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ORIGINAL RESEARCH

Limit Cycles of Discontinuous Piecewise Differential Systems Formed by Linear and Cubic Centers via Averaging Theory

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Abstract

Finding the number of limit cycles, as described by Poincaré (Memoire sur les coubes definies par une equation differentielle, Editions Jacques Gabay, Sceaux, 1993), is one of the main problems in the qualitative theory of real planar differential systems. In general, studying limit cycles is a very challenging problem that is frequently difficult to solve. In this paper, we are interested in finding an upper bound for the maximum number of limit cycles bifurcating from the periodic orbits of a given discontinuous piecewise differential system when it is perturbed inside a class of polynomial differential systems of the same degree, by using the averaging method up to third order. We prove that the discontinuous piecewise differential systems formed by a linear focus or center and a cubic weak focus or center separated by one straight line $y = 0$ can have at most 7 limit cycles.

Keywords Cubic weak focus · Limit cycle · Discontinuous piecewise differential system · Averaging theory

Mathematics Subject Classification Primary 34C05 · 34A34

Introduction

In this paper, we deal with polynomial differential systems in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where the degree of the systems is the maximum degree of P and Q. The second part of the 16th Hilbert problem [7, 9] proposes to find an upper bound for the maximum number of limit cycles and relative configurations for the differential system (1). We recall that a limit

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cycle of the differential system (1) is an isolated periodic orbit in the set of all periodic orbits of the system.

The study of the limit cycles of piecewise differential systems has recently gained much attention in the qualitative theory of differential equations. The 16th Hilbert problem, which many writers have studied, involves determining the maximum number of limit cycles that a polynomial differentiation system of a specific order can have. See for example [8].

To study the periodic solutions of differential systems, the averaging theory is a useful tool; see for instance the books of Sanders, Verhulst, and Murdock [18] and Llibre Moeckel-Simó [15]. This method has a long history, dating back to the classical works of Lagrange and Laplace, who intuitively justified the process. In 1928, Fatou [4] formalized this theory. Significant practical and theoretical contributions to the averaging method were made in 1930s by Bogoliubov-Krylov [3] and in 1945 by Bogoliubov. This technique, originally created for smooth systems, has recently been applied to research on limit cycle of discontinuous piecewise smooth systems, see [14, 19]. The authors of the articles [6, 13] developed the averaging method for discontinuous piecewise differential systems and showed a relationship between the number of limit cycles of the corresponding differential system and the zeros of the averaged functions of periodic differential equations, see for instance [5, 11]. In 2022 Baymout and Benterki [1] proved that five is the maximum number of limit cycles that can bifurcate from the discontinuous piecewise differential systems formed by an arbitrary linear focus or center and an arbitrary cubic uniform isochronous center separated by a straight line, by using the averaging theory up to seven-order.

The objective of this paper is to study the limit cycles that can bifurcate from the discontinuous piecewise differential systems separated by the straight line $y = 0$ and formed by a linear differential system having a center or focus of the form

$$\dot{x} = \alpha x + \beta y + \gamma, \qquad \dot{y} = -\beta x + \alpha y + \delta. \tag{2}$$

defined in the half-plane $y \geq 0$, where α, β, γ, and $\delta \in \mathbb{R}$, and by an arbitrary cubic weak focus or center located at the origin given by

$$\begin{aligned} \dot{x} &= -y - ax^2 - cxy - zy^2 - kx^3 - mx^2y - pxy^2 - hy^3, \\ \dot{y} &= x + by^2 + dxy + gx^2 + ly^3 + nxy^2 + qx^2y + wx^3. \end{aligned} \tag{3}$$

defined in the half-plane $y \leq 0$, where all the parameters of the system are real.

The averaging theory described in "Third Order Averaging Theory for Computing Limit Cycles" section allows to study analytically the existence of limit cycles of a non-autonomous differential system, by studying the simple zeros of the averaged function. Here we shall use the averaging theory up to the third order for studying the number of limit cycles that can bifurcate from the discontinuous piecewise differential systems formed by (2) for $y \geq 0$, when we perturb it inside the class of all polynomial differential systems of degree 1 as follows

$$\dot{x} = \sum_{i=1}^{3} P_{1i}(x, y)\varepsilon^i, \dot{y} = \sum_{i=1}^{3} Q_{1i}(x, y)\varepsilon^i, \tag{4}$$

and by the differential system (3) for $y \leq 0$ when we perturb it inside the class of all polynomial differential systems of degree 3 as follows

$$\dot{x} = \sum_{i=1}^{3} P_{3i}(x,y)\varepsilon^i, \dot{y} = \sum_{i=1}^{3} Q_{3i}(x,y)\varepsilon^i. \tag{5}$$

Here $\varepsilon > 0$ is a small parameter, $i = 1, \ldots, 3$, P_{1i} and Q_{1i}, are real polynomials of degree 1 in the variables x and y, and P_{3i}, Q_{3i} are real polynomials of degree 3 in the variables x and y.

The main result of our paper focuses on determining the maximum number of limit cycles using the averaging theory up to third order, which is presented in the following Theorem.

Theorem 1 *For $|\varepsilon| \neq 0$ sufficiently small and by using the averaging theory up to third order the maximum number of limit cycles of the discontinuous piecewise differential systems formed by linear differential focus or center (2) and the cubic weak focus or center (3) is at most seven. There are examples with exactly seven limit cycles bifurcating from the periodic orbits of these systems.*

Theorem 1 is proved in "Proof of Theorem 1" section 3.

Third Order Averaging Theory for Computing Limit Cycles

In this section we summarise the basic results of the classical averaging theory that we will use to study the number of limit cycles of discontinuous piecewise differential systems, for more details see [10].

We consider the following discontinuous differential system

$$\dot{r}(\theta) = \begin{cases} F^+(\theta, r, \varepsilon) & if \quad 0 \leq \theta \leq \pi, \\ F^-(\theta, r, \varepsilon) & if \quad \pi \leq \theta \leq 2\pi. \end{cases} \tag{6}$$

where $F^{\pm}(\theta, r, \varepsilon) = \sum_{i=0}^{3} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^4 R^{\pm}(\theta, r, \varepsilon)$, $\theta \in \mathcal{S}^1$ and $r \in D$ where D is an open interval of $\mathbb{R}^+$.

A fundamental inquiry in the investigation of discontinuous differential systems (6) revolves around comprehending which periodic orbits of the unperturbed system $\dot{r}(\theta) = F^{\pm}(\theta, r)$ persists for $|\varepsilon| \neq 0$ sufficiently small. To address this, we introduce a set of functions $f_i : D \longrightarrow \mathbb{R}$, for $i = 1, 2, \ldots, k$, called averaged functions, such that their simple zeros provide the existence of isolated periodic solutions of the differential equation (6). In [12] it was proved that these averaged functions are given by $f_i = \dfrac{y_i(2\pi, r)}{i!}$ where $y_i : \mathbb{R} \times D \to \mathbb{R}$, are defined by the following integrals

$$\begin{aligned}
y_1^{\pm}(s, r) = &\int_0^s F_1^{\pm}(t, r)dt, \\
y_2^{\pm}(s, r) = &\int_0^s \Big(2F_2^{\pm}(t, r) + 2\partial F_1^{\pm}(t, r)y_1^{\pm}(t, r)\Big)dt, \\
y_3^{\pm}(s, r) = &\int_0^s \Big(6F_3^{\pm}(t, r) + 6\partial F_2^{\pm}(t, r)y_1^{\pm}(t, z) + 3\partial^2 F_1^{\pm}(t, r)y_1^{\pm}(t, r)^2 + 3\partial F_1^{\pm}(t, r) \\
&y_2^{\pm}(t, r)\Big)dt.
\end{aligned}$$

Also, we have the functions

$$\begin{aligned} f_1^{\pm}(r) =& \int_0^{\pm\pi} F_1^{\pm}(t,r)dt, \\ f_2^{\pm}(r) =& \int_0^{\pm\pi} \Big(F_2^{\pm}(t,r)dt + \partial F_1^{\pm}(t,r)y_1^{\pm}(t,r)\Big)dt, \\ f_3^{\pm}(r) =& \int_0^{\pm\pi} \Big(F_3^{\pm}(t,r)dt + \partial F_2^{\pm}(t,r)y_1^{\pm}(t,r) + \frac{1}{2}\partial^2 F_1^{\pm}(t,r)y_1^{\pm}(t,r)^2 + \frac{1}{2}\partial F_1^{\pm}(t,r) \\ & y_2^{\pm}(t,r)\Big)dt. \end{aligned}$$

For more details see [10].

