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ORIGINAL RESEARCH

On the Number of Limit Cycles Bifurcated from Some Non-Polynomial Hamiltonian Systems

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Abstract
This paper studies the limit cycles produced by small perturbations of certain planar Hamiltonian systems. The limit cycles under consideration correspond to critical levels of the Hamiltonian, that is they are located in a small vicinity of a separatrix contour or a critical point. Two most interesting facts in the paper are that the Hamiltonian function is not a polynomial and that the system under consideration comes from a model of oscillator with a pair of irrational nonlinearities, which implies the transition from smooth to discontinuous dynamics. This model has been proposed recently by Han et al. in a paper published in 2012.

Keywords Limit cycle · Non-polynomial · Hamiltonian system · Melnikov function · Asymptotic expansion

Introduction

One of the old problems in the theory of dynamical systems is to find an upper bound for the number of limit cycles in polynomial vector fields defined in the plane, and investigate their relative positions. This problem is as know Hilbert's 16th problem. More precisely, consider a near-Hamiltonian system

$$\dot{x} = H_y + \varepsilon p(x, y, \delta), \quad \dot{y} = -H_x + \varepsilon q(x, y, \delta), \tag{1}$$

where p, q and H are real analytic functions, ε is a small positive parameter and $\delta \in D$ is a vector parameter that D is a compact subset of $\mathbb{R}^N$. Assume that the unperturbed system

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \tag{2}$$

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has a continuous family of ovals L_h defined by $H(x, y) = h$ for $h \in (h_1, h_2)$. Then, associated to perturbed system (1), we define an Abelian integral of the form

$$M(h, \delta) = \oint_{L_h} q dx - p dy.$$

By Poincaré–Pontryagin Theorem [11], the number of isolated zeros of $M(h, \delta)$, counted with multiplicity, gives an upper bound for the number of limit cycles of (1). Hence, Abelian integral plays an important role in the study of bifurcation of limit cycles from system (1). The study of the asymptotic expansion of $M(h, \delta)$ near critical values of H, in order to study the isolated zeros of Abelian integrals, is a valuable problem. There have been many studies on the limit cycle bifurcations studying the asymptotic expansion of $M(h, \delta)$ when H being a polynomial e.g. [4,8–10] and the references contained in those papers. But when the Hamiltonian function H is not a polynomial, there are very few results on this area. For instance, the authors of [3] studied the number of limit cycles for perturbed pendulum-like equations on the cylinder, in which the associated Hamiltonian is given by $H(x, y) = \frac{y^2}{2} + 1 - \cos(x)$. An excellent work is done by Villadelprat et al. in [2] based on a "computer assisted proof" using interval arithmetic. Also, the authors of [7] considered a non-polynomial potential system that the associated Hamiltonian is given by $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}(e^{-2x} + 1) - e^{-x}$. By Chebyshev criterion, they showed that the cyclicity of the period annulus of this system under the small perturbation is at most two.

Han et al. [6] proposed a novel nonlinear oscillator with strong irrational nonlinearities having smooth and discontinuous characteristics depending on the values of a smooth parameter. In fact, they considered

$$\ddot{x} + (x + \alpha)\left(1 - \frac{1}{\sqrt{(x+\alpha)^2 + \beta^2}}\right) + (x - \alpha)\left(1 - \frac{1}{\sqrt{(x-\alpha)^2 + \beta^2}}\right) = 0, \tag{3}$$

where $\alpha, \beta > 0$ are real numbers. By letting $\dot{x} = y$, Eq. (3) can be written in the following form:

$$\dot{x} = y, \quad \dot{y} = -F(x, \alpha, \beta), \tag{4}$$

where

$$F(x, \alpha, \beta) = (x + \alpha)\left(1 - \frac{1}{\sqrt{(x+\alpha)^2 + \beta^2}}\right) + (x - \alpha)\left(1 - \frac{1}{\sqrt{(x-\alpha)^2 + \beta^2}}\right).$$

System (4) is a Hamiltonian system with the Hamiltonian function

$$H(x, y) = \frac{1}{2}y^2 + x^2 - \sqrt{(x+\alpha)^2 + \beta^2} - \sqrt{(x-\alpha)^2 + \beta^2} + 2\sqrt{\alpha^2 + \beta^2} = \frac{1}{2}y^2 + H_0(x). \tag{5}$$

We see that although the above Hamiltonian is not a polynomial, its level curves are anyway branches of an algebraic curve of degree 8. More explicitly, it is

$$\left(\frac{1}{2}y^2 + x^2 - h\right)^4 - 4(x^2 + \alpha^2 + \beta^2)\left(\frac{1}{2}y^2 + x^2 - h\right)^2 + 16\alpha^2 x^2 = 0, \quad \alpha, \beta > 0.$$

The phase portraits of system (4) are shown in Fig. 1, where $C := (\frac{4\sqrt{5}}{25}, \frac{8\sqrt{5}}{25})$ and

$$\lambda_1 = \left\{(\alpha, \beta) \mid \exists (x, \alpha),\ \beta > \frac{8\sqrt{5}}{25},\ s.t.\ F = F_x = 0\right\},$$

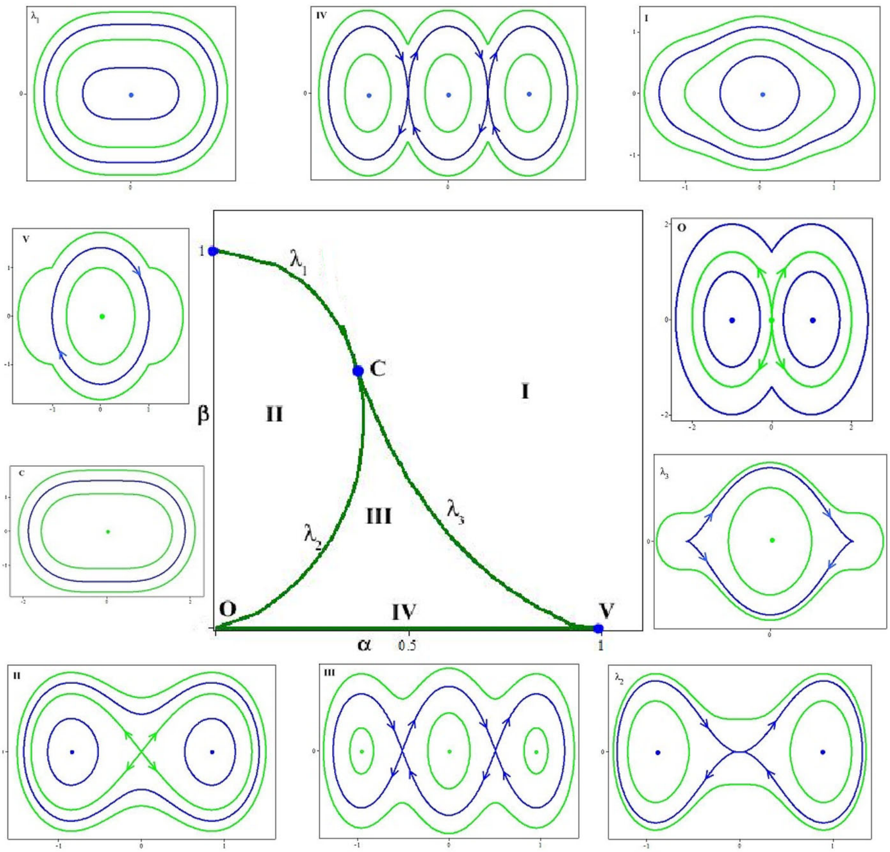

Fig. 1 Bifurcation diagram and phase portraits of system (4)

$$\lambda_2 = \left\{(\alpha, \beta) \mid \exists (x, \alpha),\ \beta < \frac{8\sqrt{5}}{25},\ s.t.\ F = F_x = 0\right\},$$

$$\lambda_3 = \{(\alpha, \beta) \mid \exists (x_i, \alpha),\ s.t.\ F = F_x = 0,\ i = 1, 2\}.$$

