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Limit Cycles for Discontinuous Piecewise Differential Systems in $\mathbb{R}^3$ Separated by a Paraboloid

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Abstract
In planar piecewise differential systems it is known that when the discontinuity curve is a straight line and both differential systems are linear centers, these piecewise differential systems have no limit cycles but if they are separated by other types of discontinuity curves, such as parabolas, then they have limit cycles. All these results are in the plane and although the qualitative theory of planar piecewise differential systems has been the subject of many research, this is not the case for piecewise differential systems in higher dimensions. In this paper, we study the maximum number of limit cycles of discontinuous piecewise differential systems in $\mathbb{R}^3$ separated by a paraboloid (elliptic or hyperbolic), and formed by what we call two linear differential centers. We prove that these systems can have at most one limit cycle and that this upper bound is reached. We also provide systems of these types without periodic solutions and with a continuum of periodic solutions.

Keywords Discontinuous piecewise linear systems · Limit cycles · First integrals

Introduction and Statement of the Main Result

The study of piecewise linear differential systems goes back to Andronov et al. [1] and still continues to receive attention from researchers. These last years a renewed interest has appeared in the mathematical community for understanding the dynamical richness of

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these piecewise linear differential systems because they are widely used to model processes appearing in electronics, mechanics, economy, etc., see for instance the books of di Bernardo et al. [5], Simpson [18], the survey of Makarenkov and Lamb [14] and the hundreds of references quoted in these works.

In the qualitative theory of dynamical systems the existence of periodic orbits and more precisely of limit cycles is very important because when they exist, they enable us to understand the dynamical behavior of differential systems. Moreover many real world phenomena are related to their existence, see for instance the Van der Pol oscillator [19, 20], among many others. In order to understand the dynamical behavior of the discontinuous piecewise differential systems it is also important to know if they have crossing periodic orbits and crossing limit cycles. In discontinuous piecewise differential systems a *crossing periodic orbit* is a periodic orbit that intersects the discontinuity manifold in a finite number of crossing points and a *crossing limit cycle* is a crossing periodic orbit that is isolated in the set of all periodic orbits of the system.

In the last years, the study of the existence and maximum number of limit cycles of planar piecewise differential systems has been a subject of intense research. Most of the studies developed in this direction were done considering piecewise linear differential systems with only two zones and separated by a straight line and only a few of them were done taking into account more zones or considering discontinuity curves with different shapes than a straight line.

In the case of planar piecewise differential systems separated by a straight line, the following interesting question emerged: discontinuous piecewise linear differential systems with only centers can create limit cycles? In 2018 Llibre and Teixeira answered this question by proving that these piecewise differential systems have no limit cycles, see [11].

In this regard, recently in [2, 4, 7, 8, 13] some authors studied the existence and the maximum number of limit cycles for discontinuous piecewise differential systems formed by differential centers that have either two or more zones, and they are separated either straight line or conics (reducible or irreducible) and they proved that these systems have limit cycles. In particular, in [13] the authors proved that discontinuous piecewise differential systems separated by a parabola and formed by two linear differential centers have at most 3 limit cycles and that this upper bound is reached. From these works, it is apparent that the shape of the discontinuity curve plays an important role in the number of limit cycles that a discontinuous piecewise differential system can have.

In this way, a natural question arises, namely, to consider discontinuous piecewise differential systems in $\mathbb{R}^3$, since although the qualitative theory of planar piecewise differential systems has been a subject of many research this is not the case for piecewise differential systems in higher dimensions. Most of the existing results are related to very specific families of systems (see for instance [12], where the authors characterized the families of periodic orbits of two discontinuous piecewise differential systems in $\mathbb{R}^3$ where the discontinuity surface is a plane).

Our objective is to study the existence and the maximum number of crossing limit cycles for discontinuous piecewise differential systems in $\mathbb{R}^3$ formed by two linear differential systems which we will call linear centers in $\mathbb{R}^3$ and separated by a paraboloid (elliptic or hyperbolic). Without loss of generality, we can consider that an elliptic paraboloid is of the form

$$P_E = \left\{(X,Y,Z) \in \mathbb{R}^3 \,:\, Z = X^2 + Y^2\right\}.$$

And a hyperbolic paraboloid is of the form

$$P_H = \{(X,Y,Z) \in \mathbb{R}^3 : Z = X^2 - Y^2\}.$$

We observe that indeed P_E and P_H divide the space $\mathbb{R}^3$ in two regions, namely

$$\mathcal{R}_E^1 = \{(X,Y,Z) \in \mathbb{R}^3 : Z - X^2 - Y^2 \geq 0\};$$
$$\mathcal{R}_E^2 = \{(X,Y,Z) \in \mathbb{R}^3 : Z - X^2 - Y^2 \leq 0\},$$

and

$$\mathcal{R}_H^1 = \{(X,Y,Z) \in \mathbb{R}^3 : Z - X^2 + Y^2 \geq 0\};$$
$$\mathcal{R}_H^2 = \{(X,Y,Z) \in \mathbb{R}^3 : Z - X^2 + Y^2 \leq 0\},$$

respectively.

We recall that a *center* of a differential system in the plane $\mathbb{R}^2$ is an equilibrium point p having a neighborhood U such that $U \setminus \{p\}$ is filled of periodic orbits. A *global* center is a center p such that $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits. The notion of a center appeared already in the works of Poincaré [15–17] in 1881 and Dulac [6] in 1908.

In $\mathbb{R}^3$ there are no centers in the sense that there are no equilibrium points p having a neighborhood U such that $U \setminus \{p\}$ is filled of periodic orbits, see for instance [3]. In the following, we introduce the notion of the linear center in $\mathbb{R}^3$ that we shall use.

One of the differential systems in $\mathbb{R}^3$ with more periodic orbits is

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = 0. \tag{1}$$

This differential system has two linearly independent first integrals, namely

$$H_1(x,y,z) = x^2 + y^2, \quad H_2(x,y,z) = z.$$

Moreover system (1) has a global center at the equilibrium point $(0, 0, z_0)$ of each plane $z = z_0$. So all its orbits are periodic except the points of the z- axis which are equilibrium points. We denote this differential system as a *linear center* in $\mathbb{R}^3$.

The aim of this paper is to study the maximum number of crossing limit cycles that the discontinuous piecewise differential systems formed by two linear centers (after applying an affine change of variables) and separated by the paraboloid either P_E or P_H can have. Moreover, we also want to show that this maximum is reached.

Our main result is as follows.

Theorem 1 *Consider discontinuous piecewise linear differential systems in* $\mathbb{R}^3$ *formed by two linear centers* (*after applying an affine change of variables*) *and separated by a paraboloid* (*elliptic or hyperbolic*). *The following statements hold*:

- (*i*) The maximum number of limit cycles in both cases is one.
- (*ii*) In both cases there are systems without crossing periodic orbits.
- (*iii*) In both cases there are systems with a continuum of crossing periodic orbits. See Fig. 1 for the case of the elliptic paraboloid and Fig. 2 for the case of the hyperbolic paraboloid.
- (*iv*) In both cases there are systems with one crossing limit cycle. See Fig. 3 for the case of the elliptic paraboloid and Fig. 4 for the case of the hyperbolic paraboloid.

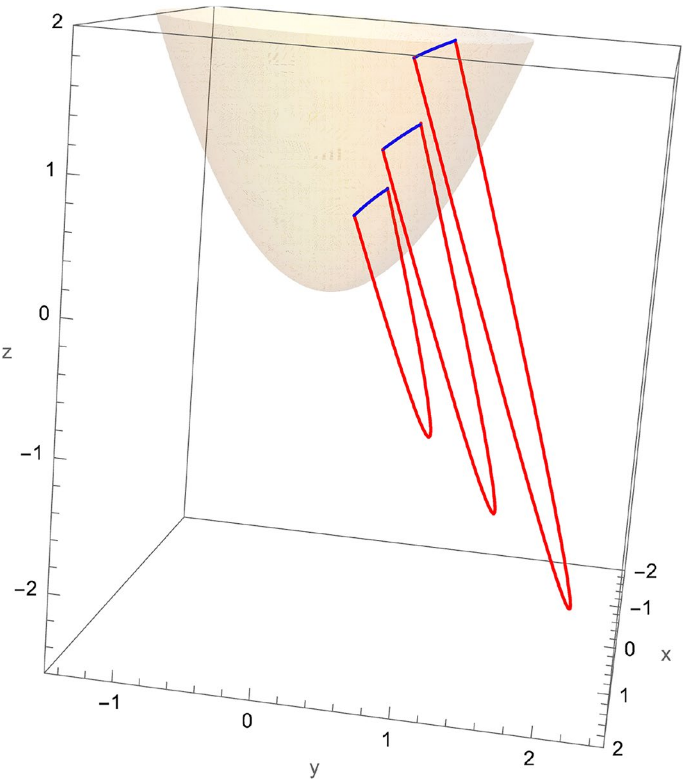

Fig. 1 The three crossing periodic orbits S^i, $i = 1, 2, 3$ of the discontinuous piecewise differential system formed by the linear systems (26) and (27)

Theorem 1 is proved in Sect. 3.

Preliminaries

The linear differential systems considered in each piece $\mathcal{R}_E^1, \mathcal{R}_E^2, \mathcal{R}_H^1$ and $\mathcal{R}_H^2$ are linear differential centers (1) after applying a general affine transformation. More precisely, we shall use the next result.

