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ORIGINAL RESEARCH

A Differential Game Problem of Many Pursuers and One Evader in the Hilbert Space ℓ_2

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Abstract

In this paper, we investigate a differential game problem of multiple number of pursuers and a single evader with motions governed by a certain system of first-order differential equations. The problem is formulated in the Hilbert space ℓ_2, with control functions of players subject to integral constraints. Avoidance of contact is guaranteed if the geometric position of the evader and that of any of the pursuers fails to coincide for all time t. On the other hand, pursuit is said to be completed if the geometric position of at least one of the pursuers coincides with that of the evader. We obtain sufficient conditions that guarantees avoidance of contact and construct evader's strategy. Moreover, we prove completion of pursuit subject to some sufficient conditions. Finally, we demonstrate our results with some illustrative examples.

Keywords Differential game · Integral constraint · Hilbert space · Avoidance of contact · Pursuit

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Introduction

In the mid twentieth century, the need for solving conflict problems brought forth an area of research in mathematics known as Differential game. It was pioneered by Isaacs [27]. Thereafter, fundamental results have been obtained and published in books such as Berkovitz [8], Friedman [14], Krasovskii and Subbotin [29, 30], Lewis [33], Petrosyan [39] and Pontryagin [40].

Evasion differential game problem is a class of differential game problem that involves finding sufficient conditions for avoidance of contacts of one (or more) dynamic object(s) from many other objects. This class of game problem amongst others has received attention from researchers and many results were obtained (see, e.g. [4, 6, 15, 34, 41, 42] and some references therein).

Pontryagin and Mischenko in [41], studied problem of evasion in linear differential game and obtained interesting results for the case of finite dimensional state spaces on an infinite time interval with geometric constraints. Azimov [6] and Mezentsev [34] investigated and extended this result to the case of integral constraints. Thereafter, Gamkrelidze and Kharatashvili [15] developed a new method of solving quasilinear evasion differential game problems.

Among the works dedicated to differential games of several players, equation of motions described by

$$\begin{cases} \dot{x}_j(t) = a(t)u_j(t), \quad x_j(0) = x_j^0,\ j = 1, 2, \dots \\ \dot{y}(t) = a(t)v(t), \quad y(0) = y^0, \end{cases} \tag{1}$$

integral or geometric constraints on players control parameters, where $a(t)$ is a scalar function defined on some intervals, were investigated in [1–3, 9, 12, 13, 16, 20–25, 28, 31, 35, 43, 44].

Alias et. al [3] investigated evasion differential game problem of countably many pursuers and countably many evaders in the Hilbert space ℓ_2 with $a(t) = 1$ and integral constraints imposed on players control functions. They proved that evasion is possible under the assumption that the total resource of evaders exceeds (or equals) that of the pursuers and initial positions of all the evaders are not limit points for initial positions of the pursuers.

In [22] and [23], pursuit-evasion differential game described by (1) was studied in the Hilbert space ℓ_2 with geometric and integral constraints on control functions of the players respectively, where $a(t) = \theta - t$, and θ is the duration of the game. In both papers, sufficient conditions for completion of pursuit as well as value of the game were obtained.

Ibragimov and Satimov [25] considered differential game problem of many pursuers and many evaders described by (1) on a nonempty convex subset of $\mathbb{R}^n$ where all players are confined within the convex set. In the paper, $a(t)$ is a scalar measurable function satisfying some conditions and the control functions of players are subjected to integral constraints. It was proven that pursuit can be completed if the total resources of the pursuers is greater than that of the evaders.

The work in [43] deals with the case when all players are endowed with equal dynamic capabilities with geometric constraints, where $a(t) = 1$. The pursuit problem was solved with the assumption that the evader's initial position must lie in the convex hull of that of the pursuers, otherwise evasion is possible. The results in [43] was later adopted in developing an efficient method of resolving functions for a linear group pursuit problem in [44].

Reducing control problems for parabolic and hyperbolic partial differential equations to infinite system of differential equations

$$\dot{z}_k(t) + \lambda_k z_k = \omega_k, \ k = 1, 2, \dots \tag{2}$$

by using decomposition method based on Fourier expansion was proposed by Chernous' ko [11], where $z_k, \omega_k \in \mathbb{R}^1$, ω_k, $k = 1, 2, \dots$ are control parameters, λ_k, $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty$ are generalized eigenvalues of the elliptic operator

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right).$$

The concept has been applied in studying differential game problems for systems described by parabolic equations but reduced to the system (2) in Ibragimov [18, 19], Ichikawa [26], Osipov [36], Satimov and Tukhtasinov [45].

Ibragimov [18] adopted the proposed decomposition method [11] in solving an optimal pursuit problem described by (2) with integral constraints on controls of players. A detailed analysis and solution of the problem is presented in explicit form.

Ibragimov and Risman [21] considered pursuit and evasion differential game described by (2) in a certain Hilbert space they introduced as ℓ_r^2 with integral constraints on control functions of the players, where the space

$$\ell_r^2 := \left\{ \alpha = (\alpha_1, \alpha_2, \dots) : \sum_{k=1}^{\infty} \lambda_k^r \alpha_k^2 < \infty \right\},$$

with inner product and norm defined by

$$\langle \alpha, \beta \rangle_r = \sum_{k=1}^{\infty} \lambda_k^r \alpha_k \beta_k, \ \ ||\alpha|| = \left(\sum_{k=1}^{\infty} \lambda_k^r \alpha_k^2 \right)^{1/2}$$

respectively, for a given fixed number r and monotonically increasing sequence of positive numbers $\{\lambda_k\}_{k \in \mathbb{N}}$. The evasion problem was studied on some disjoint subset (octants) of ℓ_r^2 and solved with the assumption that the total resources of the pursuers is less than the evader's while the pursuit problem was solved in contrary to this assumption.

There are several other techniques of analyzing pursuit-evasion scenarios involving multiple pursuers and one evader proposed in the literature, for example; situations involving attackers, defenders and one evader are analyzed via the linear quadratic differential game approach in [10, 37], scenarios where the multiple pursuers and evader are restricted within a bounded domain are investigated and analyzed via the geometric approach based on dynamic Voronoi diagram in [7, 17, 46].