Since $H_0(x)$ in (5) is an even function and $H_0(x) \sim A\,x^2 + B\,x^4 + \cdots$ near zero, where $A = 1 - \frac{\beta^2}{(\alpha^2+\beta^2)^{\frac{3}{2}}}$ and $B = \frac{\beta^2(\beta^2-4\alpha^2)}{4(\alpha^2+\beta^2)^{\frac{7}{2}}}$, then a double homoclinic loop through a triple critical point exists if and only if $A = 0,\ B < 0$. An easy calculation yields the conditions $\alpha^2 + \beta^2 - \beta^{\frac{4}{3}} = 0$, $\alpha^2 < \alpha^2 + \beta^2 < 5\alpha^2$. Therefore, λ_2 is the simple curve

$$\alpha = \beta^{\frac{2}{3}}\sqrt{1 - \beta^{\frac{2}{3}}}, \qquad \beta \in \left(0, \frac{8}{5\sqrt{5}}\right).$$

Along the curve λ_2, the phase portrait of system (4) is shown in Fig. 2.

The explicit expressions for the algebraic curves λ_1 and λ_3 are the following:

$$\lambda_1 : \alpha^6 + 3\alpha^4\beta^2 + 3\alpha^2\beta^4 + \beta^6 - \beta^4 = 0,$$
$$\lambda_3 : 256\alpha^8 + 768\alpha^6\beta^2 + 768\alpha^4\beta^4$$

$$+256\alpha^2\beta^6 - 768\alpha^6 + 2784\alpha^4\beta^2 + 96\alpha^2\beta^4 + 768\alpha^4 - 96\alpha^2\beta^2$$
$$- 27\beta^4 - 256\alpha^2 = 0.$$

In this paper, we take a codimension one case from the bifurcation diagram of the model, which corresponds to double cuspidal loop in the phase portrait. In fact, we will focus on the case $(\alpha, \beta) \in \lambda_2$. Our aim is to study the limit cycles generated by small perturbations of the non-polynomial planar Hamiltonian system (4) when $(\alpha, \beta) \in \lambda_2$. The limit cycles under consideration correspond to critical levels of the Hamiltonian, that is they are located in a small vicinity of a separatrix contour or a critical point.

The core of the present paper consists of extensive asymptotic calculations of the related line integrals which appear in the first-order approximation of the displacement map near the critical levels of the Hamiltonian. Most of the formulas are generated by computer manipulation programs such as Maple. We follow the ideas and use formulas from the paper [5] by Han Maoan et al. published in 2012, too. In Sect. 2, we perturb system (4) with $(\alpha, \beta) \in \lambda_2$, and then, we study the generated limit cycles by using the asymptotic expansions of the associated Melnikov functions. The formulation of the main result of the paper is given in Theorem 2.6 (see Sect. 2.4).
We illustrate our results on the example when $\alpha = \beta$; see Sect. 3.

Study of System (4) Under Small Perturbations

In this section, we consider the following perturbed system

$$\begin{aligned} \dot{x} &= y + \varepsilon\, p(x, y, \delta), \\ \dot{y} &= -F(x, \alpha, \beta) + \varepsilon\, q(x, y, \delta), \end{aligned} \tag{6}$$

where p, q are C^ω functions, ε is a small parameter and $\delta \in D \subset \mathbb{R}^m$ with D a compact set. Our system is a perturbation of the Hamiltonian system (4) with the Hamiltonian function (5).

System (4) with $\alpha = \beta^{\frac{2}{3}}\sqrt{1-\beta^{\frac{2}{3}}}$, $\beta \in (0, \frac{8}{5\sqrt{5}})$, has a nilpotent saddle at $A(0,0)$, two centers at $C_1(x^*, 0)$ and $C_2(-x^*, 0)$ in which the value of x^* is implicitly obtained from the equation $F(x, \alpha, \beta) = 0$ and a double homoclinic loop L_0 passing through the nilpotent saddle A. Also, system (4) has three families L_1, L_2 and L_3 of periodic orbits near $L_0 : H(x, y) = 0$, which yield three Melnikov functions as follows :

$$M(h, \delta) = \oint_{L_1} q\,dx - p\,dy, \quad for \;\; 0 < -h \ll 1$$
$$\widetilde{M}(h, \delta) = \oint_{L_2} q\,dx - p\,dy, \quad for \;\; 0 < -h \ll 1$$
$$M^*(h, \delta) = \oint_{L_3} q\,dx - p\,dy \quad for \;\; 0 < h \ll 1.$$

Our main goal in this section is to study the expansions of these Melnikov functions and use the first nonvanishing coefficients of the expansions to give a lower bound of the number of limit cycles produced near the double homoclinic loop L_0.

Before continuing the discussion, let's remind that we can write

$$M(h, \delta) = \oint_{H=h} q\,dx - p\,dy = \iint_{H\leq h} (p_x + q_y)\,dx\,dy = \oint_{H=h} \tilde{q}(x, y, \delta)\,dx,$$

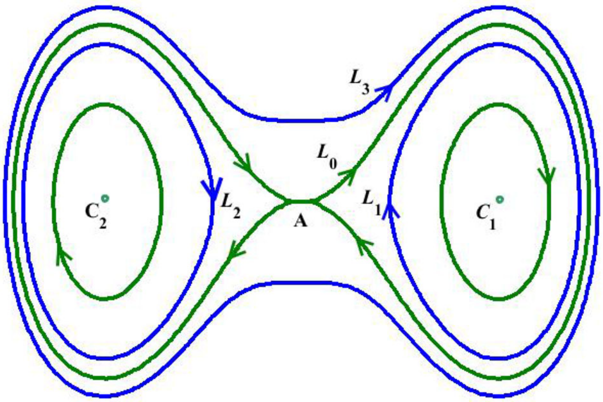

Fig. 2 Phase portrait of system (4) when $(\alpha, \beta) \in \lambda_2$

where

$$\tilde{q}(x, y, \delta) = q(x, y, \delta) - q(x, 0, \delta) + \int_0^y p_x(x, u, \delta)\, du$$

satisfies $\tilde{q}_y = p_x + q_y$ and $\tilde{q}(x, 0, \delta) = 0$. Then $\tilde{q}(x, y, \delta) = \sum_{j \ge 1} q_j(x)\, y^j$, where

$$q_{j+1}(x) = \frac{1}{(j+1)!} \frac{\partial^j}{\partial y^j}(p_x + q_y)\Big|_{\varepsilon=y=0}. \tag{7}$$

