Differential Equations and Dynamical Systems (October 2023) 31(4):729–742
https://doi.org/10.1007/s12591-020-00524-w

ORIGINAL RESEARCH

Existence of Solutions for m-Point Boundary Value Problems on an Infinite Time Scale

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Published online: 3 March 2020

Abstract

In this study, we present the existence result for the the second order m-point boundary value problems on infinite time scales. Nagumo condition, lower and upper solutions play an important role in the arguments.

Keywords Infinite interval · Lower and upper solutions · Time scales

Mathematical Subject Classification 34B15 · 39A10

Introduction

First and higher order boundary value problems (BVPs) have been studied in many papers, such as [3, 4, 10, 20, 22, 23] for multipoint BVP and [1, 2, 5, 9, 12, 15–19, 21] for infinite interval problem. Also there is a large bibliography on papers related to lower and upper solutions with nonlinear boundary value conditions for first and higher order equations. It is assumed in [11, 14] one pair of well-ordered upper and lower solutions and then applied some fixed point theorems or monotone iterative technique to obtain a solution. In the classical books of Bernfeld and Lakshmikantham [6] and Ladde et al. [13], the classical theory of the method of lower and upper solutions and the monotone iterative technique are given. This gives the solution as the limit of a monotone sequence formed by functions that solve linear problems related to the nonlinear equations considered. It is important to point out that to derive the existence of a solution a growth condition on the nonlinear part of the equation with respect to the dependence on the first derivative is imposed. The most usual condition is the so-called Nagumo condition imposes a quadratic growth in the dependence of the derivative.

In [19], Yan et al. established an upper and lower solution theory for the following second order boundary value problem

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$$\begin{cases} y''(t) + \phi(t)f(t, y(t), y'(t)) = 0, \ t \in (0, \infty), \\ ay(0) - by'(0) = y_0 \geq 0, \quad \lim_{t \to +\infty} y'(t) = k, \end{cases}$$

where $a, b, k > 0$. By using the upper and lower solutions method, the authors presented sufficient conditions for the existence of unbounded positive solutions.

In [15], Lian and Geng considered the second order Sturm–Liouville boundary value problem on the half line

$$\begin{cases} u''(t) + \phi(t)f(t, u(t), u'(t)) = 0, \ t \in (0, \infty), \\ u(0) - au'(0) = B, \quad u'(+\infty) = C, \end{cases}$$

where $a > 0, B, C \in \mathbb{R}$. They concerned with the existence of multiple unbounded solutions.

We consider the following boundary value problem on an infinite time scale;

$$\begin{cases} -[p(t)u^{\triangle}(t)]^{\nabla} + \psi(t)f\left(t, u(t), u^{[\triangle]}(t)\right) = 0, \ t \in (a, \infty), \\ \alpha u(a) - \beta u^{[\triangle]}(a) = \sum_{k=1}^{m-2} a_k u(\xi_k), \\ \gamma u(\infty) + \delta u^{[\triangle]}(\infty) = \sum_{k=1}^{m-2} b_k u(\xi_k), \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta, \xi_k, a_k, b_k$ (for $k = 1, 2, \dots, m-2$) are complex constants such that $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$ and $\xi_k \in \mathbb{T}\backslash\{a\}$, $\psi : (a, \infty) \to (0, \infty)$ is a continuous function, $p : (a, \infty) \to \mathbb{C}$ is $\nabla-$ differentiable on $(a, \infty)_\kappa$, $p(t) \neq 0$ for all $t \in (a, \infty)$, $p^{\nabla} : (a, \infty)_\kappa \to \mathbb{C}$, $f : [a, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\mathbb{T}$ is any time scale such that $\mathbb{T}^\kappa = \infty$.

Throughout this paper, we mean $u(\infty) = \lim_{t \to \infty} u(t)$ and $u^{[\triangle]}(t) = p(t)u^{\triangle}(t)$. Also, by an interval (a, ∞), we mean the intersection of the real interval (a, ∞) with the given time scale $\mathbb{T}$. Some preliminary definitions and theorems on time scales can be found in the books [7, 8].

This paper is organized as follows. In Section "The Preliminary Lemmas", some definitions and lemmas are presented. We establish an upper and lower solution theory for problem (1.1) in Section "Main Result". Sufficient conditions are given for the existence of at least one solution for this problem.

The Preliminary Lemmas

To prove the main results in this paper, we will employ several lemmas. We consider the following boundary value problem

$$\begin{cases} -[p(t)u^{\triangle}(t)]^{\nabla} = y(t), \ t \in (a, \infty), \\ \alpha u(a) - \beta u^{[\triangle]}(a) = \sum_{k=1}^{m-2} a_k u(\xi_k), \\ \gamma u(\infty) + \delta u^{[\triangle]}(\infty) = \sum_{k=1}^{m-2} b_k u(\xi_k). \end{cases} \tag{2.1}$$

Let ϕ_1 and ϕ_2 be the solutions of the linear problems

$$\begin{cases} -[p(t)\phi_1^{\triangle}(t)]^{\nabla}=0,\ t\in(a,\infty),\\ \phi_1(a)=\beta,\quad \phi_1^{[\triangle]}(a)=\alpha \end{cases}$$

and

$$\begin{cases} -[p(t)\phi_2^{\triangle}(t)]^{\nabla}=0,\ t\in(a,\infty),\\ \phi_2(\infty)=\delta,\quad \phi_2^{[\triangle]}(\infty)=-\gamma \end{cases}$$

respectively, so the functions ϕ_1 and ϕ_2 are given by $\phi_1(t)=\beta+\alpha\int_a^t\frac{\Delta s}{p(s)}$ and $\phi_2(t)=\delta+\gamma\int_t^\infty\frac{\Delta s}{p(s)}$.

