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ORIGINAL RESEARCH

Multi-term Time-Fractional Stochastic Differential Equations with Non-Lipschitz Coefficients

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Abstract
In this paper, we study the existence and uniqueness of mild solutions for a class of multi-term time-fractional stochastic differential equations in Hilbert spaces. We tend to implement fractional calculus, generalized semigroup theory and stochastic analysis techniques to obtain the main results. We come up with a new set of sufficient conditions with the coefficients in the equations satisfying some non-Lipschitz conditions and using standard Picard type iterations. Finally, an application is given to illustrate that our obtained results are valuable.

Keywords Fractional calculus · Generalized semigroup theory · Multi-term time-fractional stochastic differential equations · Mild solution · Non-Lipschitz coefficient

Mathematics Subject Classification 34A08 · 34G20 · 26A33 · 34A12 · 65C30 · 47H10

Introduction

In the last few decades, fractional differential equations have been attracted the interest of many researchers towards itself, due to demonstrate applications in widespread areas of science and engineering such as in models of medicine (modeling of human tissue under mechanical loads), electrical engineering(transmission of ultrasound waves), biochemistry (modeling of proteins and polymers) etc. It has been verified that fractional differential equations are the beneficial tools to describe dynamical behavior of the real-life phenomena more precisely. Nowadays, the multi-term time-fractional differential equations generating great interest among the mathematicians and engineers. For instance, in the papers [10] and [22] multi-term time-fractional differential equations are considered with constant and variable coefficients, respectively, which include a concrete case of fractional diffusion-wave problem. Moreover, for multi-term time-fractional diffusion equations in [16,19] the authors studied analytic solutions and numerical solutions. Recently, Pardo at al. in [29]

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studied the existence of mild solutions to the multi-term time-fractional differential equations with Caratheodory type conditions using the concept of measure of noncompactness. For fundamental concepts regarding to fractional differential equations, one can make reference to the papers [4,5,8,13,17,21,27–29,32,37], the monographs [18,25,30] and references therein.

On the other hand, noises or stochastic perturbations are unavoidable and omnipresent in nature as well as in man-made systems, so we have to move from deterministic models to stochastic models. Stochastic differential equations play an important role in formulation and analysis in mechanical, electrical, control engineering, and physical sciences. Motivated by these facts many researchers are showing great interest in investigating an appropriate system to analysis the qualitative properties such as existence, uniqueness, controllability and stability of these physical processes with the help of fractional calculus, stochastic analysis, fixed point theorems and time delay techniques. Zhang et al. [39] investigated existence and asymptotic stability for a class of fractional stochastic differential equations by virtue of some fixed point theorems. Rajivganthi et al. [31] established the existence results for mild solutions and optimal controls by applying successive approximation approach for a class of fractional neutral stochastic differential equations. Recently, Benchaabane and Sakthivel [3] obtained the existence results for Sobolev-type fractional stochastic differential equations via standard Picard type iterations. For more details, we refer to the books [11,14,24,26] and novel papers [6–9,35,38] and references therein.

Motivated by the above facts, in this paper, we investigate the existence and uniqueness results for a abstract multi-term time-fractional stochastic system using Picard type iterations. Moreover, the aim of studying such system is motivated by the fact that a integer order differential equation having n derivatives terms may be transformed into a abstract form of first order differential system, but the fractional differential equation may not have this property. So, the technique used in this paper provides the tools to study a fractional differential equation having more than one fractional derivatives.

This paper is organized as follows. In "Preliminaries" section, we will formulate the problem and recall some basics of fractional calculus and stochastic analysis which will be employ to attain our mains results. In "Main Results" section, the existence and uniqueness results of mild solution are obtained. In the next section an example is provided to show the feasibility of the theory discussed in this paper.

Preliminaries

In this section, we provide some notations, basic definitions and lemmas, which will be used throughout the paper. In particular, we recall main properties of stochastic analysis theory [24,26,36], generalized semigroup theory and well known facts in fractional calculus [30].