Lemma 1 *Doing a rescaling of the independent variable after the affine change of variables given by*

$$x = a_1X + a_2Y + a_3Z + a_4, \quad y = b_1X + b_2Y + b_3Z + b_4 \quad \text{and} \quad z = c_1X + c_2Y + c_3Z + c_4,$$

where $a_i, b_i, c_i \in \mathbb{R}$ *for* $i = 1, 2, 3, 4$ *and*

$$(a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \neq 0,$$

system (1) *becomes*

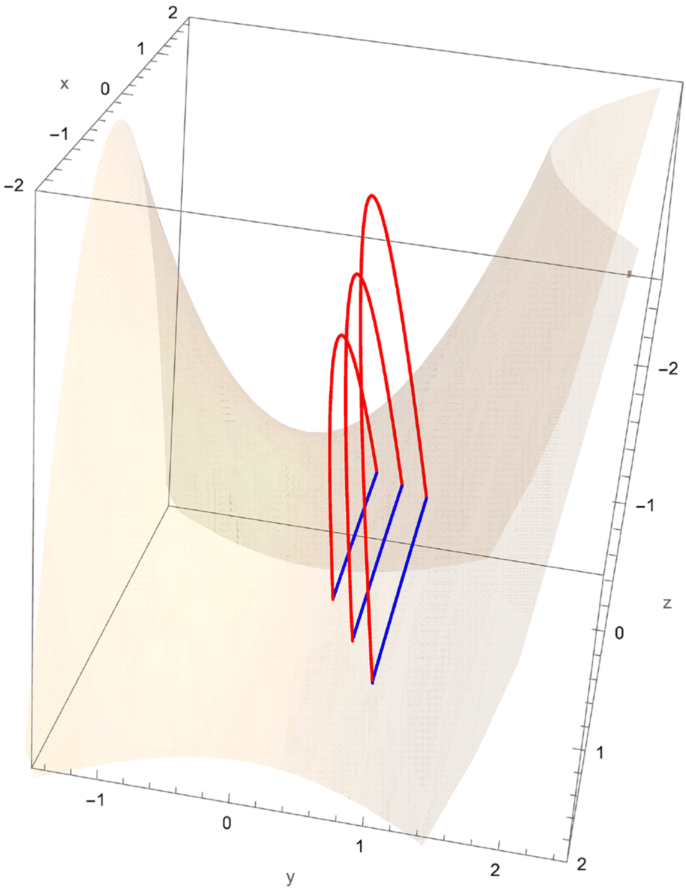

Fig. 2 The three crossing periodic orbits S^i, $i = 1, 2, 3$ of the discontinuous piecewise differential system formed by the linear systems (29) and (30)

$$\begin{aligned}\dot{X} =&((a_1a_3 + b_1b_3)c_2 - (a_1a_2 + b_1b_2)c_3)X + ((a_2a_3 + b_2b_3)c_2 - (a_2^2 + b_2^2)c_3)Y\\&+ ((a_3^2 + b_3^2)c_2 - (a_2a_3 + b_2b_3)c_3)Z + (a_3a_4 + b_3b_4)c_2 - (a_2a_4 + b_2b_4)c_3,\\\dot{Y} =&(-(a_1a_3 + b_1b_3)c_1 + (a_1^2 + b_1^2)c_3)X + (-(a_2a_3 + b_2b_3)c_1 + (a_1a_2 + b_1b_2)c_3)Y\\&+ (-(a_3^2 + b_3^2)c_1 + (a_1a_3 + b_1b_3)c_3)Z - (a_3a_4 + b_3b_4)c_1 + (a_1a_4 + b_1b_4)c_3,\\\dot{Z} =&((a_1a_2 + b_1b_2)c_1 - (a_1^2 + b_1^2)c_2)X + ((a_2^2 + b_2^2)c_1 - (a_1a_2 + b_1b_2)c_2)Y\\&+ ((a_2a_3 + b_2b_3)c_1 - (a_1a_3 + b_1b_3)c_2)Z + (a_2a_4 + b_2b_4)c_1 - (a_1a_4 + b_1b_4)c_2.\end{aligned} \tag{2}$$

Moreover, two linearly independent first integrals of the differential system (2) *are*

$$\begin{aligned}F_1(X, Y, Z) &= (a_1X + a_2Y + a_3Z + a_4)^2 + (b_1X + b_2Y + b_3Z + b_4)^2,\\F_2(X, Y, Z) &= c_1X + c_2Y + c_3Z + c_4.\end{aligned} \tag{3}$$

In general, studying crossing periodic orbits of discontinuous piecewise differential systems is a very difficult problem. And a useful tool that allows studying these periodic orbits is to verify if the differential systems that compose the piecewise

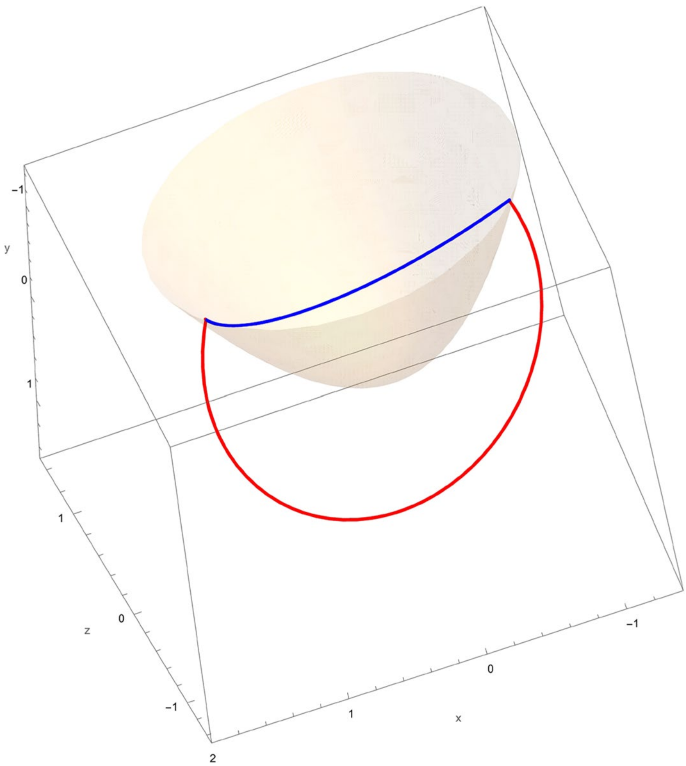

Fig. 3 One crossing limit cycle intersecting P_E

differential system are completely integrable. We recall that a differential system in $\mathbb{R}^3$ is *completely integrable* if it has two first integrals linearly independent because then we can describe an orbit that passes through a given point p as the intersection of all level surfaces to which the point p belongs. It is known that linear differential systems are always completely integrable.

Therefore in order to study crossing periodic orbits of a piecewise differential system in $\mathbb{R}^3$ formed by two linear differential systems, which intersect the discontinuity surface in the points p_0 and p_1, these points must belong to the intersection of the same level surfaces to both differential systems, this is, they must satisfy the following closing equations

$$\begin{aligned} F_1(p_0) &= F_1(p_1), \\ F_2(p_0) &= F_2(p_1), \\ G_1(p_0) &= G_1(p_1), \\ G_2(p_0) &= G_2(p_1), \end{aligned} \tag{4}$$

where $F_i(x_1, x_2, x_3)$ and $G_i(x_1, x_2, x_3)$ for $i = 1, 2$, are the linearly independent first integrals of the systems that compose the discontinuous piecewise differential system. This tool has been used in the papers [9, 10]. We use the same technique in the proof of Theorem 1.

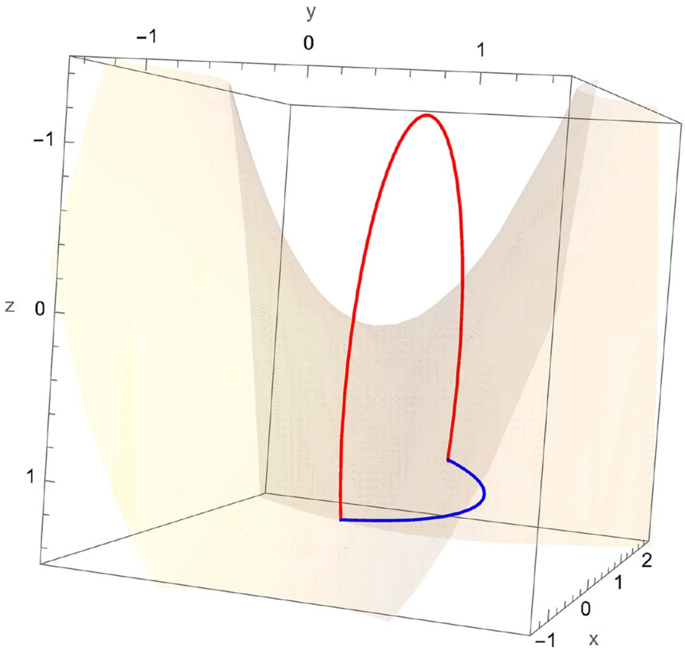

Fig. 4 One crossing limit cycle intersecting P_H

Proof of Theorem 1

Proof of statement (i) We have two cases, first when the discontinuity surface is an elliptic paraboloid (P_E) and second when the discontinuity surface is a hyperbolic paraboloid (P_H). Here we only provide all the details of the proof of statement (i) considering that the discontinuity surface is P_E, because the proof considering the paraboloid P_H is completely analogous.

By Lemma 1, we consider the discontinuous piecewise differential systems such that in the region $\mathcal{R}_E^1$ is considered the linear differential center (2), which has the two linearly independent first integrals (3) and in the region $\mathcal{R}_E^2$ we consider the linear differential center

$$\begin{aligned}\dot{X} =&((\alpha_1\alpha_3+\beta_1\beta_3)\gamma_2-(\alpha_1\alpha_2+\beta_1\beta_2)\gamma_3)X+((\alpha_2\alpha_3+\beta_2\beta_3)\gamma_2-(\alpha_2^2+\beta_2^2)\gamma_3)Y\\ &+((\alpha_3^2+\beta_3^2)\gamma_2-(\alpha_2\alpha_3+\beta_2\beta_3)\gamma_3)Z+(\alpha_3\alpha_4+\beta_3\beta_4)\gamma_2-(\alpha_2\alpha_4+\beta_2\beta_4)\gamma_3,\\ \dot{Y} =&(-(\alpha_1\alpha_3+\beta_1\beta_3)\gamma_1+(\alpha_1^2+\beta_1^2)\gamma_3)X+(-(\alpha_2\alpha_3+\beta_2\beta_3)\gamma_1+(\alpha_1\alpha_2+\beta_1\beta_2)\gamma_3)Y\\ &+((-\alpha_3^2+\beta_3^2)\gamma_1+(\alpha_1\alpha_3+\beta_1\beta_3)\gamma_3)Z-(\alpha_3\alpha_4+\beta_3\beta_4)\gamma_1+(\alpha_1\alpha_4+\beta_1\beta_4)\gamma_3,\\ \dot{Z} =&((\alpha_1\alpha_2+\beta_1\beta_2)\gamma_1-(\alpha_1^2+\beta_1^2)\gamma_2)X+((\alpha_2^2+\beta_2^2)\gamma_1-(\alpha_1\alpha_2+\beta_1\beta_2)\gamma_2)Y\\ &+((\alpha_2\alpha_3+\beta_2\beta_3)\gamma_1-(\alpha_1\alpha_3+\beta_1\beta_3)\gamma_2)Z+(\alpha_2\alpha_4+\beta_2\beta_4)\gamma_1-(\alpha_1\alpha_4+\beta_1\beta_4)\gamma_2.\end{aligned} \tag{5}$$

Where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$ and

$$(\alpha_2\beta_3 - \alpha_3\beta_2)\gamma_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\gamma_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\gamma_3 \neq 0.$$

This system has the two linearly independent first integrals

$$\begin{aligned} G_1(X,Y,Z) &= (\alpha_1 X + \alpha_2 Y + \alpha_3 Z + \alpha_4)^2 + (\beta_1 X + \beta_2 Y + \beta_3 Z + \beta_4)^2, \\ G_2(X,Y,Z) &= \gamma_1 X + \gamma_2 Y + \gamma_3 Z + \gamma_4. \end{aligned}$$