The averaged function of order k is the function $f_k(r) = f_k^{+}(r) + f_k^{-}(r)$. The simple positive real roots of the functions $f_{l+1}(r)$ which satisfy $f_l(r) = 0$ for $l \in \{1, 2\}$ but $f_{l+1}(r) \neq 0$, provide limit cycles of the piecewise differential system (6).

We need to state the following lemma and Descartes Theorem in order to demonstrate our results regarding the number of zeros in a real polynomial.

Lemma 2 *Consider $p+1$ linearly independent functions $f_i : U \subset \mathbb{R} \to \mathbb{R}, i = 0, 1, \ldots, p$*

(i) *Given p arbitrary values $x_i \in U$, $i = 1, \ldots, p$ there exist $p+1$ constants C_i, $i = 0, 1, \ldots, p$ such that*

$$f(x) := \sum_{i=0}^{p} C_i f_i(x) \tag{7}$$

is not the zero function and $f(x_i) = 0$ for $i = 0, 1, \ldots, p$.

(ii) *Furthermore, if all f_i are analytical functions on U and it exists $j \in \{1, \ldots, p\}$ such that $f_j|U$ has constant sign, it is possible to get an f given by (7), such that it has at least p simple zeroes in U.*

For a proof, see Proposition 1 of [16].

Theorem 3 (Descartes Theorem) *Consider the real polynomial $r(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \ldots + a_{i_r}x^{i_r}$ with $0 = i_1 < i_2 < \ldots < i_r$ and $a_{i_j} \neq 0$ real constant for $j \in \{1, \ldots, r\}$. When $a_{i_j}a_{i_{j+1}}<0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of the sign. If the number of variations of signs is m, then $r(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $r(x)$ in such a way that $r(x)$ has exactly $r-1$ positive real roots.*

For more details see [2].

Proof of Theorem 1

In order to apply the averaging method for studying the limit cycles for ε sufficiently small, we need to write systems in the standard form. So we have developed the parameters of the differential systems until the third order in ε. To ensure that the origin of system (2) is a center, we must add -1 with regard to the growth of β. Then in $y \geq 0$ we have the following system

$$\dot{x} = -y + \alpha x + \beta y + \gamma, \qquad \dot{y} = x - \beta x + \alpha y + \delta,$$

with $\begin{array}{ll} \alpha = \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \alpha_3\varepsilon^3, & \beta = -1 + \beta_1\varepsilon + \beta_2\varepsilon^2 + \beta_3\varepsilon^3, \\ \gamma = \gamma_1\varepsilon + \gamma_2\varepsilon^2 + \gamma_3\varepsilon^3, & \delta = \delta_1\varepsilon + \delta_2\varepsilon^2 + \delta_3\varepsilon^3. \end{array}$

Then the perturbed system of system (2) is given by

$$\begin{aligned} \dot{x} = & -2y + \varepsilon(\alpha_1 x + \beta_1 y + \gamma_1) + \varepsilon^2(\alpha_2 x + \beta_2 y + \gamma_2) + \varepsilon^3(\alpha_3 x + \beta_3 y + \gamma_3), \\ \dot{y} = & \ 2x + \varepsilon(-\beta_1 x + \alpha_1 y + \delta_1) + \varepsilon(-\beta_2 x + \alpha_2 y + \delta_2) + \varepsilon^3(-\beta_3 x + \alpha_3 y + \delta_3). \end{aligned} \tag{8}$$

According to system (4) we know that $P_{11}(x,y) = \alpha_1 x + \beta_1 y + \gamma_1$, $P_{12}(x,y) = \alpha_2 x + \beta_2 y + \gamma_2$, $P_{13}(x,y) = \alpha_3 x + \beta_3 y + \gamma_3$, $Q_{11}(x,y) = -\beta_1 x + \alpha_1 y + \delta_1$, $Q_{12}(x,y) = -\beta_2 x + \alpha_2 y + \delta_2$, $Q_{13}(x,y) = -\beta_3 x + \alpha_3 y + \delta_3$.

In $y \le 0$ we have the differential system

$$\begin{aligned} \dot{x} = & -y - ax^2 - cxy - zy^2 - kx^3 - mx^2y - pxy^2 - hy^3, \\ \dot{y} = & \ by^2 + dxy + gx^2 + ly^3 + nxy^2 + qx^2y + wx^3 + x. \end{aligned}$$

Where $\begin{array}{lll} a = a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3, & c = c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3, & p = p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3, \\ z = z_1\varepsilon + z_2\varepsilon^2 + z_3\varepsilon^3, & k = k_1\varepsilon + k_2\varepsilon^2 + k_3\varepsilon^3, & m = m_1\varepsilon + m_2\varepsilon^2 + m_3\varepsilon^3, \\ h = h_1\varepsilon + h_2\varepsilon^2 + h_3\varepsilon^3, & g = g_1\varepsilon + g_2\varepsilon^2 + g_3\varepsilon^3, & d = d_1\varepsilon + d_2\varepsilon^2 + d_3\varepsilon^3, \\ b = b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3, & w = w_1\varepsilon + w_2\varepsilon^2 + w_3\varepsilon^3, & q = q_1\varepsilon + q_2\varepsilon^2 + q_3\varepsilon^3, \\ n = n_1\varepsilon + n_2\varepsilon^2 + n_3\varepsilon^3, & l = l_1\varepsilon + l_2\varepsilon^2 + l_3\varepsilon^3, & \end{array}$

Then the perturbed system of system (3) is given by

$$\begin{aligned} \dot{x} = & -y + \varepsilon(-a_1x^2 - c_1xy - z_1y^2 - k_1x^3 - m_1x^2y - p_1xy^2 - h_1y^3) + \varepsilon^2(-a_2x^2 \\ & - c_2xy - z_2y^2 - k_2x^3 - m_2x^2y - p_2xy^2 - h_2y^3) + \varepsilon^3(-a_3x^2 - c_3xy - z_3y^2 \\ & - k_3x^3 - m_3x^2y - p_3xy^2 - h_3y^3), \\ \dot{y} = & \ x + \varepsilon(b_1y^2 + d_1xy + g_1x^2 + l_1y^3 + n_1xy^2 + q_1x^2y + w_1x^3) + \varepsilon^2(b_2y^2 + d_2xy \\ & + g_2x^2 + l_2y^3 + n_2xy^2 + q_2x^2y + w_2x^3) + \varepsilon^3(b_3y^2 + d_3xy + g_3x^2 + l_3y^3 + n_3xy^2 \\ & + q_3x^2y + w_3x^3). \end{aligned} \tag{9}$$

According to system (5) we know that
$P_{31}(x,y) = -a_1x^2 - c_1xy - z_1y^2 - k_1x^3 - m_1x^2y - p_1xy^2 - h_1y^3$,
$P_{32}(x,y) = -a_2x^2 - c_2xy - z_2y^2 - k_2x^3 - m_2x^2y - p_2xy^2 - h_2y^3$,
$P_{33}(x,y) = -a_3x^2 - c_3xy - z_3y^2 - k_3x^3 - m_3x^2y - p_3xy^2 - h_3y^3$,
$Q_{31}(x,y) = b_1y^2 + d_1xy + g_1x^2 + l_1y^3 + n_1xy^2 + q_1x^2y + w_1x^3$,
$Q_{32}(x,y) = b_2y^2 + d_2xy + g_2x^2 + l_2y^3 + n_2xy^2 + q_2x^2y + w_2x^3$,
$Q_{33}(x,y) = b_3y^2 + d_3xy + g_3x^2 + l_3y^3 + n_3xy^2 + q_3x^2y + w_3x^3$.