Let we define $d:=\gamma\beta+\alpha\delta+\alpha\gamma\int_a^\infty\frac{\Delta s}{p(s)}$

and

$$\Omega:=\begin{vmatrix} -\sum\limits_{k=1}^{m-2}\alpha_k\phi_1(\xi_k) & d-\sum\limits_{k=1}^{m-2}\alpha_k\phi_2(\xi_k)\\ d-\sum\limits_{k=1}^{m-2}\beta_k\phi_1(\xi_k) & -\sum\limits_{k=1}^{m-2}\beta_k\phi_2(\xi_k)\end{vmatrix}.$$

The following lemmas are easily proved similar to Lemmas 2.3 and 2.4 in [20].

Lemma 2.1 *Let $\Omega, d\neq 0$. Then for $y\in C[a,\infty)$, boundary value problem (2.1) has a unique solution*

$$u(t)=\int_a^\infty G(t,s)y(s)\nabla s+A(y)\phi_1(t)+B(y)\phi_2(t),$$

where

$$G(t,s)=\frac{1}{d}\begin{cases}\phi_1(s)\phi_2(t),\ a\le s\le t<\infty,\\ \phi_1(t)\phi_2(s),\ a\le t\le s<\infty,\end{cases}$$

$$A(y):=\frac{1}{\Omega}\begin{vmatrix}\sum\limits_{k=1}^{m-2}a_k\int_a^\infty G(\xi_k,s)y(s)\nabla s & d-\sum\limits_{k=1}^{m-2}a_k\phi_2(\xi_k)\\ \sum\limits_{k=1}^{m-2}b_k\int_a^\infty G(\xi_k,s)y(s)\nabla s & -\sum\limits_{k=1}^{m-2}b_k\phi_2(\xi_k)\end{vmatrix}$$

and

$$B(y):=\frac{1}{\Omega}\begin{vmatrix}-\sum\limits_{k=1}^{m-2}a_k\phi_1(\xi_k) & \sum\limits_{k=1}^{m-2}a_k\int_a^\infty G(\xi_k,s)y(s)\nabla s\\ d-\sum\limits_{k=1}^{m-2}b_k\phi_1(\xi_k) & \sum\limits_{k=1}^{m-2}b_k\int_a^\infty G(\xi_k,s)y(s)\nabla s\end{vmatrix}.$$

We assume that the following conditions are satisfied:

(H_1) $\alpha,\gamma\ge 0,\ \beta,\delta>0,\ a_k,b_k\ge 0$ for $k=1,2,\ldots,m-2,$

(H_2) $p : (a, \infty) \to (0, \infty)$ is $\nabla-$ differentiable on $(a, \infty)_k$ and $p^{\nabla} : (a, \infty)_k \to \mathbb{C}$ is continuous function,

(H_3) $\Omega < 0, d - \sum_{k=1}^{m-2} a_k \phi_2(\xi_k) > 0, d - \sum_{k=1}^{m-2} b_k \phi_1(\xi_k) > 0.$

Lemma 2.2 *Assume that* $(H_1) - (H_3)$ *hold. Then,* $0 < G(t, s) \le G(s, s)$ *for* $t, s \in [a, \infty)$.

Lemma 2.3 *If* $\int_a^{\infty} G(s, s)|y(s)|\nabla s < \infty$, *then the following inequalities are satisfied*:

$$|A(y)| \le A \int_a^{\infty} G(s, s)|y(s)|\nabla s, \quad |B(y)| \le B \int_a^{\infty} G(s, s)|y(s)|\nabla s,$$

where

$$A = \frac{1}{\Omega} \begin{vmatrix} \sum\limits_{k=1}^{m-2} a_k & d - \sum\limits_{k=1}^{m-2} a_k \phi_2(\xi_k) \\ \sum\limits_{k=1}^{m-2} b_k & - \sum\limits_{k=1}^{m-2} b_k \phi_2(\xi_k) \end{vmatrix}, \quad B = \frac{1}{\Omega} \begin{vmatrix} - \sum_{k=1}^{m-2} a_k \phi_1(\xi_k) & \sum\limits_{k=1}^{m-2} a_k \\ d - \sum\limits_{k=1}^{m-2} b_k \phi_1(\xi_k) & \sum\limits_{k=1}^{m-2} b_k \end{vmatrix}.$$

Proof It can be easily proven with simple calculation. □

Now, we present some definitions and lemmas which are essential in the proof of our main results.

Definition 2.1 A function $\underline{u} \in C^1[a, \infty) \cap C^2(a, \infty)$ is called a lower solution of (1.1) if

$$\begin{cases} [p(t)\underline{u}^{\triangle}(t)]^{\nabla} + \psi(t) f(t, \underline{u}(t), \underline{u}^{[\triangle]}(t)) \ge 0, \ t \in (a, \infty), \\ \alpha \underline{u}(a) - \beta \underline{u}^{[\triangle]}(a) \le \sum\limits_{k=1}^{m-2} a_k \underline{u}(\xi_k), \\ \gamma \underline{u}(\infty) + \delta \underline{u}^{[\triangle]}(\infty) \le \sum\limits_{k=1}^{m-2} b_k \underline{u}(\xi_k). \end{cases}$$

Definition 2.2 A function $\overline{u} \in C^1[a, \infty) \cap C^2(a, \infty)$ is called an upper solution of (1.1) if

$$\begin{cases} [p(t)\overline{u}^{\triangle}(t)]^{\nabla} + \psi(t) f(t, \overline{u}(t), \overline{u}^{[\triangle]}(t)) \le 0, \ t \in (a, \infty), \\ \alpha \overline{u}(a) - \beta \overline{u}^{[\triangle]}(a) \ge \sum\limits_{k=1}^{m-2} a_k \overline{u}(\xi_k), \\ \gamma \overline{u}(\infty) + \delta \overline{u}^{[\triangle]}(\infty) \ge \sum\limits_{k=1}^{m-2} b_k \overline{u}(\xi_k). \end{cases}$$

