly. It is also clear from Fig. 2 that when atomic injection rate increases, the amount of squeezing of the field modes also increases. This is in agreement with previous reports [33, 34].

In Fig. 3, we plot the variance of quadrature operator $\hat{c}_-$ with Ω/γ for varied values of the dephasing rates and for a fixed value of atomic injection rate. We note from this figure that the degree of squeezing is sensitive to the dephasing rates. We also observe that as the dephasing rates increase, the amplitude of the coherent pumping light for which the optimum squeezing occurs increases. Specifically, the effect of dephasing is more significant in the weak coherent pumping regime. Moreover, it is not difficult to see from this figure that the degree of squeezing diminishes with the increase in the dephasing rates. It is possible to enhance the stationary squeezing with dephasing processes by tuning other system variables.

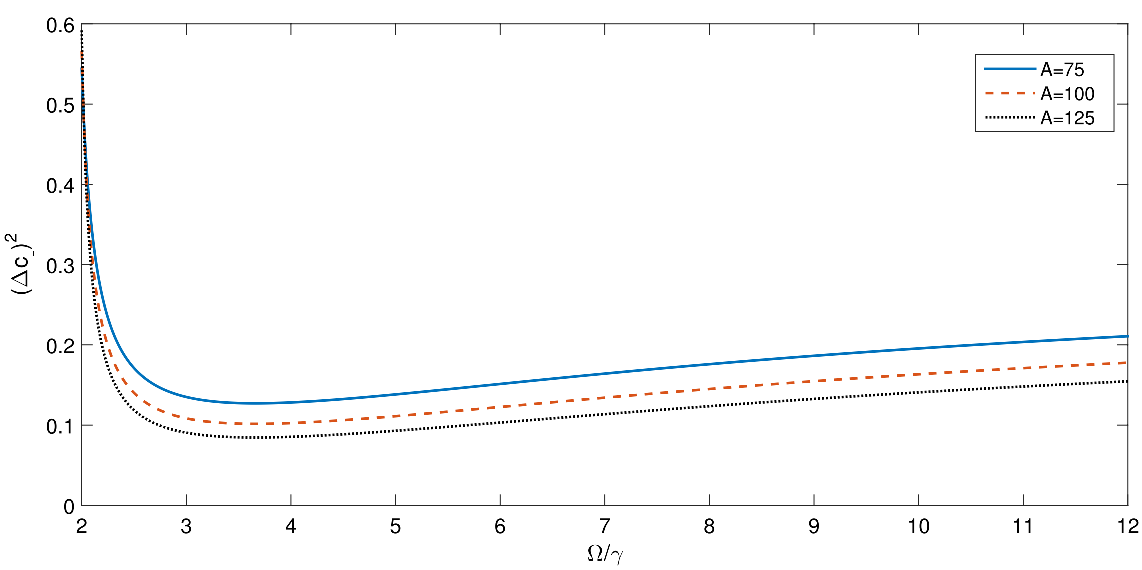

Fig. 2 Quadrature variance of the field modes versus Ω/γ, for $\kappa = 0.8$, $\gamma/\Gamma = 0.7$, and different values of A