We compute the averaged function $f_i(r)$, for $i = 1$ we get

$$f_1(r) = \frac{1}{8}\pi r^3\left(-3k_1 + 3l_1 - p_1 + q_1\right) - \frac{2}{3}r^2\left(2b_1 - c_1 + g_1\right) + \pi\alpha_1 r + 2\delta_1.$$

By using Descartes Theorem we know that the polynomial $f_1(r)$ can have at most three positive real roots, which provide three limit cycles for the discontinuous piecewise differential system (2)-(3).

In order to apply the averaging theory of second order we need that $f_1(r) \equiv 0$. So we must take $c_1 = 2b_1 + g_1$, $p_1 = -3k_1 + 3l_1 + q_1$, $\delta_1 = 0$, $\alpha_1 = 0$. Computing the function

$f_2(r)$ we get

$$f_2(r)= \frac{1}{16}\pi r^5(h_1(q_1-3k_1)+k_1(-m_1+n_1+3w_1)+2l_1m_1-2l_1n_1+m_1q_1-n_1q_1 -q_1w_1)+\frac{2}{15}r^4(a_1(3k_1-4l_1-3q_1)-4b_1m_1+4b_1n_1-3d_1k_1+2d_1l_1+2d_1q_1 +g_1(-2h_1-3m_1+2n_1+3w_1)+6k_1z_1-2q_1z_1)+\frac{1}{8}\pi r^3(2a_1b_1+3a_1g_1-b_1d_1 -d_1g_1+g_1z_1-3k_2+3l_2-p_2+q_2)-\frac{2}{3}r^2(2b_2-c_2+g_2)+\pi\alpha_2 r+2\delta_2.$$

This polynomial can have at most five positive real roots, which provide at most five limit cycles for the discontinuous piecewise differential system (2)-(3).

In order to apply the averaging theory of third order we need to have $f_2(r)\equiv 0$, for that we must take

$$w_1= -\frac{1}{g_1(3k_1-q_1)}\Big((3k_1-q_1)(a_1(3k_1-4l_1-3q_1)-4b_1m_1+4b_1n_1+d_1(2(l_1+q_1) -3k_1)-3g_1m_1+2g_1n_1+6k_1z_1-2q_1z_1)+2g_1(m_1-n_1)(k_1-2l_1-q_1)\Big),$$

$$c_2= 2b_2+g_2, p_2=a_1(2b_1+3g_1)-d_1\left(b_1+g_1\right)+g_1z_1-3k_2+3l_2+q_2, \alpha_2=0,$$

$$h_1= -\frac{1}{g_1(3k_1-q_1)}\Big(a_1(3k_1-q_1)(3k_1-4l_1-3q_1)-12b_1k_1m_1+12b_1k_1n_1 +4b_1m_1q_1-4b_1n_1q_1-d_1(3k_1-q_1)(3k_1-2(l_1+q_1))-6g_1k_1m_1+3g_1k_1n_1 -6g_1l_1m_1+6g_1l_1n_1+g_1n_1q_1+2z_1(q_1-3k_1)^2\Big), \delta_2=0.$$

For w_1 and h_1 we considered four cases $g_1(3k_1-q_1)\neq 0$, $g_1=0$ and $3k_1-q_1\neq 0$, $g_1\neq 0$ and $3k_1-q_1=0$ or $g_1=0$ and $3k_1-q_1=0$. We start with the first case $g_1(3k_1-q_1)\neq 0$.

Case 1. $g_1(3k_1-q_1)\neq 0$. Computing the function $f_3(r)$ we obtain

$$f_3(r)=A_1r^7+A_2r^6+A_3r^5+A_4r^4+A_5r^3-\frac{2}{3}\left(2b_3-c_3+g_3\right)r^2+\pi\alpha_3 r+2\delta_3.$$

Where

$$A_1= \frac{1}{64g_1\left(3k_1-q_1\right)}\Big(\pi(k_1-5l_1-2q_1)((3k_1-q_1)(m_1-n_1)(a_1(3k_1-4l_1-3q_1) -4b_1m_1+4b_1n_1+d_1(-3k_1+2l_1+2q_1)+6k_1z_1-2q_1z_1)+g_1(k_1(6l_1q_1+12m_1n_1 -6m_1^2-6n_1^2+q_1^2)-3k_1^2(3l_1+2q_1)+9k_1^3-l_1(-8m_1n_1+4m_1^2+4n_1^2+q_1^2)))\Big),$$

$$A_2= \frac{1}{105g_1\left(q_1-3k_1\right)^2}\Big(2((297k_1^4-36(9l_1+7q_1)k_1^3+3(72l_1^2+132q_1l_1-32m_1^2-32n_1^2 +23q_1^2+64m_1n_1)k_1^2-6(24q_1l_1^2+2(4m_1^2-8n_1m_1+4n_1^2+13q_1^2)l_1+q_1(-8m_1^2 +16n_1m_1-8n_1^2+q_1^2))k_1+4l_1(6l_1(2m_1^2-4n_1m_1+2n_1^2+q_1^2)+q_1(12m_1^2-24n_1m_1 +12n_1^2+5q_1^2)))g_1^2+4(m_1-n_1)(3k_1-q_1)(d_1(-15k_1^2+33l_1k_1+24q_1k_1-6q_1^2 -10l_1q_1)+a_1(12k_1^2-3(13l_1+8q_1)k_1+q_1(11l_1+6q_1))+2(3k_1-q_1)(3k_1-4l_1 -3q_1)z_1)g_1-48b_1^2(m_1-n_1)^2(q_1-3k_1)^2+2(q_1-3k_1)^2(2(3l_1+q_1)(3k_1-4l_1-3q_1) a_1^2+((3k_1-q_1)(3k_1+8l_1+q_1)z_1-d_1(9k_1-8l_1-7q_1)(3l_1+q_1))a_1+2(q_1-3k_1)^2 z_1^2+d_1^2(3l_1+q_1)(3k_1-2(l_1+q_1))-d_1(3k_1+4l_1)(3k_1-q_1)z_1)+2b_1(q_1-3k_1)^2(g_1 (27k_1^2-3(12l_1+5q_1)k_1+2(-12m_1^2+24n_1m_1-12n_1^2+q_1^2+6l_1q_1))+2(m_1-n_1) (-9d_1k_1+12z_1k_1+12d_1l_1+8d_1q_1+a_1(9k_1-24l_1-13q_1)-4q_1z_1)))\Big),$$

$$A_3 = \frac{1}{16g_1\left(3k_1-q_1\right)}\Big(\pi(2(3k_1-q_1)(b_1(3k_1-4l_1-3q_1)+g_1(8k_1-11l_1-8q_1))a_1^2$$
$$+(-8(m_1-n_1)(3k_1-q_1)b_1^2+(4(z_1(q_1-3k_1)^2+g_1(-20k_1m_1-2l_1m_1+6q_1m_1$$
$$+17k_1n_1+2l_1n_1-5n_1q_1))-d_1(9k_1-8l_1-7q_1)(3k_1-q_1))b_1+g_1(-d_1(3k_1-q_1)$$
$$(21k_1-17l_1-15q_1)+g_1(-37k_1m_1-16l_1m_1+7q_1m_1+19k_1n_1+16l_1n_1-n_1q_1)$$
$$+(3k_1-q_1)(39k_1-10l_1-17q_1)z_1))a_1+b_1^2(3k_1-q_1)(3g_1k_1+4d_1m_1-4d_1n_1$$
$$-g_1q_1)+g_1((3k_1-q_1)(5k_1-3(l_1+q_1))d_1^2+(g_1(6l_1(m_1-n_1)+k_1(9m_1-3n_1)$$
$$-(m_1+n_1)q_1)-(18k_1-5l_1-8q_1)(3k_1-q_1)z_1)d_1-9h_2k_1^2-h_2q_1^2-m_2q_1^2+n_2q_1^2$$
$$+45k_1^2z_1^2+5q_1^2z_1^2-30k_1q_1z_1^2-6k_2l_1m_1+6k_1l_2m_1-3k_1^2m_2+6k_1l_1m_2+6k_2l_1n_1$$
$$-6k_1l_2n_1+3k_1^2n_2-6k_1l_1n_2-2g_1^2(k_1+l_1)(3k_1-q_1)+6h_2k_1q_1-2k_2m_1q_1-2l_2m_1q_1$$
$$+4k_1m_2q_1-2l_1m_2q_1+2k_2n_1q_1+2l_2n_1q_1-4k_1n_2q_1+2l_1n_2q_1+2k_1m_1q_2+2l_1m_1q_2$$
$$-2k_1n_1q_2-2l_1n_1q_2+9k_1^2w_2+q_1^2w_2-6k_1q_1w_2+g_1(-10l_1(m_1-n_1)+k_1(7n_1$$
$$-13m_1)+(m_1+n_1)q_1)z_1)+b_1((3k_1-q_1)(3k_1-2(l_1+q_1))d_1^2+(g_1(25k_1m_1+4l_1m_1$$
$$-7q_1m_1-19k_1n_1-4l_1n_1+5n_1q_1)-2(q_1-3k_1)^2z_1)d_1+g_1(3k_1-q_1)(g_1(3k_1-5l_1$$
$$-2q_1)+10(n_1-m_1)z_1)))\Big),$$

