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ORIGINAL RESEARCH

Nonlocal Integro-Differential Equations Without the Assumption of Equicontinuity on the Resolvent Operator in Banach Spaces

Khalil Ezzinbi[1] · **Mohamed Ziat**[1]

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Abstract In this work, we study the existence of mild solutions for the nonlocal integro-differential equation

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)\mathrm{d}s + f(t, x_t) \quad \text{for } t \in [0, b] \\ x_0 = \phi + g(x) \in C([-r, 0]; X), \end{cases}$$

without the assumption of equicontinuity on the resolvent operator and without the assumption of separability on the Banach space X. The nonlocal initial condition is assumed to be compact. Our main result is new and its proof is based on a measure of noncompactness developed in Kamenskii et al. (Condensing multivalued maps and semilinear differential inclusions in Banach spaces. Walter De Gruyter, Berlin, 2001) together with the well-known Mönch fixed point Theorem. To illustrate our result, we provide an example in which the resolvent operator is not equicontinuous.

Keywords Measure of noncompactness · Integro-differential equation · Nonlocal conditions · Resolvent operator · Mönch's fixed point Theorem · Equicontinuity

Mathematics Subject Classification 35R09 · 47H08 · 47H10

Introduction

In the present work, we study the existence of mild solutions of the following integro-differential equation with finite delay and nonlocal initial conditions

✉ Mohamed Ziat
ziat.maths@gmail.com

Khalil Ezzinbi
ezzinbi@uca.ac.ma

[1] Département de Mathématiques, Faculté des Sciences Semlalia, Université Cadi Ayyad, B.P. 2390, Marrakesh, Morocco

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)\mathrm{d}s + f(t, x_t) & \text{for } t \in [0, b] \\ x_0 = \phi + g(x) \in C([-r, 0]; X), \end{cases} \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X with domain $D(A)$, for $t \geq 0$, $B(t)$ is a closed linear operator on X with domain $D(B(t)) \supset D(A)$ which is independent of t, $C([-r, 0]; X)$ is the Banach space of all continuous functions from $[-r, 0]$ to X endowed with the uniform topology, for $t \geq 0$, the history function $x_t : [-r, 0] \to X$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$, $\phi \in C([-r, 0]; X)$, $f : [0, b] \times C([-r, 0]; X) \longrightarrow X$ and $g : C([0, b]; X) \longrightarrow C([-r, 0]; X)$ are two continuous functions.

In physics, nonlocal initial conditions are usually more precise for physical measurements and has better effect than the classical initial condition [10]. The problems with nonlocal initial conditions have their origins in the works of Byszewski in his classical papers [7,8,10] in which he studied the existence of mild solutions for the nonlocal Cauchy problem:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) & \text{for } t \in [0, b] \\ x(0) = g(x), \end{cases} \tag{2}$$

where A is the infinitesimal generator of a semigroup $(T(t))_{t\geq 0}$ of linear operators defined on a Banach space X, and the maps f and g are suitable X-valued functions.

In the recent years, many authors have attracted much attention to the study of the existence of mild solutions to the nonlocal initial value problem. We refer to [1,5,7–10,22,27,28,30, 31]. In [5], Benchohra and Ntouyas discuss the semilinear differential equations with nonlocal conditions under compact conditions. In [7,8,10], Eq. (2) was studied when f and g satisfy Lipschitz-type conditions. In [9,22,27,28], the authors studied Eq. (2) under conditions of compactness of $(T(t))_{t\geq 0}$. Xue [30,31] studied Eq. (2) when $(T(t))_{t\geq 0}$ is equicontinuous.

On the other hand, the resolvent operator plays an important role in solving (1), it plays the role of the C_0-semigroup theory but does not satisfy the algebraic semigroup property (Remark 4). When $(R(t))_{t\geq 0}$ is a resolvent operator for Eq. (1), the mild solutions are given by the following variation of constants formula

$$x(t) = R(t)[\phi(0) + g(x)(0)] + \int_0^t R(t-s)f(s, x_s)\mathrm{d}s \quad \text{for } t \in [0, b].$$

For more details about resolvent operators we refer to [12,13,18–20,25]. A resolvent operator $(R(t))_{t\geq 0}$ is said to be equicontinuous if $t \mapsto R(t)x$ are equicontinuous at all $t > 0$ with respect to x in all bounded subsets.

In this work, we study the existence of mild solutions for Eq. (1) without the assumption of equicontinuity on the resolvent operator $(R(t))_{t\geq 0}$ and without the assumption of separability on the Banach space X. The proof of the main result is based on the application of the Mönch's fixed point Theorem combined with a measure of noncompactness with values in the cone $\mathbb{R}^2_+$ developed in [21, Example 2.1.4, page 38].

The result obtained in this work generalizes some results developed in [14,23]. In [14], the authors obtained results on the existence of mild solutions of Eq. (1) where $B = 0$ under condition of compactness of the function g and without assuming that the C_0-semigroup is equicontinuous. To achieve their goal, the authors used the theory of C_0-semigroup and the Schauder's fixed point Theorem combined with the measure of noncompactness defined as the sum of the Hausdorff measure of noncompactness and the modulus of equicontinuity (Example 9). In [23], using an adaptation of the methods described in [32], the authors studied the existence of mild solutions of Eq. (1) without delay ($r = 0$) under conditions of compactness of the function g and equicontinuity of the resolvent operator. The approach used in this work is different from [14,23].

The outline of this work is as follows. In the Preliminary Results section, we recall some preliminaries about resolvent operator and measure of noncompactness, we also give some lemmas which will be used in the proof of the main result. In the Main Result section, we establish our main result on the existence of the mild solutions for Eq. (1). We illustrate our work in the Application section by examining an example in which the resolvent operator is not equicontinuous.

Preliminary Results

In this section, we first recall some basic results about the concept of resolvent operator and the concept of measure of noncompactness in Banach spaces. Then we give some lemmas which will be used in the proof of the main results.

In the following, X is a Banach space, A and $B(t)$ are closed linear operators on X. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \quad \text{for } y \in Y.$$

Let Z and W be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the Banach space of all bounded linear operators from Z to W endowed with the operator norm and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. The notation $C([0, +\infty); Y)$ stands for the space of all continuous functions from $[0, +\infty)$ into Y. We consider the following integro-differential equation

$$\begin{cases} y'(t) = Ay(t) + \displaystyle\int_0^t B(t-s)y(s)\mathrm{d}s \quad \text{for } t \geq 0 \\ y(0) = y_0 \in X. \end{cases} \tag{3}$$

Definition 1 ([17]) A *resolvent operator* for Eq. (3) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$ having the following properties:

(a) $R(0) = I$ and $\|R(t)\| \leq \lambda e^{\beta t}$ for some constants λ and β.
(b) The operator R is strongly continuous, i.e. the map $t \mapsto R(t)x$ is continuous for every $x \in X$.
(c) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y$, $R(\cdot)x \in C^1([0, +\infty); X) \cap C([0, +\infty); Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)x\mathrm{d}s \\ &= R(t)Ax + \int_0^t R(t-s)B(s)x\mathrm{d}s \quad \text{for } t \geq 0. \end{aligned}$$

For more properties about resolvent operators theory, we refer the readers to [12,13,17]. In the sequel, we assume that A and $(B(t))_{t\geq 0}$ satisfy the following hypotheses.

(H1) A is the infinitesimal generator of a strongly continuous semigroup on X.
(H2) For all $t \geq 0$, $B(t)$ is closed linear operator from $D(A)$ to X and $B(t) \in \mathcal{L}(Y, X)$. For any $y \in Y$, the map $t \to B(t)y$ is bounded, differentiable and the derivative $t \to B'(t)y$ is bounded uniformly continuous on R^+.