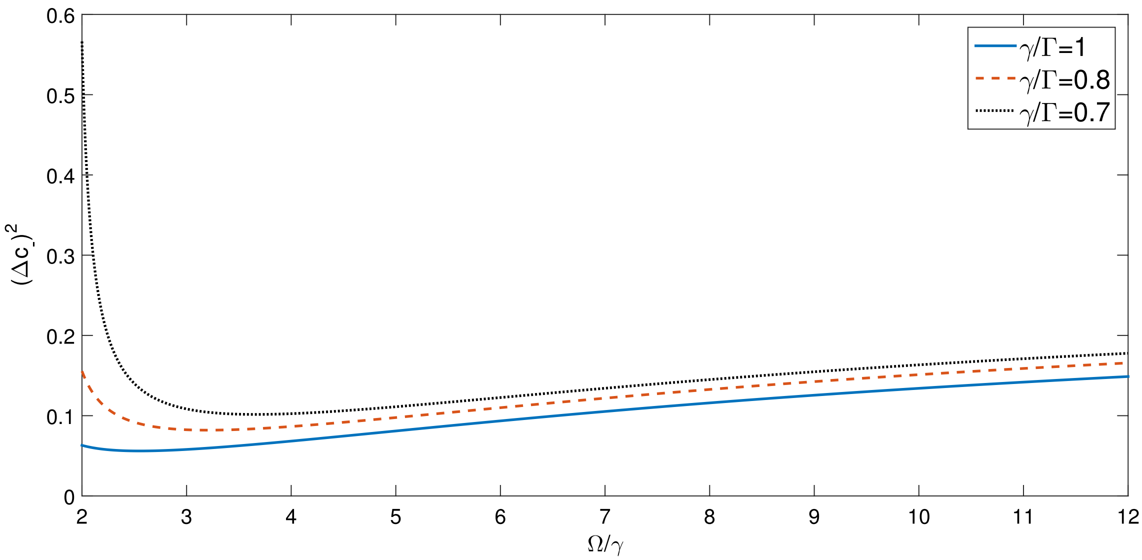

Fig. 3 Quadrature variance $(\Delta c\_)^2$ of the field modes versus Ω/γ for $\kappa = 0.8$, $A = 100$, and different values of γ/Γ

Quantum entanglement

Entanglement is one of the key resources that plays a crucial role in different quantum information processing. Thus, the quantification of the degree of entanglement is an important issue in quantum information science. The study of the quantification of entanglement in a two-photon three-level Laser has been considered [33–35, 49]. These studies reveal that the degree of entanglement depends on the atomic injection rate. One can quantify entanglement in a bipartite continuous-variable system by applying logarithmic negativity, defined mathematically by the relation [49]

$$E_N = max\left[0, -log_2 V_s\right], \tag{39}$$

in which V_s is the smallest symplectic eigenvalues of the partial transposed correlation matrix of the two field modes and is defined as $V_s = \sqrt{\frac{\zeta-\sqrt{\zeta^2-4det\mu}}{2}}$, with the symbol ζ can be expressed as ζ=$det\sigma_A + det\sigma_B - 2det\sigma_{AB}$. Here σ_A and σ_B denotes the 2×2 sub-matrices of the auto correlation of the two field modes, while σ_{AB} denotes the 2×2 sub-matrix of the cross correlation of the two field modes. The symbol μ is given by

$$\mu = \begin{bmatrix} \sigma_A & \sigma_{AB} \\ \sigma_{AB}^T & \sigma_B \end{bmatrix}. \tag{40}$$

The matrix elements of μ can be expressible as

$$\mu_{ij} = \frac{1}{2}\left\langle \hat{X}_i\hat{X}_j + \hat{X}_j\hat{X}_i \right\rangle - \left\langle \hat{X}_i \right\rangle\left\langle \hat{X}_j \right\rangle, \tag{41}$$

with $i, j = 1, 2, 3, 4$ and the quadrature operators are given by $\hat{X}_1 = \hat{a} + \hat{a}^\dagger$, $\hat{X}_2 = \frac{\hat{a}-\hat{a}^\dagger}{i}$, $\hat{X}_3 = \hat{b} + \hat{b}^\dagger$, and $\hat{X}_4 = \frac{\hat{b}-\hat{b}^\dagger}{i}$. The two field modes are entangled, provided that the logarithmic negativity (E_N) is positive. Thus, in view of Eq. (39), for E_N to be positive, $log_2 V_s$ must be negative. This leads to the condition $V_s < 1$. Hence, the condition $V_s < 1$ is a sufficient condition for the two field modes to be entangled.

It is clear from Fig. 4 that the two field modes generated by a coherently pumped cascade three-level laser are entangled for certain values of the amplitude of the pumping coherent light (Ω), the coherences decay rate (Γ), and atomic injection rate (A). It is appropriate to consider two cases of interest; the first case is when $\Omega/\gamma \approx< 50.5$ (weak coherent pumping regime) and the second case is when $\Omega/\gamma \approx> 50.5$ (strong coherent pumping regime). In Fig. 4, we plot V_s with Ω/γ for varied values of γ/Γ at a fixed value of the atomic injection rate, A and cavity dissipation rate κ. We see that the amount of entanglement increases with Ω/γ in the weak coherent pumping regime and decreases with the increase of Ω/γ in the strong coherent pumping regime.