$$A_4 = \frac{1}{15g_1}\Big(2(a_1(4b_1d_1g_1+6d_1g_1^2+3g_1k_2-3g_2k_1-4g_1l_2+4g_2l_1-3g_1q_2+3g_2q_1$$
$$-6g_1^2z_1)+3a_2g_1k_1-4a_2g_1l_1-3a_2g_1q_1-3a_1^2g_1^2+b_1(4(-g_1m_2+g_2m_1+g_1n_2$$
$$-g_2n_1+g_1^3)-2d_1^2g_1)-4b_2g_1m_1+4b_2g_1n_1+4b_1^2g_1^2-3d_2g_1k_1-3d_1g_1k_2$$
$$+3d_1g_2k_1+2d_2g_1l_1+2d_1g_1l_2-2d_1g_2l_1+2d_2g_1q_1+2d_1g_1q_2-2d_1g_2q_1$$
$$+2d_1g_1^2z_1-2d_1^2g_1^2-2g_1^2h_2+6g_1k_2z_1+6g_1k_1z_2-6g_2k_1z_1-3g_1^2m_2+2g_1^2n_2$$
$$-2g_1q_2z_1-2g_1q_1z_2+2g_2q_1z_1+3g_1^2w_2)\Big),$$

$$A_5 = \frac{\pi}{8}\Big(2a_2b_1+2a_1b_2+3a_2g_1+3a_1g_2-b_1d_2-b_2d_1-d_2g_1-d_1g_2+g_2z_1+g_1z_2$$
$$-3k_3+3l_3-p_3+q_3\Big).$$

Since the rank of the Jacobian matrix of the function $\mathcal{A}=(A_1,\dots A_5,-\frac{2}{3}\left(2b_3-c_3+g_3\right),\pi\alpha_3,2\delta_3)$ with respect to its parameters which appear in their expressions is maximal, i.e. it is 8. In view of Lemma 2, we conclude that the maximum number of real solutions of the equation $f_3(r)=0$ is at most seven. Now by using Descartes Theorem we conclude that the function $f_3(r)=0$ can have at most seven positive solutions. Therefore the averaging theory up to third order can provide at most seven limit cycles for the discontinuous piecewise differential system (2)-(3).

Now we consider the second case.

Case 2. $g_1=0$ and $q_1\neq 3k_1$. Computing the function $f_2(r)$ we obtain

$$f_2(r) = \frac{1}{16}\pi r^5(h_1(q_1-3k_1)+k_1(-m_1+n_1+3w_1)+2l_1m_1-2l_1n_1+m_1q_1-n_1q_1$$
$$-q_1w_1)+\frac{2}{15}r^4(a_1(3k_1-4l_1-3q_1)-4b_1m_1+4b_1n_1+d_1(-3k_1+2l_1+2q_1)$$
$$+6k_1z_1-2q_1z_1)+\frac{1}{8}\pi r^3\left(a_1c_1-b_1d_1-3k_2+3l_2-p_2+q_2\right)-\frac{2}{3}r^2(2b_2-c_2+g_2)$$
$$+\pi\alpha_2 r+2\delta_2.$$

So the polynomial $f_2(r)$ can have at most five positive real roots and produce at most five limit cycles for the discontinuous piecewise differential system (2)-(3) when ε is sufficiently small. In order to apply the averaging theory of third order we put $f_2(r) \equiv 0$. So we need to consider

$$c_2 = 2b_2 + g_2, p_2 = a_1c_1 - b_1d_1 - 3k_2 + 3l_2 + q_2, \alpha_2 = 0, \delta_2 = 0,$$

$$w_1 = \frac{1}{4b_1(3k_1 - q_1)}\Big((k_1 - 2l_1 - q_1)(a_1(3k_1 - 4l_1 - 3q_1) + d_1(2(l_1 + q_1) - 3k_1) + 6k_1z_1 - 2q_1z_1) + 4b_1h_1(3k_1 - q_1)\Big),$$

$$m_1 = \frac{1}{4b_1}\Big(a_1(3k_1 - 4l_1 - 3q_1) + 4b_1n_1 + d_1(2(l_1 + q_1) - 3k_1) + 6k_1z_1 - 2q_1z_1\Big).$$

From the expression of $f_2(r)$ we distinguish immediately two subcases $b_1 \neq 0$ or $b_1 = 0$.

Subcase 2.1. $b_1 \neq 0$. Computing $f_3(r)$ we get

$$f_3(r) = B_1r^7 + B_2r^6 + B_3r^5 + B_4r^4 + B_5r^3 - \frac{2}{3}\big(2b_3 - c_3 + g_3\big)r^2 + \pi\alpha_3 r + 2\delta_3.$$

Where

$$B_1 = \frac{\pi}{512b_1^2\big(3k_1 - q_1\big)}(k_1 - 5l_1 - 2q_1)(-2a_1(3k_1 - 4l_1 - 3q_1)(b_1(h_1 - n_1)(3k_1 - q_1) + l_1(d_1(3k_1 - 2(l_1 + q_1)t) + 2z_1(q_1 - 3k_1))) + a_1^2l_1(-3k_1 + 4l_1 + 3q_1)^2 + 8b_1^2(k_1 - l_1)(q_1 - 3k_1)^2 + c_1h_1 - n_1)(3k_1 - q_1)d_1(3k_1 - 2l_1 + q_1)) + 2z_1q_1 - 3k_1)) + l_1(d_1(2(l_1 + q_1) - 3k_1) + 6k_1z_1 - 2q_1z_1)^2),$$

$$B_2 = \frac{1}{105b_1(3k_1 - q_1)}\Big(- 2a_1(b_1(h_1 - n_1)(-24k_1q_1 + 27k_1^2 + 5q_1^2) + d_1(3k_1^2(8l_1 + q_1) - 3k_1(15l_1q_1 + 5l_1^2 + 3q_1^2) + q_1(29l_1q_1 + 25l_1^2 + 6q_1^2)) + z_1(3k_1 - q_1)(-33k_1l_1 - 9k_1q_1 + 3k_1^2 + 29l_1q_1 + 16l_1^2 + 6q_1^2)) + a_1^2(k_1^2(75l_1 + 36q_1) - 3k_1(50l_1q_1 + 28l_1^2 + 15q_1^2) - 9k_1^3 + q_1(75l_1q_1 + 68l_1^2 + 18q_1^2)) + 4b_1^2(q_1 - 3k_1)^2(9k_1 - 2(6l_1 + q_1)) + c_1(h_1 - n_1)(3k_1 - q_1)(d_1(9k_1 - 4q_1) + 8z_1(q_1 - 3k_1)) + 90d_1k_1l_1q_1z_1 + 6d_1^2k_1l_1q_1 - 54d_1k_1^2l_1z_1 + 48d_1k_1l_1^2z_1 - 15d_1^2k_1^2l_1 + 6d_1^2k_1l_1^2 + 42d_1k_1^2q_1z_1 - 12d_1k_1q_1^2z_1 - 24d_1^2k_1^2q_1 + 12d_1^2k_1q_1^2 - 18d_1k_1^3z_1 + 9d_1^2k_1^3 - 24d_1l_1q_1^2z_1 - 16d_1l_1^2q_1z_1 + 8d_1^2l_1q_1^2 + 8d_1^2l_1^2q_1 - 96k_1l_1q_1z_1^2 + 144k_1^2l_1z_1^2 + 16l_1q_1^2z_1^2\Big),$$