Asymptotic Expansions of the Melnikov Functions M and $\widetilde{M}$

In this section, we calculate the expansions of $M(h, \delta)$ and $\widetilde{M}(h, \delta)$. First, we start by writting

$$\begin{aligned} M(h, \delta) &= \oint_{L_1} \tilde{q}(x, y, \delta)\, dx = \int_{L_1^{(1)}} \tilde{q}\, dx + \int_{L_1 - L_1^{(1)}} \tilde{q}\, dx = I_1(h, \delta) + \int_{L_1 - L_1^{(1)}} \tilde{q}\, dx, \\ \widetilde{M}(h, \delta) &= \oint_{L_2} \tilde{q}(x, y, \delta)\, dx = \int_{L_2^{(1)}} \tilde{q}\, dx + \int_{L_2 - L_2^{(1)}} \tilde{q}\, dx = I_2(h, \delta) + \int_{L_2 - L_2^{(1)}} \tilde{q}\, dx, \end{aligned} \tag{8}$$

Where $L_1 := \{(x, y) \mid H(x, y) = \frac{1}{2}y^2 + H_0(x) = h,\ x > 0,\ 0 < -h \ll 1\}$, $L_2 := \{(x, y) \mid H(x, y) = \frac{1}{2}y^2 + H_0(x) = h,\ x < 0,\ 0 < -h \ll 1\}$ and $L_1^{(1)} = \{(x, y) \mid H(x, y) = h, \eta(h) \le x \le x_0\}$, $L_2^{(1)} = \{(x, y) \mid H(x, y) = h, x_0' \le x \le \eta'(h)\}$ (for the definitions of $x_0, x_0', \eta(h)$ and $\eta'(h)$ see Fig. 3), and the second terms in $M(h, \delta)$ and $\widetilde{M}(h, \delta)$ are analytic functions in h for $0 < -h \ll 1$.

To study the analytical properties of $I_1(h, \delta)$ and $I_2(h, \delta)$ at $h = 0$, we note that for $|h|$ small enough the equation $H(x, y) = h$ has two C^ω solutions $y^\pm = \pm\sqrt{2}w(1 + O(|x, w|))$, where $w = \sqrt{h - H_0(x)}$. Denote $u = \psi(x) = \sqrt[4]{-H_0(x)}$ and $u_0 = \psi(x_0) > 0$. Then, we have the following result on the expansion of the functions $I_1(h, \delta)$ and $I_2(h, \delta)$ near $h = 0$.

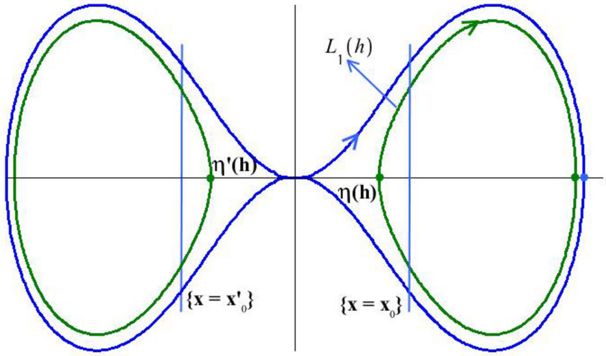

Fig. 3 Position of $\eta(h)$, $\zeta(h)$ and x_1

Lemma 2.1 *The functions $I_1(h,\delta)$ and $I_2(h,\delta)$ introduced in (8), for $0 < -h \ll 1$, can be written as*

$$I_1(h,\delta) = \chi_1(h,u_0) + \sum_{r=0}^{3} I_{1,r}^*(h)\, I_{r,0}(h,u_0),$$

$$I_2(h,\delta) = \chi_2(h,u_0) + \sum_{r=0}^{3} (-1)^r\, I_{1,r}^*(h)\, I_{r,0}(h,u_0),$$

where $\chi_1(h,u_0)$, $\chi_2(h,u_0)$ are analytic functions in h, $I_{r,0}(h,u_0) = \int_{|h|^{\frac{1}{4}}}^{u_0} u^r \sqrt{h+u^4}\, du$, and $I_{1,r}^*(h) = \sum_{m,j\ge 0} r_{4m+r,j}\alpha^*_{4m+r,j}\beta^*_{4m+r}h^{j+m}$ for $r = 0, 1, 2, 3$, with

$$\alpha^*_{k,j} = \begin{cases} 2^j \frac{(2j+1)!!}{(4j+k+3)(4j+k-1)\ldots(k+7)} & k \ge 0, \quad j \ge 1\,; \\ 1 & k \ge 0, \quad j = 0, \end{cases}$$

$$\beta^*_{4m+r} = \begin{cases} \frac{(-1)^m (r+1)(r+5)\ldots(r+4m-3)}{(r+7)(r+11)\ldots(r+4m+3)} & m \ge 1, \quad 0 \le r \le 3\,; \\ 1 & m = 0, \quad 0 \le r \le 3. \end{cases}$$

Here, the coefficients $r_{k,j}$ are given by the Taylor expansion coefficients of the functions

$$\tilde{q}_j(u) = \left.\frac{\bar{q}_j(x)}{\psi'(x)}\right|_{x=\psi^{-1}(u)} = \sum_{k=0}^{\infty} r_{k,j}u^k,$$

in u, which appear along the proof.

Proof We have that

$$\begin{aligned} I_1(h,\delta) &= \int_{L_1^{(1)}} \tilde{q}\, dx = \int_{\eta(h)}^{x_0} (\tilde{q}(x,y^+,\delta) - \tilde{q}(x,y^-,\delta))\, dx \\ &= \sum_{j\ge 0} \int_{\eta(h)}^{x_0} \bar{q}_j(x)\, w^{2j+1}\, dx, \quad \text{where} \quad \bar{q}_j(x) = 2^{j+\frac{3}{2}} q_{2j+1}(x). \end{aligned} \tag{9}$$

Therefore,

$$I_1(h,\delta) = \sum_{j\ge 0} \int_{|h|^{\frac{1}{4}}}^{u_0} \tilde{q}_j(u) w^{2j+1}\, du = \sum_{k\ge 0} r_{k,j} I_{k,j},$$

where $w=\sqrt{h+u^4}$ and

$$\tilde{q}_j(u)=\frac{\bar{q}_j(x)}{\psi'(x)}\bigg|_{x=\psi^{-1}(u)}=\sum_{k\geq 0} r_{k,j}u^k,\quad I_{k,j}=\int_{|h|^{\frac{1}{4}}}^{u_0} u^k w^{2j+1}\,du. \tag{10}$$

Similarly, we have that

$$\begin{aligned} I_2(h,\delta)&=\int_{L_2^{(1)}}\tilde{q}\,dx=\int_{x_0'}^{\eta'(h)}(\tilde{q}(x,y^+,\delta)-\tilde{q}(x,y^-,\delta))\,dx\\ &=\sum_{j\geq 0}\int_{-u_0}^{-|h|^{\frac{1}{4}}}\tilde{q}_j(u)w^{2j+1}\,du=\sum_{k\geq 0}(-1)^k r_{k,j}I_{k,j}. \end{aligned}$$