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. Let $\mathbb{H}$ and $\mathbb{K}$ be two real separable Hilbert spaces and let $\mathcal{L}(\mathbb{H}, \mathbb{K})$ be the space of bounded linear operators form $\mathbb{H}$ to $\mathbb{K}$. For convenience, without confusion we will employ the same notation $\|.\|$ to denote the norms in $\mathbb{H}$, $\mathbb{K}$ and $\mathcal{L}(\mathbb{K}, \mathbb{H})$ and $\langle \cdot, \cdot \rangle$ for inner product in $\mathbb{H}$ and $\mathbb{K}$. For a linear operator A on $\mathbb{H}$, $\mathcal{R}(A)$, $\mathcal{D}(A)$ and $\rho(A)$ represent the range, domain and resolvent of A, respectively. Let $w(t)$ be a Q-Wiener process on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions(i.e right continuous and $\{\mathcal{F}_0\}$ containing all $\mathbf{P}$-null sets) with the linear bounded covariance operator $Q \in \mathcal{L}(\mathbb{K}, \mathbb{K}) = \mathcal{L}(\mathbb{K})$ such that $tr Q < \infty$, where tr denotes the trace of the operator. Further, we assume that there exist a complete orthonormal system $\{e_n\}_{n\geq 1}$ in $\mathbb{K}$, a sequence of non-negative real numbers $\{\lambda_n\}_{n\geq 1}$

such that $Qe_n = \lambda_n e_n$, $n = 1, 2, 3, \ldots$ and a sequence $\{\zeta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle_{\mathbb{K}}\zeta_n(t), \quad e \in \mathbb{K},\ t \in \mathcal{I} = [0, T],\ T < \infty, \tag{2.1}$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$ and $\mathcal{F}_T = \mathcal{F}$. Further, assume that $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ represents the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with norm $\|\phi\|_{\mathcal{L}_2^0} = tr[\phi Q\phi^*] < \infty$, $\phi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$. Let $\mathcal{L}_2(\mathcal{F}_T, \mathbb{H})$ be the space of all $\mathcal{F}_T$ measurable $\mathbb{H}$ valued square integrable random variables. Moreover, let $\mathcal{L}_2^{\mathcal{F}}(\mathcal{I}, \mathbb{H})$ be the Hilbert space of all square integrable and $\mathcal{F}_t$ adapted processes with value in $\mathbb{H}$. We denote by $\mathbb{B}_T$ the Banach space of all $\mathbb{H}$-valued $\mathcal{F}_t$ adapted processes $y(t, \omega) : \mathcal{I} \times \Omega \to \mathbb{H}$ which are continuous in t for a.e. fixed $\omega \in \Omega$ and satisfy

$$\|y\|_{\mathbb{B}_T} = \mathbf{E}\Big(\sup_{t\in[0,T]} \|y(t, \omega)\|^p\Big)^{\frac{1}{p}} < \infty, \quad p \geq 2.$$

In this paper, we study the existence and uniqueness of mild solutions to the following multi-term time-fractional stochastic differential system

$$\begin{cases} {}^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j}y(t) = Ay(t) + F(t, y(t)) + G(t, y(t))\dfrac{dw(t)}{dt}, & t \in (0, T],\\ y(0) = \varphi, \quad y'(0) = \chi, \end{cases} \tag{2.2}$$

where ${}^cD^\eta$ stands for the Caputo fractional derivative of order $\eta > 0$, $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear operator on $\mathbb{H}$. All γ_j, $j = 1, 2, \ldots, n$ are positive real numbers such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. The functions F and G are suitable functions to be defined later. The initial given data φ, χ are in $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable independent of w with finite p moments.

To give a appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define following family of operators.

Definition 2.1 [29] Let A be a closed linear operator on a Hilbert space $\mathbb{H}$ with the domain $\mathcal{D}(A)$ and let $\beta > 0$, γ_j, α_j be the real positive numbers. Then A is called the generator of a $(\beta, \gamma_j)-$ resolvent family if there exists $\omega > 0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j} : \mathbb{R}^+ \to \mathcal{L}(\mathbb{H})$ such that $\{\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j\lambda^{\gamma_j} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^\beta\Big(\lambda^{\beta+1} + \sum_{j=1}^{n} \alpha_j\lambda^{\gamma_j} - A\Big)^{-1} y = \int_0^\infty e^{-\lambda t}\mathcal{S}_{\beta,\gamma_j}(t)y\,dt, \quad \operatorname{Re}\lambda > \omega, y \in \mathbb{H}. \tag{2.3}$$

The following result guarantee for the existence of $(\beta, \gamma_j)-$ resolvent family under some suitable conditions.

Theorem 2.2 [29] *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$ be given and let A be a generator of a bounded and strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then A generates a bounded $(\beta, \gamma_j)-$ resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$.*

Now, we recall some definitions and basic results on fractional calculus (for more details see [29,30]). Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)}t^{\eta-1}, & t > 0;\\ 0, & t \leq 0, \end{cases}$$

where Γ denotes gamma function. The function g_η has the properties $(g_a * g_b)(t) = g_{a+b}(t)$, for $a, b > 0$ and $\widehat{g_\eta}(\lambda) = \frac{1}{\lambda^\eta}$ for $\eta > 0$ and Re $\lambda > 0$, where $\widehat{(\cdot)}$ and $*$ denote the Laplace transformation and convolution, respectively.