We would like to emphasize that the dephasing rates are generally greater than the cavity dissipation rate and the spontaneous emission rates. Depasing refers to the decay of coherences, which is an inevitable quantum phenomena, and may lead to the recovery of classical properties. It is clear from Fig. 4 that the degree of entanglement is vulnerable to the dephasing processes. As displayed in Fig. 4, the dephasing rate has a diminishing effect on the amount of the photon-photon entanglement. For instance, when we increase the dephasing rates from $\Gamma = \gamma$ to $\Gamma = 2\gamma$ for $\Omega/\gamma = 42$, the

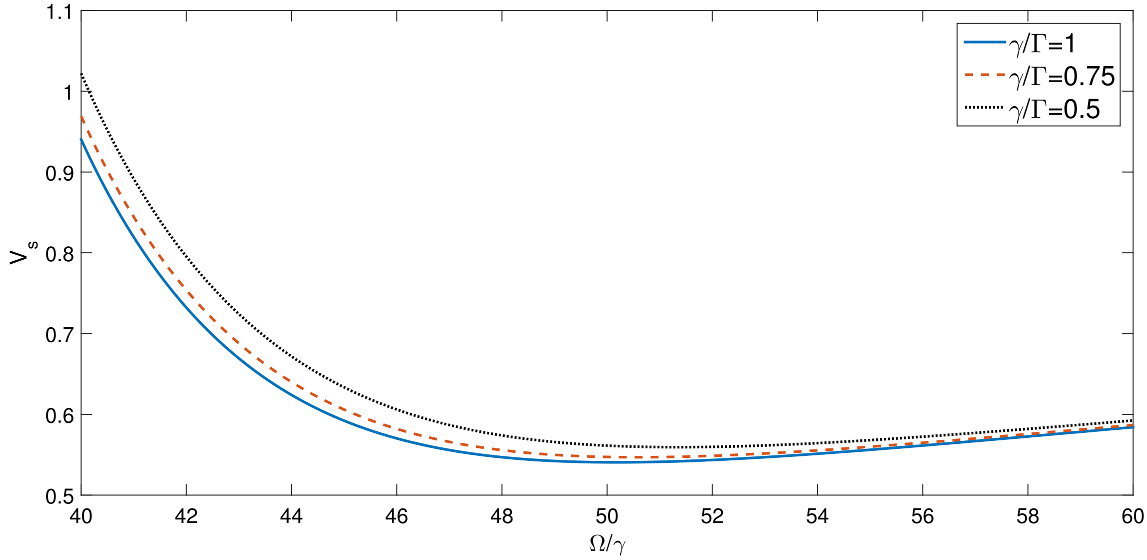

Fig. 4 Plots of V_s versus Ω/γ, for $\kappa = 0.8$, and $A = 25$, and different values of γ/Γ

maximum degree of entanglement decreases from 0.310 to 0.23. It is possible to generate robust entanglement in the presence of dephasing by tuning other system variables.

As shown in Fig. 5, the entanglement E_N between two field modes significantly depends on the atomic injection rate. We see from Eq. (26) that the coupling strength g and the spontaneous emission rates γ are constants in the expression for the linear gain coefficient. Thus, we can interpret the linear gain coefficient as the atomic injection rate. To study the effect of the atomic injection rate on the amount of entanglement of the field modes, we plot the entanglement with Ω/γ in the absence of dephasing processes ($\Gamma = \gamma$) and for $\kappa = 0.8$. Figure 5 shows that the amount of entanglement increases with atomic injection rate in the strong pumping regime. It is possible to see from Fig. 5 that the amount of entanglement rises from 0.2657 to 0.389 as atomic injection rate increases from 15 to 25 for $\Omega/\gamma = 60$.