Motivated by the results in [21], we study the same problem for a differential game described by (1) in the Hilbert space ℓ_2 (coinciding with ℓ_r^2 and $r = 0$); which amounts to solving the pursuit version of the problem in [3] and evasion version of the problem considered in [25]. More precisely, we investigate a differential game problem of avoidance of contacts (evasion) and completion of pursuit described by (1), but reduced to (2) in the Hilbert space ℓ_2. With all players control parameters subject to integral constraint, we will show that if the total energy resources of the pursuers is less than that of the evader, then avoidance of contact is guaranteed. Furthermore, we obtain sufficient condition for completion of pursuit. It should be noted that in this work, a scalar non-negative function $a(t)$ is introduced into the equation of motion of the players (similar to the problem considered in Ibragimov and Satimov [25] in a finite dimensional space but in this paper, we will adopt a different solution concept with a mild condition on $a(t)$). Also, the case considered in [3, 21] is a special case, where $a(t) = 1$.

This paper is organized as follows. Statement of the problem, notations and some basic definitions are given in Sect. 2. Section 3 constitutes the main results of this paper. It comprises three subsections, that is, Sects. 3.1, 3.2 and 3.3. In the first two subsections, we provide sufficient conditions for avoidance of contacts (Theorem 1) and completion of pursuit (Theorem 2), respectively. In addition to Sect. 3.1, we include an interesting subsection where estimation of distances between the evader and the pursuers is obtained. We illustrate our results with some examples in Sect. 3.3. Section 4 concludes the paper.

Statement of the Problem

Consider the Hilbert space

$$\ell_2 = \left\{ \alpha = (\alpha_1, \alpha_2, \dots) : \sum_{k=1}^{\infty} \alpha_k^2 < \infty \right\},$$

whose inner product and norm are respectively defined by

$$\langle \alpha, \beta \rangle = \sum_{k=1}^{\infty} \alpha_k \beta_k, \quad ||\alpha|| = \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2}, \ \alpha, \ \beta \in \ell_2.$$

Let the motions of countably many pursuers P_j, $j = 1, 2, \dots$ and an evader E be described by the equations

$$\begin{cases} P_j : & \dot{x}_j(t) = a(t)u_j(t), \quad x_j(t_0) = x_j^0, \\ E : & \dot{y}(t) = a(t)v(t), \quad y(t_0) = y^0, \ x_j^0 \neq y^0, \end{cases} \tag{3}$$

where $x_j(t), x_j^0, u_j(t), y(t), y^0, \ v(t) \in \ell_2$; $u_j(t) = (u_{j1}(t), u_{j2}(t), \dots)$ and $v(t) = (v_1(t), v_2(t), \dots)$ are control parameters of the pursuer P_j and evader E respectively. Without any loss of generality we assume that the scalar measurable function $a(t) \geq 0$ is not identically equal to zero on any interval $t_1 < t < t_2$ with $t_1 \geq 0$, and moreover $a(t)$ is such that for each $n = 1, 2,$ the integral

$$\int_{t_1}^{t_2} a^n(t)dt \to \infty \text{ as } t_2 \to \infty.$$

Suppose that $u_j(\cdot), \ v(\cdot) \in L_2(t_0, T; \ell_2)$, where $T > 0$ is the termination instant of the game (not necessarily fixed with respect to pursuit problem).

Definition 1 A function $u_j(\cdot) = (u_{j1}(\cdot), u_{j2}(\cdot), \dots)$ with Borel measurable coordinates $u_{jk}(\cdot), \ k = 1, 2, \dots$, satisfying the inequality

$$\int_{t_0}^{T} \| u_j(t) \|^2 dt \le \rho_j^{\,2}, \ \|u_j(t)\| = \left(\sum_{k=1}^{\infty} u_{jk}^2(t) \right)^{1/2}, \tag{4}$$

where ρ_j, $j = 1, 2, \dots$ are given positive numbers, is called admissible control of the j^{th} pursuer.

Definition 2 A function $v(\cdot) = (v_1(\cdot), v_2(\cdot), \dots)$ with Borel measurable coordinates $v_k(\cdot)$, satisfying the inequality

$$\int_{t_0}^{T} \| v(t) \|^2 dt \le \sigma^2, \ \|v(t)\| = \left(\sum_{k=1}^{\infty} v_k^2(t) \right)^{1/2}, \tag{5}$$

where σ is a given positive number, is called admissible control of the evader.

Remark 1 Note that the players dynamics (3) together with the integral constraints (4) (or (5)) allows the pursuer P_j (evader E, respectively) to move arbitrary far from its initial position. That is, if the control resource constraint constant is ρ_j, then applying a constant control u_j with the magnitude $\|u_j\| \equiv \rho_j/\sqrt{T}$ during a time interval with length T, the pursuer P_j spends its control resource totally and moves for distance $\rho_j\sqrt{T}$. Thus, the distance can be made arbitrary large.

Whenever the players' admissible controls $u_j(\cdot)$ and $v(\cdot)$ are chosen, the corresponding motions of the jth pursuer and evader (solutions of equation (3)) are given by

$$\begin{aligned} x_j(t) &= (x_{j1}(t), x_{j2}(t), \dots), \ x_{jk}(t) = x_{jk}^0(t) + \int_{t_0}^{t} a(s)u_{jk}(s)ds, \\ y(t) &= (y_1(t), y_2(t), \dots), \ y_k(t) = y_k^0(t) + \int_{t_0}^{t} a(s)v_k(s)ds, \end{aligned} \tag{6}$$

respectively.

Definition 3 Suppose that

$$w(\cdot) = (w_1(\cdot), w_2(\cdot), \dots) \in L_2(t_0, T : \ell_2), (z_1^0, z_2^0, \dots) \in \ell_2.$$

A function $z(t) = (z_1(t), z_2(t), \dots)$, $t_0 \le t \le T$, is called the solution of the system of equations

$$\dot{z}_k = a(t)w_k(t), \ z_k(t_0) = z_k^0, \ k = 1, 2, \dots, \tag{7}$$

if:

(1) each coordinate $z_k(t)$ is absolutely continuous function and almost everywhere on $[t_0, T]$ satisfies (7);
(2) $z(\cdot) \in C(t_0, T : \ell_2)$ where $C(t_0, T : \ell_2)$ is the space of continuous functions $z(t) = (z_1(t), z_2(t), \dots)$, $t_0 \le t \le T$ with values in ℓ_2.

Definition 4 A function

$$V(u_1, u_2, \dots), \quad V : \ell_2 \times \ell_2 \times \cdots \to \ell_2,$$

is called strategy of the evader if for any admissible controls of the pursuers $u_j(\cdot)$ the evader's control $v(t) = V(u_1(t), u_2(t), \dots)$, $[t_0, T]$, is admissible and system (3) has a unique solution after substitution of the players' controls into it.