Definition 2.3 The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}([0, \infty), \mathbb{R})$ of order $\eta > 0$ with lower limit zero is defined as follows

$$I^\eta f(t) = \int_0^t g_\eta(t-s) f(s)ds, \quad t > 0,$$

and $I^0 f(t) = f(t)$.

This fractional integral satisfies the properties $I^\eta \circ I^b = I^{\eta+b}$ for $b > 0$, $I^\eta f(t) = (g_\eta * f)(t)$ and $\widehat{I^\eta f}(t) = \frac{1}{\lambda^\eta}\widehat{f}(\lambda)$ for Re $\lambda > 0$.

Definition 2.4 Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in \mathcal{C}^m([0, \infty), \mathbb{R})$ with lower limit zero is given by

$${}^cD^\eta f(t) = I^{m-\eta} D^m f(t) = \int_0^t g_{m-\eta}(t-s) D^m f(s)ds, \quad m-1 < \eta \le m,$$

and ${}^cD^0 f(t) = f(t)$, where $D^m = \frac{d^m}{dt^m}$ and $\lceil \cdot \rceil$ is ceiling function. In addition, we have ${}^cD^\eta f(t) = (g_{m-\eta} * D^m f)(t)$ and the Laplace transformation of Caputo fractional derivative is given by

$$\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda) - \sum_{d=0}^{m-1} f^{(d)}(0)\lambda^{\eta-1-d}, \quad \lambda > 0. \tag{2.4}$$

Remark 2.5 Let $m - 1 < \eta \le m$, then

$$(I^\eta \circ {}^cD^\eta) f(t) = f(t) - \sum_{d=0}^{m-1} f^{(d)}(0) g_{d+1}(t), \quad t > 0. \tag{2.5}$$

If $f^{(d)}(0) = 0$, for $d = 1, 2, 3, \ldots, m-1$, then $(I^\eta \circ {}^cD^\eta) f(t) = f(t)$ and $\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda)$.

In order to define the concept of mild solution for the system (2.2), by comparison with the fractional differential equation given in [29], we associate system (2.2) to an integral equation. In this paper, we give the following definition of mild solution for the system (2.2).

Definition 2.6 An $\mathbb{H}$-valued stochastic process $\{y(t)\}_{t\in\mathcal{I}}$ is said to be mild solution of (2.2) if

(i) $y(t)$ is measurable and $\mathcal{F}_t$ adapted, for each $t \in \mathcal{I}$,
(ii) $y(t)$ satisfies the following equation

$$\begin{aligned} y(t) =& \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F(s, y(s))ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) G(s, y(s))dw(s), \end{aligned} \tag{2.6}$$

P-a.s. for all $t \in \mathcal{I}$, where $\mathcal{T}_{\beta,\gamma_j}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{S}_{\beta,\gamma_j}(s)ds$.

Lemma 2.7 [20], *For any $p \geq 2$ and let h be $\mathcal{L}_0^2$-valued predictable process such that* $\boldsymbol{E}\bigg(\int_0^T \|h(s)\|^p_{\mathcal{L}_0^2} ds\bigg) < +\infty$, *then we have*

$$\mathbf{E}\bigg(\sup_{s\in[0,t]}\bigg\|\int_0^s h(r)dw(r)\bigg\|^p\bigg) \leq c_p \sup_{s\in[0,t]} \mathbf{E}\bigg(\bigg\|\int_0^s h(r)dw(r)\bigg\|^p\bigg)$$
$$\leq C_p \mathbf{E}\bigg(\int_0^t \|h(r)\|^p_{\mathcal{L}_0^2} dr\bigg), \quad t \in \mathcal{I},$$

where $c_p = \Big(\frac{p}{p-1}\Big)^p$ *and* $C_p = \Big(\frac{p}{2}(p-1)\Big)^{\frac{p}{2}}\Big(\frac{p}{p-1}\Big)^{\frac{p^2}{2}}$.