Gaussian quantum discord

Entanglement and squeezing are not the only types of quantum correlations that can be realized within a quantum system's sub systems. We say two sub systems are correlated if

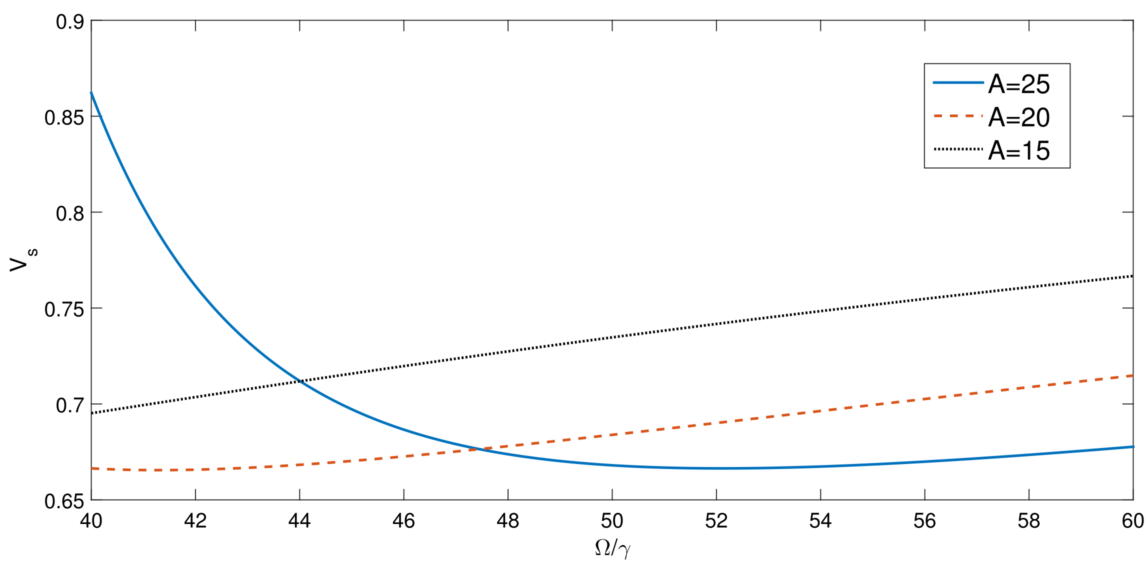

Fig. 5 Plots of V_s versus Ω/γ, for $\kappa = 0.8$, and $\gamma/\Gamma = 1$, and different values of A

the two sub systems together contain more information than taken separately. It is found that non-classical correlations can exist even in a separable bipartite state. In this regard, quantum discord is a measure of quantum correlation in the bipartite quantum state and is defined mathematically for the optical mode a by the relation [52]

$$D_A = f\left(\sqrt{det\sigma_B}\right) - f(\eta_+) - f(\eta_-) + f(\nu), \tag{42}$$

with

$$\eta_\pm = \left[\frac{\Sigma \pm \sqrt{\Sigma^2 - 4det\mu}}{2}\right], \tag{43}$$

in which

$$\Sigma = det\sigma_A + det\sigma_B + 2det\sigma_{AB},$$
$$\nu = \frac{\sqrt{det\sigma_A} + 2\sqrt{det\sigma_A(det\sigma_B)} + 2det\sigma_{AB}}{1 + 2\sqrt{det\sigma_A}}.$$

The corresponding continuous variable quantum discord for the cavity mode b is defined by

$$D_B = f\left(\sqrt{det\sigma_A}\right) - f(\eta_+) - f(\eta_-) + f(\nu') \tag{44}$$

in which

$$\nu' = \frac{\sqrt{det(\sigma_B)} + 2\sqrt{det(\sigma_A)det(\sigma_B)} + 2det(\sigma_{AB})}{1 + 2\sqrt{det(\sigma_B)}}, \tag{45}$$

and with the function f is defined by

$$f(x) = \left(x + \frac{1}{2}\right)ln\left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)ln\left(x - \frac{1}{2}\right) \tag{46}$$

If the quantum discord D has a range of values between $0 \le D < 1$, the two cavity modes of quantum states can be in a separable state. However, if the quantum discord D has a value greater than one ($D > 1$), the two cavity modes of the quantum states can not be separated. In Fig. 6, we plot the Gaussian quantum discord with Ω/γ for various values of the dephasing rates. For a given value of γ/Γ, the quantum discord decreases with increasing values of the amplitude of the pumping coherent light, and the effect of dephasing on the Gaussian quantum discord slightly diminishes as Ω/γ increases. It is not hard to see that the quantum discord increases with the decrease of the dephasing rates. The effect of atomic injection rate on the Gaussian quantum discord is depicted in Fig. 7. This Figure reveals that the atomic injection rate has a considerable effect on the degree of quantum discord. For instance, as A increases from 10 to 15 for $\Omega/\gamma = 60$, the amount of Gaussian quantum discord is enhanced by almost 8.04 %.