Definition 2.3 Given a pair of functions $\underline{u}, \overline{u} \in C^1[a, \infty)$ satisfying $\underline{u}(t) \le \overline{u}(t), t \in [a, \infty)$. A function f is said to satisfy the Nagumo condition with respect to the pair of functions $\underline{u}, \overline{u}$, if there exist a non-negative function $\varphi \in C[a, \infty)$ and a positive, increasing one $h \in C[a, \infty)$ such that

$$|f(t, u, v)| \le \varphi(t) h(|v|),$$

for all $t \in [a, \infty)$, $\underline{u}(t) \le u \le \overline{u}(t)$, $v \in \mathbb{R}$ and

$$\int_a^\infty \varphi(s)\psi(s)\nabla s < \infty, \quad \int_a^\infty \frac{s}{h(|s|)}\nabla s = \infty.$$

Lemma 2.4 *Let $g : \mathbb{T} \to \mathbb{R}$ be a strictly increasing function in $\mathbb{T}$ and $\Delta-$differentiable in $\mathbb{T}^k$, $\tilde{\mathbb{T}} = g(\mathbb{T})$ and $f : \tilde{\mathbb{T}} \to \mathbb{R}$ be rd - continuous. Then for all $a, t \in \mathbb{T}$, the following is satisfied*:

$$\int_a^t f(g(s))g^\Delta(s)\Delta s = \int_{g(a)}^{g(t)} f(r)\tilde{\Delta}r.$$

Main Result

In this section, by using upper and lower solutions, we will prove the existence of the solution of problem (1.1). We shall work in the Banach space

$$E = \left\{ u \in C^1[a, \infty) : \lim_{t\to\infty} \frac{u(t)}{1 + (t - a)} < \infty, \lim_{t\to\infty} u^{[\triangle]}(t) < \infty \right\}$$

with the norm

$$\|u\| = \max\{\|u\|_1, \|u^{[\triangle]}\|_\infty\},$$

where

$$\|u\|_1 = \sup_{t\in[a,\infty)} \frac{|u(t)|}{1 + (t - a)}, \quad \|u^{[\triangle]}\|_\infty = \sup_{t\in[a,\infty)} |u^{[\triangle]}(t)|.$$

We consider the following problem

$$\begin{cases} [p(t)u^\triangle(t)]^\nabla + \psi(t)f^*(t, u(t), u^{[\triangle]}(t)) = 0, \ t \in (a, \infty), \\ \alpha u(a) - \beta u^{[\triangle]}(a) = \sum_{k=1}^{m-2} a_k u(\xi_k), \\ \gamma u(\infty) + \delta u^{[\triangle]}(\infty) = \sum_{k=1}^{m-2} b_k u(\xi_k), \end{cases} \tag{3.1}$$

where

$$f^*(t, u, v) = \begin{cases} f_R(t, \underline{u}(t), v) - \dfrac{u - \underline{u}(t)}{1 + |u - \underline{u}(t)|}, & \underline{u}(t) > u, \\ f_R(t, u, v), & \underline{u}(t) \le u \le \overline{u}(t), \\ f_R(t, \overline{u}(t), v) - \dfrac{u - \overline{u}(t)}{1 + |u - \overline{u}(t)|}, & \overline{u}(t) < u \end{cases}$$

and

$$f_R(t,u,v)=\begin{cases} f(t,u,-R), & v<-R,\\ f(t,u,v), & -R\le v\le R,\\ f(t,u,R), & v>R.\end{cases}$$

We define the operator $F : E \to E$ by

$$Fu(t)=\int_a^\infty G(t,s)\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s + A(\psi f^*)\phi_1(t)+B(\psi f^*)\phi_2(t).$$

It is well known that the existence of the solution to the system (3.1) is equivalent to the existence of fixed point of the operator F. So we shall seek a fixed point of F .

In the rest of the paper, we assume that the following conditions are satisfied:

- (H_4) Problem (1.1) has a pair of upper and lower solutions $\overline{u}$, $\underline{u}$ with $\underline{u}(t)\le\overline{u}(t)$ for $t\in(a,\infty)$,
- (H_5) $f\in C([a,\infty)\times\mathbb{R}^2,\mathbb{R})$ satisfies the Nagumo condition with respect to $\underline{u}$ and $\overline{u}$ also $f(t, u, .)$ is non-decreasing in $\mathbb{R}$ and $f(t,u,v)<0$ for all $t\in[a,\infty)$, $\underline{u}(t)\le u\le\overline{u}(t)$ and $v\in\mathbb{R}$,
- (H_6) $\lim\limits_{t\to\infty}\psi(t)\mu(t)(\varphi(t)+1)<\infty$, where μ is graininess function.

Theorem 3.1 Assume that $(H_1)-(H_6)$ hold. Then problem (1.1) has at least one solution $u\in E$ such that $\underline{u}(t)\le u(t)\le\overline{u}(t)$ for $t\in[a,\infty)$.

Proof First, we show that the operator $F : E\to E$ is well defined. For any fixed $u\in E$, we have

$$\int_a^\infty \frac{G(t,s)}{1+(t-a)}\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s\le\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_0+1)\nabla s<\infty,$$

where $H_0=\max\limits_{0\le s\le\|u^{[\triangle]}\|_\infty} h(s)$. By Lebesgue dominated convergent theorem, we get

$$\begin{aligned}\lim_{t\to\infty}\frac{Fu(t)}{1+(t-a)} &= \lim_{t\to\infty}\left\{\int_a^\infty\frac{G(t,s)}{1+(t-a)}\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s\right.\\ &\quad + A(\psi f^*)\frac{\phi_1(t)}{1+(t-a)}\\ &\quad\left. + B(\psi f^*)\frac{\phi_2(t)}{1+(t-a)}\right\}\\ &=\int_a^\infty\lim_{t\to\infty}\frac{G(t,s)}{1+(t-a)}\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s\\ &\quad + A(\psi f^*)\lim_{t\to\infty}\frac{\phi_1(t)}{1+(t-a)}\\ &\quad + B(\psi f^*)\lim_{t\to\infty}\frac{\phi_2(t)}{1+(t-a)}\\ &<\infty,\end{aligned}$$