Main Results

In this section, we establish the existence and uniqueness results of mild solutions for the system (2.2). Throughout in this section we denote $S_0 = \sup_{t\in[0,T]}\|\mathcal{S}_{\beta,\gamma_j}(t)\|$. Moreover, we have $\|\mathcal{T}_{\beta,\gamma_j}(t)y\|_{\mathcal{L}} = \frac{S_0 t^\beta}{\Gamma(1+\beta)}\|y\|$ for $y \in \mathbb{H}$. We consider the following assumptions

(A_1) The functions $F : \mathcal{I} \times \mathbb{H} \to \mathbb{H}$, $G : \mathcal{I} \times \mathbb{H} \to \mathcal{L}_2^0$ are measurable and continuous in y for each $t \in \mathcal{I}$ and there exists a function $U : \mathcal{I} \times [0,\infty) \to [0,\infty)$ such that

$$\mathbf{E}(\|F(t,y)\|^p) + \mathbf{E}(\|G(t,y)\|^p_{\mathcal{L}_2^0}) \leq U(t, \mathbf{E}(\|y\|)^p) \tag{3.1}$$

for all $y \in L^p(\Omega, \mathcal{F}_T, \mathbb{H})$ and all $t \in \mathcal{I}$.

(A_2) For each fixed $x \in [0,\infty)$, $U(t,x)$ is locally integrable in t and non-decreasing continuous in x for each fixed $t \in \mathcal{I}$ and for all $\theta > 0$, $x_0 \geq 0$, the integral equation $x(t) = x_0 + \theta\int_0^t U(s,x(s))ds$ admits a global solution on $\mathcal{I}$.

(A_3) There exist a function $V : \mathcal{I} \times [0,\infty) \to [0,\infty)$ such that

$$\mathbf{E}(\|F(t,x) - F(t,y)\|^p) + \mathbf{E}(\|G(t,x) - G(t,y)\|^p_{\mathcal{L}_2^0}) \leq V(t, \mathbf{E}(\|x-y\|)^p) \tag{3.2}$$

for all $x, y \in L^p(\Omega, \mathcal{F}_T, \mathbb{H})$ and all $t \in \mathcal{I}$.

(A_4) For each fixed $x \in [0,\infty)$, $V(t,x)$ is locally integrable in t and non-decreasing continuous in x for each fixed $t \in \mathcal{I}$. Moreover, $V(t,0) = 0$ and if a non-negative continuous function $z(t)$, $t \in \mathcal{I}$ satisfies $z(t) \leq \sigma\int_0^t V(s,z(s))ds$ for $t \in \mathcal{I}$ subject to $z(0) = 0$ for some $\sigma > 0$, then $z(t) = 0$ for all $t \in \mathcal{I}$.

Remark 3.1 (i) For all $x \geq 0$, define $V(t,x) = Vx$, where $V > 0$ is a constant, then (A_3) implies global Lipschitz condition.

(ii) If $V(t,x)$ is concave with respect to $x > 0$ for each fixed $t \geq 0$ and

$$\|F(t,x) - F(t,y)\|^p + \|G(t,x) - G(t,y)\|^p_{\mathcal{L}_2^0}$$
$$\leq V(t, \|x-y\|^p), \quad \text{for all } x, y \in \mathbb{H}, \text{ and } t \geq 0.$$

Then by Jensen's inequality (3.2) is satisfied.

(iii) Let $V(t,x) = \xi(t)\vartheta(x)$, $t \in \mathcal{I}$, $x \geq 0$, where $\vartheta : [0,\infty) \to [0,\infty)$ is monotone non-decreasing, continuous and concave function with $\vartheta(0) = 0$, $\vartheta(x) > 0$ for all $x > 0$ and $\int_{0+} 1/\vartheta(x)dx = \infty$ and $\xi(t) \geq 0$ is locally integrable. It can be observed that ϑ satisfies (3.2) [33].

Let us give some concrete functions. For $\epsilon \in (0, 1)$ sufficiently small, we define [33]

$$\vartheta_1(x) = \begin{cases} x\log(x^{-1}), & 0 \le x \le \epsilon; \\ \epsilon\log(\epsilon^{-1}) + \vartheta_1'(\epsilon^-)(x-\epsilon), & x > \epsilon. \end{cases} \tag{3.3}$$

$$\vartheta_2(x) = \begin{cases} x\log(x^{-1})\log\log(x^{-1}), & 0 \le x \le \epsilon; \\ \epsilon\log(\epsilon^{-1})\log\log(\epsilon^{-1}) + \vartheta_2'(\epsilon^-)(x-\epsilon), & x > \epsilon. \end{cases} \tag{3.4}$$

where ϑ_1' and ϑ_2' stand for left derivatives of ϑ_1 and ϑ_2 at the point ϵ. All the functions satisfy $\int_{0+} 1/\vartheta_i(x)dx = \infty$, $i = 1, 2$ and concave and nondecreasing. It should be noted that the proposed conditions are more general than the Lipschitz conditions.