The following theorem gives an existence result of the resolvent operator for Eq. (3).

Theorem 2 ([17]) *Assume that **(H1)** and **(H2)** hold. Then there exists a unique resolvent operator of Eq. (3).*

Example 3 ([13]) Let $X = \mathbb{R}$, $Ay = y$, and $B(t) = -2y$ in Eq. (3). Then we have

$$R(t)x_0 = e^t(\cos t + \sin t)x_0 \quad \text{and} \quad T(t)x_0 = e^{2t}x_0.$$

Remark 4 The above example also shows that, in general, the resolvent operator $(R(t))_{t\geq 0}$ for Eq. (3) does not satisfy the algebraic semigroup property, namely,

$$R(t+s) \neq R(t)R(s) \quad \text{for some } t, s \geq 0.$$

Definition 5 Let $(R(t))_{t\geq 0}$ be a resolvent operator on X. We say that $(R(t))_{t\geq 0}$ is *equicontinuous* (we say also norm continuous or immediately norm continuous) if $\{t \to R(t)x : x \in B\}$ is equicontinuous at any $t > 0$ for all bounded subsets B in X. This is equivalent to say that

$$\lim_{t\to t'} \|R(t) - R(t')\| = 0 \quad \text{for } t' \geq 0.$$

In order to give in the last section an example illustrating the theory in which the resolvent operator is not equicontinuous, we need to use the following theorem.

Theorem 6 ([16]) *Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and let $(B(t))_{t\geq 0}$ satisfy (**H2**). Then the resolvent operator $(R(t))_{t\geq 0}$ for Eq. (3) is equicontinuous for $t > 0$ if and only if $(T(t))_{t\geq 0}$ is equicontinuous for $t > 0$.*

Remark 7 Throughout the remainder of this work, M_b denotes the constant $M_b = \sup_{t\in[0,b]} \|R(t)\|$.

Next, we give some definitions, properties and examples about measure of noncompactness. More details about these facts can be found in the monographs [2–4,21].

Definition 8 Let E^+ be a positive cone of an ordered Banach space $(E, \leq)$. A function Ψ defined on the set of all bounded subsets of a Banach space X with values in E^+ is called a *measure of noncompactness* if $\Psi(\overline{co}\Omega) = \Psi(\Omega)$ for all bounded subset $\Omega \subset X$, where $\overline{co}\Omega$ stands for the closed convex hull of Ω.

A measure of noncompactness Ψ is said to be:

1. *monotone* if for all bounded subsets Ω_1, Ω_2 of X, $\Omega_1 \subset \Omega_2$ implies $\Psi(\Omega_1) \leq \Psi(\Omega_2)$,
2. *nonsingular* if $\Psi(\{a\} \cup \Omega) = \Psi(\Omega)$ for every $a \in X$ and every nonempty subset $\Omega \subset X$,
3. *algebraically semiadditive* if $\Psi(\Omega_1 + \Omega_2) \leq \Psi(\Omega_1) + \Psi(\Omega_2)$ for all bounded subsets Ω_1, Ω_2 of X,
4. *regular* if $\Psi(\Omega) = 0$ if and only if Ω is relatively compact in X,
5. *semi-homogeneous:* if $\Psi(t\Omega) = |t|\Psi(\Omega)$ for every $t \in \mathbb{R}$ and all bounded subset Ω of X.

Example 9 We give some examples of measure of noncompactness.

1. The *Hausdorff measure of noncompactness* $\chi(\cdot)$ defined on each bounded subset of the Banach space X is given by

$$\chi(\Omega) = \inf\left\{\varepsilon > 0 : \Omega \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon),\ x_i \in X \text{ for } i = 1, \ldots, n\right\}.$$

   χ is monotone, nonsingular, algebraically semiadditive and regular.

2. The modulus of fiber noncompactness χ_1 defined for bounded set $\Omega \subset C([a,b];X)$ is given by
$$\chi_1(\Omega) = \sup_{t\in[a,b]} \chi(\Omega(t))$$
where $\Omega(t) = \{x(t) : x \in \Omega\}$ for $t \in [a,b]$.
3. The *modulus of equicontinuity*
$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{x\in\Omega} \max_{|t_1-t_2|\le\delta} \|x(t_1) - x(t_2)\|$$
where $\Omega \subset C([a,b];X)$ is bounded. Note that $\mathrm{mod}_C(\Omega) = 0$ if and only if Ω is equicontinuous.

The measure of noncompactness χ_1 and mod_C are not regular.

Now, we give another measure of noncompactness in the space $C([a,b];X)$ with values in the cone $\mathbb{R}^2_+$ which is a very powerful tool in this work.

Example 10 ([21, Example 2.1.4, page 38]). We consider the measure of noncompactness
$$\nu(\Omega) = \max_{D\in\Delta(\Omega)} \Big(\gamma(D), \mathrm{mod}_C(D)\Big) \tag{4}$$
in $C([a,b];X)$ where $\Delta(\Omega)$ is the collection of all countable subsets of Ω, mod_C is given in Example 9 and γ is the measure of noncompactness defined by
$$\gamma(D) = \sup_{t\in[a,b]} e^{-Lt}\chi(D(t)) \tag{5}$$
where L is a constant. The range for the measure of noncompactness ν is the cone $\mathbb{R}^2_+$, max is taken in the sense of the ordering induced by this cone. ν is monotone, nonsingular and regular. Furthermore, there exists $\widetilde{D} \in \Delta(\Omega)$ such that the maximum on the right-hand side of (4) is achieved on $\widetilde{D}$.

Lemma 11 ([6, page 125]) *Let X be a Banach space. If $W \subset X$ is a bounded subset, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{+\infty} \subset W$ such that*
$$\Phi(W) \le 2\Phi(\{u_n\}_{n=1}^{+\infty}) + \varepsilon$$
where Φ is any measure of noncompactness.

Lemma 12 ([24, Lemma 1.3, page 25]) *Suppose that X is a Banach space and f is an integrable function from J to X. Then*
$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}s \in \overline{\mathrm{co}}\left(\{f(t) : t \in [a,b]\}\right)$$
for all $a, b \in J$ with $a < b$.

Definition 13 A set of functions $\{f_n\}_{n=1}^{+\infty} \subset L^1([0,b];X)$ is said to be *uniformly integrable* (*integrably bounded*) if there exists a positive function $v \in L^1([0,b];\mathbb{R}^+)$ such that for all $n \ge 1$
$$\|f_n(t)\| \le v(t) \quad \text{a.e. } t \in [0,b].$$

Lemma 14 ([14]) *If* $\{u_n\}_{n=1}^{+\infty} \subset L^1(a, b, X)$ *is uniformly integrable, then* $\chi(\{u_n(t)\})_{n\geq 1}^{+\infty})$ *is measurable and*

$$\chi\left(\left\{\int_a^t u_n(s)\mathrm{d}s\right\}_{n=1}^{+\infty}\right) \leq 2\int_a^t \chi\left(\{u_n(s)\}_{n=1}^{+\infty}\right)\mathrm{d}s.$$

For an abstract operator $S : L^1([0, b]; X) \to C([0, b]; X)$ we consider the following properties taken from [21].

(**S1**) There exists $N > 0$ such that

$$\|Sf(t) - Sg(t)\| \leq N\int_0^t \|f(s) - g(s)\|\mathrm{d}s$$

for every $f, g \in L^1([0, b]; X)$ and $t \in [0, b]$.