In Fig. 8, we plot quantum correlations such as quadrature squeezing, bipartite entanglement, and Gaussian quantum discord of two cavity modes with Ω/γ for fixed values of κ, A, and γ/Γ. We see that squeezing, entanglement, and quantum discord exhibit a similar evolution pattern, i.e, the quantum correlations decrease with the increase of Ω/γ. One can also note from the same figure that the Gaussian quantum discord is stronger than the squeezing and entanglement, while squeezing is the weakest of all the quantum correlations.

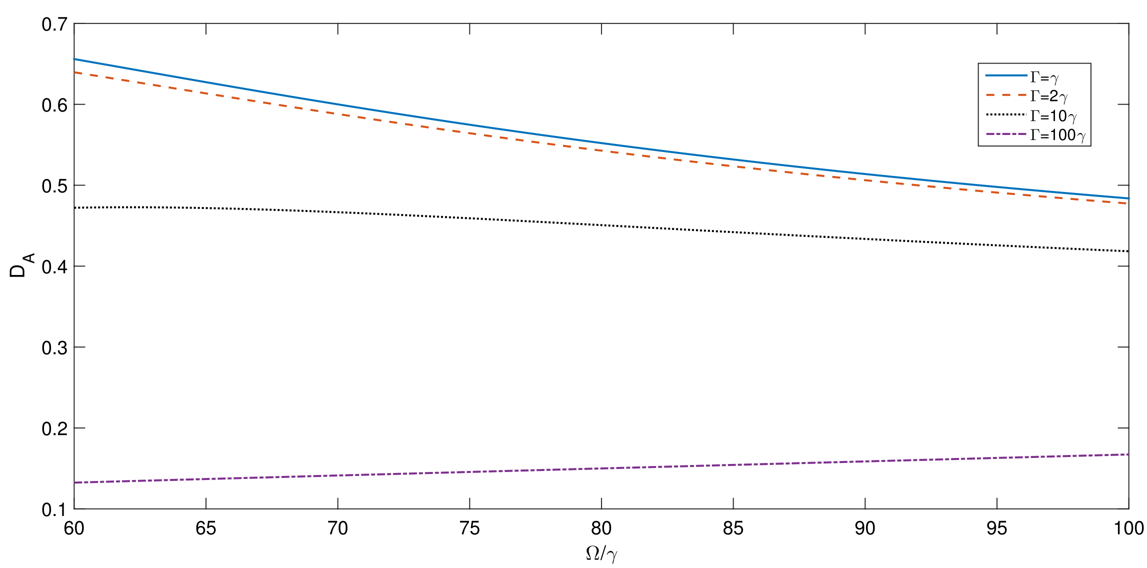

Fig. 6 Plots of quantum discord D_A versus Ω/γ, for $\kappa = 0.8$, $A = 25$, and different values of Γ