Taking into account the aforementioned definitions and lemmas, we give the following existence and uniqueness results of mild solutions for the system (2.2).

Theorem 3.2 *Assume that the assumptions* $(A_1)-(A_4)$ *are hold, then the system* (2.2) *admits a unique mild solution in* $\mathbb{B}_T$.

First, we prove the existence part of Theorem 3.2 based on the Picard type approximation technique. Let us construct a sequence of stochastic processes $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ defined by

$$\begin{cases} y_0(t) = \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \displaystyle\sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ y_{n+1}(t) = \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \displaystyle\sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ \qquad +B_1(y_n)(t) + B_2(y_n)(t), \end{cases} \tag{3.5}$$

where

$$B_1(y_n)(t) = \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_n(s))ds, \tag{3.6}$$

$$\text{and} \qquad B_2(y_n)(t) = \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)G(s, y_n(s))dw(s). \tag{3.7}$$

In order to establish existence results of the Theorem 3.2, we are required the following lemmas.

Lemma 3.3 *Under the assumptions* $(A_1)-(A_4)$*, the sequence* $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ *is well defined. Moreover, it is bounded in* $\mathbb{B}_T$ *i.e.* $\sup_{n\in\mathbb{N}\cup\{0\}}\|y_n\|_{\mathbb{B}_T} \le C$*, where* $C > 0$ *is a constant.*

Proof From (3.5), we have

$$\begin{aligned} \mathbf{E}\|y_{n+1}(t)\|^p \le & 5^{p-1}\mathbf{E}\|\mathcal{S}_{\beta,\gamma_j}(t)\varphi\|^p + 5^{p-1}\mathbf{E}\|(g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi\|^p \\ & + 5^{p-1}\mathbf{E}\left\|\sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds\right\|^p \\ & + 5^{p-1}\mathbf{E}\|B_1(y_n)(t)\|^p + 5^{p-1}\mathbf{E}\|B_2(y_n)(t)\|^p. \end{aligned} \tag{3.8}$$

Using (3.5), Hölder inequality and monotonicity of U, we get

$$\begin{aligned}\mathbf{E}\|B_1(y_n)(t)\|^p \leq & \frac{S_0^p}{(\Gamma(1+\beta))^p}\left(\frac{p-1}{\beta p+p-1}\right)^{p-1} T^{\beta p+p-1}\int_0^t \mathbf{E}(\|F(s, y_n(s))\|^p)ds \\ \leq & C_1\int_0^t U(s, \mathbf{E}\|y_n(s)\|^p)ds \\ \leq & C_1\int_0^t U(s, \|y_n\|^p_{\mathbb{B}_s})ds,\end{aligned}$$

where $C_1 = \frac{S_0^p}{(\Gamma(1+\beta))^p}\left(\frac{p-1}{\beta p+p-1}\right)^{p-1} T^{\beta p+p-1}$.

Again, using Lemma 2.7, Hӧlder inequality and monotonicity of U, we get

$$\begin{aligned}\mathbf{E}\|B_2(y_n)(t)\|^p \leq & C_p\mathbf{E}\left(\int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|^2\|G(s, y_n(s))\|^2_{\mathcal{L}_2^2}ds\right)^{\frac{p}{2}} \\ \leq & C_p\left(\frac{S_0}{\Gamma(1+\beta)}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\beta p+p-2}\right)^{\frac{p-2}{2}} T^{2\beta p+p-2}\int_0^t \mathbf{E}(\|G(s, y_n(s))\|^p_{\mathcal{L}_2^2})ds \\ \leq & C_2\int_0^t U(s, \mathbf{E}\|y_n(s)\|^p)ds \\ \leq & C_2\int_0^t U(s, \|y_n\|^p_{\mathbb{B}_s})ds,\end{aligned}$$

where $C_2 = C_p\left(\frac{S_0}{\Gamma(1+\beta)}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\beta p+p-2}\right)^{\frac{p-2}{2}} T^{2\beta p+p-2}$.