Definition 5 Avoidance of contact is said to be guaranteed in the game described by (3)-(5) with initial positions $\{x_1^0, x_2^0, \dots, x_m^0, \dots, y^0\}$, $x_j^0, y^0 \in \ell_2$, if there exists a strategy V of the evader such that for all admissible controls of the pursuers $u_j(\cdot)$, $j = 1, 2, \dots$, the relation $x_j(t) \neq y(t)$ holds, for all $t \in [t_0, T]$.

Before stating the research problem, let introduce a dummy variable $z_j(t) = y(t) - x_j(t)$ so that the players dynamics (3) reduces to

$$\dot{z}_j(t) = a(t)(v(t) - u_j(t)), \; z_j(t_0) = y^0 - x_j^0.$$

Therefore, the system (3) is represented by the infinite system of differential equations of the form

$$\dot{z}_j(t) = a(t)w_j(t), \; z_j(t_0) = z_j^0, \; j = 1, 2, \dots, \tag{8}$$

where $w_j(t) = (w_{j1}(t), w_{j2}(t), \dots)$, $z_j^0 = (z_{j1}^0, z_{j2}^0, \dots) \neq 0$, $w_{ji} = v_i - u_{ji}$, $z_{ji}^0 = y_i^0 - x_{ji}^0$, $i, j = 1, 2, \dots$. By this notation, the solution to the system (8) becomes

$$z_j(t) = (z_{j1}(t), z_{j2}(t), \dots), \; z_{ji}(t) = z_{ji}^0 + \int_{t_0}^{t} a(s)w_{ji}(s)ds, \tag{9}$$

with $z_j(t)$ satisfying the conditions in Definition (3), $u_j(t)$ and $v(t)$ satisfying (1) and (5), respectively, for all $j = 1, 2, \dots$. We also denote that

$$A_n(t_0, t) = \int_{t_0}^{t} a^n(s)ds; \; t \neq t_0, \; n = 1, 2 \; ; \rho^2 = \Sigma_{j=1}^{\infty} \rho_j^2 < \infty.$$

Remark 2 In view of the properties of the function $a(\cdot)$, it is worth noting that the integral $A_n(t_0, t) > 0$ if $t > t_0$ and $A_n(t_0, t) < 0$ if $t < t_0$ for $n = 1, 2$.

Remark 3 The problem of finding sufficient conditions for completion of pursuit in a pursuit problem described by the system (2) often requires conditions on the parameters λ_k, $k = 1, 2, \dots$. However, by considering the system (8) with a generalized scalar function $a(\cdot)$, these conditions have been circumvented, except the condition $\rho > \sigma$. That is, we examine the case where

$$\sum_{j=1}^{\infty} \rho_j^2 > \sigma^2. \tag{10}$$

The fact that (10) implies the existence of $m > 0$, such that

$$\sum_{j=1}^{m}\rho_j^2 > \sigma^2$$

motivated the following definitions:

Definition 6 A function $u_j(t, z_j(t), \sigma(t), v(t)),\ j = \{1, 2, \dots, m\},\ u_j : [t_0, T] \times \ell_2 \times [0, \sigma^2] \times \ell_2 \to \ell_2$, is called a strategy of the j^{th} pursuer $j \in \{1, 2, \dots, m\}$ if there exists a unique absolutely continuous vector-function $(\sigma(\cdot), z_j(\cdot)),\ z_j(\cdot) \in C(t_0, T : \ell_2),\ t \in [t_o, T]$, satisfying the system

$$\begin{cases} \dot{\sigma}(t) = -\|v(t)\|^2,\ \sigma(t_0) = \sigma^2, \\ \dot{z}_{jk}(t) = \ a(t)w_k(t), \quad z_{jk}(t_0) = z^0_{jk},\ k = 1, 2, \dots \end{cases} \tag{11}$$

at $u_j = u_j(t, z_j(t), \sigma(t), v(t))$ and $v = v(t)$ almost everywhere on $[0, T]$, where $v(t),\ t_0 \le t \le T$, is an arbitrary admissible control of the evader. The strategy of the j^{th} pursuer is called admissible if each control formed by this strategy is admissible.

To guarantee the existence and uniqueness of the solution to the system (11), we assume Lipschitzian (or even linear) dependence of the pursuers' strategies $u_j(t, z_j(t), \sigma(t), v(t))$ on the phase coordinate $z_{jk}(t),\ k = 1, 2, \dots$.

Note that we have only defined strategies of the first m pursuers, for $j > m$, we set $u_{jk}(t) = 0,\ k = 1, 2, \dots$.

Definition 7 Pursuit is said to be completed in the game described by (8) not later than time $t(z^0) \ge t_0$, from the initial positions

$$z^0 = \{z^0_1, z^0_2, \dots\},\ (z^0_j = z^0_{j1}, z^0_{j2}, \dots),\ z^0_j \in \ell_2,\ j = 1, 2, \dots, m,$$

if there exists admissible strategies $u_1 = u_1(t, z_1, \sigma, v),\ u_2 = u_2(t, z_2, \sigma, v), \dots, u_m = u_m(t, z_m, \sigma, v)$, of pursuers $P_j,\ j = 1, 2, \dots, m$, such that for any admissible control $v = v(t)$ of the evader at some $j \in \{1, 2, \dots, m\}$, the solution $z_j(t)$ of the system (8) with $u_j = u_j(t, z_j, \sigma, v),\ v = v(t),\ t_0 \le t \le t(z^0)$ satisfies the equality $z_j(\tau) = 0$, for some $\tau \in [t_0, t(z^0)]$.

Remark 4 In view of *Remark* 1, we further emphasize here that, if the pursuer P_j has control resource equal ρ_j, and that of the evader E equal to σ, and $\rho_j > \sigma$, then under discrimination of the evader (it declares its instantaneous control to the pursuer), the pursuer P_j can complete pursuit of the evader E for any of their initial mutual location by the parallel approach strategy. The parallel approach strategy (also called π-strategy and characterized with the property that straight lines through positions of the pursuer and the evader are parallel) is an effective strategy that has been applied in solving simple motion (i.e. $a(t) = 1$) pursuit-evasion differential game problems of many pursuers one evader (see e.g. [5, 32, 38])

Research Questions: In the game (3)–(5), find sufficient conditions for

i avoidance of contact,
ii completion of pursuit.

Main Results

In this section, we present the main results of the research work.