In order to have a crossing periodic orbit that intersects the discontinuity surface P_E in two points $p_0 = (X_0, Y_0, Z_0)$ and $p_1 = (X_1, Y_1, Z_1)$, using (4) we obtain that these points must satisfy the following equivalent system

$$\begin{array}{rc} e_1: & (a_4 + a_1X_0 + a_2Y_0 + a_3Z_0)^2 + (b_4 + b_1X_0 + b_2Y_0 + b_3Z_0)^2 \\ & - (a_4 + a_1X_1 + a_2Y_1 + a_3Z_1)^2 - (b_4 + b_1X_1 + b_2Y_1 + b_3Z_1)^2 = 0, \\ e_2: & c_1X_0 - c_1X_1 + c_2Y_0 - c_2Y_1 + c_3Z_0 - c_3Z_1 = 0, \\ e_3: & (X_0\alpha_1 + Y_0\alpha_2 + Z_0\alpha_3 + \alpha_4)^2 + (X_0\beta_1 + Y_0\beta_2 + Z_0\beta_3 + \beta_4)^2 \\ & - (X_1\alpha_1 + Y_1\alpha_2 + Z_1\alpha_3 + \alpha_4)^2 - (X_1\beta_1 + Y_1\beta_2 + Z_1\beta_3 + \beta_4)^2 = 0, \\ e_4: & X_0\gamma_1 - X_1\gamma_1 + Y_0\gamma_2 - Y_1\gamma_2 + Z_0\gamma_3 - Z_1\gamma_3 = 0, \\ & Z_0 - X_0^2 - Y_0^2 = 0, \\ & Z_1 - X_1^2 - Y_1^2 = 0. \end{array} \tag{6}$$

We study two cases: $c_3\gamma_3 \neq 0$ and $c_3\gamma_3 = 0$.

Case 1: $c_3\gamma_3 \neq 0$. Setting $E_1 = \gamma_3 e_2 - c_3 e_4$, we obtain

$$E_1 = (c_3\gamma_1 - c_1\gamma_3)(X_1 - X_0) - (c_3\gamma_2 - c_2\gamma_3)(Y_1 - Y_0). \tag{7}$$

We have two subcases: $c_3\gamma_1 - c_1\gamma_3 = 0$, or $c_3\gamma_1 - c_1\gamma_3 \neq 0$.

Subcase 1.1: $c_3\gamma_1 - c_1\gamma_3 = 0$.

We have $E_1 = (c_3\gamma_2 - c_2\gamma_3)(Y_0 - Y_1) = 0$. We have two subcases: $c_3\gamma_2 - c_2\gamma_3 \neq 0$, or $c_3\gamma_2 - c_2\gamma_3 = 0$.

Subcase 1.1.1: $c_3\gamma_2 - c_2\gamma_3 \neq 0$. From E_1 we obtain that

$$Y_1 = Y_0. \tag{8}$$

Then equations e_1, e_3 and e_4 in system (6) reduce to

$$\begin{aligned} e_1 :& (X_0 - X_1)\big(2a_1a_4 + 2b_1b_4 + (a_1^2 + 2a_3a_4 + b_1^2 + 2b_3b_4)X_0 + 2(a_1a_3 + b_1b_3)X_0^2 \\ &+ (a_3^2 + b_3^2)X_0^3 + (a_1^2 + 2a_3a_4 + b_1^2 + 2b_3b_4)X_1 + 2(a_1a_3 + b_1b_3)X_0X_1 \\ &+ (a_3^2 + b_3^2)X_0^2X_1 + 2(a_1a_3 + b_1b_3)X_1^2 + (a_3^2 + b_3^2)X_0X_1^2 + (a_3^2b_3^2)X_1^3 \\ &+ 2(a_1a_2 + b_1b_2)Y_0 + 2(a_2a_3 + b_2b_3)X_0Y_0 + 2(a_2a_3 + b_2b_3)X_1Y_0 + 2(a_1a_3 \\ &+ b_1b_3)Y_0^2 + 2(a_3^2 + b_3^2)X_0Y_0^2 + 2(a_3^2 + b_3^2)X_1Y_0^2\big), \\ e_3 :& (X_0 - X_1)\big((X_0 + X_1)\alpha_1^2 + 2Y_0\alpha_1\alpha_2 + 2(X_0^2 + X_0X_1 + X_1^2 + Y_0^2)\alpha_1\alpha_3 \\ &+ 2(X_0 + X_1)Y_0\alpha_2\alpha_3 + X_0^3\alpha_3^2 + X_0^2X_1\alpha_3^2 + X_0X_1^2\alpha_3^2 + X_1^3\alpha_3^2 + 2X_0Y_0^2\alpha_3^2 \\ &+ 2X_1Y_0^2\alpha_3^2 + 2\alpha_1\alpha_4 + 2(X_0 + X_1)\alpha_3\alpha_4 + (X_0 + X_1)\beta_1^2 + 2Y_0\beta_1\beta_2 + 2X_0^2\beta_1\beta_3 \\ &+ 2(X_0X_1 + X_1^2 + Y_0^2)\beta_1\beta_3 + 2X_0Y_0\beta_2\beta_3 + 2X_1Y_0\beta_2\beta_3 + X_0^3\beta_3^2 + X_0^2X_1\beta_3^2 \\ &+ X_0X_1^2\beta_3^2 + X_1^3\beta_3^2 + 2X_0Y_0^2\beta_3^2 + 2X_1Y_0^2\beta_3^2 + 2\beta_1\beta_4 + 2X_0\beta_3\beta_4 + 2X_1\beta_3\beta_4\big), \\ e_4 :& (X_0 - X_1)(\gamma_1 + X_0\gamma_3 + X_1\gamma_3). \end{aligned}$$

We observe that we can consider $X_0 \neq X_1$, because if $X_0 = X_1$ from (8), we would have that $p_0 = p_1$, and consequently we would not have limit cycles. From equation e_4 we get

$$X_1 = \frac{-\gamma_1 - X_0\gamma_3}{\gamma_3}. \tag{9}$$

Now we introduce X_1 into e_1 and e_3 and we obtain

$$\begin{aligned}
e_1 : \; & -((a_3^2+b_3^2)\gamma_1^3) + 2(a_1a_3+b_1b_3)\gamma_1^2\gamma_3 - (a_1^2+2a_3a_4+b_1^2+2b_3b_4)\gamma_1\gamma_3^2 + 2(a_1a_4 \\
& + b_1b_4)\gamma_3^3 + 2\gamma_3X_0\gamma_1(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3) + 2Y_0(-a_2a_3\gamma_1\gamma_3^2 \\
& - b_2b_3\gamma_1\gamma_3^2 + a_1a_2\gamma_3^3 + b_1b_2\gamma_3^3) + 2\gamma_3^2X_0^2(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3) \\
& + 2\gamma_3^2Y_0^2(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3), \\
e_3 : \; & -((\alpha_3^2+\beta_3^2)\gamma_1^3) + 2(\alpha_1\alpha_3+\beta_1\beta_3)\gamma_1^2\gamma_3 - (\alpha_1^2+2\alpha_3\alpha_4+\beta_1^2+2\beta_3\beta_4)\gamma_1\gamma_3^2 \\
& + 2(\alpha_1\alpha_4+\beta_1\beta_4)\gamma_3^3 + 2\gamma_1\gamma_3X_0(-\alpha_3^2\gamma_1 - \beta_3^2\gamma_1 + \alpha_1\alpha_3\gamma_3 + \beta_1\beta_3\gamma_3) \\
& + 2Y_0(-\alpha_2\alpha_3\gamma_1\gamma_3^2 - \beta_2\beta_3\gamma_1\gamma_3^2 + \alpha_1\alpha_2\gamma_3^3 + \beta_1\beta_2\gamma_3^3) + 2\gamma_3^2X_0^2(-\alpha_3^2\gamma_1 - \beta_3^2\gamma_1 \\
& + \alpha_1\alpha_3\gamma_3^2 + \beta_1\beta_3\gamma_3^2) + 2\gamma_3^2Y_0^2(-\alpha_3^2\gamma_1 - \beta_3^2\gamma_1 + \alpha_1\alpha_3\gamma_3^2 + \beta_1\beta_3\gamma_3^2).
\end{aligned} \tag{10}$$

Analyzing equation e_1 we have two subcases: $\gamma_1(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3) \neq 0$, or $\gamma_1(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3) = 0$.

Subcase 1.1.1.1: $\gamma_1(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3) \neq 0$. From equation e_1 we can isolate the variable X_0 and we get

$$\begin{aligned}
X_0^{\pm} =& \frac{1}{4(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3)}\Big(2(a_3^2+b_3^2)\gamma_1^2\gamma_3 - 2(a_1a_3+b_1b_3)\gamma_1\gamma_3^2 \\
& \pm \sqrt{\big(-2(a_3^2+b_3^2)\gamma_1^2\gamma_3 + 2a_1a_3\gamma_1\gamma_3^2 + 2b_1b_3\gamma_1\gamma_3^2\big)^2 - 4(-2a_3^2\gamma_1\gamma_3^2 - 2b_3^2\gamma_1\gamma_3^2} \\
& + 2(a_1a_3+b_1b_3)\gamma_3^3)(-(a_3^2+b_3^2)\gamma_1^3 + 2(a_1a_3+b_1b_3)\gamma_1^2\gamma_3 - (a_1^2+2a_3a_4)\gamma_1\gamma_3^2 \\
& + (-b_1^2 - 2b_3b_4 - 2a_2a_3Y_0 - 2b_2b_3Y_0 - 2a_3^2Y_0^2 - 2b_3^2Y_0^2)\gamma_1\gamma_3^2 + 2(a_1a_4 \\
& + 2b_1b_4 + 2a_1a_2Y_0 + 2b_1b_2Y_0 + 2a_1a_3Y_0^2 + 2b_1b_3Y_0^2)\gamma_3^3))\Big).
\end{aligned}$$

We observe that

$$X_0^+ + X_0^- = -\frac{\gamma_1}{\gamma_3}. \tag{11}$$

Consider X_0^-. From (9) we have $X_1^- = (-\gamma_1 - X_0^-\gamma_3)/\gamma_3$ and from equation e_3 in (10), we obtain

$$\begin{aligned}
Y_0^- =& \big((\gamma_3((a_1^2+b_1^2)\gamma_1 - 2(a_1a_4+b_1b_4)\gamma_3) + 2(a_3a_4+b_3b_4)\gamma_1\gamma_3)((\alpha_3^2+\beta_3^2)\gamma_1 \\
& - (\alpha_1\alpha_3+\beta_1\beta_3)\gamma_3) + (a_3^2\gamma_1 + b_3^2\gamma_1)((\alpha_1\alpha_3+\beta_1\beta_3)\gamma_1^2 - (\alpha_1^2 + 2\alpha_3\alpha_4 + \beta_1^2 \\
& + 2\beta_3\beta_4)\gamma_1\gamma_3 + 2(\alpha_1\alpha_4+\beta_1\beta_4)\gamma_3^2) + (a_1a_3+b_1b_3)(-((\alpha_3^2+\beta_3^2)\gamma_1^3) + (\alpha_1^2 \\
& + 2\alpha_3\alpha_4 + \beta_1^2 + 2\beta_3\beta_4)\gamma_1\gamma_3^2 - 2(\alpha_1\alpha_4+\beta_1\beta_4)\gamma_3^3)\big)/\big(2\gamma_3((a_3^2\gamma_1 + b_3^2\gamma_1 - a_3a_1\gamma_3 \\
& - b_3b_1\gamma_3)(\alpha_2\alpha_3\gamma_1 + \beta_2\beta_3\gamma_1 - \alpha_1\alpha_2\gamma_3 - \beta_1\beta_2\gamma_3) + ((a_1a_2+b_1b_2)\gamma_3 - a_3a_2\gamma_1 \\
& - b_3b_2\gamma_1)((\alpha_3^2+\beta_3^2)\gamma_1 - (\alpha_1\alpha_3+\beta_1\beta_3)\gamma_3))\big).
\end{aligned}$$

Using this and also (8), we have that

$$(X_0^-, Y_0^-, Z_0^-, X_1^-, Y_0^-, Z_1^-) \tag{12}$$

is a solution of system (6).