$$B_3 = \frac{-\pi}{64b_1(3k_1 - q_1)}\Big(a_1(b_1(2z_1(3k_1 - q_1)(5k_1 - 4l_1 - 5q_1) - d_1(-26k_1l_1 - 22k_1q_1 + 9k_1^2 + 22l_1q_1 + 16l_1^2 + 9q_1^2)) + 8b_1^2(h_1 + n_1)(3k_1 - q_1) - 2(3k_1 - 4l_1 - 3q_1)(-k_2(3l_1 + q_1) + 3k_1l_2 + k_1q_2 - l_2q_1 + l_1q_2)) + 4a_1^2b_1(-5k_1(l_1 + q_1) + 3k_1^2 + 5l_1q_1 + 4l_1^2 + 2q_1^2) + b_1(2d_1z_1(3k_1 - q_1)(2k_1 + 2l_1 + q_1) + d_1^2(-8k_1l_1 - 3k_1q_1 - 3k_1^2 + 6l_1q_1 + 4l_1^2 + 2q_1^2) + 4(3k_1 - q_1)(3h_2k_1 - h_2q_1 + k_1m_2 - k_1n_2 - 3k_1w_2 - 2l_1m_2 + 2l_1n_2 - m_2q_1 + n_2q_1 + q_1w_2)) - 4b_1^2d_1(h_1 + n_1)(3k_1 - q_1) - 4b_1^3(q_1 - 3k_1)^2 + 2(k_2(3l_1 + q_1) - k_1(3l_2 + q_2) + l_2q_1 - l_1q_2)(d_1(2(l_1 + q_1) - 3k_1) + 6k_1z_1 - 2q_1z_1)\Big),$$

$$B_4 = \frac{1}{15b_1(3k_1 - q_1)}\Big(a_1(8b_1^2d_1(3k_1 - q_1) + (3k_1 - 4l_1 - 3q_1)(-b_2(6k_1 - 2q_1) - 3g_2(k_1 + l_1)) + c_1(3k_1 - q_1)(3k_2 - 4l_2 - 3q_2)) + c_1(3k_1 - q_1)(a_2(3k_2 - 4l_1 - 3q_1) - 3d_1k_2 + 2d_1l_2 + 2d_1q_2 - 3d_2k_1 + 2d_2l_1 + 2d_2q_1 + g_2h_1 - n_1) + 6k_1z_2 + 6k_2z_1 - 2q_1z_2 - 2q_2z_1) + (b_2(6k_1 - 2q_1) + 3g_2(k_1 + l_1))(d_1(3k_1 - 2(l_1 + q_1)) + 2z_1(q_1 - 3k_1)) - 4b_1^2(3k_1 - q_1)(d_1^2 + 2m_2 - 2n_2)\Big),$$

$$B_5 = \frac{\pi}{8}(2a_1b_2 + 3a_1g_2 + a_2c_1 - b_2d_1 - b_1d_2 - d_1g_2 + g_2z_1 - 3k_3 + 3l_3 - p_3 + q_3).$$

In the similar way in the proof of Case 1 and according to Descartes Theorem, we know that the function $f_3(r)$ can have at most seven positive real roots, which provide at most seven limit cycles.

Subcase 2.2. If $b_1 = 0$ the polynomial function $f_2(r)$ becomes

$$\begin{aligned} f_2(r) = {} & \frac{1}{16}\pi r^5(h_1(q_1 - 3k_1) + k_1(-m_1 + n_1 + 3w_1) + 2l_1m_1 - 2l_1n_1 + m_1q_1 - n_1q_1 \\ & - q_1w_1) + \frac{2}{15}r^4(a_1(3k_1 - 4l_1 - 3q_1) + d_1(-3k_1 + 2l_1 + 2q_1) + 6k_1z_1 - 2q_1z_1) \\ & + \frac{1}{8}\pi r^3(-3k_2 + 3l_2 - p_2 + q_2) - \frac{2}{3}r^2(2b_2 - c_2 + g_2) + \pi\alpha_2 r + 2\delta_2. \end{aligned}$$

This function can have at most five positive real roots. We should have $f_2(r) \equiv 0$ to apply the averaging theory of the third order. So we need to take $z_1 = \dfrac{a_1\big(-3k_1 + 4l_1 + 3q_1\big) + d_1\big(3k_1 - 2\big(l_1 + q_1\big)\big)}{6k_1 - 2q_1}, \alpha_2 = 0, \delta_2 = 0,$

$c_2 = 2b_2 + g_2, p_2 = -3k_2 + 3l_2 + q_2, w_1 = h_1 + \dfrac{2\big(k_1 + l_1\big)\big(m_1 - n_1\big)}{q_1 - 3k_1} + m_1 - n_1.$

Computing $f_3(r)$ we get

$$f_3(r) = C_1r^7 + C_2r^6 + C_3r^5 + C_4r^4 + C_5r^3 - \frac{2}{3}\big(2b_3 - c_3 + g_3\big)r^2 + \pi\alpha_3 r + 2\delta_3.$$

Where

$$\begin{aligned} C_1 = {} & \frac{1}{64\big(3k_1 - q_1\big)}\Big(\pi(k_1 - 5l_1 - 2q_1)(k_1(-3h_1m_1 + 3h_1n_1 + 6l_1q_1 + 3m_1n_1 - 3n_1^2 + q_1^2) \\ & + q_1(h_1 - n_1)(m_1 - n_1) - 3k_1^2(3l_1 + 2q_1) + 9k_1^3 + l_1(-4m_1n_1 + 2m_1^2 + 2n_1^2 - q_1^2))\Big), \\ C_2 = {} & \frac{1}{105\big(3k_1 - q_1\big)}\Big(2(a_1(h_1(3k_1 - q_1)(3k_1 - 16l_1 - 7q_1) + 6k_1(3l_1m_1 + 5l_1n_1 + 3m_1q_1 \\ & + n_1q_1) - 3k_1^2(2m_1 + n_1) - 10l_1m_1q_1 + 32l_1^2m_1 - 6l_1n_1q_1 - 32l_1^2n_1 - 12m_1q_1^2 \\ & + 5n_1q_1^2) + d_1(-h_1(3k_1 - q_1)(3k_1 - 4(2l_1 + q_1)) + 3k_1(10l_1m_1 - 18l_1n_1 + 4m_1q_1 \\ & - 9n_1q_1) + k_1^2(15n_1 - 6m_1) - 4(2l_1 + q_1)(2l_1m_1 - 2l_1n_1 - n_1q_1)))\Big), \\ C_3 = {} & -\frac{1}{64\big(3k_1 - q_1\big)}\Big(\pi(-2a_1d_1(3k_1l_1 + q_1(3l_1 + 2q_1)) + a_1^2(k_1 - q_1)(3k_1 - 12l_1 - 7q_1) \\ & + d_1^2k_1(-3k_1 + 6l_1 + 4q_1) - 4(h_2(q_1 - 3k_1)^2 + 2k_2(3l_1 + q_1)(m_1 - n_1) - 6k_1l_2m_1 \\ & - 6k_1l_1m_2 + 6k_1l_2n_1 + 6k_1l_1n_2 - 4k_1m_2q_1 - 2k_1m_1q_2 + 3k_1^2m_2 + 4k_1n_2q_1 + 2k_1n_1q_2 \\ & - 3k_1^2n_2 + 6k_1q_1w_2 - 9k_1^2w_2 + 2l_2m_1q_1 + 2l_1m_2q_1 - 2l_1m_1q_2 - 2l_2n_1q_1 - 2l_1n_2q_1 \\ & + 2l_1n_1q_2 + m_2q_1^2 - n_2q_1^2 - q_1^2w_2))\Big), \\ C_4 = {} & -\frac{1}{(45k_1 - 15q_1)}\Big(2(-a_2(3k_1 - q_1)(3k_1 - 4l_1 - 3q_1) + 12a_1k_1l_2 - 12a_1k_2l_1 + 6a_1k_1q_2 \\ & - 6a_1k_2q_1 - 4a_1l_2q_1 + 4a_1l_1q_2 + 12b_2k_1m_1 - 12b_2k_1n_1 - 4b_2m_1q_1 + 4b_2n_1q_1 \\ & - 6d_2k_1l_1 - 6d_1k_1l_2 + 6d_1k_2l_1 - 9d_2k_1q_1 - 3d_1k_1q_2 + 3d_1k_2q_1 + 9d_2k_1^2 + 2d_2l_1q_1 \\ & + 2d_1l_2q_1 - 2d_1l_1q_2 + 2d_2q_1^2 + g_2(h_1(q_1 - 3k_1) + 6k_1m_1 - 3k_1n_1 + 6l_1m_1 - 6l_1n_1 \\ & - n_1q_1) + 12k_1q_1z_2 - 18k_1^2z_2 - 2q_1^2z_2)\Big), \\ C_5 = {} & \frac{1}{(48k_1 - 16q_1)}\Big(\pi(a_1(12b_2k_1 - 4b_2q_1 + 15g_2k_1 + 4g_2l_1 - 3g_2q_1) + 2b_2d_1(q_1 - 3k_1) \\ & - 3d_1g_2k_1 - 2d_1g_2l_1 + 18k_1l_3 - 6k_1p_3 + 6k_1q_3 + 6k_3q_1 - 18k_3k_1 - 6l_3q_1 + 2p_3q_1 \\ & - 2q_1q_3)\Big). \end{aligned}$$