(**S2**) For any compact $K \subset X$ and sequence $\{f_n\}_{n=1}^{+\infty} \subset L^1([0, b]; X)$ such that $\{f_n(t)\}_{n=1}^{+\infty} \subset K$ for a.e. $t \in [0, b]$ the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $Sf_n \to Sf_0$.

Consider the operator $\Phi : L^1([0, b]; X) \to C([0, b]; X)$ defined by

$$(\Phi f)(t) = \int_0^t R(t-s)f(s)\mathrm{d}s \quad \text{for } t \in [0, b]. \tag{6}$$

The next Lemma plays a key role in this work and it is a generalization of [21, Lemma 4.2.1, page 111]. Note that the proof of [21, Lemma 4.2.1, page 111] is based on the use of the algebraic semigroup property which is not satisfied, in general, for a resolvent operator as mentioned in Remark 4.

Lemma 15 *The operator* Φ *defined by* (6) *satisfies the conditions (**S1**) and (**S2**).*

Proof Let $f, g \in L^1([0, b]; X)$ and $t \in [0, b]$. We have

$$\begin{aligned}\|\Phi f(t) - \Phi g(t)\| &= \left\|\int_0^t R(t-s)(f(s) - g(s))\mathrm{d}s\right\| \\ &\leq M_b\int_0^t \|f(s) - g(s)\|\mathrm{d}s,\end{aligned} \tag{7}$$

which implies that (**S1**) holds. Now let $K \subset X$ be a compact set and $\{f_n\}_{n=1}^{+\infty} \subset L^1([0, b]; X)$ a sequence such that $\{f_n(t)\}_{n=1}^{+\infty} \subset K$ for a.e. $t \in [0, b]$. Then, there exists $F > 0$ such that $\|f_n(t)\| \leq F$ for $n \geq 1$ and a.e. $t \in [0, b]$. By (7), Φ is a bounded linear operator from the space $L^1([0, b]; X)$ into $C([0, b]; X)$. Consequently, we have

$$f_n \rightharpoonup f_0 \Longrightarrow \Phi f_n \rightharpoonup \Phi f_0. \tag{8}$$

To end the proof, we claim that the sequence $\{\Phi f_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0, b]; X)$. By Arzelà–Ascoli's Theorem, it suffices to prove that the sequence $\{\Phi f_n\}_{n=1}^{+\infty} \subset C([0, b]; X)$ is equicontinuous on $[0, b]$ and $\{\Phi f_n(t)\}_{n=1}^{+\infty}$ is relatively compact in X for each $t \in [0, b]$. For the first assertion, we start by the equicontinuity at 0. For $n \geq 1$ and $t \in [0, b]$ we have

$$\|\Phi f_n(t) - \Phi f_n(0)\| \leq M_b F t,$$

which gives that $\|\Phi f_n(t) - \Phi f_n(0)\| \to 0$ as $t \to 0$ uniformly with respect to $n \geq 1$. Now, let $0 < t_1 < t_2 \leq b$ and $n \geq 1$. We have

$$\begin{aligned}\|\Phi f_n(t_2) - \Phi f_n(t_1)\| &\leq \int_{t_1}^{t_2} \|R(t_2 - s) f_n(s)\| \mathrm{d}s \\ &\quad + \int_0^{t_1} \|R(t_2 - s) f_n(s) - R(t_1 - s) f_n(s)\| \mathrm{d}s \\ &\leq (t_2 - t_1) M_b F \\ &\quad + \int_0^{t_1} \sup_{x \in K} \left\| R(t_2 - t_1 + u)x - R(u)x \right\| \mathrm{d}u\end{aligned}$$

The strong continuity of the resolvent operator $(R(t))_{t\geq 0}$ ensures that

$$\sup_{x \in K} \left\| R(t_2 - t_1 + u)x - R(u)x \right\| \to 0 \text{ as } t_2 \to t_1.$$

Since

$$\sup_{x \in K} \left\| R(t_2 - t_1 + u)x - R(u)x \right\| \leq 2M_b F,$$

the Lebesgue dominated convergence theorem leads to

$$\int_0^{t_1} \sup_{x \in K} \left\| R(t_2 - t_1 + u)x - R(u)x \right\| \mathrm{d}u \to 0 \text{ as } t_2 \to t_1.$$

We deduce that

$$\lim_{t_1 \to t_2} \|\Phi f_n(t_2) - \Phi f_n(t_1)\| = 0 \tag{9}$$

uniformly with respect to $n \geq 1$, which means that $\{\Phi f_n\}_{n=1}^{+\infty}$ is equicontinuous on $[0, b]$. Now, we prove the second assertion. Let $t \in [0, b]$, $(\sigma_n) \in [0, t]$ and $(x_n) \in K$. Sine $[0, t]$ and K are compact, there exist two subsequences (σ_{n_k}) and (x_{n_k}) of (σ_n) and (x_n) respectively and $\sigma_0 \in [0, t]$ and $x_0 \in K$ such that $\sigma_{n_k} \to \sigma_0$ and $x_{n_k} \to x_0$ as $k \to \infty$. Now we have

$$\begin{aligned}\|R(\sigma_{n_k})x_{n_k} - R(\sigma_0)x_0\| &\leq \|R(\sigma_{n_k})x_{n_k} - R(\sigma_{n_k})x_0\| + \|R(\sigma_{n_k})x_0 - R(\sigma_0)x_0\| \\ &\leq M_b \|x_{n_k} - x_0\| + \|R(\sigma_{n_k})x_0 - R(\sigma_0)x_0\|.\end{aligned}$$

Using the strong continuity of the resolvent operator $(R(t))_{t\geq 0}$, we get that $\|R(\sigma_{n_k})x_0 - R(\sigma_0)x_0\| \to 0$ as $k \to \infty$. Consequently

$$\lim_{k \to \infty} \|R(\sigma_{n_k})x_{n_k} - R(\sigma_0)x_0\| = 0. \tag{10}$$

We conclude that the set $\{R(\sigma)x : \sigma \in [0, t],\ x \in K\}$ is compact in X. Hence, the set

$$\overline{\text{co}}\Big(R(\sigma)x : \sigma \in [0, t],\ x \in K \Big) \tag{11}$$

is also compact in X. According to Lemma 12, we have

$$\int_0^t R(t - s) f_n(s) \mathrm{d}s \in t\overline{\text{co}}\Big(R(\sigma)x : \sigma \in [0, t],\ x \in K \Big).$$

Afterwards

$$\{\Phi f_n(t)\}_{n=1}^{+\infty} \subset t\overline{\text{co}}\Big(R(\sigma)x : \sigma \in [0, t],\ x \in K \Big), \tag{12}$$

which implies that the set $\{\Phi f_n(t)\}_{n=1}^{+\infty}$ is relatively compact in X for each $t \in [0, b]$. Thus, in view of the Arzelà–Ascoli Theorem, $\{\Phi f_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0, b]; X)$ and so the convergence in (8) is strong. i.e., $\Phi f_n \to \Phi f_0$. □

Lemma 16 ([21, Theorem 4.2.2, page 112]). *Let $\{f_n\}_{n=1}^{+\infty} \subset L^1([0, b]; X)$ be uniformly integrable. Assume that*

$$\chi(\{f_n(t)\}_{n=1}^{+\infty}) \le q(t)$$

*for a.e. $t \in [0, b]$ where $q \in L^1([0, b])$. If S satisfies conditions (**S1**) and (**S2**) then*

$$\chi(\{(Sf_n)(t)\}_{n=1}^{+\infty}) \le 2N \int_0^t q(s)\mathrm{d}s \quad \text{for } t \in [0, b], \tag{13}$$

*where $N > 0$ is the constant in condition (**S1**).*

Definition 17 ([21]) The sequence $\{f_n\}_{n=1}^{+\infty} \subset L^1([0, b]; X)$ is *semicompact* if it is uniformly integrable and the set $\{f_n(t)\}_{n=1}^{+\infty}$ is relatively compact for almost every $t \in [0, b]$.