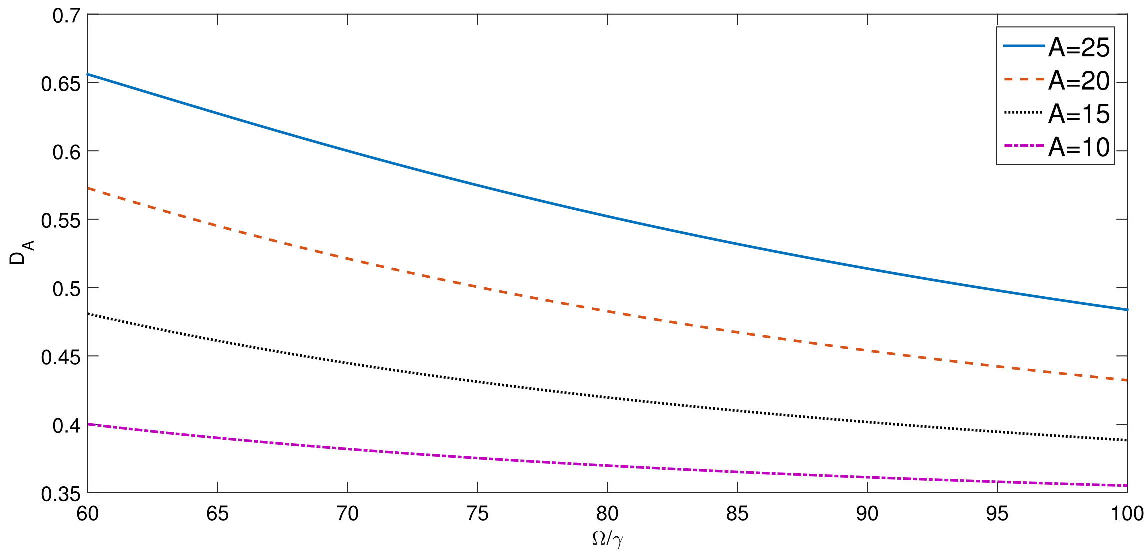

Fig. 7 Plots of quantum discord D_A versus Ω/γ, for $\kappa = 0.8$, $\gamma/\Gamma = 1$, and different values of A

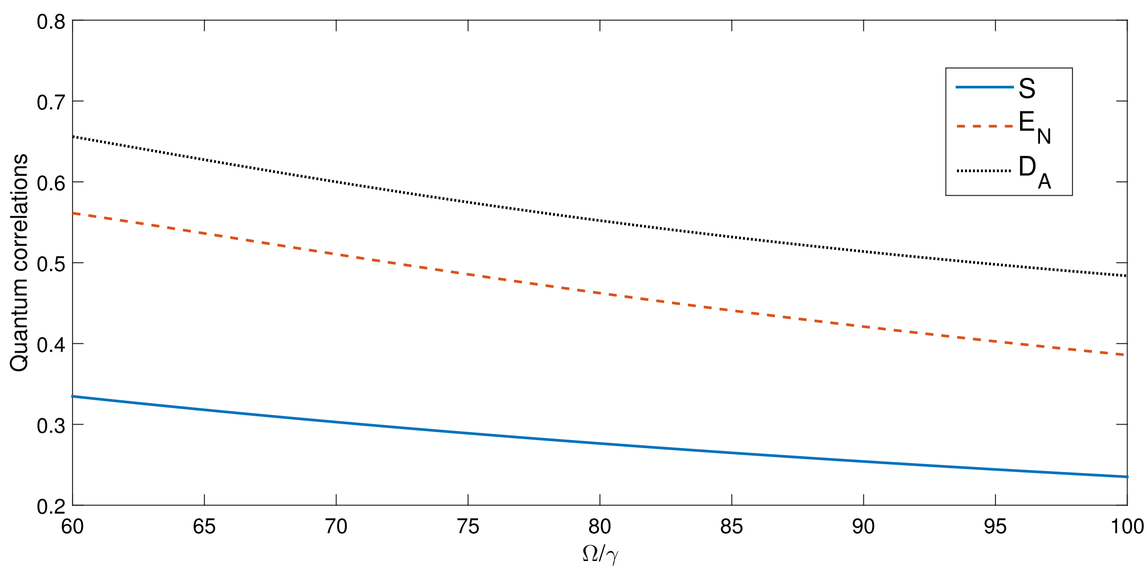

Fig. 8 Plots of quantum correlation versus Ω/γ, for $\kappa = 0.8$, $A = 25$, and $\gamma/\Gamma = 1$

Conclusion

We have examined the photon-photon quantum correlations (entanglement, quadrature squeezing, and Gaussian quantum discord) produced by a coherently pumped cascade three-level laser. We have obtained the master equation by taking into account the dephasing processes and using the adiabatic approximation scheme. Then the steady-state solutions of the cavity mode operators are determined with the help of the master equation. We have employed entanglement, quadrature squeezing, and Gaussian quantum discord to quantify the photon-photon quantum correlation. Specifically, we apply logarithmic negativity to measure the photon-photon entanglement generated by the scheme under consideration. We have shown that quantum coherence decays with the dephasing processes that lead to the destruction of quantum correlation between the two field modes, while the increase in the rate of atomic injection enhances the photon-photon quantum correlations. Moreover, we demonstrated that the quadrature squeezing and entanglement increase with the coherent pumping amplitude in the weak pumping regime and decrease with the coherent pumping amplitude in the strong pumping regime. In general, the amount of quantum discord is found to be higher than the amount of squeezing and entanglement for the same choice of system variables.

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