Avoidance of Contact Problem

The following theorem gives sufficient conditions for avoidance of contact in the game described by (8):

Theorem 1 *If $\sigma > \rho$, then avoidance of contact is guaranteed in the game described by (8), for any initial position $z^0 = \{z_1^0, z_2^0, \dots, z_m^0, \dots\}$, $z_j^0 \in \ell_2$, $j = 1, 2, \dots$.*

Proof *1. Construction of evader's Strategy*

We first define octants of the space ℓ_2 as follows:

$$X(I,J) = \left\{ z = (z_1, z_2, \dots) : z \in \ell_2, z_i > 0,\ i \in I,\ z_k < 0, k \in J \right\}, \tag{12}$$

where $I \subset \mathbb{N}$ and $J = \mathbb{N} \setminus I$. By this definition (12), the intersection of any two distinct octants is null. Since the cardinality of the collection of all subsets of any set is greater than the cardinality of the set itself, then the cardinality of the set of octants of the space ℓ_2, that is $|2^{\mathbb{N}}|$, is greater than $|\mathbb{N}|$. In view of this and the fact that the set of points z_j^0, $j = 1, 2, \dots$, is countable, then there exists an octant that do not contain the points z_j^0, $j = 1, 2, \dots$. Due to this fact and without loss of generality, we assume that the points z_j^0, $j = 1, 2, \dots$, are not contained in the octant defined by:

$$X(\emptyset, \mathbb{N}) = \left\{ z = (z_1, z_2, \dots) : z \in \ell_2, z_i < 0, i \in \mathbb{N} \right\}. \tag{13}$$

This implies z_j^0 is a vector in ℓ_2, with at least one nonnegative coordinate, for each $j \in \mathbb{N}$.

Let the evader use the strategy $v(\cdot) = (v_1(\cdot), v_2(\cdot), \dots)$ with

$$v_i(t) = \begin{cases} \left(\sum_{j \in I_i} u_{ji}^2(t) + \dfrac{a(t)(\sigma^2 - \rho^2)}{2^i A_1(t_0, T)} \right)^{1/2}, & t_0 \le t \le T \\ 0, & t > T, \end{cases} \tag{14}$$

where $I_i = \{j : z_{jk}^0 < 0,\ k = 1, 2, \dots, i-1; z_{ji}^0 \ge 0\}$. If $l \in I_i$ then $z_l^0 = (z_{l1}^0, z_{l2}^0, \dots, z_{li}^0, \dots)$ has the coordinates $z_{lr}^0 < 0$, for all $r = 1, 2, \dots, i-1$.

According to the strategy (14) which is similar to the parallel approach method, for each coordinate, the evader applies a control which guarantees keeping its distance on this coordinate from all pursuers, which can move towards the evader on this coordinate at the current instant, plus some addition. This addition exploits the difference between the control resource of the evader and the total control resource of the pursuers and allows the evader even to increase the distance.

Furthermore, the set $\{I_r : r = 1, 2, \dots\}$ is a collection of pairwise disjoint sets. That is, $I_m \cap I_n = \emptyset$, for $m \neq n$. We prove this claim by contradiction. Suppose $p \in I_m \cap I_n$ and $m \neq n$. Without loss of generality, let $m < n$. We have

$$\begin{aligned} z_{pl}^0 &< 0,\ l = 1, 2, 3, \dots, m-1;\ z_{pm} \geq 0 \\ z_{pl}^0 &< 0,\ l = 1, 2, 3, \dots, n-1;\ z_{pn} \geq 0. \end{aligned} \tag{15}$$

The second line in (15) implies that $z_{pm} < 0$, this is in contradiction with the first line. Hence, the claim follows. That is, $I_m \cap I_n = \emptyset$, if $m \neq n$.

We now show that the evader's strategy (14) is admissible. That is, it satisfies (5) as follows

$$\begin{aligned} \int_{t_0}^{T} \|v(s)\|^2 ds &= \int_{t_0}^{T} \sum_{i=1}^{\infty} v_i^2(s) ds \\ &= \int_{t_0}^{T} \sum_{i=1}^{\infty} \left(\sum_{j \in I_i} u_{ji}^2(s) + \frac{a(s)(\sigma^2 - \rho^2)}{2^i A_1(t_0, T)} \right) ds \\ &\leq \int_{t_0}^{T} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} u_{ji}^2(s) + \frac{a(s)(\sigma^2 - \rho^2)}{2^i A_1(t_0, T)} \right) ds \\ &= \sum_{j=1}^{\infty} \int_{t_0}^{T} \sum_{i=1}^{\infty} u_{ji}^2(s) ds + \frac{(\sigma^2 - \rho^2)}{A_1(t_0, T)} \int_{t_0}^{T} a(s) ds \sum_{i=1}^{\infty} \frac{1}{2^i}. \end{aligned}$$

Thus,

$$\int_{t_0}^{T} \|v(s)\|^2 ds \leq \sum_{j=1}^{\infty} \rho_j^2 + \sigma^2 - \rho^2 = \sigma^2.$$

2. Avoidance of Contact

Lastly, we show that avoidance of contact is possible for any given initial positions of the players $z^0 = \{z_1^0, z_2^0, \dots, z_m^0, \dots\}$, $z_j^0 \in \ell_2$, $j = 1, 2, \dots$. That is, we show that $x_j(t) \neq y(t)$ holds, for all $t \in [t_0, T]$, $j = 1, 2, \dots$. To achieve our goal, it suffice to show $z_j(t) \neq 0$, for all $t \in [t_0, T]$, $j = 1, 2, \dots$. We take an arbitrary point $z_p(t), p \in I_i$, where the index $i \in \mathbb{N}$ is chosen in such a way that z_p^0 has coordinates $z_{pk}^0 < 0$, $k = 1, 2, 3, \dots, i-1$ and $z_{pi} \geq 0$. It is also easy to see that

$$\sum_{j \in I_i} u_{ji}^2(s) \geq u_{pi}^2(s), \tag{16}$$

because $u_{pi}^2(s)$ is among the considered $u_{ji}^2(s)$.