Lemma 18 ([21, Theorem 5.1.1, page 122]). *Let $S : L^1([0, b]; X) \to C([0, b]; X)$ be an operator satisfying the conditions (**S1**) and (**S2**). Then for every semicompact sequence $\{f_n\}_{n=1}^{+\infty} \subset L^1([0, b]; X)$ the sequence $\{Sf_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0, b]; X)$.*

Main Result

In this section, we give some existence results for the nonlocal integro-differential equation (1). Let us first give the definition of the mild solution.

Definition 19 A continuous function $x : [-r, b] \to X$ is said to be a mild solution of the nonlocal Eq. (1) if $x_0 = \phi + g(x)$ and

$$x(t) = R(t)[\phi(0) + g(x)(0)] + \int_0^t R(t-s) f(s, x_s)\mathrm{d}s \quad \text{for } \ t \in [0, b].$$

Equation (1) will be studied under the hypotheses (**H1**), (**H2**) and the following hypotheses.

(**H3**) (i) $f : [0, b] \times C([-r, 0]; X) \to X$ satisfies the Carathéodory-type condition, i.e., $f(\cdot, \varphi) : [0, b] \to X$ is measurable for all $\varphi \in C([-r, 0]; X)$ and $f(t, \cdot) : C([-r, 0]; X) \to X$ is continuous for a.e. $t \in [0, b]$.

(ii) There exist a function $m \in L^1([0, b]; \mathbb{R}^+)$ and a nondecreasing continuous function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, \varphi)\| \le m(t)\Omega(\|\varphi\|) \tag{14}$$

for a.e. $t \in [0, b]$ and all $\varphi \in C([-r, 0]; X)$.

(iii) There exists a function $\eta \in L^1([0, b]; \mathbb{R}^+)$ such

$$\chi(f(t, D)) \le \eta(t) \sup_{-r \le \theta \le 0} \chi(D(\theta)) \tag{15}$$

for a.e. $t \in [0, b]$ and any bounded subset $D \subset C([-r, 0]; X)$.

(**H4**) $g : C([0, b]; X) \to C([-r, 0]; X)$ is continuous, compact. Moreover

$$\|g(\varphi)\| \le c\|\varphi\| + d, \tag{16}$$

for all $\varphi \in C([0, b]; X)$.

The class of the functions satisfying (**H3**) (iii) is not empty. The next example gives a function verifying a such hypothesis. A second function will be given in the next section.

Example 20 Let X be a Banach space and f be defined by

$$f(t,\varphi) = e^{-t}\int_{-r}^{0} F(\theta,\varphi(\theta))\mathrm{d}\theta$$

for $t \in [0,b]$ and $\varphi \in C([-r,0];X)$, where $F : [-r,0]\times X \to X$ satisfies the following condition.

(**C**) (i) There exist $\alpha \in L^1([-r,0];\mathbb{R}^+)$ and $\beta \in L^1([-r,0];\mathbb{R}^+)$ such that

$$\|F(\theta,x)\| \le \alpha(\theta)\|x\| + \beta(\theta)$$

for $\theta \in [-r,0]$ and $x \in X$.

(ii) There exists $\xi > 0$ such that for any bounded subset $D \subset X$

$$\chi(F(\theta,D)) \le \xi\chi(D) \quad \text{for a.e. } \theta \in [-r,0].$$

Then f satisfies Hypothesis (**H3**) (iii). In fact, let D be a bounded subset of $C([-r,0];X)$. From (**C**) (ii) we have

$$\chi(F(\theta,D(\theta))) \le \xi\chi(D(\theta)) \quad \text{for a.e. } \theta \in [0,b],$$

where $F(\theta,D(\theta)) = \{F(\theta,\varphi(\theta)) \in X,\ \ \varphi \in D\}$. Using (**C**) (i), we obtain $\|f(t,\varphi)\| \le \|\alpha\|_{L^1}\|\varphi\| + \|\beta\|_{L^1}$ for all $t \in [0,b]$ and $\varphi \in C([-r,0];X)$. Therefore, for each $t \in [0,b]$, $f(t,D)$ is bounded. Let $\varepsilon > 0$ be fixed. By means of Lemma 11, there exists a sequence $\{\varphi_n\}_{n=1}^{+\infty} \subset D$ such that

$$\begin{aligned}\chi(f(t,D)) &\le 2\chi(f(t,\{\varphi_n\}_{n=1}^{+\infty})) + \varepsilon\\ &= 2\chi\left(e^{-t}\int_{-r}^{0} F(\theta,\{\varphi_n(\theta)\}_{n=1}^{+\infty})\mathrm{d}\theta\right) + \varepsilon.\end{aligned}$$

On the other hand, using again (**C**) (i), it follows that

$$\|F(\theta,\varphi_n(\theta))\| \le \varrho\alpha(\theta) + \beta(\theta) \quad \text{for } n \ge 1 \text{ and } \theta \in [-r,0],$$

where $\varrho = \sup_{\psi\in D}\|\psi\|$. Thus, $F(\theta,\{\varphi_n(\theta)\}_{n=1}^{+\infty})$ is uniformly integrable. From Lemma 14, we get

$$\begin{aligned}\chi(f(t,D)) &\le 4e^{-t}\int_{-r}^{0}\chi\left(F(\theta,\{\varphi_n(\theta)\}_{n=1}^{+\infty})\right)\mathrm{d}\theta + \varepsilon\\ &\le 4e^{-t}\int_{-r}^{0}\chi(F(\theta,D(\theta)))\mathrm{d}\theta + \varepsilon\\ &\le 4\xi e^{-t}\int_{-r}^{0}\chi(D(\theta))\mathrm{d}\theta + \varepsilon\\ &\le 4\xi r e^{-t}\sup_{-r\le\theta\le 0}\chi(D(\theta))) + \varepsilon\end{aligned}$$

for $t \in [0,b]$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\chi(f(t,D)) \le 4\xi r e^{-t}\sup_{-r\le\theta\le 0}\chi(D(\theta))).$$

Hence the function f satisfies Hypothesis (**H3**) (iii) with $\eta(t) = 4\xi r e^{-t}$ for $t \in [0,b]$.

The key tool in our approach is the following theorem due to Mönch [26].

Theorem 21 ([26]) *Let D be a closed convex subset of a Banach space E and $0 \in D$. Assume that $F : D \to E$ is a continuous map which satisfies Mönch's condition, that is, ($M \subseteq D$ is countable, $M \subseteq \overline{\text{co}}(\{0\} \cup F(M)) \Longrightarrow \overline{M}$ is compact). Then F has a fixed point in D.*

The first result of this work is the following.