Now if we consider X_0^+, by (9) we have $X_1^+ = (-\gamma_1 - X_0^+\gamma_3)/\gamma_3$ and by equation e_3 we get Y_0^+, which satisfies that

$$Y_0^+ = Y_0^-.$$

Moreover by (9) and (11) we have that

$$X_1^+ = -\frac{\gamma_1}{\gamma_3} - X_0^+ = X_0^- \quad \text{and} \quad X_1^- = -\frac{\gamma_1}{\gamma_3} - X_0^- = X_0^+.$$

From the above conditions and by (8), the second solution is given by

$$(X_0^+, Y_0^+, Z_0^+, X_1^+, Y_0^+, Z_1^+) = (X_1^-, Y_0^-, Z_1^-, X_0^-, Y_0^-, Z_0^-).$$

That is, the second solution provides the same periodic orbit than the solution (12). Therefore in this case we have proved that it is possible to have at most one limit cycle intersecting the paraboloid P_E in two different points p_0 and p_1.

Subcase 1.1.1.2: $\gamma_1(-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3) = 0$. We have three subcases: $\gamma_1 = 0$ and $-2a_3^2\gamma_1 - 2b_3^2\gamma_1 + 2a_1a_3\gamma_3 + 2b_1b_3\gamma_3 \neq 0$, or $\gamma_1 \neq 0$ and $-2a_3^2\gamma_1 - 2b_3^2\gamma_1 + 2a_1a_3\gamma_3 + 2b_1b_3\gamma_3 = 0$, or $\gamma_1 = 0$ and $-2a_3^2\gamma_1 - 2b_3^2\gamma_1 + 2a_1a_3\gamma_3 + 2b_1b_3\gamma_3 = 0$.

Subcase 1.1.1.2.1: $\gamma_1 = 0$ *and* $-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3 \neq 0$. When $\gamma_1 = 0$, from e_1 in (10) we obtain

$$X_0^\pm = \pm\frac{\sqrt{-a_1a_4 - b_1b_4 - a_1a_2Y_0 - b_1b_2Y_0 - a_1a_3Y_0^2 - b_1b_3Y_0^2}}{\sqrt{a_1a_3 + b_1b_3}}.$$

If we consider X_0^-, from (9) and e_3 we obtain X_1^- and Y_0^-, respectively. Similarly to Subcase 1.1.1.1, if we consider X_0^+ we get X_1^+ and Y_0^+. But then we have that

$$X_0^- = X_1^+, \; X_1^- = X_0^+, \quad \text{and} \quad Y_0^- = Y_0^+.$$

So we only have one solution of system (6) and therefore at most one limit cycle.

Subcase 1.1.1.2.2: $\gamma_1 \neq 0$ and $-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3 = 0$. If $a_3^2 + b_3^2 \neq 0$, from $-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3 = 0$ we get

$$\gamma_1 = \frac{(a_1a_3 + b_1b_3)\gamma_3}{a_3^2 + b_3^2}.$$

Substituting into equation e_1, we get

$$e_1 : \frac{\gamma_3^3(a_3b_1 - a_1b_3)}{(a_3^2 + b_3^2)}\left(\frac{(a_1a_3 + b_1b_3)(a_1b_3 - a_3b_1)}{(a_3^2 + b_3^2)} + 2(a_3b_4 - a_4b_3) + 2(a_3b_2 - a_2b_3)Y_0\right).$$

If $a_1b_3 - a_3b_1 = 0$ we have that $e_1 \equiv 0$ and equation $e_3 = 0$ has two unknowns X_0 and Y_0. Then if there are solutions for this equation, we would have a continuum of solutions that would generate a continuum of crossing periodic solutions, and so we would not have limit cycles.

If $a_1b_3 - a_3b_1 \neq 0$ and $a_3b_2 - a_2b_3 = 0$ then equation $e_1 = 0$ only provides conditions for the parameters a_i, b_i for $i = 1, 3, 4$, and moreover to solve system (6) it is equivalent to solve equation $e_3 = 0$ which has two unknowns X_0 and Y_0. As before, we cannot have limit cycles.

If $a_1b_3 - a_3b_1 \neq 0$ and $a_3b_2 - a_2b_3 \neq 0$, then we have

$$Y_0 = \frac{(a_1a_3 + b_1b_3)(a_3b_1 - a_1b_3) + 2(a_4b_3 - a_3b_4)(a_3^2 + b_3^2)}{2(a_3b_2 - a_2b_3)(a_3^3 + b_3^2)}.$$

Substituting Y_0 into equation e_3 we obtain $X_0^{\pm}$. Similarly to the above case, if we consider X_0^-, using (9) we obtain X_1^- and then we have the solution $(X_0^-, Y_0, Z_0^-, X_1^-, Y_0, Z_1^-)$. Considering X_0^+ we get X_1^+, but we have that

$$X_0^- = X_1^+, \ X_1^- = X_0^+.$$

Then the second solution provides the same periodic orbit as the first one. Therefore in this case we only have one solution for system (6), and hence at most one limit cycle intersecting the paraboloid P_E in two points.

If $a_3^2 + b_3^2 = 0$, equation e_1 in (10) becomes

$$e_1 \ : \ 2\gamma_3^3(a_1a_2 + b_1b_2)Y_0 + \gamma_3^2(2\gamma_3(a_1a_4 + b_1b_4) - (a_1^2 + b_1^2)\gamma_1).$$

If $a_1a_2 + b_1b_2 = 0$, equation $e_1 = 0$ provides conditions for the parameters $a_1, a_4, b_1, b_4, \gamma_1, \gamma_3$, and similarly to the above case, equation $e_3 = 0$ has two unknowns X_0 and Y_0 and as before, we cannot have limit cycles.

Considering that $a_1a_2 + b_1b_2 \neq 0$, from e_1 we get

$$Y_0 = \frac{a_1^2\gamma_1 + b_1^2\gamma_1 - 2a_1a_4\gamma_3 - 2b_1b_4\gamma_3}{2(a_1a_2 + b_1b_2)\gamma_3}.$$

Substituting it in equation e_3 we obtain $X_0^{\pm}$, and similarly to the above cases, the two possible solutions generate the same periodic orbit, and so we have at most one limit cycle.

Subcase 1.1.1.2.3: $\gamma_1 = 0$ *and* $-a_3^2\gamma_1 - b_3^2\gamma_1 + a_1a_3\gamma_3 + b_1b_3\gamma_3 = 0$. This condition is equivalent to $\gamma_1 = 0$ and

$$a_1a_3 + b_1b_3 = 0. \tag{13}$$

From (13) we obtain that when $a_3 \neq 0$ then $a_1 = -b_1b_3/a_3$. In this case equation e_1 becomes

$$e_1 \ : \ \frac{2b_1\gamma_3^3}{a_3}\Big((a_3b_4 - a_4b_3) + (a_3b_2 - a_2b_3)Y_0\Big).$$

If $b_1 = 0$ then $e_1 \equiv 0$ and as in the above case, equation $e_3 = 0$ has two unknowns X_0 and Y_0, and if there are solutions for this equation, we would have a continuum of crossing periodic orbits and so we do not have limit cycles.

If $b_1 \neq 0$ and $a_3b_2 - a_3b_3 = 0$, equation $e_1 = 0$ provides conditions for the parameters a_3, a_4, b_3 and b_4, and again $e_3 = 0$ has two unknowns X_0 and Y_0, and so we cannot have limit cycles.

If $b_1 \neq 0$ and $a_3b_2 - a_3b_3 \neq 0$ then from equation e_1, we get $Y_0 = (a_4b_3 - a_3b_4)/(a_3b_2 - a_2b_3)$. In this case, substituting it in equation e_3 we obtain $X_0^{\pm}$, and as in the above cases the two possible solutions generate only one periodic orbit and then we have at most one limit cycle.

When $a_3 = 0$, from the condition (13) we obtain three possibilities, namely either $b_1 = 0$ and $b_3 \neq 0$, or $b_1 \neq 0$ and $b_3 = 0$, or $b_1 = 0 = b_3$. In all cases, we obtain the expression for Y_0 by equation e_1 in (10) and substituting it in equation e_3 we obtain $X_0^{\pm}$. These points satisfy $X_0^+ = X_1^-$ and $X_0^- = X_1^+$, that is, the two possible solutions $(X_0^{\pm}, Y_0, Z_0^{\pm}, X_1^{\mp}, Y_0, Z_1^{\mp})$ generate the same periodic orbit. Therefore we can have at most one crossing limit cycle.

Subcase 1.1.2: $c_3\gamma_2 - c_2\gamma_3 = 0$. In this case, we have that equations e_2 and e_4 in (6) satisfy,

$$e_2 = \frac{c_3}{\gamma_3} e_4,$$

this is, system (6) reduces to system which has three polynomial equations and four unknowns X_0, X_1, Y_0, Y_1, if there are solutions for this system, then it would have a continuum of solutions that produce a continuum of crossing periodic solutions, then we cannot have crossing limit cycles.

Subcase 1.2: $c_3\gamma_1 - c_1\gamma_3 \neq 0$. From (7) we get

$$X_0 = \frac{(c_3\gamma_1 - c_1\gamma_3)X_1 - (c_3\gamma_2 - c_2\gamma_3)(Y_1 - Y_0)}{c_3\gamma_1 - c_1\gamma_3}. \tag{14}$$

Substituting this expression of X_0 into system (6), we obtain that equations e_1, e_3 and e_4, have as common factor $Y_1 - Y_0$, and we observed that this factor can be eliminated because if $Y_1 = Y_0$, by (14) we would have that $p_0 = p_1$, and consequently we would not have limit cycles.