In view of (9), (14) and (16), we have

$$\begin{aligned} z_{pi}(t) &= z_{pi}^0 + \int_{t_0}^{t} a(s)w_{pi}(s)ds \\ &= z_{pi}^0 + \int_{t_0}^{t} a(s)(v_i(s) - u_{pi}(s))ds \\ &\geq \int_{t_0}^{t} a(s)\left(\left(\sum_{j\in I_i} u_{ji}^2(s) + \frac{a(s)(\sigma^2-\rho^2)}{2^i A_1(t_0,T)}\right)^{1/2} - u_{pi}(s)\right)ds \\ &\geq \int_{t_0}^{t} a(s)\left(\left(u_{pi}^2(s) + \frac{a(s)(\sigma^2-\rho^2)}{2^i A_1(t_0,T)}\right)^{1/2} - u_{pi}(s)\right)ds \\ &> \int_{t_0}^{t} a(s)\left(\left(u_{pi}^2(s)\right)^{1/2} - u_{pi}(s)\right)ds = 0. \end{aligned} \tag{17}$$

Thus, $z_{pi}(t) > 0$ for all $t \geq t_0$. Consequently, $x_{pi}(t) < y_i(t),\ t_0 \leq t \leq T$. This implies that the distance $\|z_j(t)\| \neq 0$ for all $j = 1, 2, \ldots,\ t_0 \leq t \leq T$. Estimation of this distance is further discussed in the subsequent subsection.

3. Estimation of distances between the evader and the pursuers

By virtue of the evader's strategy (14), we estimate the distance $\|z_j(t)\|$ of the evader from the pursuers P_j, $j = 1, 2, \ldots$ on the time interval $[t_0, T]$ as follows:

Since $z_j(t) \neq 0$ for all $j = 1, 2, \ldots$, let $x_{ji}(t) < y_i(t)$ for all $i, j = 1, 2, \ldots,\ t_0 \leq t \leq T$ and $\|z_j^0\|$ denotes the initial distance of the evader from the j^{th} pursuer.

Then, by Cauchy–Schwartz inequality

$$\begin{aligned} \|z_j(t)\| &= \left\| z_j^0 + \int_{t_0}^{t} a(s)w_j(s)ds \right\| \\ &\geq \|z_j^0\| - \left\| \int_{t_0}^{t} a(s)v(s)ds \right\| - \left\| \int_{t_0}^{t} a(s)u_j(s)ds \right\| \\ &\geq \|z_j^0\| - \sigma\sqrt{A_2(t_0,T)} - \rho\sqrt{A_2(t_0,T)} \\ &\geq \|z_j^0\| - 2\sigma\sqrt{A_2(t_0,T)}. \end{aligned} \tag{18}$$

On the other hand, since $\|z_j(t)\| \geq |z_{ji}(t)|$ for any fixed $j = 1, 2, \ldots$ and

$$z_{ji}(t) = z_{ji}^0 + \int_{t_0}^{t} a(s)w_{ji}(s)ds = z_{ji}^0 + \int_{t_0}^{t} a(s)(v_i(s) - u_{ji}(s))ds \geq z_{ji}^0,$$

then

$$\|z_j(t)\| \geq |z_{ji}^0|. \tag{19}$$

Set

$$d_j = \begin{cases} |z_{js}^0|, & if\, \|z_j^0\| \leq 2\sigma\sqrt{A_2(t_0,T)}, \\ \min\{|z_{js}^0|, \|z_j^0\| - 2\sigma\sqrt{A_2(t_0,T)}\}, & \text{if } \|z_j^0\| > 2\sigma\sqrt{A_2(t_0,T)}, \end{cases} \tag{20}$$

for some $s = 1, 2, \ldots$. Then in view of (18) and (19), we have $\|z_j(t)\| \geq d_j,\ t_0 \leq t \leq T$. That is, the value d_j is the smallest distance the evader can preserve from the j^{th} pursuer in the time interval $[t_0, T]$. This completes the proof of the theorem. □

Pursuit Problem

Here, we provide a theorem which constitutes sufficient conditions for completion of pursuit as well as its proof.

Theorem 2 *Suppose that* $\sum_{j=1}^{m} \rho_j^2 > \sigma^2$ *and* $A_2(t_0, t) \to +\infty$ *as* $t \to +\infty$, *then pursuit can be completed for the finite time* $t = t(z^0)$ *from any initial positions of players in the game described by* (8).

Proof Let $\sigma_j = \frac{\sigma \rho_j}{\rho}$, where $\rho = (\rho_1^2 + \ldots + \rho_m^2)^{1/2}$ and

$$F_j(\tau, t) = \sum_{k=1}^{\infty} \frac{z_{jk}^2(\tau)}{A_2(\tau, t)},\ t > \tau,\ j = 1, 2, \ldots, m. \tag{21}$$

Roughly put, the value $F_j(\tau, t)$ (which actually depends also on a point z_j) is the square of the resource necessary to guide an object with dynamics (3) from the position (τ, z_j) to the position $(t, 0)$. That is, it denotes the amount of control resources to be spent by the j^{th} pursuer to capture the evader at any given time instant from some of their initial positions if the evader will stay in its place.

For a fixed positive real number τ, we claim that the function (21) is endowed with the following properties:

(i) $F_j(\tau, t),\ t > \tau$, is a decreasing function of t;
(ii) $F_j(\tau, t) \to +\infty$ as $t \to \tau^+$ (that is, t approaches τ from the right);
(iii) $F_j(\tau, t) \to 0$ as $t \to +\infty$.

Property (i) follows directly from the fact that each term of the series $F_j(\tau, t)$ is a decreasing function for all $t > \tau$. To prove the second property, recall that $z_j(\cdot) \in C(t_0, t : \ell_2)$, which implies, the series $\sum_{k=1}^{\infty} z_{jk}^2(\tau) < \infty$. Then, in view of the fact that $A_2^{-1}(\tau, t) \to +\infty$ as $t \to \tau^+$, we must have $F_j(\tau, t) \to +\infty$ as $t \to \tau^+$. Using similar arguments in the proof of property (ii) along with hypothesis of the theorem, we have $F_j(\tau, t) \to 0$ as $t \to +\infty$, i.e., property (iii) holds.

Hence, the equation

$$F_1^{1/2}(t_0, t) = \rho_1 - \sigma_1$$

with $\rho_1 > \sigma_1$, has a unique solution $t = \theta_1$. Moreover, since $F_1(t_0, t) < 0$ for $t < t_0$ (in view of Remark 1), then the root θ_1 exists only in the semiaxis $t > t_0$, where θ_1 is the instant when the first pursuer will capture the evader if the evader will spent control resource σ_1 for avoidance from the first pursuer.

1. Construction of the pursuers Strategies

The pursuers strategies we construct here suggest that the pursuers act one after another in the order of their enumeration. That is, each pursuer in its turn applies the parallel approach control discriminating the evader (it declares its instantaneous control value). And the excess of the pursuer's control resource is spent in such a way that the total resource will be exhausted at some preliminarily computed instant if the evader spends some prescribed portion of its control resource. If the evader spends more than its planned, then the pursuer will spend totally its control resource earlier than the precomputed instant. And if the evader will spend not greater than this planned portion, then the evader will be caught already by this pursuer without involving the ones with greater numbers.