Now, using the above inequalities in (3.8), we acquire

$$\begin{aligned}\mathbf{E}\|y_{n+1}(t)\|^p \leq & 5^{p-1}S_0^p\mathbf{E}(\|\varphi\|^p) + 5^{p-1}S_0^pT^p\mathbf{E}(\|\chi\|^p) + 5^{p-1}\left(\sum_{j=1}^n \frac{S_0\alpha_jT^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\right)^p\mathbf{E}(\|\varphi\|^p) \\ & + 5^{p-1}(C_1+C_2)\int_0^t U(s, \|y_n\|^p_{\mathbb{B}_s})ds \\ \leq & k_1 + k_2\int_0^t U(s, \|y_n\|^p_{\mathbb{B}_s})ds,\end{aligned}$$

where $k_1 = 5^{p-1}\left[S_0^p\mathbf{E}(\|\varphi\|^p) + S_0^pT^p\mathbf{E}(\|\chi\|^p) + \left(\sum_{j=1}^n \frac{S_0\alpha_jT^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\right)^p\mathbf{E}(\|\varphi\|^p)\right]$ and $k_2 = 5^{p-1}(C_1+C_2)$.

Therefore,

$$\|y_{n+1}\|^p_{\mathbb{B}_t} \leq k_1 + k_2\int_0^t U(s, \|y_n\|^p_{\mathbb{B}_s})ds. \tag{3.9}$$

Now, we consider the following integral equation

$$z(t) = k_1 + k_2\int_0^t U(s, z(s))ds. \tag{3.10}$$

By the assumption (A_2), (3.10) admits a global solution $z(\cdot)$ on $\mathcal{I}$. Next, we show by applying induction argument that $\|y_n\|^p_{\mathbb{B}_t} \leq z(t)$, for all $t \in \mathcal{I}$. For all $t \in \mathcal{I}$, we have

$$\|y_0\|_{\mathbb{B}_t}^p \leq 3^{p-1}S_0^p\mathbf{E}(\|\varphi\|^p) + 3^{p-1}S_0^pT^p\mathbf{E}(\|\chi\|^p) + 3^{p-1}\left(\sum_{j=1}^{n}\frac{S_0\alpha_jT^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\right)^p\mathbf{E}(\|\varphi\|^p) \leq k_1 \leq z(t).$$

Now, let us assume that $\|y_n(t)\|_{\mathbb{B}_t}^p \leq z(t)$ for all $t \in \mathcal{I}$. Then by (3.9),(3.10) and non-decreasing property on U in second variable, we obtain

$$z(t) - \|y_{n+1}\|_{\mathbb{B}_t}^p \geq k_2\int_0^t (U(s,z(s)) - U(s,\|y_n\|_{\mathbb{B}_s}^p))ds, \quad \forall t \in \mathcal{I}. \tag{3.11}$$

In particular, $\sup_{n\in\mathbb{N}\cup\{0\}}\|y_n\|_{\mathbb{B}_T} \leq z(T)^{1/p}$ i.e. $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ is well defined. □

Lemma 3.4 *Under the assumptions* $(A_1)-(A_4)$, *the sequence* $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ *is a Cauchy sequence in* $\mathbb{B}_T$.

Proof Let us define $\delta_n(t) = \sup_{n\leq m}\|y_m - y_n\|_{\mathbb{B}_t}^p$. For all $m, n \in \mathbb{N}\cup\{0\}$, we obtain

$$y_m(t) - y_n(t) \leq \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)[F(s,y_m(s)) - F(s,y_n(s))]ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)[G(s,y_m(s)) - G(s,y_n(s))]dw(s).$$

Now, recalling the same argument as in Lemma 3.3, we obtain

$$\|y_m - y_n\|_{\mathbb{B}_t}^p \leq C_3\int_0^t V(s,\|y_{m-1} - y_{n-1}\|_{\mathbb{B}_s}^p)ds \tag{3.12}$$

where $C_3 = 2^{p-1}\left[\frac{S_0^p}{(\Gamma(1+\beta))^p}\left(\frac{p-1}{\beta p+p-1}\right)^{p-1}T^{\beta p+p-1} + C_p\left(\frac{S_0}{\Gamma(1+\beta)}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\beta p+p-2}\right)^{\frac{p-2}{2}} T^{2\beta p+p-2}\right]$. This shows that

$$\delta_n(t) \leq C_3\int_0^t V(s,\delta_{n-1}(s))ds. \tag{3.13}$$

It is clear that the functions δ_n are well defined for all $n \geq 0$, categorically monotone non-decreasing and uniformly bounded due to Lemma 3.3. Since $\{\delta_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ is a monotonic non-increasing sequence for each $t \in \mathcal{I}$, there exists a monotone non-decreasing function δ such that $\lim_{n\to\infty}\delta_n(t) \to \delta(t)$. Now, by virtue of Lebesgue convergence theorem, we follow from the inequality (3.13) that

$$\delta(t) \leq C_3\int_0^t V(s,\delta(t))ds, \qquad \text{as} \quad n \to \infty. \tag{3.14}$$