and similarly, we find

$$\begin{aligned}\lim_{t\to\infty}(Fu)^{[\triangle]}(t) &= \lim_{t\to\infty}\left\{\frac{\phi_2^{[\triangle]}(t)}{d}\int_a^t \phi_1(s)\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s\right.\\&\quad+\frac{\phi_1^{[\triangle]}(t)}{d}\int_t^\infty \phi_2(s)\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s+A(\psi f^*)\phi_1^{[\triangle]}(t)\\&\quad\left.+B(\psi f^*)\phi_2^{[\triangle]}(t)+\mu(t)\psi(\sigma(t))f^*(\sigma(t),u(\sigma(t)),u^{[\triangle]}(\sigma(t)))\right\}\\&=\lim_{t\to\infty}\frac{\phi_2^{[\triangle]}(t)}{d}\int_a^t \phi_1(s)\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s\\&\quad+A(\psi f^*)\lim_{t\to\infty}\phi_1^{[\triangle]}(t)+B(\psi f^*)\lim_{t\to\infty}\phi_2^{[\triangle]}(t)\\&\quad+\lim_{t\to\infty}\mu(t)\psi(\sigma(t))f^*(\sigma(t),u(\sigma(t)),u^{[\triangle]}(\sigma(t)))\\&=\frac{-\gamma}{d}\int_a^\infty \phi_1(s)\psi(s)f^*\left(s,u(s),u^{[\triangle]}(s)\right)\nabla s+\alpha A(\psi f^*)-\gamma B(\psi f^*)\\&\quad+\lim_{t\to\infty}\mu(t)\psi(\sigma(t))f^*\left(\sigma(t),u(\sigma(t)),u^{[\triangle]}(\sigma(t))\right)\\&<\infty,\end{aligned}$$

so $Fu\in E$.

Now, we shall prove that $F:E\to E$ is continuous. For any convergent sequence $u_n\to u$, there exists $r_1>0$ such that $\|u_n\|,\|u\|\le r_1$. From then, we have

$$\begin{aligned}&\int_a^\infty \frac{G(t,s)}{1+(t-a)}\psi(s)|f^*\left(s,u_n(s),u_n^{[\triangle]}(s)\right)\\&-f^*\left(s,u(s),u^{[\triangle]}(s)\right)|\nabla s\le 2\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_{r_1}+1)\nabla s<\infty,\end{aligned}$$

where $H_{r_1}=\max_{0\le s\le r_1}h(s)$.

And then, for all $t\in[a,\infty)$, we get

$$\begin{aligned}\frac{|Fu_n(t)-Fu(t)|}{1+(t-a)} &= \frac{1}{1+(t-a)}\left|\int_a^\infty G(t,s)\psi(s)\left[f^*\left(s,u_n(s),u_n^{[\triangle]}(s)\right)\right.\right.\\&\quad\left.-f^*\left(s,u(s),u^{[\triangle]}(s)\right)\right]\nabla s\\&\quad+\left[A\left(\psi f^*\left(s,u_n,u_n^{[\triangle]}\right)\right)-A\left(\psi f^*\left(s,u,u^{[\triangle]}\right)\right)\right]\frac{\phi_1(t)}{1+(t-a)}\\&\quad\left.+\left[B\left(\psi f^*\left(s,u_n,u_n^{[\triangle]}\right)\right)-B\left(\psi f^*\left(s,u,u^{[\triangle]}\right)\right)\right]\frac{\phi_2(t)}{1+(t-a)}\right|\\&\le\int_a^\infty \sup_{a\le t<\infty}\frac{G(t,s)}{1+(t-a)}\psi(s)\left|f^*\left(s,u_n(s),u_n^{[\triangle]}(s)\right)\right.\\&\quad-f^*(s,u(s),u^{[\triangle]}(s))|\nabla s\\&\quad+\left|A\left(\psi\left[f^*\left(s,u_n,u_n^{[\triangle]}\right)-f^*\left(s,u,u^{[\triangle]}\right)\right]\right)\right|\sup_{a\le t<\infty}\frac{\phi_1(t)}{1+(t-a)}\\&\quad+\left|B\left(\psi\left[f^*\left(s,u_n,u_n^{[\triangle]}\right)-f^*\left(s,u,u^{[\triangle]}\right)\right]\right)\right|\sup_{a\le t<\infty}\frac{\phi_2(t)}{1+(t-a)}\end{aligned}$$