To calculate $I_{k,j}$, by using the formula (27) given in [5], namely,

$$\int u^k(h+u^4)^{j+\frac{1}{2}}\,du=\frac{u^{k+1}(h+u^4)^{j+\frac{1}{2}}}{4j+k+3}+\frac{4\left(j+\frac{1}{2}\right)h}{4j+k+3}\int u^k(h+u^4)^{j-\frac{1}{2}}\,du,$$

we have that

$$I_{k,j}(h,u_0)=\varphi_{k,j}(u_0,h)+\frac{4\left(j+\frac{1}{2}\right)h}{4j+k+3}I_{k,j-1}(h,u_0),$$

where

$$\varphi_{k,j}(u_0,h)=\frac{u_0^{k+1}(h+u_0^4)^{j+\frac{1}{2}}}{4j+k+3}\in C^\omega.$$

It follows that

$$I_{k,j}=\bar{\varphi}_{k,j}+\alpha_{k,j}^* h^j\, I_{k,0},\quad k\geq 0,\quad j\geq 1, \tag{11}$$

where $\bar{\varphi}_{k,j}\in C^\omega$, and

$$\alpha_{k,j}^*=\begin{cases} 2^j\frac{(2j+1)!!}{(4j+k+3)(4j+k-1)\ldots(k+7)} & k\geq 0,\quad j\geq 1;\\ 1 & k\geq 0,\quad j=0. \end{cases}$$

Further, using the formula (29) given in [5], namely,

$$\int u^k(h+u^4)^{\frac{1}{2}}\,du=\frac{u^{k-3}(h+u^4)^{\frac{3}{2}}}{k+3}-\frac{(k-3)h}{k+3}\int u^{k-4}(h+u^4)^{\frac{1}{2}}\,du,$$

we have that

$$I_{k,0}=\psi_k-\frac{k-3}{k+3}h\,I_{k-4,0},\quad k\geq 4,\quad \psi_k\in C^\omega.$$

It follows that

$$I_{4m+r,0}=\tilde{\psi}_{4m+r}+\beta_{4m+r}^* h^m\, I_{r,0},\quad k\geq 0,\quad j\geq 1, \tag{12}$$

where $\tilde{\psi}_{4m+r}\in C^\omega$ and

$$\beta_{4m+r}^*=\begin{cases} \frac{(-1)^m(r+1)(r+5)\ldots(r+4m-3)}{(r+7)(r+11)\ldots(r+4m+3)} & m\geq 1,\quad 0\leq r\leq 3\,;\\ 1 & m=0,\quad 0\leq r\leq 3. \end{cases}$$

By (11) and (12), we get that

$$I_{k,j}=\bar{\varphi}_{k,j}+\alpha_{k,j}^*\tilde{\psi}_k h^j+\alpha_{k,j}^*\beta_k^*\,h^{j+m}\,I_{r,0},\qquad for\ \ k=4m+r,\quad m\geq 0,\quad 0\leq r\leq 3. \tag{13}$$

Hence, by (9), (11) and (13) we get

$$
\begin{aligned}
I_1(h,\delta) =& \sum_{k\geq 0} r_{k,0}\, I_{k,0} + \sum_{\substack{0\leq k\leq 3\\ j\geq 1}} r_{k,j}\, I_{k,j} + \sum_{\substack{k=4m+r\\ 0\leq r\leq 3\\ m,j\geq 1}} r_{k,j}\, I_{k,j} \\
=& \sum_{\substack{m\geq 1\\ 0\leq r\leq 3}} r_{4m+r,0}\tilde{\psi}_{4m+r} + \sum_{r=0}^{3}\left(\sum_{m\geq 0} r_{4m+r,0}\beta^*_{4m+r}\, h^m\right) I_{r,0} + \sum_{k=0}^{3}\sum_{j\geq 1} r_{k,j}\bar{\varphi}_{k,j} \\
&+ \sum_{k=0}^{3}\left(\sum_{j\geq 1} r_{k,j}\alpha^*_{k,j}\, h^j\right) I_{k,0} + \sum_{r=0}^{3}\sum_{m,j\geq 1} r_{4m+r,j}\left(\bar{\varphi}_{4m+r,j} + \alpha^*_{4m+r,j}\tilde{\psi}_{4m+r}h^j\right) \\
&+ \sum_{r=0}^{3}\left(\sum_{m,j\geq 1} r_{4m+r,j}\alpha^*_{4m+r,j}\beta^*_{4m+r}h^{j+m}\right) I_{r,0}.
\end{aligned}
$$

Thus,

$$
I_1(h,\delta) = \chi_1(h,u_0) + \sum_{r=0}^{3} I^*_{1,r}(h)\, I_{r,0}(h,u_0), \tag{14}
$$

and in a similar way,

$$
I_2(h,\delta) = \chi_2(h,u_0) + \sum_{r=0}^{3}(-1)^r\, I^*_{1,r}(h)\, I_{r,0}(h,u_0), \tag{15}
$$

with $\chi_1(h,u_0), \chi_2(h,u_0) \in C^\omega$ and $I^*_{1,r}(h) = \sum_{m,j\geq 0} r_{4m+r,j}\alpha^*_{4m+r,j}\beta^*_{4m+r}h^{j+m}$ for $r = 0, 1, 2, 3$. □

To gain the analytical properties of the functions $I_{r,0}(h)$, we let $v = |h|^{\frac{1}{4}}/u$ in (10) for $(k,j) = (r,0)$, and obtain

$$
I_{r,0}(h) = |h|^{\frac{r}{4}+\frac{3}{4}} \int_{|h|^{\frac{1}{4}}/u_0}^{1} v^{-r-4}\sqrt{1-v^4}\, dv.
$$

Note that for $0 \leq v \leq 1$ we have the following convergent series

$$
\sqrt{1-v^4} = \sum_{j\geq 0} c_j v^{4j} = 1 - \frac{v^4}{2} - \frac{v^8}{8} - \frac{v^{12}}{16} + O\left(v^{16}\right).
$$

Then, for $r = 1$, we have

$$
I_{1,0}(h,u_0) = |h|\, c_1\left(-\ln|h|^{\frac{1}{4}} + \ln|u_0|\right) + \hat{\varphi}_1(h,u_0) = -\frac{1}{8}h\ln|h| + \tilde{\varphi}_1(h,u_0), \tag{16}
$$

where

$$
\tilde{\varphi}_1(h,u_0) = -\frac{1}{2}|h|\ln|u_0| + \sum_{\substack{j\geq 0\\ j\neq 1}} \frac{c_j\left(|h| - u_0^{4(1-j)}|h|^j\right)}{4(j-1)}.
$$

For $r \neq 1$, we have

$$
I_{r,0}(h,u_0) = |h|^{\frac{r}{4}+\frac{3}{4}} \sum_{j\geq 0} c_j \int_{|h|^{\frac{1}{4}}/u_0}^{1} v^{-r+4j-4}\, dv = \tilde{A}_r|h|^{\frac{r}{4}+\frac{3}{4}} + \tilde{\varphi}_r(h,u_0), \tag{17}
$$

where $\tilde{A}_r = \sum_{j\geq 0} \frac{c_j}{4j-r-3}$ and $\tilde{\varphi}_r(h, u_0) = -\sum_{j\geq 0} \frac{c_j u_0^{r-4j+3}}{4j-r-3} |h|^j$.