Theorem 22 *Assume that **(H1)**–**(H4)** hold. Then, for each $\phi \in C([-r, 0]; X)$, the nonlocal problem (1) has at least one mild solution on $[-r, b]$ provided that there exists a constant $\mathrm{R}_0 > 0$ satisfying*

$$M_b \left(\|\phi\| + c\mathrm{R}_0 + d + \Omega(\mathrm{R}_0)\|m\|_{L^1}\right) \leq \mathrm{R}_0. \tag{17}$$

Proof For each $x \in C([-r, b]; X)$ the restriction of x on $[0, b]$, $x|_{[0,b]} \in C([0, b]; X)$. For simplicity, we write $g(x|_{[0,b]})$ as $g(x)$. Our goal is to show that the operator solution $\mathbb{K} : C([-r, b]; X) \to C([-r, b]; X)$ defined by the following

$$(\mathbb{K}x)(t) = \begin{cases} \underbrace{\phi(t) + g(x)(t)}_{=(K_1x)(t)} & \text{for } t \in [-r, 0], \\ \underbrace{R(t)(\phi(0) + g(x)(0)) + \int_0^t R(t-s) f(s, x_s)\mathrm{d}s}_{=(K_2x)(t)} & \text{for } t \in [0, b] \end{cases}$$

has a fixed point in the closed ball $\overline{B}_{\mathrm{R}_0} = \{x \in C([-r, b]; X) : \|x\| \leq \mathrm{R}_0\}$ where R_0 is the constant appearing in the inequality (17). In order to apply Theorem 21, we divide the proof into three steps.

Step 1 We claim that the operator solution $\mathbb{K}$ maps $\overline{B}_{\mathrm{R}_0}$ into itself. For every $x \in \overline{B}_{\mathrm{R}_0}$, we have for $t \in [-r, 0]$

$$\begin{aligned} \|(\mathbb{K}x)(t)\| &\leq \|\phi(t)\| + \|g(x)(t)\| \\ &\leq \|\phi\| + \|g(x)\| \\ &\leq \|\phi\| + c\|x\| + d \\ &\leq M_b\left(\|\phi\| + c\mathrm{R}_0 + d\right). \end{aligned} \tag{18}$$

For $t \in [0, b]$

$$\begin{aligned} \|(\mathbb{K}x)(t)\| &\leq \|R(t)(\phi(0) + g(x)(0))\| + \left\| \int_0^t R(t-s) f(s, x_s) \right\| \\ &\leq M_b(\|\phi\| + \|g(x)\|) + M_b \int_0^b \Omega(\|x_s\|)m(s)\mathrm{d}s \\ &\leq M_b(\|\phi\| + c\|x\| + d) + M_b\Omega(\|x\|)\|m\|_{L^1} \\ &\leq M_b\left(\|\phi\| + c\mathrm{R}_0 + d + \Omega(\mathrm{R}_0)\|m\|_{L^1}\right). \end{aligned} \tag{19}$$

It follows from the inequalities (17)–(19) that $\mathbb{K}\overline{B}_{\mathrm{R}_0} \subseteq \overline{B}_{\mathrm{R}_0}$.

Step 2 We claim that the operator $\mathbb{K}$ is continuous on $\overline{B}_{\mathrm{R}_0}$. Let $(x^n)_{n\geq 1}$ be a sequence in $\overline{B}_{\mathrm{R}_0}$ such that

$$\lim_{n\to+\infty} \|x^n - x\| = 0. \tag{20}$$

We have for each $s \in [0, b]$

$$
\begin{aligned}
\|x_s^n - x_s\| &= \sup_{\theta \in [-r,0]} \|x_s^n(\theta) - x_s(\theta)\| \\
&= \sup_{\theta \in [-r,0]} \|x^n(\theta + s) - x(\theta + s)\| \\
&\le \sup_{\theta' \in [-r,b]} \|x^n(\theta') - x(\theta')\| \\
&= \|x^n - x\|.
\end{aligned}
$$

Using (20), we get that

$$\lim_{n \to +\infty} \|x_s^n - x_s\| = 0 \ \text{ for } s \in [0, b].$$

Hypothesis (**H3**) (i) implies that

$$\lim_{n \to +\infty} \|f(s, x_s^n) - f(s, x_s)\| = 0 \ \text{ for } s \in [0, b].$$

Due to (**H3**) (ii) we get that

$$\|f(s, x_s^n) - f(s, x_s)\| \le 2\Omega(\mathrm{R}_0)m(s) \ \text{ for } s \in [0, b].$$

In view of the Lebesgue dominated convergence theorem we have that

$$\lim_{n \to +\infty} \int_0^t \|f(s, x_s^n) - f(s, x_s)\| \mathrm{d}s = 0 \ \text{ for } s \in [0, b]. \tag{21}$$

For $t \in [0, b]$ we have

$$\|(\mathbb{K}x^n)(t) - (\mathbb{K}x)(t)\| \le M_b \|g(x^n) - g(x)\|_{C([-r,0];X)} + M_b \int_0^b \|f(s, x_s^n) - f(s, x_s)\|_X \mathrm{d}s.$$

Taking into account (**H4**) and (21) we get that

$$\lim_{n \to +\infty} \|\mathbb{K}x^n - \mathbb{K}x\|_{C([0,b];X)} = 0.$$

On the other hand

$$\lim_{n \to +\infty} \|\mathbb{K}x^n - \mathbb{K}x\|_{C([-r,0];X)} = \lim_{n \to +\infty} \|g(x^n) - g(x)\|_{C([-r,0];X)} = 0.$$

The proof of step 2 is now complete.

Step 3 We claim that the solution operator $\mathbb{K}$ satisfies Mönch's condition.

Suppose $A \subset B_{\mathrm{R}_0}$ is countable and $A \subset \overline{\mathrm{co}}(\{0\} \cup \mathbb{K}(A))$. We need to show that A is relatively compact by using the measure of noncompactness ν defined in Example 10 where the constant $L > 0$ is chosen such that

$$2M_b \sup_{t \in [0,b]} \int_0^t e^{-L(t-s)} \eta(s) \mathrm{d}s < 1. \tag{22}$$

Since

$$\nu(\mathbb{K}(A)) = \max_{D \in \Delta(\mathbb{K}(A))} \Big(\gamma(D), \mathrm{mod}_C(D) \Big), \tag{23}$$

there exists a countable set $\{y_n\}_{n=1}^{+\infty}$ such that $\{y_n\}_{n=1}^{+\infty} \subset \mathbb{K}(A)$ and $\{y_n\}_{n=1}^{+\infty}$ achieves the maximum $\nu(\mathbb{K}(A))$, namely,

$$\nu(\mathbb{K}(A)) = \Big(\gamma(\{y_n\}_{n=1}^{+\infty}), \mathrm{mod}_C(\{y_n\}_{n=1}^{+\infty}) \Big). \tag{24}$$

So, there exists a set $\{x_n\}_{n=1}^{+\infty} \subset A$ such that

$$y_n(t) = (\mathbb{K}x_n)(t) \quad \text{for } n \geq 1 \text{ and } t \in [-r, b]. \tag{25}$$

According to (**H3**) (iii) and (5) we have for $s \in [0, b]$

$$\begin{aligned}
\chi\left(\{f(s, x_s^n)\}_{n=1}^{+\infty}\right) &\leq \eta(s) \sup_{-r\leq\theta\leq 0} \chi\left(\{x_s^n(\theta)\}_{n=1}^{+\infty}\right) \\
&\leq \eta(s) \sup_{-r\leq\theta\leq 0} \chi\left(\{x^n(s+\theta)\}_{n=1}^{+\infty}\right) \\
&\leq \eta(s) \sup_{-r\leq\tau\leq s} \chi\left(\{x^n(\tau)\}_{n=1}^{+\infty}\right) \\
&\leq \eta(s) \sup_{-r\leq\tau\leq s} e^{L\tau}\gamma\left(\{x^n\}_{n=1}^{+\infty}\right) \\
&\leq \eta(s)e^{Ls}\gamma\left(\{x^n\}_{n=1}^{+\infty}\right).
\end{aligned} \tag{26}$$