By the assumption (A_4) and Lemma 2.2 in [1] that $\delta = 0$, $\forall t \in \mathcal{I}$. Since $0 \leq \|y_m - y_n\|_{\mathbb{B}_T}^p \leq \delta_n(T)$ and $\lim_{n\to\infty}\delta_n(t) \to \delta(t)$, therefore as a result $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in $\mathbb{B}_T$. □

Proof of Theorem 3.2. **Existence:** Form Lemma 3.4, let us denote y as a limit of the sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$. Now, similar as in the proof of Lemma 3.4, we can show that the right side of the sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ given by (3.5) tends to

$$\mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\varphi ds$$
$$+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y(s))ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)G(s, y(s))dw(s), \qquad \text{as} \quad n \to \infty.$$

□

Uniqueness: Let $x, y \in \mathbb{B}_T$ be two mild solutions of the system (2.2). Now following the proof of Lemma 3.4, similar as (3.12) we can obtain

$$\|x-y\|^p_{\mathbb{B}_t} \leq C_3 \int_0^t V(s, \|x-y\|^p_{\mathbb{B}_s})ds. \tag{3.15}$$

By using the assumption (A_4), similar as in proof of Lemma 3.4, we get $\|x-y\|^p_{\mathbb{B}_T} \to 0$, which shows that $x = y$. This completes the proof.

Remark 3.5 In this paper, in order to obtain existence and uniqueness results a Picard type iterations technique is employed in place of fixed point theorem. In this technique, we avoided the compactness conditions [29], Lipschitz continuity of nonlinear functions and some inequality conditions (as given in Theorem 3.5 in [29], Theorem 3.2 and Theorem 3.3 in [36]).

Example

The fractional order diffusion wave equations have great applications in varies fields of science and engineering. These equations represent propagation of mechanical waves through viscoelastic media, charge transport in amorphous semiconductors [15,23], and may be used in thermodynamics and the flow of fluid through fissured rocks [2].

We provide a concrete example to illustrate the feasibility of the established results. Let $\beta, \gamma_j > 0$, $j = 1, 2, 3, \ldots, n$ be given such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. Let $\mathbb{H} = L^2([0,\pi])$. We consider the following system

$$^cD^{1+\beta}z(t,x) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j}z(t,x) = \frac{\partial^2}{\partial x^2}z(t,x) + \widehat{F}(t, z(t,x)) + \widehat{G}(t, z(t,x))\frac{dw(t)}{dt}, \tag{4.1}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in [0,1], \tag{4.2}$$

$$z(0,x) = z_0(x), \quad \frac{\partial z(t,x)}{\partial t}|_{t=0} = z_1(x), \ 0 \leq x \leq \pi, \tag{4.3}$$

where $w(t)$ denotes one dimensional $\mathbb{R}$-valued Brownian motion and $z_0(x), z_1(x) \in L^2([0,\pi])$ are $\mathcal{F}_0$ measurable and satisfy $\mathbf{E}\|z_0\|^2 \leq \infty, \mathbf{E}\|z_1\|^2 \leq \infty$, here we consider $p = 2$. Let $w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\zeta_n(t)e_n$ $(\lambda_n > 0)$, where $\zeta_n(t)$ are one dimensional standard Brownian motion mutually independent on a usual complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$. Define a operator $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$Au = u'', \quad u \in \mathcal{D}(A),$$

where $\mathcal{D}(A) := \{u \in \mathbb{H} : u, u' \text{ are absolutely continuous}, u'' \in \mathbb{H}, u(0) = u(\pi) = 0\}$. Then the operator A has spectral representation given by

$$Au = \sum_{n=1}^{\infty} -n^2\langle u, u_n\rangle u_n,$$

where $u_n(x) = (\sqrt{2/\pi}) \sin nx$, $n = 1, 2, \ldots$, is the orthogonal set of eigenfunctions corresponding to the eigenvalues $\lambda_n = -n^2$ of A. Then A will be a generator of cosine family such that

$$C(t)u = \sum_{n=1}^{\infty} \cos nt \langle u, u_n \rangle u_n,$$

Thus A generates a strongly continuous cosine family. Then, for $\beta, \gamma_j > 0$, $j = 1, 2, 3, \ldots, n$ such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$, by Theorem 2.2, we conclude that A generates a bounded $(\beta, \gamma_j)-$ resolvent family