Furthermore, for the constants $\tilde{A}_r$ in (17), since $\tilde{A}_r$ is independent of u_0, we can take $u_0 = 1$. Then

$$\tilde{\varphi}_r(h, 1) = -\sum_{j\geq 0} \frac{c_j}{4j-r-3} |h|^j = \frac{1}{r+3} + \frac{|h|}{2r-2} + O(|h|^2).$$

Thus, by (10), for $0 < -h \ll 1$ we have

$$\frac{\partial I_{r,0}(h, u_0)}{\partial h} = -\left(\frac{r}{4} + \frac{3}{4}\right) \tilde{A}_r |h|^{\frac{r}{4}-\frac{1}{4}} + \frac{1}{2r-2} + O(|h|). \tag{18}$$

On the other hand, by (10), we have that

$$\frac{\partial I_{r,0}(h, u_0)}{\partial h} = \frac{1}{2} \int_{|h|^{\frac{1}{4}}}^{1} \frac{u^r \, du}{\sqrt{h+u^4}} = \frac{1}{2} |h|^{\frac{r}{4}-\frac{1}{4}} \int_{|h|^{\frac{1}{4}}}^{1} \frac{v^{-r} \, du}{\sqrt{1-v^4}}. \tag{19}$$

- For $r = 0$, comparing (18) and (19) gives

$$\bar{A}_0 = -\frac{2}{3} \lim_{h\to 0} \frac{\frac{\partial I_{r,0}(h,u_0)}{\partial h}}{|h|^{-\frac{1}{4}}} = -\frac{2}{3} \int_0^1 \frac{dv}{\sqrt{1-v^4}} = -0.8740191850.$$

- For $r = 2$, note that

$$\int_{|h|^{\frac{1}{4}}}^{1} \frac{v^{-2}}{\sqrt{1-v^4}} \, dv = \int_{|h|^{\frac{1}{4}}}^{1} v^{-2} \left[1 + \left(\frac{1}{\sqrt{1-v^4}} - 1\right)\right] dv$$
$$= -(1 - |h|^{-\frac{1}{4}}) + \int_{|h|^{\frac{1}{4}}}^{1} \frac{v^2 \, dv}{\sqrt{1-v^4}(1+\sqrt{1-v^4})}.$$

Therefore, by substituting the above into (19), we get

$$\frac{\partial I_{r,0}(h, u_0)}{\partial h} = -\frac{1}{2}|h|^{\frac{1}{4}} + \frac{1}{2} + \frac{1}{2}|h|^{\frac{1}{4}} \int_{|h|^{\frac{1}{4}}}^{1} \frac{v^2 \, dv}{\sqrt{1-v^4}(1+\sqrt{1-v^4})}.$$

Consequently,

$$\bar{A}_2 = -\frac{4}{5} \lim_{h\to 0} \frac{\frac{\partial I_{r,0}(h,u_0)}{\partial h}}{|h|^{\frac{1}{4}}} = -\frac{4}{5} \left[-\frac{1}{2} + \frac{1}{2} \int_0^1 \frac{v^2 \, dv}{\sqrt{1-v^4}(1+\sqrt{1-v^4})}\right] = 0.2396280472.$$

Proposition 2.2 *Let $L_0 = \bar{L}_0 \cup \tilde{L}_0$ be a double homoclinic loop defined by $H(x, y) = 0$, where $\bar{L}_0 = L_0|_{x\geq 0}$ and $\tilde{L}_0 = L_0|_{x\leq 0}$. Then for the functions $M(h, \delta)$ and $\widetilde{M}(h, \delta)$ given in* (8)*, we have*

$$M(h, \delta) = c_0 + c_1|h|^{\frac{3}{4}} + c_2 \, h \ln |h| + c_3 \, h + c_4|h|^{\frac{5}{4}} + c_5|h|^{\frac{7}{4}} + c_6 \, h^2 \ln |h| + O(|h|^2),$$
$$\widetilde{M}(h, \delta) = \tilde{c}_0 + c_1|h|^{\frac{3}{4}} - c_2 \, h \ln |h| + \tilde{c}_3 \, h + c_4|h|^{\frac{5}{4}} + c_5|h|^{\frac{7}{4}} - c_6 \, h^2 \ln |h| + O(|h|^2), \tag{20}$$

in which

$$c_0 = M(0, \delta) = \oint_{\bar{L}_0} q \, dx - p \, dy|_{\varepsilon=0}, \quad \tilde{c}_0 = \widetilde{M}(0, \delta) = \oint_{\tilde{L}_0} q \, dx - p \, dy|_{\varepsilon=0},$$
$$c_1 = \bar{A}_0 \, r_{00}, \quad c_2 = -\frac{1}{8} r_{10},$$

$$c_3 = \oint_{\bar{L}_0} \left[(p_x + q_y)|_{\varepsilon=0} - a_0 - a_1 x \right] dt + O_1(c_1) + O_1(c_2),$$
$$\tilde{c}_3 = \oint_{\tilde{L}_0} \left[(p_x + q_y)|_{\varepsilon=0} - a_0 - a_1 x \right] dt + O_1(c_1) + O_1(c_2),$$
$$c_4 = \bar{A}_2 r_{20}, \quad c_5 = -\bar{A}_0 \left(\frac{6}{7} r_{01} - \frac{1}{7} r_{40} \right), \quad c_6 = -\frac{1}{32}(3 r_{11} - r_{50}), \tag{21}$$

with $a_0 = (p_x + q_y)|_{\varepsilon=x=y=0}$, $a_1 = (p_{xx} + q_{yx})|_{\varepsilon=x=y=0}$, $O_1(c)$ *denotes c times a constant, and* r_{ij} *will be introduced in the proof.*

Proof By (8), (14), (15), (16) and (17), for $0 < -h \ll 1$ we have

$$M(h,\delta) = \varphi_1(h,\delta) + \bar{A}_0 I^*_{10}(h)|h|^{\frac{3}{4}} - \frac{1}{8} h \ln|h|\, I^*_{11}(h) + \bar{A}_2 I^*_{12}(h)|h|^{\frac{5}{4}},$$
$$\widetilde{M}(h,\delta) = \varphi_2(h,\delta) + \bar{A}_0 I^*_{10}(h)|h|^{\frac{3}{4}} + \frac{1}{8} h \ln|h|\, I^*_{11}(h) + \bar{A}_2 I^*_{12}(h)|h|^{\frac{5}{4}}, \tag{22}$$

where

$$I^*_{10} = r_{00}\alpha^*_{00}\beta^*_0 + \left(r_{40}\alpha^*_{40}\beta^*_4 + r_{01}\alpha^*_{01}\beta^*_0\right) h + O(h^2),$$
$$I^*_{11} = r_{10}\alpha^*_{10}\beta^*_1 + \left(r_{50}\alpha^*_{50}\beta^*_5 + r_{11}\alpha^*_{11}\beta^*_1\right) h + O(h^2),$$
$$I^*_{12} = r_{20}\alpha^*_{20}\beta^*_2 + O(h),$$

and

$$\alpha^*_{00} = \alpha^*_{10} = \alpha^*_{20} = \alpha^*_{40} = \alpha^*_{50} = 1, \quad \alpha^*_{01} = \frac{6}{7}, \quad \alpha^*_{11} = \frac{3}{4},$$
$$\beta^*_0 = \beta^*_1 = \beta^*_2 = 1, \quad \beta^*_4 = -\frac{1}{7}, \quad \beta^*_5 = -\frac{1}{4}.$$