The polynomial $f_3(r)$ can have at most seven positive real roots, and generate when ε is sufficiently small at most seven limit cycles for the discontinuous piecewise differential system (2)-(3).

Case 3. $g_1 \neq 0$ and $q_1 = 3k_1$. The polynomial $f_2(r)$ is written as

$$\begin{aligned} f_2(r) = &\frac{1}{8}\pi r^5(k_1+l_1)(m_1-n_1) - \frac{2}{15}r^4(6a_1k_1+4a_1l_1+4b_1m_1-4b_1n_1-3d_1k_1-2d_1l_1 \\ &+g_1(2h_1+3m_1-2n_1-3w_1)) + \frac{1}{8}\pi r^3(2a_1b_1+3a_1g_1-b_1d_1-d_1g_1+g_1z_1-3k_2 \\ &+3l_2-p_2+q_2) - \frac{2}{3}r^2(2b_2-c_2+g_2)+\pi\alpha_2 r+2\delta_2. \end{aligned}$$

This polynomial can have at most five positive real roots. In order to apply the averaging theory of third order we must have $f_2(r) \equiv 0$, and in order to eliminate the coefficient of r^5 we need to have $m_1 = n_1$ or $k_1 = -l_1$. Here also we have two subcases.

Subcase 3.1. We consider

$$z_1 = \frac{-a_1\left(2b_1+3g_1\right)+d_1\left(b_1+g_1\right)+3k_2-3l_2+p_2-q_2}{g_1}, \alpha_2 = 0, \delta_2 = 0,$$

$$w_1 = \frac{\left(2a_1-d_1\right)\left(3k_1+2l_1\right)+g_1\left(2h_1+n_1\right)}{3g_1}, k_1 \neq -l_1, m_1 = n_1, c_2 = 2b_2+g_2.$$

Computing $f_3(r)$ we get

$$f_3(r) = D_1r^6+D_2r^5+D_3r^4+D_4r^3-\frac{2}{3}\left(2b_3-c_3+g_3\right)r^2+\pi\alpha_3r+2\delta_3.$$

Where

$$\begin{aligned} D_1 = &\frac{1}{315g_1}\Big(2(-2g_1(2a_1-d_1)(h_1-m_1)(15k_1+8l_1)-10l_1(d_1-2a_1)^2(2l_1+q_1)+g_1^2 \\ &(-8h_1m_1+4h_1^2+72k_1l_1+45k_1^2+72l_1^2+4m_1^2))\Big), \end{aligned}$$

$$\begin{aligned} D_2 = &-\frac{1}{48g_1}\Big(\pi(a_1(6b_1(2d_1l_1+g_1(m_1-h_1))+d_1g_1(2l_1-15k_1)-7g_1^2(h_1-m_1) \\ &+2(9k_2l_1-9k_1l_2+5l_1p_2-3l_1q_2-15l_2l_1+p_2q_1))+2a_1^2(-6b_1l_1+15g_1k_1 \\ &+2g_1l_1)-3b_1(d_1g_1(m_1-h_1t)+d_1^2l_1-g_1^2(5l_1+q_1))+2d_1g_1^2h_1-2d_1^2g_1l_1 \\ &-9d_1k_2l_1+9d_1k_1l_2-3d_1k_1p_2-5d_1l_1p_2+15d_1l_1l_2+d_1p_1q_2+18g_1h_1k_2 \\ &+5g_1h_1p_2-6g_1h_1q_2-6g_1k_1m_2+6g_1k_1n_2+6g_1^3k_1-3g_1l_2m_1-6g_1l_1m_2 \\ &+6g_1l_1n_2+6g_1^3l_1+g_1m_1p_2-15g_1h_1l_2-2d_1g_1^2m_1)\Big), \end{aligned}$$

$$\begin{aligned} D_3 = &\frac{1}{15g_1}\Big(2(a_1(-2b_1(3d_1g_1+6k_2-2q_2)-6d_1g_1^2-33g_1k_2+6g_2k_1+14g_1l_2 \\ &-6g_1p_2+9g_1q_2)+3a_1^2g_1(4b_1+5g_1)-6a_2g_1k_1-4a_2g_1l_1+b_1(6d_1k_2-2d_1q_2 \\ &-4g_1m_2+4g_1n_2+4g_1^3)+4b_1^2g_1^2-3d_1g_2k_1+9d_1g_1k_2+2d_2g_1l_1-2d_1g_2l_1 \\ &-4d_1g_1l_2+2d_1g_1p_2-2d_1g_1q_2+d_2g_1q_1-2g_1^2h_2-3g_1^2m_2+2g_1^2n_2+3g_1^2w_2 \\ &-18k_2l_2+6k_2p_2-12k_2q_2+18k_2^2+6l_2q_2-2p_2q_2+2q_2^2+4g_2l_1)\Big), \end{aligned}$$

$$\begin{aligned} D_4 = &\frac{1}{8g_1}(\pi(2a_1b_2g_1+a_2g_1(2b_1+3g_1)-2a_1b_1g_2-b_2d_1g_1-b_1d_2g_1+b_1d_1g_2-d_2g_1^2 \\ &-3g_1k_3+3g_2k_2+3g_1l_3-3g_2l_2-g_1p_3+g_2p_2+g_1q_3-g_2q_2+g_1^2z_2)). \end{aligned}$$

This polynomial can have at most six positive real roots, which provide at most six limit cycles.

Now we analyze the second subcase.