On the other hand, using (**H3**) (ii) and the fact that $\{x^n\}_{n=1}^{+\infty} \subset A \subset B_{\mathrm{R}_0}$, we get for all $n \geq 1$

$$\|f(s, x_s^n)\| \leq \Omega(\mathrm{R}_0)m(s) \quad \text{for } s \in [0, b]. \tag{27}$$

Thanks to (**H3**) (ii), (**H3**) (iii), (26), (27) and Lemma 15, all the assumptions of Lemma 16 hold. Consequently we find that

$$\chi\left(\{(\Phi f(\bullet, x_\bullet^n)(t)\}_{n=1}^{+\infty}\right) \leq 2M_b\gamma\left(\{x^n\}_{n=1}^{+\infty}\right)\int_0^t e^{Ls}\eta(s)\mathrm{d}s \tag{28}$$

for $t \in [0, b]$. Since (**H4**) holds, by the strong continuity of the resolvent operator $(R(t))_{t\geq 0}$ and the Arzelà–Ascoli Theorem, we have that the sets

$$\{(K_1x^n)(t) : n \geq 1\} = \{\phi(t) + g(x^n)(t) : n \geq 1\} \quad \text{for } t \in [-r, 0],$$

and

$$\{\phi(0) + g(x^n)(0) : n \geq 1\}$$

are relatively compact in X. Consequently

$$\chi\left(\{(K_1x^n)(t)\}_{n=1}^{+\infty}\right) = 0 \quad \text{for } t \in [-r, 0], \tag{29}$$

and

$$\chi\left(\left\{R(t)\big(\phi(0) + g(x^n)(0)\big)\right\}_{n=1}^{+\infty}\right) = 0 \quad \text{for } t \in [0, b]. \tag{30}$$

Taking into account (25), (28)–(30) we get that

$$\begin{aligned}\gamma\left(\{y_n\}_{n=1}^{+\infty}\right) &= \sup_{t\in[-r,b]} e^{-Lt}\chi\left(\{y_n(t)\}_{n=1}^{+\infty}\right)\\ &= \sup_{t\in[0,b]} e^{-Lt}\chi\left(\{(K_2x^n)(t)\}_{n=1}^{+\infty}\right)\\ &\le \sup_{t\in[0,b]} e^{-Lt}\chi\left(\{R(t)(\phi(0)+g(\{x^n\}_{n=1}^{+\infty})(0)\big)+(\Phi f(\cdot,x^n_\cdot)(t))\}_{n=1}^{+\infty}\right)\\ &\le \sup_{t\in[0,b]} e^{-Lt}2M_b\gamma\left(\{x^n\}_{n=1}^{+\infty}\right)\int_0^t e^{Ls}\eta(s)\mathrm{d}s\\ &\le \gamma\left(\{x^n\}_{n=1}^{+\infty}\right)2M_b\sup_{t\in[0,b]}\int_0^t e^{-L(t-s)}\eta(s)\mathrm{d}s.\end{aligned}$$

On the other hand

$$\begin{aligned}\gamma\left(\{x^n\}_{n=1}^{+\infty}\right) \le \gamma(A) \le \gamma\left(\overline{\mathrm{co}}\big(\{0\}\cup\mathbb{K}(A)\big)\right) \le \gamma\left(\{y_n\}_{n=1}^{+\infty}\right)\\ \le \gamma\left(\{x^n\}_{n=1}^{+\infty}\right)2M_b\sup_{t\in[0,b]}\int_0^t e^{-L(t-s)}\eta(s)\mathrm{d}s.\end{aligned}$$

The inequality (22) gives

$$\gamma\left(\{x^n\}_{n=1}^{+\infty}\right)=0. \tag{31}$$

Combining (26) and (31) we get the following

$$\chi\left(\{f(s,x^n_s)\}_{n=1}^{+\infty}\right)=0 \quad \text{for all } s\in[0,b]. \tag{32}$$

This gives that the set $\{f(\bullet,x^n_\bullet)\}_{n=1}^{+\infty}\subset L^1([0,b];X)$ is semicompact. In view of Lemma 18, the set $\Phi(\{f(\bullet,x^n_\bullet)\}_{n=1}^{+\infty})$ is relatively compact in $C([0,b];X)$ where Φ is the operator defined by (6).

From (**H4**), the set $\{K_1x^n : n\ge 1\}$ is relatively compact in $C([-r,0];X)$. Since $\{\phi(0)+g(x^n)(0) : n\ge 1\}$ is relatively compact in X, the family of functions $\{R(\cdot)(\phi(0)+g(x^n)(0)) : n\ge 1\}$ can be shown to be equicontinuous by using the strong continuity of the resolvent operator $(R(t))_{t\ge 0}$. Using (30) together with the Arzelà–Ascoli Theorem, we obtain that the set $\{R(\bullet)(\phi(0)+g(x^n)(0)) : n\ge 1\}$ is relatively compact in $C([0,b];X)$. Finally we get that $\{K_2x^n : n\ge 1\}$ is relatively compact in $C([0,b];X)$ because $(K_2x^n)(t)=R(t)(\phi(0)+g(x^n)(0))+\Phi f(\bullet,x^n_\bullet)(t)$ for $t\in[0,b]$.

Now, let us prove that $\{y_n\}_{n=1}^{+\infty}=\{\mathbb{K}x^n\}_{n=1}^{+\infty}$ is relatively compact in $C([-r,b];X)$. To do this, we use the Arzelà–Ascoli Theorem. That is, we need to prove that for each $t\in[-r,b]$, the set $\big\{(\mathbb{K}x^n)(t), n\ge 1\big\}$ is relatively compact in X and the set $\{\mathbb{K}x^n\}_{n=1}^{+\infty}$ is equicontinuous on $[-r,b]$. The relative compactness follows by the fact that for $t\in[-r,b]$ we have

$$\big\{(\mathbb{K}x^n)(t), n\ge 1\big\} = \begin{cases}\big\{(K_1x^n)(t), n\ge 1\big\} & \text{for } t\in[-r,0],\\ \big\{(K_2x^n)(t), n\ge 1\big\} & \text{for } t\in[0,b].\end{cases}$$

For the equicontinuity, it suffices to use the equicontinuity of $\{K_1x^n\}_{n=1}^{+\infty}$ and $\{K_2x^n\}_{n=1}^{+\infty}$ and the fact that $(\mathbb{K}x^n)(0)=(K_1x^n)(0)=(K_2x^n)(0)$.

It follows that $\gamma(\{y_n\}_{n=1}^{+\infty})=0$ and $\mathrm{mod}_C(\{y_n\}_{n=1}^{+\infty})=0$.

From the monotonicity and the nonsingularity of the measure of noncompactness ν and the condition $A \subset \overline{\text{co}}(\{0\} \cup \mathbb{K}(A))$ we have

$$\nu(A) \leq \nu\Big(\overline{\text{co}}(\{0\} \cup \mathbb{K}(A))\Big) \leq \nu(\mathbb{K}(A)) = \Big(\gamma(\{y_n\}_{n=1}^{+\infty}), \text{mod}_C(\{y_n\}_{n=1}^{+\infty})\Big) = (0, 0).$$

Since ν is regular, A is relatively compact. In view of the Theorem 21, the operator $\mathbb{K}$ has at least one fixed point $x \in \overline{B}_{\mathrm{R}_0}$, which is a mild solution of the nonlocal problem (1). □

Remark 23 Theorem 22 can be also proved by using the measure of noncompactness defined in Example 10 together with the condensing operators. For more details, see the proof of Theorem 3 in [11] when the authors studied the existence of mild solutions of $x'(t) \in A(t)x(t) + F(t, x(t))$ for $t \in [0, b]$ and $x(0) = x_0$.