The expression for equation e_4 is

$$\begin{aligned} e_4 \ : \ & -(c_2\gamma_1 - c_1\gamma_2)(c_3\gamma_1 - c_1\gamma_3) + 2X_1(c_3\gamma_2 - c_2\gamma_3)(c_3\gamma_1 - c_1\gamma_3) + Y_0(-c_3^2(\gamma_1^2 + \gamma_2^2) \\ & + 2c_3(c_1\gamma_1 + c_2\gamma_2)\gamma_3 - (c_1^2 + c_2^2)\gamma_3^2) - Y_1((c_3(\gamma_1 - \gamma_2) + (c_2 - c_1)\gamma_3)(c_3(\gamma_1 + \gamma_2) \\ & - (c_1 + c_2)\gamma_3)). \end{aligned} \tag{15}$$

Then we have two subcases, either $c_3\gamma_2 - c_2\gamma_3 = 0$, or $c_3\gamma_2 - c_2\gamma_3 \neq 0$.

Subcase 1.2.1: $c_3\gamma_2 - c_2\gamma_3 = 0$. We obtain that $c_2 = c_3\gamma_2/\gamma_3$. Moreover by (14), we get that

$$X_0 = X_1. \tag{16}$$

And from (15) we obtain

$$Y_0 = -Y_1 - \frac{\gamma_2}{\gamma_3}. \tag{17}$$

Substituting these expressions into equations e_1 and e_3, we have

$$
\begin{aligned}
e_1 = {} & (a_3^2+b_3^2)\gamma_2^3 - 2(a_2a_3+b_2b_3)\gamma_2^2\gamma_3 + (a_2^2+2a_3a_4+b_2^2+2b_3b_4)\gamma_2\gamma_3^2 - 2(a_2a_4 \\
& + b_2b_4)\gamma_3^3 + 2\gamma_2\gamma_3Y_1((a_3^2+b_3^2)\gamma_2 - (a_2a_3+b_2b_3)\gamma_3) + 2\gamma_3^2X_1(a_1a_3\gamma_2 \\
& + b_1b_3\gamma_2 - a_1a_2\gamma_3 - b_1b_2\gamma_3) + 2\gamma_3^2X_1^2((a_3^2+b_3^2)\gamma_2 - (a_2a_3+b_2b_3)\gamma_3) \\
& + 2\gamma_3^2Y_1^2((a_3^2+b_3^2)\gamma_2 - (a_2a_3+b_2b_3)\gamma_3), \\
e_3 = {} & (\alpha_3^2+\beta_3^2)\gamma_2^3 - 2(\alpha_2\alpha_3+\beta_2\beta_3)\gamma_2^2\gamma_3 + (\alpha_2^2+2\alpha_3\alpha_4+\beta_2^2+2\beta_3\beta_4)\gamma_2\gamma_3^2 \\
& - 2(\alpha_2\alpha_4+\beta_2\beta_4)\gamma_3^3 + 2\gamma_2\gamma_3Y_1((\alpha_3^2+\beta_3^2)\gamma_2 - (\alpha_2\alpha_3+\beta_2\beta_3)\gamma_3) + 2\gamma_3^2X_1 \\
& (\alpha_1\alpha_3\gamma_2 + \beta_1\beta_3\gamma_2 - \alpha_1\alpha_2\gamma_3 - \beta_1\beta_2\gamma_3) + 2\gamma_3^2X_1^2((\alpha_3^2+\beta_3^2)\gamma_2 - (\alpha_2\alpha_3 \\
& + \beta_2\beta_3)\gamma_3) + 2\gamma_3^2Y_1^2((\alpha_3^2+\beta_3^2)\gamma_2 - (\alpha_2\alpha_3+\beta_2\beta_3)\gamma_3).
\end{aligned}
\tag{18}
$$

By equation e_1 we can have two subcases: $(a_3^2+b_3^2)\gamma_2 - (a_2a_3+b_2b_3)\gamma_3 = 0$, or $(a_3^2+b_3^2)\gamma_2 - (a_2a_3+b_2b_3)\gamma_3 \neq 0$.

Subcase 1.2.1.1: $(a_3^2+b_3^2)\gamma_2 - (a_2a_3+b_2b_3)\gamma_3 = 0$. If we consider $a_3^2+b_3^2 \neq 0$, by $(a_3^2+b_3^2)\gamma_2 - (a_2a_3+b_2b_3)\gamma_3 = 0$, we obtain

$$
\gamma_2 = \frac{(a_2a_3+b_2b_3)\gamma_3}{a_3^2+b_3^2}. \tag{19}
$$

Hence from e_1 we get

$$
X_1 = -\frac{2(a_3b_1-a_1b_3)(a_3b_2-a_2b_3)\gamma_3^2}{a_3^2+b_3^2}.
$$

Now we introduce X_1 into e_3 and we obtain two options for Y_1, namely Y_1^+ and Y_1^-, which satisfy

$$
Y_1^+ + Y_1^- = -\frac{a_2a_3+b_2b_3}{a_3^2+b_3^2}. \tag{20}
$$

Then we have two real solutions, namely $S^\pm = (X_0^\pm, Y_0^\pm, Z_0^\pm, X_1^\pm, Y_1^\pm, Z_1^\pm)$, but from (16), we have that $X_0^\pm = X_1^\pm$. Moreover from equation (17) we obtain two options for Y_0, this is,

$$
Y_0^\pm = -Y_1^\pm - \frac{\gamma_2}{\gamma_3}. \tag{21}
$$

Then by equations (19), (20) and (21), we have that

$$
\begin{aligned}
Y_0^+ &= -Y_1^+ - \frac{\gamma_2}{\gamma_3} = Y_1^- + \frac{a_2a_3+b_2b_3}{a_3^2+b_3^2} - \frac{\gamma_2}{\gamma_3} = Y_1^-, \\
Y_0^- &= -Y_1^- - \frac{\gamma_2}{\gamma_3} = Y_1^+ + \frac{a_2a_3+b_2b_3}{a_3^2+b_3^2} - \frac{\gamma_2}{\gamma_3} = Y_1^+.
\end{aligned}
$$

Then the solution

$$
\begin{aligned}
S^- = (X_0^-, Y_0^-, Z_0^-, X_1^-, Y_1^-, Z_1^-) &= (X_0, Y_0^-, Z_0^-, X_0, Y_1^-, Z_1^-) \\
&= (X_0, Y_1^+, Z_1^+, X_0, Y_0^+, Z_0^+),
\end{aligned}
$$

generates the same periodic orbit than the solution S^+. Therefore we have proved that it is possible to have at most one real solution of system (6) and so at most one limit cycle that intersects the paraboloid P_E in two points.

If $a_3^2+b_3^2=0$, we can find the expression for X_1 from equation e_1 and substituting in equation e_3 we get two options for Y_1, namely $Y_1^{\pm}$. Moreover these points satisfy that $Y_1^+ + Y_1^- = -\gamma_2/\gamma_3$ and similarly to the above case, we obtain that $Y_0^+ = Y_1^-$ and $Y_0^- = Y_1^+$ and so the two possible solutions $S^{\pm} = (X_0^{\pm}, Y_0^{\pm}, Z_0^{\pm}, X_1^{\pm}, Y_1^{\pm}, Z_1^{\pm})$ generate the same periodic orbit. In short, we can have at most one limit cycle.

Subcase 1.2.1.2: $(a_3^2+b_3^2)\gamma_2-(a_2a_3+b_2b_3)\gamma_3 \neq 0$. By equation e_1 in (18) we obtain

$$
\begin{aligned}
Y_1^{\pm} =& \frac{-1}{2\gamma_3^2((a_3^2+b_3^2)\gamma_2-(a_2a_3+b_2b_3)\gamma_3)}\Big(\gamma_2\gamma_3((a_3^2+b_3^2)\gamma_2-(a_2a_3+b_2b_3)\gamma_3)\\
&\pm\sqrt{(}-\gamma_3^2((a_3^2+b_3^2)\gamma_2-(a_2a_3+b_2b_3)\gamma_3)((a_3^2+b_3^2)\gamma_2^3-3(a_2a_3+b_2b_3)\gamma_2^2\gamma_3\\
&+2(a_2^2+b_2^2+2a_3^2X_1^2+2a_3(a_4+a_1X_1)+2b_3(b_4+X_1(b_1+b_3X_1)))\gamma_2\gamma_3^2\\
&-4(a_2(a_4+X_1(a_1+a_3X_1))+b_2(b_4+X_1(b_1+b_3X_1)))\gamma_3^3))\Big).
\end{aligned}
$$

Moreover we observe that

$$
Y_1^+ + Y_1^- = -\frac{\gamma_2}{\gamma_3}. \tag{22}
$$

Then we have two reals solution, namely $S^{\pm} = (X_0^{\pm}, Y_0^{\pm}, Z_0^{\pm}, X_1^{\pm}, Y_1^{\pm}, Z_1^{\pm})$. Substituting $Y_1^{\pm}$ in equation e_3 in (18) we get expressions for $X_1^{\pm}$. From (17) we have $Y_0^{\pm} = -Y_1^{\pm} - \gamma_2/\gamma_3$. More precisely from (17) and (22) we have that

$$
\begin{aligned}
Y_0^+ &= -Y_1^+ - \frac{\gamma_2}{\gamma_3} = Y_1^-,\\
Y_0^- &= -Y_1^- - \frac{\gamma_2}{\gamma_3} = Y_1^+.
\end{aligned}
$$

Then for conditions above and by (16) the solution $S^- = (X_0, Y_0^-, Z_0^-, X_0, Y_1^-, Z_1^-) = (X_0, Y_1^+, Z_1^+, X_0, Y_0^+, Z_0^+)$, generates the same periodic orbit than the solution S^+. Therefore we only have one real solution of system (6), and so at most one limit cycle.