Subcase 3.2. We consider the following values of the parameters of the function $f_3(r)$
$z_1 = \dfrac{-a_1(2b_1+3g_1)+d_1(b_1+g_1)+3k_2-3l_2+p_2-q_2}{g_1}, k_1=-l_1, c_2=2b_2+g_2,$
$w_1 = \dfrac{-2a_1l_1-2n_1(2b_1+g_1)+4b_1m_1+d_1l_1+2g_1h_1+3g_1m_1}{3g_1}, \alpha_2=0, \delta_2=0, m_1\neq n_1.$
$f_3(r)$ becomes

$$f_3(r) = F_1r^7+F_2r^6+F_3r^5+F_4r^4+F_5r^3-\frac{2}{3}(2b_3-c_3+g_3)r^2+\pi\alpha_3r+2\delta_2.$$

Where
$F_1 = \dfrac{\pi l_1}{192g_1}((m_1-n_1)(5(-2a_1l_1+4b_1m_1-4b_1n_1+d_1l_1)+g_1(-5h_1+12m_1-7n_1))),$
$F_2 = \dfrac{1}{315g_1}\Big(2(2g_1(2a_1l_1(7h_1-45m_1+38n_1)-8b_1(m_1-n_1)(2h_1+3m_1-5n_1)+d_1l_1$
$(-7h_1+24m_1-17n_1))+2(-4a_1l_1(17b_1(m_1-n_1)+5d_1l_1)+20a_1^2l_1^2+34b_1d_1l_1$
$(m_1-n_1)-8(m_1-n_1)(5b_1^2(m_1-n_1)-9k_2l_1+3l_1(3l_2-p_2+q_2))+5d_1^2l_1^2)$
$+g_1^2(-8h_1(3m_1-2n_1)+4h_1^2+45l_1^2+4n_1(6m_1-5n_1)))\Big),$
$F_3 = \dfrac{1}{48g_1}\Big(\pi(a_1(-2b_1(6d_1l_1+g_1(-3h_1+2m_1+n_1))+12b_1^2(m_1-n_1)-17d_1g_1l_1+g_1^2$
$(7h_1-12m_1+5n_1)+2l_1(-9k_2+6l_2-2p_2+3q_2))+2a_1^2l_1(6b_1+13g_1)+b_1(d_1g_1$
$(-3h_1-4m_1+7n_1)+3d_1^2l_1-2(3g_1^2l_1+(m_1-n_1)(9k_2-15l_2+5p_2-3q_2)))$
$+6b_1^2d_1(n_1-m_1)-2d_1g_1^2h_1+2d_1^2g_1l_1+2d_1g_1^2n_1+9d_1k_2l_1+2d_1l_1p_2-3d_1l_1q_2$
$-6d_1l_1l_2-18g_1h_1k_2+15g_1h_1l_2-5g_1h_1p_2+6g_1h_1q_2-3g_1k_2m_1+3g_1k_2n_1$
$+15g_1l_2m_1-12g_1l_2n_1-3g_1m_1p_2+3g_1m_1q_2+2g_1n_1p_2-3g_1n_1q_2)\Big),$
$F_4 = \dfrac{1}{15g_1}\Big(2(-a_1(2b_1(3d_1g_1+6k_2-2q_2)+6d_1g_1^2+33g_1k_2-14g_1l_2+2g_2l_1+6g_1p_2$
$-9g_1q_2)+3a_1^2g_1(4b_1+5g_1)+2a_2g_1l_1+b_1(6d_1k_2-2d_1q_2-4g_1m_2+4g_2m_1+4g_1n_2$
$-4g_2n_1+4g_1^3)-4b_2g_1m_1+4b_2g_1n_1+4b_1^2g_1^2+9d_1g_1k_2-d_2g_1l_1+d_1g_2l_1-4d_1g_1l_2$
$+2d_1g_1p_2-2d_1g_1q_2-2g_1^2h_2-3g_1^2m_2+2g_1^2n_2+3g_1^2w_2-18k_2l_2+6k_2p_2-12k_2q_2$
$+18k_2^2+6l_2q_2-2p_2q_2+2q_2^2\Big),$
$F_5 = \dfrac{1}{8g_1}\Big(\pi(2a_1b_2g_1+a_2g_1(2b_1+3g_1)-2a_1b_1g_2-b_2d_1g_1-b_1d_2g_1+b_1d_1g_2-d_2g_1^2$
$-3g_1k_3+3g_2k_2+3g_1l_3-3g_2l_2-g_1p_3+g_2p_2+g_1q_3-g_2q_2+g_1^2z_2\Big).$

Then the polynomial $f_3(r)$ can have at most seven positive real roots.

Case 4. $g_1 = 0$ and $q_1 = 3k_1$. Computing the function $f_2(r)$ we obtain

$$f_2(r) = \frac{1}{8}\pi r^5(k_1 + l_1)(m_1 - n_1) - \frac{2}{15}r^4(6a_1k_1 + 4a_1l_1 + 4b_1m_1 - 4b_1n_1 - 3d_1k_1 - 2d_1l_1) + \frac{1}{8}\pi r^3(a_1c_1 - b_1d_1 - 3k_2 + 3l_2 - p_2 + q_2) - \frac{2}{3}r^2(2b_2 - c_2 + g_2) + \pi\alpha_2 r + 2\delta_2.$$

This polynomial can have at most five positive real roots. Now we apply the averaging theory of third order by considering $f_2(r) \equiv 0$. We see that to remove the coefficient of r^5 we need to have $k_1 = -l_1$ or $m_1 = n_1$. Here we also have two subcases.

Subcase 4.1. We consider $k_1 = -l_1$, $c_2 = 2b_2 + g$, $q_2 = -2a_1b_1 + b_1d_1 + 3k_2 - 3l_2 + p_2$, $m_1 = \dfrac{l_1\big(2a_1 - d_1\big)}{4b_1} + n_1$, $\alpha_2 = 0$, $\delta_2 = 0$, $b_1 \neq 0$, $m_1 \neq n_1$, and we distinguish another two subcases $b_1 \neq 0$ or $b_1 = 0$.

Subcase 4.1.1. For $b_1 \neq 0$. Computing $f_3(r)$ we get

$$f_3(r) = G_1r^7 + G_2r^6 + G_3r^5 + G_4r^4 + G_5r^3 - \frac{2}{3}\big(2b_3 - c_3 + g_3\big)r^2 + \pi\alpha_3 r + 2\delta_3.$$

Where

$$\begin{aligned}
G_1 &= \frac{1}{1024b_1^2}(-\pi l_1^2\big(2a_1 - d_1\big)\big(l_1\big(2a_1 - d_1\big) + 20b_1\big(h_1 - w_1\big)\big)),\\
G_2 &= \frac{1}{210b_1}(l_1\big(2a_1 - d_1\big)\big(l_1\big(10a_1 - 11d_1 + 16z_1\big) + 12b_1\big(4h_1 + n_1 - 5w_1\big)\big)),\\
G_3 &= -\frac{1}{64b_1}\pi\Big(2a_1(b_1(6l_1z_1 - 5d_1l_1) + 4b_1^2(h_1 + n_1) + l_1(-2k_2 + l_2 - p_2)) + 8a_1^2b_1l_1\\
&\quad + b_1(-6d_1l_1z_1 + d_1^2p_1 + 4(h_1 - w_1)(3l_2 - p_2)) - 4b_1^2d_1(h_1 + n_1) + d_1l_1(2k_2 - l_2 + p_2)\Big),\\
G_4 &= \frac{1}{30b_1}\Big(-2a_1(b_1^2(6d_1 - 8z_1) + l_1(4b_2 + 3g_2) + c_1(6k_2 - 5l_2 + 3p_2)) - 4b_1(-2a_2l_1\\
&\quad - 3d_1k_2 + d_2l_1 + 4d_1l_2 - 2d_1p_2 + g_2(2h_1 + n_1 - 3w_1) - 6l_2z_1 + 2p_2z_1) + 24a_1^2b_1^2\\
&\quad + d_1l_1(4b_2 + 3g_2) - 8b_1^2(d_1z_1 + 2m_2 - 2n_2)\Big),\\
G_5 &= \frac{1}{8}\pi\Big(2a_1b_2 + 3a_1g_2 + a_2c_1 - b_2d_1 - b_1d_2 - d_1g_2 + g_2z_1 - 3k_3 + 3l_3 - p_3 + q_3\Big).
\end{aligned}$$

This polynomial can have at most seven positive real roots, consequently at most seven limit cycles for the discontinuous piecewise differential system (2)-(3).

Subcase 4.1.2. If $b_1 = 0$ the polynomial $f_2(r)$ is written as

$$f_2(r) = \frac{1}{8}\pi r^5(k_1 + l_1)(m_1 - n_1) - \frac{2}{15}r^4(2a_1 - d_1)(3k_1 + 2l_1) + \frac{1}{8}\pi r^3(-3k_2 + 3l_2 - p_2 + q_2) - \frac{2}{3}r^2(2b_2 - c_2 + g_2) + \pi\alpha_2 r + 2\delta_2.$$

In this case the function $f_2(r)$ can have at most five positive real roots. We set $f_2(r) \equiv 0$, and to delete the coefficients of r^4 we need another two subcases $3k_1 + 2l_1 = 0$ or $d_1 = 2a_1$. We start with the first subcase $3k_1 + 2l_1 = 0$.