Now, we give another existence theorem without condition (16). It is a generalization of Theorem 3.5 given in [14].

Theorem 24 *Assume that **(H1)**–**(H4)** hold except for condition* (16)*. Then, for each* $\phi \in C([-r, 0]; X)$*, the nonlocal problem* (1) *has at least one mild solution on* $[-r, b]$ *provided that*

$$\limsup_{k\to\infty} \frac{M_b}{k}\Big(\|\phi\| + g_k + \Omega(k)\|m\|_{L^1}\Big) < 1, \tag{33}$$

where $g_k = \sup\{\|g(\varphi)\| : \|\varphi\| \leq k\}$.

Proof We should only find a closed convex subset $W \subset C([-r, b]; X)$ such that $\mathbb{K}$ maps W into itself and complete the proof similarly to Theorem 22. From the inequality (33), there exists a constant $k > 0$ such that

$$\|\phi\| + g_k < k$$

and

$$M_b\Big(\|\phi\| + g_k + \Omega(k)\|m\|_{L^1}\Big) < k.$$

Let $W = \{x \in C([-r, b]; X) : \|x\|\} \leq k\}$. For every $x \in W$, we have

$$\begin{aligned}\|(\mathbb{K}x)(t)\| &\leq \|\phi(t)\| + \|g(x)(t)\| \\ &\leq \|\phi\| + \|g(x)\| \\ &\leq \|\phi\| + g_k \\ &\leq k\end{aligned}$$

for $t \in [-r, 0]$, and

$$\begin{aligned}\|(\mathbb{K}x)(t)\| &\leq \|R(t)(\phi(0) + g(x)(0))\| + \left\|\int_0^t R(t-s)f(s, x_s)\right\| \\ &\leq M_b(\|\phi\| + \|g(x)\|) + M_b\int_0^b \Omega(\|x_s\|)m(s)\mathrm{d}s \\ &\leq M_b\left(\|\phi\| + g_k + \Omega(k)\|m\|_{L^1}\right) \\ &\leq k\end{aligned}$$

for $t \in [0, b]$. We conclude that $\mathbb{K}$ maps W into itself. □

Application

In this section, we apply the results obtained in the previous section to the following partial integro-differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,y) = \dfrac{\partial}{\partial y}u(t,y) + \displaystyle\int_0^t \zeta(t-s)\dfrac{\partial}{\partial y}u(s,y)\mathrm{d}s + \kappa(t)h(u(t-1,y)) \\ \qquad\qquad \text{for } t \in [0,1],\ \ y \in [0,1], \\ u(t,1) = 0 \quad \text{for } t \in [0,1], \\ u(\theta,y) = \cos\left(\dfrac{\pi}{2}y\right) + \lambda \displaystyle\int_{-1}^{\theta} (1-y)\sin(u(0,-\eta))\mathrm{d}\eta \quad \text{for } \theta \in [-1,0] \text{ and } y \in [0,1], \end{cases} \tag{34}$$

where λ is a positive constant and ζ, κ and h are functions satisfying the following assumptions:

(A1) $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded and C^1 function such that ζ' is bounded and uniformly continuous on $\mathbb{R}^+$.
(A2) $\kappa : [0,1] \to \mathbb{R}$ is integrable.
(A3) $h : \mathbb{R} \to \mathbb{R}$ is Lipschitzian with a Lipschitz constant L_h and $h(0) = 0$.

To rewrite Eq. (34) in the abstract form, we introduce the Banach space $X = \{f \in C([0,1];\mathbb{R}) : f(1) = 0\}$ of continuous functions from $[0,1]$ to $\mathbb{R}$ vanishing at 1, equipped with the supremum norm.
We define the operator $A : D(A) \subset X \to X$ by

$$\begin{cases} D(A) = \{f \in C^1([0,1];\mathbb{R}) : f'(1) = f(1) = 0\} \\ Af = f'. \end{cases}$$

It is well known from [29, page 44] that A is the generator of the so-called C_0- semigroup of left translations $(T(t))_{t\geq 0}$ on X, given by

$$\begin{cases} (T(t)f)(s) = f(t+s) & \text{for } t+s \leq 1 \\ (T(t)f)(s) = 0 & \text{for } t+s > 1, \end{cases}$$

which implies that (**H1**) is satisfied. Furthermore, from [15, page 120], the C_0-semigroup $(T(t))_{t\geq 0}$ is not equicontinuous.
Let $B : D(A) \subset X \to X$ be the operator defined by

$$B(t)f = \zeta(t)Af \quad \text{for } t \in [0,1] \text{ and } f \in D(A).$$

Then (**H2**) follows from (A1). Define the functions $f : [0,1] \times C([-1,0];X) \longrightarrow X$, $g : C([0,1];X) \longrightarrow C([-1,0];X)$ and ϕ by

$$f(t,\varphi)(y) = \kappa(t)h(\varphi(-1)(y)) \quad \text{for } t \in [0,1] \text{ and } y \in [0,1], \tag{35}$$

$$g(\psi)(\theta)(y) = \lambda \int_{-1}^{\theta} (1-y)\sin(\psi(0)(-\eta))\mathrm{d}\eta \quad \text{for } \theta \in [-1,0] \text{ and } y \in [0,1],$$

$$\phi(\theta,y) = \cos\left(\frac{\pi}{2}y\right) \quad \text{for } \theta \in [-1,0] \text{ and } y \in [0,1]. \tag{36}$$

If we take $x(\cdot)(y) = u(\cdot,y)$, then Equation (34) takes the following abstract form

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)\mathrm{d}s + f(t,x_t) \quad \text{for } \ t \in [0,1] \\ x_0 = \phi + g(x) \in C([-1,0];X). \end{cases} \tag{37}$$

In view of Theorem 2, Eq. (34) has a unique resolvent operator $(R(t))_{t\geq 0}$. According to Theorem 6, the resolvent operator $(R(t))_{t\geq 0}$ is *not equicontinuous*.
The function f is well defined. In fact, let $t \in [0, 1]$ and $\varphi \in C([-1, 0]; X)$. If $y_n \to y$ in $[0, 1]$ then we have that $\varphi(-1)(y_n) \to \varphi(-1)(y)$. From (A3), it follows that $h(\varphi(-1)(y_n)) \to h(\varphi(-1)(y))$. Consequently

$$f(t, \varphi)(y_n) \to f(t, \varphi)(y).$$

Using (A3) ($h(0) = 0$) we get $f(t, \varphi)(1) = \kappa(t)h(\varphi(-1)(1)) = \kappa(t)h(0) = 0$. We conclude that $f(t, \varphi) \in X$ which means that f is well defined.
Thanks to (A2) and (A3), we can also prove that the function f satisfies (**H3**) (i).
Assumption (A3) implies for $t \in [0, 1]$ and $\varphi \in C([-1, 0], X)$ that

$$\begin{aligned}\|f(t,\varphi)\| = \sup_{y\in[0,1]} |f(t,\varphi)(y)| &\leq |\kappa(t)| \sup_{y\in[0,1]} |h(\varphi(-1)(y))| \\ &= |\kappa(t)| \sup_{y\in[0,1]} |h(\varphi(-1)(y)) - h(0)| \\ &\leq |\kappa(t)| L_h \sup_{y\in[0,1]} |\varphi(-1)(y)| \\ &= |\kappa(t)| L_h \|\varphi(-1)\| \\ &\leq |\kappa(t)| L_h \|\varphi\|.\end{aligned}$$