Subcase 1.2.2: $c_3\gamma_2 - c_2\gamma_3 \neq 0$. From equation e_4 in (15) we get

$$
\begin{aligned}
X_1 =& \frac{1}{2(c_3\gamma_1-c_1\gamma_3)(c_3\gamma_2-c_2\gamma_3)}\big((c_2\gamma_1-c_1\gamma_2)(c_3\gamma_1-c_1\gamma_3)+Y_1(c_3(\gamma_1-\gamma_2)\\
&+(c_2-c_1)\gamma_3)(c_3(\gamma_1+\gamma_2)-(c_1+c_2)\gamma_3)+Y_0(c_3^2(\gamma_1^2+\gamma_2^2)-2c_3(c_1\gamma_1\\
&+c_2\gamma_2)\gamma_3+(c_1^2+c_2^2)\gamma_3^2)\big).
\end{aligned}
$$

Then substituting this expression into equations e_1 and e_3, they become

$$
\begin{aligned}
e_1 &: k_0 + k_1(Y_0+Y_1) + k_2(Y_0^2+Y_1^2) + k_3Y_0Y_1,\\
e_3 &: \eta_0 + \eta_1(Y_0+Y_1) + \eta_2(Y_0^2+Y_1^2) + \eta_3Y_0Y_1.
\end{aligned}
$$

Where the expressions of the coefficients k_i for $i = 0, 1, 2, 3$ are

$$
\begin{aligned}
k_0 =&(c_3\gamma_1 - c_1\gamma_3)^2(a_3^2(c_2\gamma_1 - c_1\gamma_2)^3 + b_3^2(c_2\gamma_1 - c_1\gamma_2)^3 + 2(c_3\gamma_2 - c_2\gamma_3)(-a_1a_2(c_2\gamma_1\\
&- c_1\gamma_2)(c_3\gamma_1 - c_1\gamma_3) - b_1b_2(c_2\gamma_1 - c_1\gamma_2)(c_3\gamma_1 - c_1\gamma_3) - 2(a_2a_4 + b_2b_4)(c_3\gamma_1\\
&- c_1\gamma_3)(c_3\gamma_2 - c_2\gamma_3) + 2a_1a_4(c_3\gamma_2 - c_2\gamma_3)^2 + 2b_1b_4(c_3\gamma_2 - c_2\gamma_3)^2 - a_1^2(c_2\gamma_1\\
&- c_1\gamma_2)(-c_3\gamma_2 + c_2\gamma_3) - b_1^2(c_2\gamma_1 - c_1\gamma_2)(-c_3\gamma_2 + c_2\gamma_3)) - a_3(c_2\gamma_1 - c_1\gamma_2)\\
&(a_2(c_2\gamma_1 - c_1\gamma_2)(c_3\gamma_1 - c_1\gamma_3) + (-c_3\gamma_2 + c_2\gamma_3)(3a_1c_2\gamma_1 - 3a_1c_1\gamma_2 + 4a_4c_3\gamma_2\\
&- 4a_4c_2\gamma_3)) - b_3(c_2\gamma_1 - c_1\gamma_2)(b_2(c_2\gamma_1 - c_1\gamma_2)(c_3\gamma_1 - c_1\gamma_3) + (-c_3\gamma_2 + c_2\gamma_3)\\
&(3b_1c_2\gamma_1 - 3b_1c_1\gamma_2 + 4b_4c_3\gamma_2 - 4b_4c_2\gamma_3))),\\
k_1 =&- 2(c_3\gamma_1 - c_1\gamma_3)^2(-a_3^2(c_2\gamma_1 - c_1\gamma_2)^2(c_3\gamma_1 - c_1\gamma_3) - b_3^2(c_2\gamma_1 - c_1\gamma_2)^2(c_3\gamma_1\\
&- c_1\gamma_3) - a_3(c_2\gamma_1 - c_1\gamma_2)(2a_1(c_3\gamma_1 - c_1\gamma_3)(c_3\gamma_2 - c_2\gamma_3) - a_2(c_3(\gamma_1 - \gamma_2)\\
&+ (-c_1 + c_2)\gamma_3)(c_3(\gamma_1 + \gamma_2) - (c_1 + c_2)\gamma_3)) - (c_3\gamma_2 - c_2\gamma_3)(a_1^2(c_3\gamma_1 - c_1\gamma_3)\\
&(c_3\gamma_2 - c_2\gamma_3) + b_1^2(c_3\gamma_1 - c_1\gamma_3)(c_3\gamma_2 - c_2\gamma_3) - (a_2^2 + b_2^2)(c_3\gamma_1 - c_1\gamma_3)(c_3\gamma_2\\
&- c_2\gamma_3) - b_1b_2(c_3\gamma_1 - c_3\gamma_2 - c_1\gamma_3 + c_2\gamma_3)(c_3(\gamma_1 + \gamma_2) - (c_1 + c_2)\gamma_3) - a_1a_2(c_3\\
&(\gamma_1 - \gamma_2) + (-c_1 + c_2)\gamma_3)(c_3(\gamma_1 + \gamma_2) - (c_1 + c_2)\gamma_3)) - b_3(c_2\gamma_1 - c_1\gamma_2)(2b_1\\
&(c_3\gamma_1 - c_1\gamma_3)(c_3\gamma_2 - c_2\gamma_3) - b_2(c_3(\gamma_1 - \gamma_2) + (-c_1 + c_2)\gamma_3)(c_3(\gamma_1 + \gamma_2)\\
&- (c_1 + c_2)\gamma_3))),\\
k_2 =&(c_3^2(\gamma_1^2 + \gamma_2^2) - 2c_3(c_1\gamma_1 + c_2\gamma_2)\gamma_3 + (c_1^2 + c_2^2)\gamma_3^2)^2(a_3^2(c_2\gamma_1 - c_1\gamma_2) + a_3(-a_2c_3\gamma_1\\
&+ a_1c_3\gamma_2 + a_2c_1\gamma_3 - a_1c_2\gamma_3) + b_3(b_3c_2\gamma_1 - b_2c_3\gamma_1 - b_3c_1\gamma_2 + b_1c_3\gamma_2 + b_2c_1\gamma_3\\
&- b_1c_2\gamma_3)),\\
k_3 =&2(c_3(\gamma_1 - \gamma_2) + (-c_1 + c_2)\gamma_3)(c_3(\gamma_1 + \gamma_2) - (c_1 + c_2)\gamma_3)(c_3^2(\gamma_1^2 + \gamma_2^2) - 2c_3(c_1\gamma_1\\
&+ c_2\gamma_2)\gamma_3 + (c_1^2 + c_2^2)\gamma_3^2)(a_3^2(c_2\gamma_1 - c_1\gamma_2) + a_3(-a_2c_3\gamma_1 + a_1c_3\gamma_2 + a_2c_1\gamma_3\\
&- a_1c_2\gamma_3) + b_3(b_3c_2\gamma_1 - b_2c_3\gamma_1 - b_3c_1\gamma_2 + b_1c_3\gamma_2 + b_2c_1\gamma_3 - b_1c_2\gamma_3)).
\end{aligned}
$$

The expressions of η_i for $i = 0, 1, 2, 3$ are equals, only interchanging a_j by α_j and b_j by β_j for $j = 1, 2, 3, 4$.

We know that if (Y_0, Y_1) is a solution of the system $e_1 = 0$, $e_3 = 0$, the point Y_0 must be a root of the resultant of e_1 and e_3 with respect to the variable Y_1. We denote such a resultant as $R(Y_0)$. Moreover the point Y_1 must be a root of the resultant of e_1 and e_3 with respect to the variable Y_0, which we denote it as $R(Y_1)$. In this case, it is possible to verify that by interchanging the variable Y_0 by Y_1, the expressions of the resultants $R(Y_0)$ and $R(Y_1)$ are the same. Moreover in this case the resultants are polynomials of degree 2. Then we would have at most two reals solutions Y_0, Y_1 for system $e_1 = 0 = e_3$, and consequently two real solutions $S^i = (X_0^i, Y_0^i, Z_0^i, X_1^i, Y_1^i, Z_1^i)$ for system (6). But we can observe that if $(X_0, Y_0, Z_0, X_1, Y_1, Z_1)$ is a solution of system (6) then $(X_1, Y_1, Z_1, X_0, Y_0, Z_0)$ is also a solution, and this last solution generates the same periodic orbit than the first one. Therefore we can conclude that system (6) has at most one real solution and consequently the discontinuous piecewise differential system formed by the linear differential systems (2) and (5) has at most one limit cycle.

We do not give the explicit expressions of the polynomials $R(Y_0)$ and $R(Y_1)$ because their expressions are huge. These resultants can be computed immediately using an algebraic manipulator, such as Mathematica or Maple.

Case 2: $c_3\gamma_3 = 0$. We have three subcases: $c_3 = \gamma_3 = 0$, or $c_3 = 0$ and $\gamma_3 \neq 0$, or $c_3 \neq 0$ and $\gamma_3 = 0$.

Subcase 2.1: $c_3 = \gamma_3 = 0$. Equation e_2 in system (6) becomes

$$e_2 \, : \, c_1(X_0 - X_1) + c_2(Y_0 - Y_1). \tag{23}$$

then we have two subcases $c_2 \neq 0$, or $c_2 = 0$.

Subcase 2.1.1: $c_2 \neq 0$. From (23), we get

$$Y_1 = \frac{c_1(X_0 - X_1) + c_2 Y_0}{c_2}. \tag{24}$$

Then equation e_4 in system (6) reduces to

$$e_4 \, : \, \frac{(X_0 - X_1)(c_2\gamma_1 - c_1\gamma_2)}{c_2}.$$

First, we assume that $c_2\gamma_1 - c_1\gamma_2 \neq 0$. Then from e_4 we get $X_0 = X_1$, and from (24) we get $Y_0 = Y_1$. So we have that $p_0 = p_1$ and we do not have limit cycles.

Second we consider that $c_2\gamma_1 - c_1\gamma_2 = 0$, that is, $\gamma_1 = c_1\gamma_2/c_2$. Then $e_4 \equiv 0$, and equation e_1 becomes

$$\begin{aligned} e_1 \, :& (a_4 + a_1X_0 + a_2Y_0 + a_3(X_0^2 + Y_0^2))^2 + (b_4 + b_1X_0 + b_2Y_0 + b_3(X_0^2 + Y_0^2))^2 \\ & - \left(a_4 + a_1X_1 + a_2\left(\frac{c_1(X_0 - X_1)}{c_2} + Y_0\right) + a_3\left(X_1^2 + \frac{(c_1(X_0 - X_1) + c_2Y_0)^2}{c_2^2}\right)\right)^2 \\ & - \left(b_4 + b_1X_1 + b_2\left(\frac{c_1(X_0 - X_1)}{c_2} + Y_0\right) + b_3\left(X_1^2 + \frac{(c_1(X_0 - X_1) + c_2Y_0)^2}{c_2^2}\right)\right)^2. \end{aligned}$$

Moreover, the expression for equation e_3 are the same, changing $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4)$. This means that we must solve system $e_1 = 0 = e_3$, which has two polynomial equations and three unknowns X_0, X_1, Y_0. Hence, if there are solutions for this system, then it would have a continuum of solutions that produce a continuum of crossing periodic solutions and therefore we cannot have crossing limit cycles.

Subcase 2.1.2: $c_2 = 0$. Equation (23) becomes

$$e_2 \, : \, c_1(X_0 - X_1).$$

First, we consider that $c_1 \neq 0$. Then we have that $X_0 = X_1$ and equation e_4 in system (6) reduces to

$$e_4 = \gamma_2(Y_0 - Y_1).$$

We observe that we can consider $Y_0 \neq Y_1$, because if $Y_0 = Y_1$ we have that $p_0 = p_1$ and we do not have limit cycles. Hence $\gamma_2 = 0$ and equation

$$\begin{aligned} e_1 \, :& (a_4 + a_1X_0 + a_2Y_0 + a_3(X_0^2 + Y_0^2))^2 + (b_4 + b_1X_0 + b_2Y_0 + b_3(X_0^2 + Y_0^2))^2 \\ & - (a_4 + a_1X_0x + a_2Y_1 + a_3(X_0^2 + Y_1^2))^2 - (b_4 + b_1X_0 + b_2Y_1 + b_3(X_0^2 + Y_1^2))^2. \end{aligned}$$

Moreover the expression for equation e_3 is the same changing a_i by α_i and b_i by β_i, respectively, for $i = 1, 2, 3, 4$. Then similar arguments used for Subcase 2.1.1 show that if system $e_1 = 0 = e_3$ has real solutions then it has a continuum of solutions and therefore we do not have limit cycles.