Therefore, we can obtain the expansion of $M(h,\delta)$ by inserting the above into (22) with $c_0 = \varphi_1(0,\delta) = M(0,\delta)$ and $\tilde{c}_0 = \varphi_2(0,\delta) = \widetilde{M}(0,\delta)$ given by (21), and

$$c_1 = \bar{A}_0 r_{00}, \quad c_2 = -\frac{1}{8} r_{10}, \quad c_4 = \bar{A}_2 r_{20},$$
$$c_5 = -\bar{A}_0 \left(\frac{6}{7} r_{01} - \frac{1}{7} r_{40} \right), \quad c_6 = -\frac{1}{32}(3 r_{11} - r_{50}).$$

Note that, by Taylor expansion, we obtain $x = \psi^{-1}(u) = \tau_0 u + \tau_1 u^3 + \tau_2 u^5 + O(u^7)$, where

$$\tau_0 = \frac{\sqrt{2}\sqrt[3]{\beta}}{\sqrt[4]{-5\,\beta^{2/3} + 4}}, \quad \tau_1 = -\frac{21\,\beta^{11/3} + 8\,\beta^{7/3} - 28\,\beta^3}{\left(-2\sqrt{-\beta^{2/3}\left(5\,\beta^{2/3} - 4\right)}\beta\right)^{3/2} \beta^{2/3}\left(5\,\beta^{2/3} - 4\right)},$$
$$\tau_2 = -\frac{1}{32}\sqrt{-2\sqrt{-\beta^{2/3}\left(5\,\beta^{2/3}-4\right)}\beta}\sqrt{-\beta^{2/3}\left(5\,\beta^{2/3}-4\right)}\left(40125\,\beta^{\frac{37}{3}} - 192300\,\beta^{\frac{35}{3}} + 379440\,\beta^{11}\right.$$
$$\left. -385664\,\beta^{\frac{31}{3}} + 198592\,\beta^{\frac{29}{3}} - 33024\,\beta^9 - 11264\,\beta^{\frac{25}{3}} + 4096\,\beta^{\frac{23}{3}}\right)\left(78125\,\beta^{\frac{43}{3}} - 437500\,\beta^{\frac{41}{3}}\right.$$
$$\left. -1400000\,\beta^{\frac{37}{3}} + 1120000\,\beta^{\frac{35}{3}} + 143360\,\beta^{\frac{31}{3}} - 16384\,\beta^{\frac{29}{3}} + 1050000\,\beta^{13} - 537600\,\beta^{11}\right)^{-1},$$

and $\frac{1}{\psi'(x)} = \lambda_0 + \lambda_1 x^2 + \lambda_2 x^4 + O(x^6)$, where

$$\lambda_0 = -\frac{\sqrt{2}\beta^{-\frac{20}{3}}}{5\,\beta^{2/3} - 4}\left(-\beta^{\frac{28}{3}}\left(5\,\beta^{2/3} - 4\right)\right)^{3/4},$$

$$\lambda_1 = \frac{3}{8}\,\frac{\sqrt{2}\beta^{2/3}\left(105\,\beta^{8/3} - 224\,\beta^2 + 152\,\beta^{4/3} - 32\,\beta^{2/3}\right)}{\left(5\,\beta^{2/3}-4\right)^2\sqrt[4]{-\beta^{\frac{28}{3}}\left(5\,\beta^{2/3}-4\right)}},$$

$$\lambda_2 = -\frac{\sqrt{2}}{128}\Big(862125\,\beta^{16/3} - 4313100\,\beta^{14/3} + 9080880\,\beta^4 \\ -10357888\,\beta^{10/3} + 6830528\,\beta^{8/3} - 2545920\,\beta^2 \\ +472064\,\beta^{4/3} - 28672\,\beta^{2/3}\Big)\left(\left(-5\,\beta + 4\sqrt[3]{\beta}\right)^2\left(5\,\beta^{2/3}-4\right)^3\sqrt[4]{-\beta^{\frac{28}{3}}\left(5\,\beta^{2/3}-4\right)}\right)^{-1}.$$

Suppose that $p(x,y) = \sum_{i+j\geq 0} a_{ij}\,x^i\,y^j$ and $q(x,y) = \sum_{i+j\geq 0} b_{ij}\,x^i\,y^j$. Now, we calculate r_{ij} in (10). Note that, by (7), we have

$$\bar{q}_0 = 2\sqrt{2}\,q_1 = 2\sqrt{2}\left(p_x + q_y\right)\Big|_{\varepsilon=y=0} = 2\sqrt{2}\sum_{i=0}^{\infty}\left[(i+1)\,a_{i+1,0} + b_{i1}\right]x^i,$$

$$\bar{q}_1 = 4\sqrt{2}\,q_3 = \frac{2}{3}\sqrt{2}\left(p_{xyy} + q_{yyy}\right)\Big|_{\varepsilon=y=0} = \frac{4}{3}\sqrt{2}\sum_{i=0}^{\infty}\left[(i+1)\,a_{i+1,2} + 3\,b_{i3}\right]x^i.$$

Then

$$r_{00} = \tilde{q}_0(0) = 2\sqrt{2}\,\lambda_0\,\left(a_{10}+b_{01}\right), \qquad r_{10} = 2\sqrt{2}\,\lambda_0\,\tau_0\,\left(2\,a_{20}+b_{11}\right),$$
$$r_{20} = 2\sqrt{2}\,\tau_0^2\left(\lambda_0\,\left(3\,a_{30}+b_{21}\right) + \lambda_1\,\left(a_{10}+b_{01}\right)\right),$$
$$r_{40} = 2\sqrt{2}\Big(\lambda_0\left(2\,\left(3\,a_{30}+b_{21}\right)\tau_0\tau_1 + \left(5\,a_{50}+b_{41}\right)\tau_0{}^4\right) + \lambda_1\tau_0{}^4\left(3\,a_{30}+b_{21}\right) \\ + \left(\lambda_2\tau_0{}^4 + 2\,\lambda_1\tau_0\tau_1\right)\left(a_{10}+b_{01}\right)\Big),$$
$$r_{50} = 2\sqrt{2}\Big(\lambda_0\left(\left(2\,a_2+b_{11}\right)\tau_2 + 3\,\left(4\,a_{40}+b_{31}\right)\tau_0{}^2\tau_1 + \left(6\,a_{60}+b_{51}\right)\tau_0{}^5\right) + \lambda_1\tau_0{}^2\left(\left(2\,a_{20}+b_{11}\right)\tau_1 \\ + \left(4\,a_{40}+b_{31}\right)\tau_0{}^3\right) + \left(\lambda_2\tau_0{}^4 + 2\,\lambda_1\tau_0\tau_1\right)\left(2\,a_{20}+b_{11}\right)\tau_0\Big),$$
$$r_{01} = \frac{4}{3}\sqrt{2}\,\lambda_0\,\left(a_{12}+3\,b_{03}\right), \qquad r_{11} = \frac{4}{3}\sqrt{2}\,\lambda_0\,\tau_0\,\left(2\,a_{22}+3\,b_{13}\right). \tag{23}$$