from which, we find

$$\lim_{n\to\infty} \|Fu_n - Fu\|_1 = 0.$$

And similarly, for all $t \in [a, \infty)$, we get

$$\begin{aligned}
\left|(Fu_n)^{[\triangle]}(t) - (Fu)^{[\triangle]}(t)\right| =& \Bigg| \frac{\phi_2^{[\triangle]}(t)}{d} \int_a^t \phi_1(s)\psi(s)\Big[f^*\Big(s, u_n(s), u_n^{[\triangle]}(s)\Big) \\
&-f^*\Big(s, u(s), u^{[\triangle]}(s)\Big)\Big]\nabla s \\
&+ \frac{\phi_1^{[\triangle]}(t)}{d} \int_t^\infty \phi_2(s)\psi(s)\Big[f^*\Big(s, u_n(s), u_n^{[\triangle]}(s)\Big) \\
&-f^*\Big(s, u(s), u^{[\triangle]}(s)\Big)\Big]\nabla s \\
&+ \Big[A\Big(\psi f^*\Big(s, u_n, u_n^{[\triangle]}\Big)\Big) - A\Big(\psi f^*\Big(s, u, u^{[\triangle]}\Big)\Big)\Big]\phi_1^{[\triangle]}(t) \\
&+ \Big[B\Big(\psi f^*\Big(s, u_n, u_n^{[\triangle]}\Big)\Big) - B\Big(\psi f^*\Big(s, u, u^{[\triangle]}\Big)\Big)\Big]\phi_2^{[\triangle]}(t) \\
&+ \mu(t)\psi(\sigma(t))\Big[f^*\Big(\sigma(t), u_n(\sigma(t)), u_n^{[\triangle]}(\sigma(t))\Big) \\
&-f^*\Big(\sigma(t), u(\sigma(t)), u^{[\triangle]}(\sigma(t))\Big)\Big]\Bigg| \\
\le& \sup_{a\le t<\infty} \frac{\phi_2^{[\triangle]}(t)}{d} \int_a^t \phi_1(s)\psi(s)\Big|f^*\Big(s, u(s), u^{[\triangle]}(s)\Big) \\
&- f^*\Big(s, u(s), u^{[\triangle]}(s)\Big)\Big|\nabla s \\
&+ \sup_{a\le t<\infty} \frac{\phi_1^{[\triangle]}(t)}{d} \int_t^\infty \phi_2(s)\psi(s)\Big|f^*\Big(s, u_n(s), u_n^{[\triangle]}(s)\Big) \\
&- f^*\Big(s, u(s), u^{[\triangle]}(s)\Big)\Big|\nabla s \\
&+ \Big|A\Big(\psi\Big[f^*\Big(s, u_n, u_n^{[\triangle]}\Big) - f^*\Big(s, u, u^{[\triangle]}\Big)\Big]\Big)\Big| \sup_{a\le t<\infty} \phi_1^{[\triangle]}(t) \\
&+\Big|B\Big(\psi\Big[f^*\Big(s, u_n, u_n^{[\triangle]}\Big) - f^*\Big(s, u, u^{[\triangle]}\Big)\Big]\Big)\Big| \sup_{a\le t<\infty} \phi_2^{[\triangle]}(t) \\
&+\mu(t)\psi(\sigma(t))\Big|f^*\Big(\sigma(t), u_n(\sigma(t)), u_n^{[\triangle]}(\sigma(t))\Big) \\
&-f^*\Big(\sigma(t), u(\sigma(t)), u^{[\triangle]}(\sigma(t))\Big)\Big]\Big|,
\end{aligned}$$

this implies

$$\lim_{n\to\infty} \left\|(Fu_n)^{[\triangle]}(t) - (Fu)^{[\triangle]}(t)\right\|_\infty = 0.$$

Thus $F : E \to E$ is continuous.

Let $D \subset E$ be any bounded set, then there exists $r > 0$ such that for any $u \in D$, it holds $\|u\| \le r$. Then for any $u \in D$ and $t \in [a, \infty)$, we find

$$\begin{aligned}\frac{|Fu(t)|}{1+(t-a)} =&\frac{1}{1+(t-a)}\left| \int_a^\infty G(t,s)\psi(s)f^*\Big(s,u(s),u^{[\triangle]}(s)\Big)\nabla s + A(\psi f^*)\phi_1(t) \right.\\ &\left. + B(\psi f^*)\phi_2(t)\right| \\ \le& \int_a^\infty \sup_{a\le t<\infty}\frac{G(t,s)}{1+(t-a)}\psi(s)\left|f^*\Big(s,u(s),u^{[\triangle]}(s)\Big)\right|\nabla s \\ &+ |A(\psi f^*)| \sup_{a\le t<\infty}\frac{\phi_1(t)}{1+(t-a)} + |B(\psi f^*)| \sup_{a\le t<\infty}\frac{\phi_2(t)}{1+(t-a)} \\ \le&(1+A\phi_1(\infty)+B\phi_2(a))\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_r+1)\nabla s < \infty,\end{aligned}$$

where $H_r = \max_{0\le s\le r} h(s)$. And by the similar way, for all $t \in [a, \infty)$, we get

$$\begin{aligned}\left|(Fu)^{[\triangle]}(t)\right| =&\left|\frac{\phi_2^{[\triangle]}(t)}{d}\int_a^t \phi_1(s)\psi(s)f^*\Big(s,u(s),u^{[\triangle]}(s)\Big)\nabla s\right.\\ &+ \frac{\phi_1^{[\triangle]}(t)}{d}\int_t^\infty \phi_2(s)\psi(s)f^*\Big(s,u(s),u^{[\triangle]}(s)\Big)\nabla s \\ &+ A(\psi f^*)\phi_1^{[\triangle]}(t) + B(\psi f^*)\phi_2^{[\triangle]}(t) \\ &\left.+ \mu(t)\psi(\sigma(t))f^*\Big(\sigma(t),u(\sigma(t)),u^{[\triangle]}(\sigma(t))\Big)\right| \\ \le& \sup_{a\le t<\infty}\frac{\phi_2^{[\triangle]}(t)}{d}\int_a^t \phi_1(s)\psi(s)(\varphi(s)H_r+1)\nabla s \\ &+ \mu(t)\psi(\sigma(t))(\varphi(\sigma(t))H_r+1) \\ &+ \sup_{a\le t<\infty}\frac{\phi_1^{[\triangle]}(t)}{d}\int_t^\infty \phi_2(s)\psi(s)(\varphi(s)H_r+1)\nabla s \\ &+ \left[A\sup_{a\le t<\infty}\left|\phi_1^{[\triangle]}(t)\right|\right. \\ &\left.+B\sup_{a\le t<\infty}\left|\phi_2^{[\triangle]}(t)\right|\right]\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_r+1)\nabla s < \infty,\end{aligned}$$

that is, $F(D)$ is uniformly bounded. Moreover, we can easily show that $F(D)$ is equicontinuous.