Second, if we consider $c_1 = 0$, then we have $c_2 = 0 = c_3$. In this case, we obtain that the linear differential system (2) considered in the region $\mathcal{R}_E^1$, becomes $\dot{X} = 0$, $\quad \dot{Y} = 0$, $\quad \dot{Z} = 0$ and so we cannot have limit cycles.

Subcase 2.2: $c_3 = 0$ *and* $\gamma_3 \neq 0$. Equation e_2 in system (6) reduces to equation (23). Then we have two subcases $c_2 \neq 0$, or $c_2 = 0$.

Subcase 2.2.1: $c_2 \neq 0$. From equation e_2, we obtain the expression for Y_1 given in (24). Then equation e_4 in system (6) becomes

$$e_4 \; : \; \frac{(X_0 - X_1)(c_1^2(X_1 - X_0)\gamma_3 + c_2^2(\gamma_1 + (X_0 + X_1)\gamma_3) - c_1c_2(\gamma_2 + 2Y_0\gamma_3))}{c_2^2}. \tag{25}$$

We can consider $X_0 \neq X_1$, because if $X_0 = X_1$, from equation (24), we get $Y_0 = Y_1$ and then $p_0 = p_1$ implying that we cannot have limit cycles. So, considering $X_0 \neq X_1$ in e_4 (25), we get

$$X_1 = \frac{-c_2^2\gamma_1 + c_1c_2\gamma_2 + c_1^2X_0\gamma_3 - c_2^2X_0\gamma_3 + 2c_1c_2Y_0\gamma_3}{(c_1^2 + c_2^2)\gamma_3}.$$

Substituting Y_1 and X_1 in equations e_1 and e_3 of system (6), we have

$$\begin{aligned} e_1 =& \frac{1}{(c_1^2 + c_2^2)^2\gamma_3^2}\Big(\lambda_0(c_1^2 + c_2^2)\gamma_3\Big(X_0^3 - \frac{c_1}{c_2}Y_0^3 + Y_0^2X_0 - \frac{c_1}{c_2}Y_0X_0^2\Big) - 2\lambda_1Y_0^2 \\ &+ 2\lambda_2X_0^2 + \lambda_3X_0Y_0 - \frac{2}{\gamma_3}\lambda_4X_0 - \frac{2}{\gamma_3}\lambda_5Y_0 - \frac{(c_2\gamma_1 - c_1\gamma_2)}{\gamma_3}\lambda_6\Big), \\ e_3 =& \frac{1}{(c_1^2 + c_2^2)^2\gamma_3^2}\Big(\delta_0(c_1^2 + c_2^2)\gamma_3\Big(X_0^3 - \frac{c_1}{c_2}Y_0^3 + Y_0^2X_0 - \frac{c_1}{c_2}Y_0X_0^2\Big) - 2\delta_1Y_0^2 \\ &+ 2\delta_2X_0^2 + \delta_3X_0Y_0 - \frac{2}{\gamma_3}\delta_4X_0 - \frac{2}{\gamma_3}\delta_5Y_0 - \frac{(c_2\gamma_1 - c_1\gamma_2)}{\gamma_3}\delta_6\Big), \end{aligned}$$

and the expressions of the coefficients λ_i for $i = 0, 1, 2, 3, 4, 5, 6$ are

$$
\begin{aligned}
\lambda_0 =& -4(a_3^2+b_3^2)c_2(c_2\gamma_1-c_1\gamma_2)+4c_2(-a_2a_3c_1-b_2b_3c_1+a_1a_3c_2+b_1b_3c_2)\gamma_3,\\
\lambda_1 =&(a_3^2+b_3^2)(3c_1^2+c_2^2)(c_2\gamma_1-c_1\gamma_2)^2-(a_2a_3c_1(-3c_1^2+c_2^2)+b_2b_3c_1(c_2^2-3c_1^2)\\
&+(a_1a_3+b_1b_3)c_2(5c_1^2+c_2^2))(c_2\gamma_1-c_1\gamma_2)\gamma_3+2c_1c_2(-((a_1a_2+b_1b_2)c_1^2)\\
&+(a_1^2-a_2^2+b_1^2-b_2^2)c_1c_2+(a_1a_2+b_1b_2)c_2^2)\gamma_3^2,\\
\lambda_2 =& -((a_3^2+b_3^2)(c_1^2+3c_2^2)(c_2\gamma_1-c_1\gamma_2)^2)+((a_1a_3+b_1b_3)c_2(c_1^2-3c_2^2)+a_2a_3c_1(c_1^2\\
&+5c_2^2)+b_2b_3c_1(c_1^2+5c_2^2))(-c_2\gamma_1+c_1\gamma_2)\gamma_3+2c_1c_2(-((a_1a_2+b_1b_2)c_1^2)+(a_1^2\\
&-a_2^2+b_1^2-b_2^2)c_1c_2+(a_1a_2+b_1b_2)c_2^2)\gamma_3^2,\\
\lambda_3 =&(8(a_3^2+b_3^2)c_1c_2(c_2\gamma_1-c_1\gamma_2)^2-4(a_1a_3c_1(c_1^2-3c_2^2)+b_1b_3c_1(c_1^2-3c_2^2)-(a_2a_3\\
&+b_2b_3)c_2(-3c_1^2+c_2^2))(-c_2\gamma_1+c_1\gamma_2)\gamma_3+4(c_1-c_2)(c_1+c_2)((a_1a_2+b_1b_2)c_1^2\\
&+(a_2^2-a_1^2-b_1^2+b_2^2)c_1c_2-(a_1a_2+b_1b_2)c_2^2)\gamma_3^2,\\
\lambda_4 =&2(a_3^2+b_3^2)c_2(c_2\gamma_1-c_1\gamma_2)^3+((a_1a_3+b_1b_3)c_1^2+4(a_2a_3+b_2b_3)c_1c_2-3(a_1a_3\\
&+b_1b_3)c_2^2)(c_2\gamma_1-c_1\gamma_2)^2\gamma_3+((a_1a_2+b_1b_2)c_1^3-(a_1^2-2a_2^2-2a_3a_4+b_1^2-2(b_2^2\\
&+b_3b_4))c_1^2c_2-3(a_1a_2+b_1b_2)c_1c_2^2+(a_1^2+2a_3a_4+b_1^2+2b_3b_4)c_2^3)(c_2\gamma_1-c_1\gamma_2)\gamma_3^2\\
&-2c_2(-a_2a_4c_1-b_2b_4c_1+a_1a_4c_2+b_1b_4c_2)(c_1^2+c_2^2)\gamma_3^3,\\
\lambda_5 =&2(a_3^2+b_3^2)c_1(-c_2\gamma_1+c_1\gamma_2)^3+\gamma_3(4(a_1a_3+b_1b_3)c_1c_2-3(a_2a_3+b_2b_3)c_1^2+(a_2a_3\\
&+b_2b_3)c_2^2)(c_2\gamma_1-c_1\gamma_2)^2+((a_2^2+2a_3a_4+b_2^2+2b_3b_4)c_1^3-3(a_1a_2+b_1b_2)c_1^2c_2\\
&+(2a_1^2-a_2^2+2a_3a_4+2b_1^2-b_2^2+2b_3b_4)c_1c_2^2+(a_1a_2+b_1b_2)c_2^3)(c_1\gamma_2-c_2\gamma_1)\gamma_3^2\\
&-2c_1(a_2a_4c_1+b_2b_4c_1-(a_1a_4+b_1b_4)c_2)(c_1^2+c_2^2)\gamma_3^3,\\
\lambda_6 =&(a_3^2+b_3^2)(c_2\gamma_1-c_1\gamma_2)^3+2(a_2a_3c_1+b_2b_3c_1-(a_1a_3+b_1b_3)c_2)(c_2\gamma_1-c_1\gamma_2)^2\\
&\gamma_3+((a_2^2+2a_3a_4+b_2^2+2b_3b_4)c_1^2-2(a_1a_2+b_1b_2)c_1c_2+(a_1^2+2a_3a_4+b_1^2\\
&+2b_3b_4)c_2^2)(c_2\gamma_1-c_1\gamma_2)\gamma_3^2+2(a_2a_4c_1+b_2b_4c_1-(a_1a_4+b_1b_4)c_2)(c_1^2+c_2^2)\gamma_3^3.
\end{aligned}
$$

Moreover the expressions of δ_i for $i=0,1,2,3,4,5,6$ are equal, only interchanging a_j by α_j and b_j by β_j, respectively, for $j=1,2,3,4$. We know that if (X_0,Y_0) is a solution of the system $e_1=0$, $e_3=0$, the point X_0 must be a root of the resultant of e_1 and e_3 with respect to the variable Y_0, that we denote it by $R(X_0)$. Moreover the point Y_0 must be a root of the resultant of e_1 and e_3 with respect to the variable X_0, which we denote by $R(Y_0)$. In this case, it is possible to verify that $R(Y_0)=R(X_0)=0$. Therefore we cannot have limit cycles.

Subcase 2.2.2: $c_2=0$. Equation e_2 given in (23) reduces to

$$e_2=c_1(X_0-X_1).$$

First if we consider that $c_1\neq 0$, then $X_0=X_1$ and equation e_4 in (6) becomes

$$e_4=(Y_0-Y_1)(\gamma_2+(Y_0+Y_1)\gamma_3).$$

We observe that we can consider $Y_0\neq Y_1$, because if $Y_0=Y_1$ we obtain that $p_0=p_1$ and we cannot have limit cycles. Then from e_4, we get $Y_0=-(\gamma_2-Y_1\gamma_3)/\gamma_3$, and substituting them in equations e_1 and e_3 we obtain two polynomials of degree 4 whose resultants are zero. Therefore we cannot have limit cycles.

Second, we consider that $c_1=0$. But then since $c_2=c_3=0$, we obtain that the linear differential system (2) considered in the region $\mathcal{R}_E^1$ becomes $\dot{X}=0, \quad \dot{Y}=0, \quad \dot{Z}=0$, and so we cannot have limit cycles.