Subcase 4.1.2.1. $c_2 = 2b_2 + g_2,\ p_2 = -3k_2 + 3l_2 + q_2,\ \alpha_2 = 0,\ \delta_2 = 0,\ l_1 = 0,\ k_1 = 0$ and $d_1 \neq 2a_1$. Computing the function $f_3(r)$ we obtain

$$\begin{aligned} f_3(r) = &\ \frac{1}{16}\pi r^5(h_1(q_2 - 3k_2) + k_2(-m_1 + n_1 + 3w_1) + 2l_2m_1 - 2l_2n_1 + m_1q_2 - n_1q_2 \\ &- q_2w_1) + \frac{2}{15}r^4(a_1(3k_2 - 4l_2 - 3q_2) - 4b_2m_1 + 4b_2n_1 - 3d_1k_2 + 2d_1l_2 + 2d_1q_2 \\ &+ g_2(-2h_1 - 3m_1 + 2n_1 + 3w_1) + 6k_2z_1 - 2q_2z_1) + \frac{1}{8}\pi r^3(2a_1b_2 + 3a_1g_2 - b_2d_1 \\ &- d_1g_2 + g_2z_1 - 3k_3 + 3l_3 - p_3 + q_3) - \frac{2}{3}r^2(2b_3 - c_3 + g_3) + \pi\alpha_3 r + 2\delta_3. \end{aligned}$$

Then the polynomial $f_3(r)$ can have at most five positive real roots.

Now we compute $f_3(r)$ for the second case $d_1 = 2a_1$.

Subcase 4.1.2.2. $c_2 = 2b_2 + g_2,\ \alpha_2 = 0,\ c_2 = 2b_2 + g_2,\ \delta_2 = 0,\ d_1 = 2a_1,\ k_1 = -l_1$ and $3k_1 + 2l_1 \neq 0$. Computing the function $f_3(r)$ we obtain

$$\begin{aligned} f_3(r) = &-\frac{1}{64}\pi l_1 r^7(m_1 - n_1)(5h_1 + m_1 - n_1 - 5w_1) - \frac{8}{105}l_1r^6(3a_1 - 4z_1)(m_1 - n_1) + \frac{1}{16} \\ &\pi r^5(h_1(q_2 - 3k_2) + k_2(-m_1 + n_1 + 3w_1) + 2l_2m_1 - 2l_2n_1 + m_1q_2 - n_1q_2 - q_2w_1) \\ &- \frac{2}{15}r^4(3a_1k_2 - 2a_2l_1 - a_1q_2 + 4b_2m_1 - 4b_2n_1 + d_2l_1 + g_2(2h_1 + 3m_1 - 2n_1 - 3w_1) \\ &- 6k_2z_1 + 2q_2z_1) + \frac{1}{8}\pi r^3(a_1g_2 + g_2z_1 - 3k_3 + 3l_3 - p_3 + q_3) - \frac{2}{3}r^2(2b_3 - c_3 + g_3) \\ &+ \pi\alpha_3 r + 2\delta_3. \end{aligned}$$

This polynomial function can have at most seven positive real roots. Now taking the second subcase $m_1 = n_1$.

Subcase 4.2. $m_1 = n_1,\ c_2 = 2b_2 + g_2,\ p_2 = 2a_1b_1 - b_1d_1 - 3k_2 + 3l_2 + q_2,\ \alpha_2 = 0,\ \delta_2 = 0$ and $k_1 = -\dfrac{2l_1}{3}$. Computing $f_3(r)$ we get

$$\begin{aligned} f_3(r) = &\ \frac{4}{105}l_1r^6(2a_1 - d_1)(4h_1 + n_1 - 5w_1) - \frac{1}{48}\pi r^5(a_1(6(b_1(n_1 + w_1) + l_1z_1) - 3d_1l_1) \\ &+ 2a_1^2l_1 - 3b_1d_1n_1 - 3b_1d_1w_1 - 3d_1l_1z_1 + d_1^2l_1 + h_1(9k_2 - 3q_2) - 9k_2w_1 - 2l_1m_2 \\ &+ 2l_1n_2 + 3q_2w_1) - \frac{2}{15}r^4(a_1(-4b_1d_1 - 3k_2 + 4l_2 + 3q_2) + 2b_1(d_1^2 + 2m_2 - 2n_2) \\ &+ 3d_1k_2 - 2d_1l_2 - 2d_1q_2 + 2g_2h_1 + g_2n_1 - 3g_2w_1 - 6k_2z_1 + 2q_2z_1) + \frac{1}{8}\pi r^3(2a_2b_1 \\ &+ 2a_1b_2 + 3a_1g_2 - b_1d_2 - b_2d_1 - d_1g_2 + g_2z_1 - 3k_3 + 3l_3 - p_3 + q_3) - \frac{2}{3}r^2(2b_3 \\ &- c_3 + g_3) + \pi\alpha_3 r + 2\delta_3. \end{aligned}$$

This polynomial can have at most six positive real roots. In general, in all the cases mentioned above, the polynomials $f_i(r)$, with $i = 1, 2, 3$ can have at most 3, 5, 6, and 7 real positive roots. Thus the maximum number of limit cycles that can be obtained via the averaging theory up to third order is seven.

Now we are going to reach our result by giving an example with exactly seven limit cycles.

Example with seven limit cycles.

In the half plane $y \geq 0$, we consider the linear differential system (8) with the values $\{\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \alpha_3, \beta_3, \gamma_3, \delta_3\} \to \{0, 1, 2, 0, 0, 1, 1, 0, 13,068/\pi, -1, -1, -2520\}$.

Now in the half plane $y \leq 0$, we consider the cubic weak focus or center (9) with $\{a_1, b_1, c_1, d_1, g_1, h_1, k_1, l_1, m_1, n_1, p_1, q_1, w_1, z_1, a_2, b_2, c_2, d_2, g_2, h_2, k_2, l_2, m_2, n_2, p_2, q_2, w_2, z_2,$

$a_3, b_3, c_3, d_3, g_3, h_3, k_3, l_3, m_3, n_3, p_3, q_3, w_3, z_3\} \to \{0, H_1/K_1, H_1/K_1 + 1, 0, 1, -2\sqrt{25 + \frac{64}{\pi}} - 9, 2, 0, 1,$

$1, (-5\pi - \sqrt{64\pi + 25\pi^2})/(2\pi), (7\pi - \sqrt{64\pi + 25\pi^2})/(2\pi), -2\sqrt{25 + \frac{64}{\pi}} - 9, 2, 0, 1, 1, 1, -1, -1,$

$-1302.26, 0, \frac{1}{2}, -\frac{1}{2}, 3907.78, -1, 163.431, -1, 1, -20, 0, -1, 19, 738, -1, 1, 1, -1, -1, -54, 152/\pi,$

$H_2/K_2, -2, 1\}$, with $H_1 = -23\sqrt{\pi(64 + 25\pi)} - 880\pi - 128$, $K_1 = 8(5\pi + 4 + \sqrt{\pi(64 + 25\pi)})$, $H_2 = 9\left(\sqrt{\pi(64 + 25\pi)} - 80\pi\right)$ and $K_2 = 8\left(\sqrt{\pi(64 + 25\pi)} + 5\pi + 4\right)$.

An exhausting computation shows that $f_1(r) \equiv f_2(r) \equiv 0$ and

$$f_3(r) = (r-1)(r-2)(r-3)(r-4)(r-5)(r-6)(r-7).$$

Then for these systems we have seven limit cycles bifurcating from the periodic orbits of the discontinuous piecewise differential system (2)-(3).

Moreover, in polar coordinates (r, θ) the periodic orbits that bifurcate are $r = 1, 2, 3, 4, 5, 6, 7$. This completes the proof of the Theorem 1.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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