Finally, we show that $F(D)$ is equiconvergent at infinity. In fact, we have

$$\begin{aligned}
&\left|\frac{Fu(t)}{1+(t-a)}-\lim_{t\to\infty}\frac{Fu(t)}{1+(t-a)}\right|\\
&=\Bigg|\int_a^\infty\left[\frac{G(t,s)}{1+(t-a)}-\lim_{t\to\infty}\frac{G(t,s)}{1+(t-a)}\right]\psi(s)f^*(s,u(s),u^{[\triangle]}(s))\nabla s\\
&\quad+A(\psi f^*)\left[\frac{\phi_1(t)}{1+(t-a)}-\lim_{t\to\infty}\frac{\phi_1(t)}{1+(t-a)}\right]\\
&\quad+B(\psi f^*)\left[\frac{\phi_2(t)}{1+(t-a)}-\lim_{t\to\infty}\frac{\phi_2(t)}{1+(t-a)}\right]\Bigg|\\
&\le\int_a^\infty\left|\frac{G(t,s)}{1+(t-a)}-\lim_{t\to\infty}\frac{G(t,s)}{1+(t-a)}\right|\psi(s)(\varphi(s)H_r+1)\nabla s\\
&\quad+A\left|\frac{\phi_1(t)}{1+(t-a)}-\lim_{t\to\infty}\frac{\phi_1(t)}{1+(t-a)}\right|\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_r+1)\nabla s\\
&\quad+B\left|\frac{\phi_2(t)}{1+(t-a)}-\lim_{t\to\infty}\frac{\phi_2(t)}{1+(t-a)}\right|\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_r+1)\nabla s\\
&\quad\to 0,\ as\, t\to\infty,
\end{aligned}$$

and similarly, we get

$$\begin{aligned}
&\left|(Fu)^{[\triangle]}(t)-(Fu)^{[\triangle]}(\infty)\right|\\
&\le\quad\int_a^\infty\left|G^{[\triangle]}(t,s)-\lim_{t\to\infty}G^{[\triangle]}(t,s)\right|\psi(s)(\varphi(s)H_r+1)\nabla s\\
&\qquad+A\left|\phi_1^{[\triangle]}(t)-\lim_{t\to\infty}\phi_1^{[\triangle]}(t)\right|\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_r+1)\nabla s\\
&\qquad+B\left|\phi_2^{[\triangle]}(t)-\lim_{t\to\infty}\phi_2^{[\triangle]}(t)\right|\int_a^\infty G(s,s)\psi(s)(\varphi(s)H_r+1)\nabla s\\
&\qquad+|\mu(t)\psi(\sigma(t))-\lim_{t\to\infty}\mu(t)\psi(\sigma(t))|(\varphi(t)H_r+1)\\
&\qquad\to 0,\ as\, t\to\infty.
\end{aligned}$$

Then, we obtain that $F:E\to E$ is completely continuous. By using the Schäuder fixed point theorem, we find that F has at least one fixed point $u\in E$.

Now, we shall demonstrate that the function u satisfies $\underline{u}(t)\le u(t)\le\overline{u}(t)$ and $|u^{[\triangle]}(t)|\le R$ for $t\in[a,\infty)$ and positive real R. For this purpose, firstly, we show that $\underline{u}(t)\le u(t), t\in[a,\infty)$. Therefore, let $N>0$ and choose

$$\eta\ge\left(\sup_{t\in[a,\infty)}p(t)\right)\max\left\{\sup_{t\in[N,\infty)}\frac{\overline{u}(t)-\underline{u}(a)}{t-a},\sup_{t\in[N,\infty)}\frac{\overline{u}(a)-\underline{u}(t)}{t-a}\right\},$$
$$\max\left\{\sup_{t\in[a,\infty)}\left|\underline{u}^{[\triangle]}(t)\right|,\sup_{t\in[a,\infty)}\left|\overline{u}^{[\triangle]}(t)\right|\right\}\le\eta,$$

and $R>\eta$ such that

$$\int_\eta^R\frac{s}{h(|s|)}\nabla s>\left(\sup_{t\in[a,\infty)}p(t)\psi(t)\varphi(t)\right)\left(\sup_{t\in[a,\infty)}\overline{u}(t)-\inf_{t\in[a,\infty)}\underline{u}(t)\right).$$

Moreover, define $g(t)=\underline{u}(t)-u(t)$. Suppose that $g(t)>0$ for some $t\in[a,\infty)$. Hence, there exists $t_0\in[a,\infty)$ such that $g(t_0)=\sup\limits_{t\in[a,\infty)} g(t)$. There are three cases to consider.

Case 1. If $t_0\in(a,\infty)$, then, from [8, Lemma 6.17], we know

$$g^{\triangle}(t_0)\le 0 \text{ and } [p(t_0)g^{\triangle}(t_0)]^{\nabla}\le 0,$$

from which, the assumption (H_5) and $R>\sup\limits_{t\in[a,\infty)}|\underline{u}^{[\triangle]}(t)|$, we obtain

$$\begin{aligned}
[p(t_0)g^{\triangle}(t_0)]^{\nabla} =&[p(t_0)\underline{u}^{\triangle}(t_0)]^{\nabla}-[p(t_0)u^{\triangle}(t_0)]^{\nabla}\\
\ge&-\psi(t)f\Big(t_0,\underline{u}(t_0),\underline{u}^{[\triangle]}(t_0)\Big)+\psi(t)f^{*}\Big(t_0,u(t_0),u^{[\triangle]}(t_0)\Big)\\
=&-\psi(t)f\Big(t_0,\underline{u}(t_0),\underline{u}^{[\triangle]}(t_0)\Big)+\psi(t)f_R\Big(t_0,\underline{u}(t_0),u^{[\triangle]}(t_0)\Big)\\
&-\psi(t)\frac{u(t_0)-\underline{u}(t_0)}{1+|u(t_0)-\underline{u}(t_0)|}\\
>&\psi(t)\Big[f_R\Big(t_0,\underline{u}(t_0),u^{[\triangle]}(t_0)\Big)-f\Big(t_0,\underline{u}(t_0),\underline{u}^{[\triangle]}(t_0)\Big)\Big]\\
>&0,
\end{aligned}$$

which is a contradiction.