This means that (**H3**) (ii) holds with $\Omega(z) = L_h z$ for $z \in \mathbb{R}^+$ and $m(t) = |\kappa(t)|$ for $t \in [0, 1]$.
On the other hand, due to (A3), we get

$$\|f(t,\varphi) - f(t,\psi)\| \leq L_h|\kappa(t)|\|\varphi(-1) - \psi(-1)\|. \tag{38}$$

Lemma 25 *The function f satisfies Hypothesis (**H3**) (iii) with $\eta(\cdot) = L_h|\kappa(\cdot)|$.*

Proof Let D be a bounded subset of $C([-1, 0], X)$ and $t \in [0, 1]$. Then $D(-1)$ is a bounded subset of X. Let us pose $\lambda = \chi(D(-1))$ and fix some $\varepsilon > 0$. Then there exists $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$D(-1) \subset \bigcup_{i=1}^{n} B(x_i, \lambda + \varepsilon). \tag{39}$$

We can find $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subset C([-1, 0]; X)$ such that $\varphi_i(-1) = x_i$ for each $i \in \{1, 2, \ldots, n\}$. Put $z_i = f(t, \varphi_i)$ for $i \in \{1, 2, \ldots, n\}$. Let $z \in f(t, D)$. Then there exists $\varphi \in D$ such that $z = f(t, \varphi)$. Since $\varphi(-1) \in D(-1)$, by (39) there exists $i_0 \in \{1, \ldots, n\}$ such that

$$\|\varphi(-1) - x_{i_0}\| < \lambda + \varepsilon. \tag{40}$$

By virtue of (38) and (40), it follows that

$$\begin{aligned}\|z - z_{i_0}\| = \|f(t,\varphi) - f(t,\varphi_{i_0})\| \leq L_h|\kappa(t)|\|\varphi(-1) - \varphi_{i_0}(-1)\| = L_h|\kappa(t)|\|\varphi(-1) \\ -x_{i_0}\| < L_h|\kappa(t)|[\lambda + \varepsilon].\end{aligned}$$

We have proved that for each $z \in f(t, D)$ there exists $i_0 \in \{1, \ldots, n\}$ such that $z \in B(z_{i_0}, L_h|\kappa(t)|[\lambda + \varepsilon])$. Hence we conclude that

$$f(t, D) \subset \bigcup_{i=1}^{n} B\left(z_i, L_h|\kappa(t)|[\lambda + \varepsilon]\right). \tag{41}$$

On the basis of the definition of the Hausdorff measure of noncompactness we get

$$\chi(f(t, D)) < L_h|\kappa(t)|[\chi(D(-1)) + \varepsilon].$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\chi(f(t, D)) \leq L_h|\kappa(t)|\chi(D(-1)).$$

Afterwards

$$\chi(f(t, D)) \leq L_h|\kappa(t)| \sup_{\theta\in[-1,0]} \chi(D(\theta)). \tag{42}$$

This means that (**H3**) (iii) is satisfied with $\eta(\cdot) = L_h|\kappa(\cdot)|$ and the proof of Lemma 25 is now complete. □

Lemma 26 *The function g defined by* (36) *satisfies Hypothesis (**H4**).*

Proof For the sake of convenience, we divide the proof into several steps.

Step 1. We show that the function g is well defined. Let $u \in C([0, 1], X)$, $\theta, \theta' \in [-1, 0]$ with $\theta' \leq \theta$ and $y, y' \in [0, 1]$.
We have

$$\begin{aligned} |g(u)(\theta)(y) - g(u)(\theta)(y')| &\leq \lambda \left(\int_{-1}^{\theta} |\sin(u(0)(-\eta))|\mathrm{d}\eta\right)|y - y'| \leq \lambda(\theta + 1)|y - y'| \\ &\leq \lambda|y - y'|. \end{aligned} \tag{43}$$

Clearly, $g(u)(\theta)(1) = 0$ which together with the inequality (43) lead to $g(u)(\theta) \in X$. On the other hand

$$\begin{aligned} |g(u)(\theta)(y) - g(u)(\theta')(y)| &\leq \lambda(1 - y) \int_{\theta'}^{\theta} |\sin(u(0)(-\eta))|\mathrm{d}\eta \\ &\leq \lambda|\theta - \theta'|. \end{aligned}$$

Hence

$$\|g(u)(\theta) - g(u)(\theta')\| \leq \lambda|\theta - \theta'|. \tag{44}$$

We conclude that $g(u) \in C([-1, 0]; X)$.

Step 2. We prove that the function g is continuous. Let $u, u' \in C([0, 1]; X)$.
For each $\theta \in [-1, 0]$ and $y \in [0, 1]$ we have

$$\begin{aligned} |g(u)(\theta)(y) - g(u')(\theta)(y)| &\leq \lambda(1 - y) \int_{-1}^{\theta} |\sin(u(0)(-\eta)) - \sin(u'(0)(-\eta))|\mathrm{d}\eta \\ &\leq \lambda(1 - y) \int_{-1}^{\theta} |u(0)(-\eta) - u'(0)(-\eta)|\mathrm{d}\eta \\ &\leq \lambda(1 - y)(\theta + 1)\|u - u'\| \\ &\leq \lambda\|u - u'\|, \end{aligned} \tag{45}$$

which yields that

$$\|g(u) - g(u')\| \leq \lambda\|u - u'\|. \tag{46}$$

We conclude that the function g is continuous on $C([0, 1]; X)$.

Step 3. We prove that the function g satisfies inequality (16).
Observing that $g(0) = 0$, it follows from inequality (46) that $\|g(u)\| \leq \lambda\|u\|$ for each $u \in C([0, 1]; X)$. Hence (16) holds with $c = \lambda$ and $d = 0$.

Step 4. We show that the function g is compact.
Let $B \subset C([0, 1]; X)$ be a bounded subset of diameter $\delta(B)$. We have to prove that $g(B)$ is relatively compact in $C([-1, 0]; X)$. To do this, we must prove that:

(i) The family $g(B)$ is equicontinuous on $[-1, 0]$.
(ii) The family $g(B)(\theta)$ is relatively compact in X for each $\theta \in [-1, 0]$.

Assertion (i) follows from (44). Next, we show that (ii) holds. Firstly, the equicontinuity of the family $g(B)(\theta)$ on $[0, 1]$ follows from (43). Secondly, using (45) we deduce that $\delta(g(B)(\theta)(y)) \leq \lambda\delta(B)$ which means that the family $g(B)(\theta)(y)$ is bounded. The proof of Lemma 26 is now complete. □

Now, the inequality (17) of Theorem 22 is equivalent to the existence of $R_0 > 0$ such that

$$R_0\left(1 - M_1(\lambda + L_h\|\kappa\|_{L^1})\right) \geq M_1\|\phi\| = M_1, \tag{47}$$

where $M_1 = \sup_{t\in[0,1]} \|R(t)\|$. This is equivalent to say that

$$M_1(\lambda + L_h\|\kappa\|_{L^1}) < 1. \tag{48}$$

At this point, if we suppose that (48) holds, then all the assumptions of Theorem 22 are fulfilled. Thus, we have the following result.

Proposition 27 *Under the assumptions* (A1)–(A3)*, the nonlocal problem* (34) *has at least one mild solution on* $[-1, 1]$ *provided that* (48) *holds.*

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