To construct the pursuers strategies, we introduce the following notations.

In the game described by (8), let

τ_j, $j = 1, 2, \dots, m$ denotes the instant when the evader will spend totally the portion of its control resource planned for the j^{th} pursuer. Such a time τ_j is finite (if it exist) since the evader's control resources is limited. On the other hand, if τ_j fails to exist, then the evader never had the chance to spend all its control resources planned on avoiding the j^{th} pursuer (that is, the evader gets caught by the j^{th} pursuer). From the moment the evader gets caught by pursuer P_j, its left over control resource eventually becomes time invariant (thus, making it reasonable to put $\tau_j = \infty$ in this case). Also let t_j, $j = 1, 2, \dots, m$ denotes the minimum time planned by the j^{th} pursuer to complete pursuit. At this same time t_j, the pursuer P_j expects the evader to have spent its total control resource on avoiding contact from it.

We now define the strategy of the pursuers P_j, $j = 1, 2, \dots, m$ as follows.

Set

$$u_{1k}(t, z_1(t), \sigma(t), v(t)) = \begin{cases} \dfrac{a(t)}{A_2(t_0, \theta_1)} z_{1k}(t_0) + v_k(t), & t_0 \le t \le t_1, \\ 0, & t > t_1, \end{cases} \tag{22}$$

$u_{jk}(t, z_1(t), \sigma(t), v(t)) = 0,\ t_0 \le t \le t_1,\ k = 1, 2, \dots; j = 2, \dots, m$, where time

$$t_1 = \begin{cases} \theta_1, & \tau_1 \ge \theta_1, \\ \tau_1, & \tau_1 < \theta_1, \end{cases}$$

and τ_1 is the first time when $\int_{t_0}^{\tau_1} \|v(s)\|^2 ds = \sigma_1^2$ which may or may not exists.

Consequently

$$\sigma(\tau_1) = \sigma^2 - \int_{t_0}^{\tau_1} \|v(s)\|^2 ds = \sigma^2 - \sigma_1^2.$$

If such time τ_1 fails to exists, then we set $\tau_1 = +\infty$.

The case $\tau_1 = +\infty$ yields the inequality $\sigma(t) > \sigma^2 - \sigma_1^2$, for all $t \ge t_0$ which follows from

$$\sigma(t) = \sigma^2 - \int_{t_0}^{t} \|v(s)\|^2 ds > \sigma^2 - \int_{t_0}^{\infty} \|v(s)\|^2 ds = \sigma^2 - \sigma_1^2,\ \ t \ge t_0.$$

Hence, we have

$$\int_{t_0}^{\tau_1} \|v(s)\|^2 ds = \int_{t_0}^{\infty} \|v(s)\|^2 ds \le \sigma_1^2.$$

But if $\tau_1 \geq \theta_1$, that is, $t_1 = \theta_1$, then we claim pursuit can be completed at time t_1. Observe that, for this case,

$$\int_{t_0}^{\theta_1} \|v(s)\|^2 ds = \int_{t_0}^{\tau_1} \|v(s)\|^2 ds \leq \sigma_1^2.$$

2. Completion of pursuit

To show completion of pursuit, we first establish the admissibility the pursuers strategy (22), that is,

$$\begin{aligned}
\left(\int_{t_0}^{t_1} \|u_1(s)\|^2 ds\right)^{\frac{1}{2}} &= \left(\int_{t_0}^{\theta_1} \|u_1(s)\|^2 ds\right)^{\frac{1}{2}} \\
&\leq \left(\int_{t_0}^{\theta_1} \left\|\frac{a(s)}{A_2(t_0,\theta_1)} z_1(t_0)\right\|^2 ds\right)^{\frac{1}{2}} + \left(\int_{t_0}^{\theta_1} \|v(s)\|^2 ds\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty} \frac{z_{1k}^2(t_0)}{A_2^2(t_0,\theta_1)} \int_{t_0}^{\theta_1} a^2(s) ds\right) + \sigma_1 \\
&= \left(\sum_{k=1}^{\infty} \frac{z_{1k}^2(t_0)}{A_2(t_0,\theta_1)}\right) + \sigma_1 = F_1^{\frac{1}{2}}(t_0,\theta_1) + \sigma_1 = \rho_1,
\end{aligned} \tag{23}$$

where $u_1(t) = (u_{11}(t), \dots, u_{1k}(t), \dots)$. Hence, in accordance with Definition (1), we conclude that the strategy (22) is admissible for the case $\tau_1 \geq \theta_1$. And if the pursuers adopt the strategy (22), then for $k = 1, 2, \dots$, we have

$$\begin{aligned}
z_{1k}(t_1) &= z_{1k}(t_0) + \int_{t_0}^{\theta_1} a(t)(v_k(t) - u_{1k}(t)) dt \\
&= z_{1k}(t_0) - \int_{t_0}^{\theta_1} a(t)\left(\frac{a(t)}{A_2(t_0,\theta_1)} z_{1k}(t_0)\right) dt \\
&= z_{1k}(t_0) - \frac{z_{1k}(t_0)}{A_2(t_0,\theta_1)} \int_{t_0}^{\theta_1} a^2(t) dt = 0.
\end{aligned}$$

This implies that pursuit can be completed in the game (3)–(5) at time $t_1 = \theta_1$. Thus, if the pursuers apply the admissible strategies (22), then either the evader spend the resource of control less than or equal to σ_1^2 or pursuit will be completed. □

If we assume that pursuit fails to be completed on the time interval $[t_0, t_1]$, then we must have $\tau_1 < \theta_1$ which implies $[t_0, t_1] = [t_0, \tau_1]$.

Using mathematical induction, we define the numbers $\theta_j,\ \tau_j; j = 1, 2, \dots, m$ as follows:

Let the number $\theta_1, \tau_1, \theta_2, \tau_2, \dots, \theta_j, \tau_j,\ (j = 1, 2, \dots)$ be defined subject to the following conditions

(i) $F_j(\tau_{j-1}, t)$, where $\tau_0 = t_0$, has a unique solution at $t = \theta_j$;
(ii) τ_j is the first instant when $\int_{\tau_{j-1}}^{\tau_j} \|v(s)\|^2 ds = \sigma_j^2$, for all $j \geq 1$. That is, $\sigma(\tau_j) = \sigma^2 - \sum_{i=2}^{j} \sigma_i^2$, such time may exist or not. For the latter case, we let $\tau_j = +\infty$;
(iii) $\tau_k < \theta_k,\ k = 1, 2, \dots, j$, and pursuit fails to be completed on $[t_0, \tau_j]$. In particular, $\tau_k < \infty,\ k = 1, 2, \dots, j$.