Case 2. If $t_0=a$, then, by the boundary condition, we have

$$0\le\alpha(\underline{u}(a)-u(a))\le\beta p(a)\Big(\underline{u}^{\triangle}(a)-u^{\triangle}(a)\Big),$$

which is a contradiction.

Case 3. If $g(\infty)=\sup\limits_{t\in[a,\infty)} g(t)$, then, similarly, using the boundary condition, we get

$$0\le\gamma(\underline{u}(\infty)-u(\infty))\le\delta\Big(u^{[\triangle]}(\infty)-\underline{u}^{[\triangle]}(\infty)\Big),$$

which is also a contradiction. Consequently, $\underline{u}(t)\le u(t)$ holds for all $t\in[a,\infty)$. Similarly, we can show that $u(t)\le\overline{u}(t)$. Now, we demonstrate that $|u^{[\triangle]}(t)|\le R$ for $t\in[a,\infty)$. There are three cases to be considered.

Case 1. If $|u^{[\triangle]}(t)|\le\eta$, $\forall t\in[a,\infty)$, we take $R=\eta$ and we complete the proof.

Case 2. If $|u^{[\triangle]}(t)|>\eta$, $\forall t\in[a,\infty)$, without loss of generality, we suppose $u^{[\triangle]}(t)>\eta$ for $t\in[a,\infty)$. While for any $T>N$, we get

$$\begin{aligned}
\sup_{t\in[a,\infty)} p(t)\left(\frac{\overline{u}(T)-\underline{u}(a)}{T-a}\right)\ge&\sup_{t\in[a,\infty)} p(t)\left(\frac{u(T)-u(a)}{T-a}\right)\\
=&\frac{1}{T-a}\int_a^T\sup_{t\in[a,\infty)} p(t)u^{\triangle}(s)\Delta s\\
>&\eta,
\end{aligned}$$

which is a contradiction.

Case 3. There exist $t_0, t_1 \in [a, \infty)$ such that $|u^{[\triangle]}(t_1)| > \eta$ and $|u^{[\triangle]}(t_0)| \leq \eta$.

In this case, firstly, suppose that $u^{[\triangle]}(t_1) > \eta$ and $|u^{[\triangle]}(t_0)| \leq \eta$, then by a convenient change of variable, we have

$$\begin{aligned}
\int_{u^{[\triangle]}(t_0)}^{u^{[\triangle]}(t_1)} \frac{s}{h(|s|)} \nabla s &= \int_{t_0}^{t_1} \frac{u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)} \Big[p(t)u^{\triangle}(t)\Big]^{\nabla} \nabla t \\
&= \int_{t_0}^{t_1} \frac{-u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)} \psi(t) f^*\Big(t, u(t), u^{[\triangle]}(t)\Big) \nabla t \\
&= \int_{t_0}^{c} \frac{-u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)} \psi(t) f^* \nabla t + \int_{c}^{t_1} \frac{-u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)} \psi(t) f^* \nabla t \\
&\leq \int_{c}^{t_1} \frac{-u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)} \psi(t) f^*\Big(t, u(t), u^{[\triangle]}(t\Big)) \nabla t \\
&\leq \int_{c}^{t_1} \frac{u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)} \psi(t) \varphi(t) h(|u^{[\triangle]}(t)|) \nabla t \\
&= \int_{c}^{t_1} p(t) u^{\triangle}(t) \psi(t) \varphi(t) \nabla t \\
&\leq \left(\sup_{t\in[a,\infty)} p(t)\psi(t)\varphi(t) \right) \int_{c}^{t_1} u^{\triangle}(t) \nabla t \\
&= \left(\sup_{t\in[a,\infty)} p(t)\psi(t)\varphi(t) \right) \int_{c}^{t_1} u^{\nabla}(\sigma(t)) \nabla t \\
&= \left(\sup_{t\in[a,\infty)} p(t)\psi(t)\varphi(t) \right) (u(\sigma(t_1) - u(\sigma(c))) \\
&\leq \left(\sup_{t\in[a,\infty)} p(t)\psi(t)\varphi(t) \right) \left(\sup_{t\in[a,\infty)} \overline{u}(t) - \inf_{t\in[a,\infty)} \underline{u}(t) \right) \\
&< \int_{\eta}^{R} \frac{s}{h(|s|)} \nabla s,
\end{aligned}$$

this implies, $u^{[\triangle]}(t_1) < R$, where $c = \inf\{t \in [t_0, t_1] : u^{[\triangle]}(t) > 0\}$.

Secondly, we suppose that $-u^{[\triangle]}(t_1) > \eta$ and $|u^{[\triangle]}(t_0)| \leq \eta$. Similarly, we get