To prove the formulas of c_3 and $\tilde{c}_3$ in (21) see [5]. □

Asymptotic Expansion of the Melnikov Function M^*

In this section, we calculate the expansion of $M^*(h,\delta)$. We start by writting

$$M^*(h,\delta) = \oint_{L_3}\tilde{q}(x,y,\delta)\,dx = \int_{L_3^{(1)}}\tilde{q}\,dx + \int_{L_3^{(2)}}\tilde{q}\,dx + \int_{L_3-L_3^{(1)}-L_3^{(2)}}\tilde{q}\,dx \\ = I_3^{(1)}(h,\delta) + I_3^{(2)}(h,\delta) + \int_{L_3-L_3^{(1)}-L_3^{(2)}}\tilde{q}\,dx, \tag{24}$$

where $L_3 := \{(x,y)\,|\,H(x,y) = \frac{1}{2}y^2 + H_0(x) = h,\ 0 < h \ll 1\}$, $L_3^{(1)} = \{(x,y)\,|\,H(x,y) = h, x_0' \leq x \leq x_0, y > 0\}$ and $L_3^{(2)} = \{(x,y)\mid H(x,y) = h, x_0' \leq x \leq x_0, y < 0\}$ (for the definitions of x_0 and x_0' see Fig. 4) and the third term in $M^*(h,\delta)$ is an analytic function in h for $0 < h \ll 1$.

To study the analytical properties of the functions $I_3^{(1)}(h,\delta)$ and $I_3^{(2)}(h,\delta)$ at $h = 0$, we have the following result.

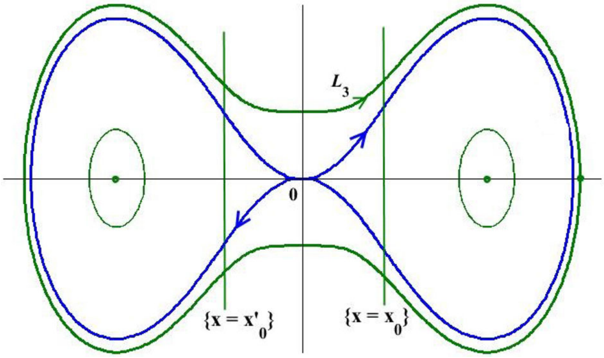

Fig. 4 The line segment $\{x = x_0\}$ and $\{x = x_0'\}$

Lemma 2.3 *Suppose that* $u = \psi(x) = \sqrt[4]{-H_0(x)}$, $u_0 = \psi(x_0) > 0$ *and* $w = \sqrt{h + u^4}$. *Then,*

$$I_3^{(1)}(h, u_0) = \chi_3(h, u_0) + \sum_{r=0}^{1} \tilde{I}_{1,r}^*(h)\, \tilde{I}_{r,1}(h, u_0),$$

$$I_3^{(2)}(h, u_0) = \chi_4(h, u_0) + \sum_{r=0}^{1} \tilde{I}_{1,r}^*(h)\, \tilde{I}_{r,1}(h, u_0),$$

where $\chi_3(h, u_0)$, $\chi_4(h, u_0)$ *are some analytic functions in* h, $\tilde{I}_{r,1}(h, u_0) = \int_0^{u_0} u^{2r}\sqrt{h+u^4}du$, *and* $\tilde{I}_{1,r}^*(h) = \sum_{\substack{k=2m+r\\ m\geq 0,\ j\geq 1\ odd}} \tilde{r}_{k,j}\tilde{\alpha}_{k,j}\tilde{\beta}_k\, h^{m+[\frac{j}{2}]}$ *for* $r = 0, 1$, *with*

$$\tilde{\alpha}_{k,j} = \begin{cases} \frac{6.10.\cdots.2j}{(2k+7)(2k+11)...(2k+2j+1)} & k \geq 0, \quad j \geq 3\ odd\ ; \\ 1 & k \geq 0, \quad j = 1, \end{cases}$$

$$\tilde{\beta}_k = \begin{cases} \frac{(-1)^m(2k-3)(2k-7)...(2k+1-4m)}{(2k+3)(2k-1)...(2k-4m+7)} & k = 2m + r, \quad m \geq 1, \quad r = 0, 1\ ; \\ 1 & k = 0, 1. \end{cases}$$

Here, the coefficients $\tilde{r}_{k,j}$ *are given by the Taylor expansion coefficients of the functions*

$$\tilde{q}_j(u) + \tilde{q}_j(-u) = \sum_{k=0}^{\infty} \tilde{r}_{k,j} u^{2k},$$

in u*, which appear along the proof.*

Proof In view of (24), we can write

$$I_3^{(1)}(h, \delta) = \int_{L_3^{(1)}} \tilde{q}\, dx = \int_{x_0'}^{x_0} \tilde{q}\left(x, y^+, \delta\right)\, dx = \sum_{j\geq 1}\int_{x_0'}^{x_0} \bar{q}_j\, w^j\, dx = \sum_{j\geq 1}\int_{u_0'}^{u_0} \tilde{q}_j(u)\, w^j\, du$$

$$= \sum_{j\geq 1}\int_0^{u_0} \left[\tilde{q}_j(u) + \tilde{q}_j(-u)\right]\, w^j\, du = \sum_{j\geq 1, k\geq 0} \tilde{r}_{k,j}\tilde{I}_{k,j}, \tag{25}$$

where

$$\tilde{q}_i(u) = \frac{\bar{q}_i(x)}{\psi'(x)}\bigg|_{x=\psi^{-1}(u)}, \quad \tilde{q}_i(u) + \tilde{q}_i(-u) = \sum_{k=0}^{\infty} \tilde{r}_{k,i} u^{2k}, \quad \tilde{I}_{k,j} = \int_0^{u_0} u^{2k} w^j \, du. \quad (26)$$

In the same way, we get,

$$\begin{aligned} I_3^{(2)}(h,\delta) &= \int_{L_3^{(2)}} \tilde{q}\, dx = \int_{x_0}^{x_0'} \tilde{q}(x, y^-, \delta)\, dx = \sum_{j\geq 1} \int_{x_0}^{x_0'} (-1)^j \bar{q}_j\, w^j\, dx \\ &= -\sum_{j\geq 1} \int_{u_0'}^{u_0} (-1)^j \tilde{q}_j(u)\, w^j\, du \\ &= -\sum_{j\geq 1} \int_0^{u_0} (-1)^j \left[\tilde{q}_j(u) + \tilde{q}_j(-u)\right] w^j\, du = -\sum_{j\geq 1, k\geq 0} (-1)^j \tilde{r}_{k,j} \tilde{I}_{k,j}. \end{aligned}$$