We now define $t=\theta_{j+1}$ as a unique solution of the equation

$$F_{j+1}^{\frac{1}{2}}(\tau_j,t)=\rho_{j+1}-\sigma_{j+1},$$

where $F_{j+1}(\tau_j,t)=\sum_{k=1}^{\infty}\frac{z_{j+1,k}^2(\tau_j)}{A_2(\tau_j,\theta_{j+1})}\int_{\tau_j}^{t}a^2(s)ds.$

The strategies of the pursuers for all $t\geq\tau_j$, is defined as follows:

Set

$$u_{j+1,k}(t,z_{j+1}(t),\sigma(t),v(t))=\begin{cases}0, & t<\tau_j,\\ \dfrac{a(t)}{A_2(\tau_j,\theta_{j+1})}z_{j+1,k}(\tau_j)+v_k(t), & \tau_j\leq t\leq t_{j+1},\\ 0, & t>t_{j+1},\end{cases} \tag{24}$$

$u_{ik}(t,z_{j+1}(t),\sigma(t),v(t))=0,\ \tau_j\leq t\leq t_{j+1},\ i=1,2,\dots,j,j+2,\dots,m,$ where time

$$t_{j+1}=\begin{cases}\theta_{j+1}, & \tau_{j+1}\geq\theta_{j+1},\\ \tau_{j+1}, & \tau_{j+1}<\theta_{j+1},\end{cases}$$

and τ_{j+1} is the first time when $\int_{\tau_j}^{\tau_{j+1}}\|v(s)\|^2ds=\sigma_{j+1}^2$ which may or may not exists. Consequently

$$\sigma(\tau_{j+1})=\sigma^2-\int_{\tau_0}^{\tau_{j+1}}\|v(s)\|^2ds=\sigma^2-(\sigma_1^2+\ldots+\sigma_{j+1}^2).$$

If τ_{j+1} fails to exist, we let $\tau_{j+1}=+\infty$, which yields

$$\int_{\tau_j}^{\tau_{j+1}}\|v(s)\|^2ds=\sigma(\tau_j)-\sigma(\tau_{j+1})\leq\sigma_{j+1}^2.$$

For the case $\tau_{j+1}\geq\theta_{j+1}$ (that is, $t_{j+1}=\theta_{j+1}$), observe that

$$\int_{\tau_j}^{\theta_{j+1}}\|v(s)\|^2ds\leq\int_{\tau_j}^{\tau_{j+1}}\|v(s)\|^2ds\leq\sigma_{j+1}^2.$$

Admissibility of the pursuers strategy (24) can verified using similar argument in (23). That is,

$$\begin{aligned}\left(\int_{\tau_j}^{\theta_{j+1}}\|u_{j+1}(s)\|^2ds\right)^{\frac{1}{2}} &\leq\left(\int_{\tau_j}^{\theta_{j+1}}\left\|\frac{a(s)}{A_2(\tau_j,\theta_{j+1})}z_{j+1}(\tau_j)\right\|^2ds\right)^{\frac{1}{2}}\\ &\quad+\left(\int_{\tau_j}^{\theta_{j+1}}\|v(s)\|^2ds\right)^{\frac{1}{2}}\\ &=\left(\sum_{k=1}^{\infty}\frac{z_{j+1,k}^2(\tau_j)}{A_2(\tau_j,\theta_{j+1})}\right)^{\frac{1}{2}}+\sigma_{j+1}\\ &=F_j^{\frac{1}{2}}(\tau_j,\theta_{j+1})+\sigma_{j+1}=\rho_{j+1}.\end{aligned}$$

In addition, the strategy (24) ensures $z_{j+1}(\theta_{j+1}) = 0$ if adopted by the pursuers on the time interval $[\tau_j, \tau_{j+1}]$. That is, pursuit will be completed at time θ_{j+1} and so on. And if pursuit goes on until time τ_{m-1} without completion, then let θ_m be the unique solution of

$$F_m^{1/2}(\tau_{m-1}, t) = \rho_m - \sigma_m,$$

where

$$F_m(\tau_{m-1}, t) = \sum_{k=1}^{\infty} \frac{z_{mk}^2(\tau_{m-1})}{A_2(\tau_{m-1}, \theta_m)} \int_{\tau_{m-1}}^{t} a^2(s)ds.$$

Since we have $\sigma(\tau_{m-1}) = \sigma^2 - (\sigma_1^2 + \ldots + \sigma_{m-1}^2) = \sigma_m^2$, then

$$\int_{\tau_{m-1}}^{\theta_m} \|v(s)\|^2 ds = \sigma(\tau_{m-1}) - \sigma(\tau_m) \leq \sigma(\tau_{m-1}) = \sigma_m^2.$$

Let

$$u_{mk}(t, z_m(t), \sigma(t), v(t)) = \begin{cases} 0, & t < \tau_{m-1}, \\ \dfrac{a(t)}{A_2(\tau_{m-1}, \theta_m)} z_{mk}(\tau_{m-1}) + v_k(t), & \tau_{m-1} \leq t \leq \theta_m, \\ 0, & t > \theta_m, \end{cases} \tag{25}$$

$u_{ik}(t, z_m(t), \sigma(t), v(t)) = 0,\ t > \tau_{m-1},\ k = 1, 2, \ldots;\ i = 1, 2, \ldots, m-1.$

Admissibility of the pursuers strategy (25) can easily be verified using similar argument in the admissibility of (24). Thus,

$$\left(\int_{\tau_{m-1}}^{\theta_m} \|u_m(t)\|^2 dt \right)^{\frac{1}{2}} \leq \rho_m.$$

It can also be verified that the strategy (25) ensures the equality $z_m(\theta_m) = 0$ if adopted by the pursuers on the time interval $[\tau_{m-1}, \tau_m]$. That is, pursuit is completed at time θ_m. Hence, the conclusion of the theorem follows.

Remark 5 At first glance, the pursuers strategies (22) and (24) may seem to be independent on $\sigma(\cdot)$ as stated in Definition 6. But, of course, they do since they are defined by usage of instances θ_j, which depend on $\sigma(\cdot)$. In addition, the question of optimality of the strategies (22) is addressed in the recent work of Ahmed et. al. [1] wherein the authors, using the pursuers strategies (22) estimated the value of the game described by (8). The game value is value of the payoff function at the instant of the termination of the game and when players are using their optimal strategies. To estimate this value, the authors [1] not only constructed the optimal strategies (22) but also presented the tools (that is, players attainability domain and also the strategies of some fictitious pursuers) required to construct (22). In this paper, we present a more detailed application of the strategy in showing completion of pursuit and moreover, we address evasion problem under the same dynamic equations of players.