$$
\begin{aligned}
\int_{-u^{[\triangle]}(t_0)}^{-u^{[\triangle]}(t_1)} \frac{s}{h(|s|)}\nabla s &= \int_{t_0}^{t_1} \frac{u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)}\Big[p(t)u^{\triangle}(t)\Big]^{\nabla}\nabla t \\
&= \int_{t_1}^{t_0} \frac{u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)}\psi(t)f^*\Big(t, u(t), u^{[\triangle]}(t)\Big)\nabla t \\
&= \int_{t_1}^{k} \frac{u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)}\psi(t)f^*\nabla t + \int_{k}^{t_0} \frac{u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)}\psi(t)f^*\nabla t \\
&\le \int_{t_1}^{k} \frac{u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)}\psi(t)f^*\Big(t, u(t), u^{[\triangle]}(t)\Big)\nabla t \\
&\le \int_{t_1}^{k} \frac{-u^{[\triangle]}(t)}{h(|u^{[\triangle]}(t)|)}\psi(t)\varphi(t)h(|u^{[\triangle]}(t)|)\nabla t \\
&= \int_{t_1}^{k} -p(t)u^{\triangle}(t)\psi(t)\varphi(t)\nabla t \\
&\le \left(\sup_{t\in[a,\infty)} p(t)\psi(t)\varphi(t)\right)\int_{t_1}^{k} -u^{\triangle}(t)\nabla t \\
&= \left(\sup_{t\in[a,\infty)} p(t)\psi(t)\varphi(t)\right)\int_{t_1}^{k} -u^{\nabla}(\sigma(t))\nabla t \\
&\le \left(\sup_{t\in[a,\infty)} p(t)\psi(t)\varphi(t)\right)\left(\sup_{t\in[a,\infty)} \overline{u}(t) - \inf_{t\in[a,\infty)} \underline{u}(t)\right) \\
&< \int_{\eta}^{R} \frac{s}{h(|s|)}\nabla s,
\end{aligned}
$$

hence, $-u^{[\triangle]}(t_1) < R$, where $k = \sup\{t \in [t_0, t_1] \,:\, u^{[\triangle]}(t) < 0\}$. Since t_0 and t_1 can be taken arbitrary, we conclude that if there exists a $t \in [a, \infty)$ with $|u^{[\triangle]}(t)| \ge \eta$, then $|u^{[\triangle]}(t)| \le R$. Therefore,

$$[p(t)u^{\triangle}(t)]^{\nabla} = -\psi(t)f^*(t, u(t), u^{[\triangle]}(t)) = -\psi(t)f(t, u(t), u^{[\triangle]}(t)),$$

that is, u is a solution of problem (1.1). This completes the proof. □

References

1. Agarwal, R.P., O'Regan, D.: Infinite interval problems modeling phenomena ahich arise in the theory of plasma and electrical potential theory. Stud. Appl. Math. **111**, 339–358 (2003)
2. Agarwal, R.P., O'Regan, D.: Nonlinear boundary value problems on the semi-infinite interval: an upper and lower solution approach. Mathematika **49**, 129–140 (2002)
3. Anderson, D.R., Zhai, C.: Positive solutions to semi-positone second-order three-point problems on time scales. Appl. Math. Comput. **215**, 3713–3720 (2010)
4. Asif, N.A., Khan, R.A.: Positive solutions to singular system with four-point coupled boundary conditions. J. Math. Anal. Appl. **386**, 848–861 (2012)
5. Bai, C., Li, C.: Unbounded upper and lower solution method for third-order boundary-value problems on the half-line. Electron. J. Differ. Equ. **119**, 1–12 (2009)
6. Bernfeld, S.R., Lakshmikantham, V.: An Introduction to Nonlinear Boundary Value Problems, Mathematics in Science and Engineering, vol. 10. Academic Press, New York (1974)

7. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales, An Introduction with Applications. Birkhauser, Boston (2001)
8. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston (2003)
9. Countryman, M., Kannan, R.: Nonlinear boundary value problems on semi-infinite intervals. Comput. Math. Appl. **28**, 59–75 (2009)
10. Du, Z., Liu, W., Lin, X.: Multiple solutions to a three-point boundary value problem for higher-order ordinary differential equations. J. Math. Anal. Appl. **335**, 1207–1218 (2007)
11. Ehme, J., Eloe, P.W., Henderson, J.: Upper and lower solution methods for fully nonlinear boundary value problems. J. Differ. Equ. **180**, 51–64 (2002)
12. Eloe, P.W., Kaufmann, E.R., Tisdell, C.C.: Multiple solutions of a boundary value problem on an unbounded domain. Dyn. Syst. Appl. **15**(1), 53–63 (2006)
13. Ladde, G.S., Lakshmikantham, V., Vatsala, A.S.: Monotone Iterative Techniques for Nonlinear Differential Equations. Pitman, Boston (1985)
14. Lian, H., Wang, P., Ge, V.: Unbounded upper and lower solutions method for Sturm-Liouville boundary-value problem on infinite intervals. Nonlinear Anal. **70**, 2627–2633 (2009)
15. Lian, H., Geng, F.: Multiple unbounded solutions for a boundary value problem on infinite intervals. Bound. Value Probl. **51**, 8 (2011)
16. Lian, H., Zhao, J.: Existence of unbounded solutions for a third-order boundary value problem on infinite intervals. Discrete Dyn. Nat. Soc. **20**, 357697 (2012)
17. Philos, C.G.: A boundary value problem on the half-line for higher-order nonlinear differential equations. Proc. R. Soc. Edinb. **139**, 1017–1035 (2009)
18. Umamahesearam, S., Venkata Rama, M.: Multipoint focal boundary value problems on infinite interval. J. Appl. Math. Stoch. Anal. **5**, 283–290 (1992)
19. Yan, B., O'Regan, D., Agarwal, R.P.: Unbounded solutions for singular boundary value problems on the semi-infinite interval: upper and lower solutions and multiplicity. J. Comput. Appl. Math. **197**, 365–386 (2006)
20. Yaslan Karaca, I.: Multiple positive solutions for dynamic m-point boundary value problems. Dyn. Syst. Appl. **17**, 25–42 (2008)
21. Zhao, Y., Chen, H., Xu, C.: Existence of multiple solutions for three-point boundary-value problems on infinite intervals in Banach spaces. Electron. J. Differ. Equ. **44**, 1–11 (2012)
22. Zhou, Y., Xu, Y.: Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations. J. Math. Anal. Appl. **320**, 78–590 (2006)
23. Zhao, J., Lian, H., Ge, W.: Existence of positive solutions for nonlinear m -point boundary value problems on time scales. Bound. Value Probl. **4**, 15 (2012)