Subcase 2.3: $c_3\neq 0$ *and* $\gamma_3=0$. The proof in this case is analogous to the Subcase 2.2

□

Proof of statement (ii) If the discontinuity surface is P_E it is sufficient to consider the discontinuous piecewise linear differential system formed by a linear differential center (1) in the region $\mathcal{R}_E^1$ and an arbitrary linear differential center (5) in the region $\mathcal{R}_E^2$. We have that this discontinuous piecewise differential system has no crossing periodic orbits. If the discontinuity surface is P_H we consider the linear differential systems (2) and (5) in $\mathcal{R}_H^1$ and $\mathcal{R}_H^2$, respectively, considering $c_3 = 0 = \gamma_3$, $c_2 \neq 0$, and $c_2\gamma_1 - c_1\gamma_2 \neq 0$, as in the proof of Subcase 2.1.1 in a statement (i). In this case, we obtain that the unique real solution for system (4) is $(X_0, Y_0, Z_0, X_0, Y_0, Z_0)$ for $X_0, Y_0, Z_0 \in \mathbb{R}$, which do not generates crossing periodic orbits.

Proof of statement (iii) First, we provide a discontinuous piecewise differential system that has a continuum of crossing periodic orbits when the discontinuity surface is P_E. For this we consider in the region $\mathcal{R}_E^1$ the linear differential system

$$\dot{X} = -\frac{15}{8}(-1 + 2Y + 4Z), \quad \dot{Y} = -\frac{X}{20}, \quad \dot{Z} = \frac{X}{10}, \tag{26}$$

and in the region $\mathcal{R}_E^2$ the linear differential system

$$\dot{X} = \frac{319(15Y + 16(Z - 15))}{19200}, \quad \dot{Y} = \frac{29}{4}X, \quad \dot{Z} = -\frac{87}{4}X. \tag{27}$$

Considering $Z_0 = X_0^2 + Y_0^2$ and $Z_1 = X_1^2 + Y_1^2$ we get that system (6) is equivalent to

$$\begin{aligned} e_1 &: \frac{1}{20}(X_0^2 - X_1^2)(-49 + 100X_0^2 + 100X_1^2 + 100Y_1 + 200Y_1^2), \\ e_2 &: -X_0^2 + X_1^2 - 2Y_0 - Y_0^2 + 2Y_1 + Y_1^2, \\ e_3 &: \frac{29(X_0^2 - X_1^2)(210 + 8X_0^2 + 8X_1^2 + 15Y_1 + 16Y_1^2)}{1800}, \\ e_4 &: (X_0^2 - X_1^2). \end{aligned} \tag{28}$$

We observe that the points $p_{0,1} = (\pm X_0, Y_0, Z_0)$ for $X_0, Y_0, Z_0 \in \mathbb{R}$ are solutions of system (28). Therefore, the discontinuous piecewise linear differential system formed by the differential systems (26) and (27) has a continuum of crossing periodic solutions, which intersect the paraboloid P_E at the two points p_0 and p_1. See the three crossing periodic solutions $S^i = (p_0^i, p_1^i)$ for $i = 1, 2, 3$, of this continuum of crossing periodic solutions in Fig. 1, where

$$S^1 = \left(-1, \frac{1}{2}, \frac{5}{4}, 1, \frac{1}{2}, \frac{5}{4}\right), \quad S^2 = \left(-\frac{6}{5}, \frac{7}{10}, \frac{193}{100}, \frac{6}{5}, \frac{7}{10}, \frac{193}{100}\right),$$
$$S^3 = \left(-\frac{4}{5}, \frac{3}{10}, \frac{73}{100}, \frac{4}{5}, \frac{3}{10}, \frac{73}{100}\right).$$

Now we consider that the discontinuity surface is P_H and we provide a discontinuous piecewise differential system that has a continuum of crossing periodic orbits. In the region $\mathcal{R}_H^1$ we consider the linear differential system

$$\dot{X} = -\frac{15}{8}(-1 + 2Y + 4Z), \quad \dot{Y} = -\frac{X}{20}, \quad \dot{Z} = \frac{X}{10}, \tag{29}$$

and in the region $\mathcal{R}_H^2$ the linear differential system

$$\dot{X} = \frac{1073(15Y + 16(Z - 15))}{38400}, \quad \dot{Y} = -\frac{29}{8}X, \quad \dot{Z} = -\frac{87}{4}X. \tag{30}$$

Considering $Z_0 = X_0^2 - Y_0^2$ and $Z_1 = X_1^2 - Y_1^2$ we get that system (4) is equivalent to

$$\begin{aligned}
e_1 &: \frac{1}{20}(X_0^2 - X_1^2)(-49 + 100X_0^2 + 100X_1^2 + 100Y_1 - 200Y_1^2),\\
e_2 &: -X_0^2 + X_1^2 - 2Y_0 + 2Y_1 + Y_0^2 - Y_1^2,\\
e_3 &: \frac{29(X_0^2 - X_1^2)(210 + 8X_0^2 + 8X_1^2 + 15Y_1 - 16Y_1^2)}{1800},\\
e_4 &: \frac{-(X_0^2 - X_1^2)}{2}.
\end{aligned} \tag{31}$$

We observe that the points $p_{0,1} = (\pm X_0, Y_0, Z_0)$ for $X_0, Y_0, Z_0 \in \mathbb{R}$ are solutions of system (31). Then, the discontinuous piecewise linear differential system formed by the differential systems (29) and (30) has a continuum of crossing periodic solutions, which intersect the paraboloid P_H at the two points p_0 and p_1. See the three crossing periodic solutions $S^i = (p_0^i, p_1^i)$ for $i = 1, 2, 3$, of this continuum of crossing periodic solutions in Fig. 2, where

$$S^1 = \left(1, \frac{1}{2}, \frac{3}{4}, -1, \frac{1}{2}, \frac{3}{4}\right), \quad S^2 = \left(\frac{6}{5}, \frac{7}{10}, \frac{19}{20}, -\frac{6}{5}, \frac{7}{10}, \frac{19}{20}\right),$$
$$S^3 = \left(\frac{7}{5}, \frac{9}{10}, \frac{23}{20}, -\frac{7}{5}, \frac{9}{10}, \frac{23}{20}\right).$$

Proof of statement (iv) We provide two examples of discontinuous piecewise differential systems separated by either P_E or P_H and which have one crossing limit cycle. With these examples we can conclude that the upper bound found is reached.

First, we provide a discontinuous piecewise differential system whose discontinuity surface is P_E and that has one crossing limit cycle. In the region $\mathcal{R}_E^1$ we consider the linear differential system

$$\begin{aligned}
\dot{X} =& \frac{5(-77131 + 7\sqrt{146892247})X - 16(1040 + 5385Y + 1818Z)}{1200},\\
\dot{Y} =& \frac{1}{14400}\Big((307751207 - 25325\sqrt{146892247})X - 16(74045 - 5\sqrt{146892247}\\
&+ 15(-9748 + \sqrt{146892247})Y + 9(-62959 + 5\sqrt{146892247})Z)\Big),\\
\dot{Z} =& \frac{1}{28800}\Big((154454086 - 12715\sqrt{146892247})X - 40(14185 - \sqrt{146892247}\\
&+ 3(-10825 + \sqrt{146892247})Y + 9(-12713 + \sqrt{146892247})Z)\Big),
\end{aligned}$$

and in the region $\mathcal{R}_E^2$ the linear differential system

$$\begin{aligned}\dot{X} =& \frac{(-984749+1676\sqrt{291071})X-16(-9790+17066Y+17485Z)}{8000},\\ \dot{Y} =& \frac{1}{32000}\Big((2953428-5327\sqrt{291071})X-16(5990-10\sqrt{291071}+(-7996\\ &+14\sqrt{291071})Y+(-8660+15\sqrt{291071})Z)\Big),\\ \dot{Z} =& \frac{1}{128000}\Big((319954973-576992\sqrt{291071})X-16(656710-1080\sqrt{291071}\\ &+(-880634+1512\sqrt{291071})Y+(-952765+1620\sqrt{291071})Z)\Big).\end{aligned}$$

With these linear differential systems, system (6) has only one real solution which generates one limit cycle that intersects the paraboloid P_E in two different points $p_0=(X_0,Y_0,Z_0)$ and $p_1=(X_1,Y_1,Z_1)$, namely

$$p_0=\Big(1,\frac{1}{4},\frac{17}{16}\Big)\quad\text{and}\quad p_1=\Big(-\frac{17}{16},\frac{1}{4},\frac{305}{256}\Big).$$

See this crossing limit cycle in Fig. 3.

Now we provide a discontinuous piecewise differential system whose discontinuity surface is P_H and that has one crossing limit cycle. We consider in the region $\mathcal{R}_H^1$ the linear differential system

$$\begin{aligned}\dot{X} =& \frac{5(-67051+7\sqrt{112701463})X-16(1040+5385Y+1818Z)}{1200},\\ \dot{Y} =& \frac{1}{14400}\Big((239166023-22445\sqrt{112701463})X-16(66845-5\sqrt{112701463}\\ &+15(-8308+\sqrt{112701463})Y+9(-55759+5\sqrt{112701463})Z)\Big),\\ \dot{Z} =& \frac{1}{28800}\Big((120085894-11275\sqrt{112701463})X-40(12745-\sqrt{112701463}\\ &+3(-9385+\sqrt{112701463})Y+9(-11273+\sqrt{112701463})Z)\Big),\end{aligned}$$

and in the region $\mathcal{R}_H^2$ the linear differential system

$$\begin{aligned}\dot{X} =& \frac{(-783629+1676\sqrt{166911})X-16(-9790+17066Y+17485Z)}{8000},\\ \dot{Y} =& \frac{1}{32000}\Big(11(170468-397\sqrt{166911})X+32(3158-7\sqrt{166911})Y\\ &-80(958-2\sqrt{166911}+(-1372+3\sqrt{166911})Z)\Big),\\ \dot{Z} =& \frac{1}{128000}\Big((203299613-473312\sqrt{166911})X-16(527110-1080\sqrt{166911}\\ &+(-699194+1512\sqrt{166911})Y+(-758365+1620\sqrt{166911})Z)\Big).\end{aligned}$$

With these linear differential systems, system (4) has only one real solution which generates one crossing limit cycle that intersects the paraboloid P_H in two different points $p_0=(X_0,Y_0,Z_0)$ and $p_1=(X_1,Y_1,Z_1)$, namely

$$p_0 = \left(1, \frac{1}{4}, \frac{15}{16}\right) \quad \text{and} \quad p_1 = \left(-\frac{17}{16}, \frac{1}{4}, \frac{273}{256}\right).$$

See this crossing limit cycle in Fig. 4.

This completes the proof of Theorem 1.

Conclusions

In Theorem 1 we have solved the extension of the 16th Hilbert problem to the discontinuous piecewise linear differential systems formed by linear centers and separated by a paraboloid (elliptic or hyperbolic) restricted to the crossing limit cycles which intersect the quadric in two points. We recall that this problem was studied intensively in the plane but this is not the case for piecewise differential systems in higher dimensions.

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Data Availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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