Illustrative Examples

Here, we present some examples to illustrate our results.

Example 3.3.1 Consider the motions of countably many pursuers $P_j, j \in I = \{1, 2, 3, \dots\}$ and an evader E in the space ℓ_2 governed by the equations

$$\begin{cases} \dot{x}_j(t) = e^{\lambda t}\, u_j(t),\ x_j(0) = x_j^0 \\ \dot{y}(t) = e^{\lambda t}\, v(t), \quad y(0) = y^0,\ \lambda > 0 \end{cases}$$

where all the variables are defined as in Sect. 2 with control functions $u_j(\cdot)$ and $v(\cdot)$ satisfying the inequalities

$$\int_0^\infty \| u_j(t) \|^2 dt \le 3/2^j,$$
$$\int_0^\infty \| v(t) \|^2 dt \le 4.$$

Given the initial position of the pursuers P_j and evader E as $x_j^0 = (2^{-1}, 2^{-2}, 2^{-3}, \dots, 2^{-(i-1)},\ -2^{-i},\ -2^{-(i+1)}, \dots)$ and $y^0 = (0, 0, 0, \dots)$ respectively, where the number -2^{-i} is in the $(j+1)^{th}$ coordinate of point x_j^0. Let $T = 4$, observe that $A_1(0, 4) = (e^{4\lambda} - 1)/\lambda,\ \rho = \sqrt{\sum_{j=1}^\infty \rho_j^2} = \sqrt{3}$ and $z_j^0 = y^0 - x_j^0$ is not contained in the octant $X(\emptyset, \mathbb{N})$ defined in (13) for each $j \in I$. Since $\rho < \sigma = 2$, then by Theorem 1, if the evader adopt the admissible strategy $v(t) = (v_1(t), v_2(t), \dots)$ where

$$v_i(t) = \begin{cases} \left(\sum_{j \in I_i} u_{ji}^2(t) + \dfrac{\lambda(\sigma^2 - \rho^2)e^{\lambda t}}{2^i(e^{4\lambda} - 1)} \right)^{1/2}, & 0 \le t \le 4 \\ 0, & t > 4, \end{cases}$$

avoidance of contact from all the pursuers P_j is guaranteed for all $t > 0$. That is, for any arbitrary point $z_p(t) = (z_{p1}(t), z_{p2}(t), \dots),\ p \in I_i$, we have

$$\begin{aligned} z_{pi}(t) &= z_{pi}^0 + \int_0^t e^{\lambda s} w_{pi}(s))ds \\ &= z_{pi}^0 + \int_0^t e^{\lambda s}(v_i(s) - u_{pi}(s))ds \\ &\ge \int_0^t e^{\lambda s} \left(\left(\sum_{j \in I_i} u_{ji}^2(s) + \frac{\lambda(\sigma^2 - \rho^2)e^{\lambda s}}{2^i(e^{4\lambda} - 1)} \right)^{1/2} - u_{pi}(s) \right) ds \\ &> \int_0^t e^{\lambda s} \left(\left(u_{pi}^2(s) \right)^{1/2} - u_{pi}(s) \right) ds = 0. \end{aligned}$$

Thus, $z_{pi}(t) > 0$ for all $t \ge 0$. Consequently, $x_{pi}(t) \ne y_i(t),\ t \ge 0$.

Example 3.3.2 For the pursuit problem, we consider the case of a particular pursuer P_s for some $s \in I$ and evader E with motions govern by the equations in Example 3.3.1 and initial positions

$$x_s(0) = (0, -\ln 2, 0, -(\ln 2)^2, 0, \dots), y(0) = (\sqrt{\ln 2}, 0, \sqrt{(\ln 2)^3}, 0, \dots)$$

respectively. Let the players control functions be subjected to the integral constraints

$$\int_0^\infty \| u_s(t) \|^2 dt \le (\ln 4)^2,$$
$$\int_0^\infty \| v(t) \|^2 dt \le (\ln 2)^2.$$

Clearly, the hypothesis of Theorem 2 is satisfied since $\rho_s = \ln 4 > \ln 2 = \sigma$ and $A_2(0,t) = (e^{2\lambda t} - 1)/2\lambda \to \infty$ as $t \to \infty$. Given $\lambda = 1/2$ and $z_s^0 = y^0 - x_s^0 = (\sqrt{\ln 2}, \ln 2, \sqrt{(\ln 2)^3}, (\ln 2)^2, \ldots)$. Then, from the conclusion of Theorem 2, the pursuer's admissible strategy

$$u_s(t) = \begin{cases} \dfrac{e^{t/2}}{(e^{1.9} - 1)} z_s^0 + v(t), & 0 \le t \le 1.9, \\ 0, & t > 1.9 \end{cases}$$

guarantees completion of pursuit at time $t_s = 1.9$ as follows

$$\begin{aligned} z_s(1.9) &= z_s^0 + \int_0^{1.9} e^{t/2}(v(t) - u_s(t))dt \\ &= z_s^0 - \frac{z_s^0}{(e^{1.9} - 1)} \int_0^{1.9} e^t dt = 0. \end{aligned}$$

Thus, $z_s(1.9) = 0$. Consequently, $x_s(1.9) = y(1.9)$.

Conclusion

We have studied a differential game problem of avoiding contact and completing pursuit in the Hilbert space ℓ_2 with integral constraints on all players control functions, where the scalar function $a(t)$ introduced is such that it is identically non-zero on the interval (t_1, t_2). By virtue of the players dynamics we considered, the conditions often imposed on the parameters $\lambda_k,\ k = 1, 2, \ldots$ in the system of differential Eq. (2) is redundant. With a mild condition on $a(t)$, we have proved that if the total energy resources of the pursuers is less than that of the evader, then avoidance of contact is guaranteed through out the game. In addition, we estimated the smallest possible distance the evader can preserve from the pursuers through out the game. For the pursuit problem, we constructed strategies of the pursuers and showed that pursuit can be completed for a finite time $t = t(z^0)$ from any initial positions z^0 of players. The problem studied in this paper with the coefficient $a(t)$ replaced by the coefficients $a_j(t),\ j = 1, 2, \ldots$ is an open problem.

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