$$\mathcal{S}_{\beta,\gamma_j}(t)u = \int_0^{\infty} \frac{1}{t^{\frac{(1+\beta)}{2}}} \Phi_{\frac{(1+\beta)}{2}}(st^{-\frac{(1+\beta)}{2}})C(s)uds, \quad t \in [0, 1],$$

where

$$\Phi_{\frac{(1+\beta)}{2}}(v) = \sum_{n=0}^{\infty} \frac{(-v)^n}{n!\Gamma(-(\beta(n+1)) - n)}, v \in \mathbb{C},$$

is the Wright functions. Let us denote $y(t)(x) = z(t, x)$ and $\varphi = z_0(x)$, $\chi = z_1(x)$ for $t \in [0, 1], x \in [0, \pi]$. Then, $Ay(t) = \frac{\partial^2}{\partial x^2} z(t, x)$ and for the functions $F, G : [0, 1] \times \mathbb{H} \to \mathbb{H}$, we have

$$F(t, y(t))(x) = \widehat{F}(t, z(t, x)), \ \ G(t, y(t))(x) = \widehat{G}(t, z(t, x)).$$

Then the system (4.1)–(4.3) has a abstract form of the system

$$^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} y(t) = Ay(t) + F(t, y(t)) + G(t, y(t))\frac{dw(t)}{dt}, \quad t \in (0, 1], \tag{4.4}$$

$$y(0) = \varphi, \qquad y'(0) = \chi. \tag{4.5}$$

Now, by the Theorem 3.2 we may conclude that if the functions F and G satisfy the assumptions $(A_1)-(A_4)$, then the system (4.1)–(4.3) has a unique mild solution.

Remark 4.1 Since, the mathematical models involving nonlocal conditions may describe many real life problems more precisely rather than standard initial conditions, for instance, Deng [12] explained that the diffusion phenomena of a small amount of gas in a transparent tube can be described efficiently using nonlocal conditions rather than local conditions. Therefore there has been significant development in study of differential equations with nonlocal conditions [12,29]. In this remark, we consider the existence and uniqueness results for mild solutions to the following class of multi-term time-fractional stochastic differential system

$$\begin{cases} ^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} y(t) = Ay(t) + F(t, y(t)) + G(t, y(t))\frac{dw(t)}{dt}, \quad t \in (0, T], \\ y(0) = g_1(y), \quad y'(0) = g_2(y), \end{cases} \tag{4.6}$$

where $g_1, g_2 : \mathcal{C}([0, T], \mathbb{H}) \to \mathbb{H}$ are suitable functions and other functions are defined in (2.2). In particular, the nonlocal conditions in (4.6) may be applied in physics for more realistic results than the classical initial conditions $y(0) = \varphi, \quad y'(0) = \chi$. For example g_1, g_2 may be express as

$$g_1(y) = \sum_{i=1}^{m} a_i y(t_i), \quad g_2 = \sum_{i=1}^{m} b_i y(t_i),$$

where $a_i, b_i (i = 1, 2, 3, \ldots, m)$ are given constants and $0 < t_1 < \cdots < t_n \leq T$. The established results for the system (2.2) may be extended to investigate the existence and uniqueness of mild solutions of (4.6) with nonlocal conditions by applying the same technique as used in Theorem 3.2.

Remark 4.2 On the other hand, the theory of fractional impulsive differential equations also has been generated a great interest among the researchers, because many physical processes and phenomena which are effected by abrupt changes in the state at certain moments are naturally described by fractional impulsive differential equations. These changes occur due to disturbances, changing operational conditions and component failures of the state. For example, mechanical and biological models subject to shocks. Generally the abrupt changes in the state for instant period in evolution process are formulated by impulsive differential equations. Since, in addition to stochastic effects in fractional system, impulsive effects likewise exists in real process. Therefore, the fractional stochastic differential equations have been widely investigated with impulsive effects, see [4,13,38]. Since, the study of existence of mild solutions for the system (2.2) with impulsive effects is left open. Anticipating a wide interest in such problems, one may contributes in filling this important gap.

Conclusion

The available literature regarding to multi-term time-fractional differential systems has been reported with the method of separation of variables [10,16,22], Caratheodory type conditions and measure of noncompactness technique [28,29,37], to obtain the main outcomes in deterministic case. But, in this paper, we established the existence and uniqueness results for multi-term time-fractional stochastic differential system with the coefficients in the equations satisfying some non-Lipschitz conditions and using standard Picard type iterations. Here, it should be noticed that the Lipschitz condition is a special case of the proposed conditions. By adopting the ideas developed in this paper, one may establish some stability results [34] with impulsive effects which are very effective in study of a phenomenon with discontinuous jumps.

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