To calculate $\tilde{I}_{k,j}$, we see that by (26) $\tilde{I}_{k,j} \in C^{\omega}$ for $j > 0$ and even. For $j \geq 3$ and odd, similar to (11), we obtain that

$$\tilde{I}_{k,j} = \tilde{\varphi}_{k,j} + \tilde{\alpha}_{k,j} h^{[\frac{j}{2}]}\, \tilde{I}_{k,1}, \quad k \geq 0, \quad (27)$$

where $\tilde{\varphi}_{k,j} \in C^{\omega}$ and

$$\tilde{\alpha}_{k,j} = \begin{cases} \frac{6.10.\cdots.2j}{(2k+7)(2k+11)\ldots(2k+2j+1)} & k \geq 0, \quad j \geq 3 \; odd\,; \\ 1 & k \geq 0, \quad j = 1. \end{cases}$$

Also, similar to (12), we get that

$$\tilde{I}_{k,1} = \bar{\psi}_k + \tilde{\beta}_k h^m\, \tilde{I}_{r,1}, \quad (28)$$

for $2k = 4m + 2r$ with $r = 0, 1, m \geq 1$ and

$$\tilde{\beta}_k = \begin{cases} \frac{(-1)^m (2k-3)(2k-7)\ldots(2k+1-4m)}{(2k+3)(2k-1)\ldots(2k-4m+7)} & k = 2m + r, \quad m \geq 1, \quad r = 0, 1\,; \\ 1 & k = 0, 1. \end{cases}$$

Therefore, by (25), (27) and (28), we have that

$$\begin{aligned} I_3^{(1)}(h, u_0) &= \sum_{\substack{k\geq 0 \\ j\geq 1\; odd}} \tilde{r}_{k,j}\, \tilde{I}_{k,j} + \cdots = \sum_{k\geq 0} \tilde{r}_{k,1}\, \tilde{I}_{k,1} + \sum_{\substack{k\geq 0 \\ j\geq 3\; odd}} \tilde{r}_{k,j}\, \tilde{I}_{k,j} + \cdots \\ &= \sum_{k\geq 0} \tilde{r}_{k,1} \tilde{I}_{k,1} + \sum_{\substack{k\geq 0 \\ j\geq 3\; odd}} \tilde{r}_{i,j} \tilde{\alpha}_{i,j} h^{[\frac{j}{2}]} \tilde{I}_{k,1} + \cdots = \sum_{\substack{k\geq 0 \\ j\geq 1\; odd}} \tilde{r}_{i,j} \tilde{\alpha}_{i,j} h^{[\frac{j}{2}]} \tilde{I}_{k,1} + \cdots \\ &= \sum_{\substack{0\leq k\leq 1 \\ j\geq 1\; odd}} \tilde{r}_{i,j} \tilde{\alpha}_{i,j} h^{[\frac{j}{2}]} \tilde{I}_{k,1} + \sum_{\substack{k=2m+r \\ m\geq 1,\; 0\leq r\leq 1 \\ j\geq 1\; odd}} \tilde{r}_{i,j} \tilde{\alpha}_{i,j} \tilde{\beta}_i h^{m+[\frac{j}{2}]} \tilde{I}_{k,1} + \cdots \\ &= \sum_{\substack{k=2m+r \\ m\geq 0,\; 0\leq r\leq 1 \\ j\geq 1\; odd}} \tilde{r}_{i,j} \tilde{\alpha}_{i,j} \tilde{\beta}_i h^{m+[\frac{j}{2}]} \tilde{I}_{k,1} + \cdots \\ &= \sum_{r=0}^{1} \tilde{I}_{1,r}^{*} \tilde{I}_{r,1}(h, u_0) + \cdots, \end{aligned}$$

where "$\cdots$" in each equation denotes a C^ω function and $\tilde{I}^*_{1,r}(h) = \sum_{\substack{k=2m+r\\ m\geq 0,\ j\geq 1\ odd}} \tilde{r}_{k,j}\tilde{\alpha}_{k,j}$ $\tilde{\beta}_k h^{m+[\frac{j}{2}]}$. Hence,

$$I_3^{(1)}(h, u_0) = \chi_3(h, u_0) + \sum_{r=0}^{1} \tilde{I}^*_{1,r}(h)\, \tilde{I}_{r,1}(h, u_0), \tag{29}$$

where $\chi_3(h, u_0) \in C^\omega$. Similarly, we have that

$$I_3^{(2)}(h, u_0) = \chi_4(h, u_0) + \sum_{r=0}^{1} \tilde{I}^*_{1,r}(h)\, \tilde{I}_{r,1}(h, u_0), \tag{30}$$

where $\chi_4(h, u_0) \in C^\omega$. □

To obtain the analytical properties of the functions $\tilde{I}_{r,1}(h, u_0)$, we let $v = u/h^{\frac{1}{4}}$ in (26) for $k = r$ and $j = 1$, and we get

$$\tilde{I}_{r,1}(h, u_0) = h^{\frac{r}{2}+\frac{3}{4}} \int_0^{u_0 h^{-\frac{1}{4}}} v^{2r}(1+v^4)^{\frac{1}{2}}\, dv = h^{\frac{r}{2}+\frac{3}{4}} \left[B_{N_r} + \psi_{N_r}(h, u_0)\right],$$

where

$$B_{N_r} = \int_0^N v^{2r}(1+v^4)^{\frac{1}{2}}\, dv, \qquad N > 1,$$
$$\psi_{N_r}(h, u_0) = \int_N^{u_0/h^{1/4}} v^{2r}(1+v^4)^{\frac{1}{2}}\, dv = \int_{h^{1/4}/u_0}^{1/N} v^{-2r-4}\sqrt{1+v^4}\, dv.$$

Note that for $0 \leq v \leq 1$ we have the following convergent series

$$\sqrt{1+v^4} = \sum_{j\geq 0} \tilde{c}_j v^{4j} = 1 + \frac{v^4}{2} - \frac{v^8}{8} + \frac{v^{12}}{16} + O\left(v^{16}\right).$$

So,

$$\psi_{N_r}(h, u_0) = \sum_{j\geq 0} \tilde{c}_j \int_{h^{1/4}/u_0}^{1/N} v^{4j-2r-4}\, dv.$$

Let $j_r = \frac{r}{2} + \frac{3}{4}$. Then $4j - 2r - 4 = 4(j - j_r) - 1$ and

$$\tilde{I}_{r,1}(h, u_0) = h^{j_r}\left[B_{N_r} + \sum_{j\geq 0} \tilde{c}_j \frac{N^{4(j_r-j)}}{4(j_r-j)}\right] - \sum_{j\geq 0} \tilde{c}_j \frac{u_0^{4(j_r-j)}}{4(j_r-j)} h^j = h^{j_r} \bar{A}_r + \tilde{\varphi}_r(h, u_0), \tag{31}$$

where $\tilde{\varphi}_r$ is analytic for $0 \leq h \ll 1$.

To determine the constants $\bar{A}_r$ in (31), as $\bar{A}_r$ is independent of u_0, we can take $u_0 = 1$. Thus, by (31), we get

$$\bar{\varphi}_r(h, 1) = -\sum_{j\geq 0} \tilde{c}_j \frac{h^j}{4(j - j_r)} = \frac{1}{2r+3} + \frac{h}{4r-2} + O(h^2).$$

Then, by (26), for $0 < h \ll 1$ we have

$$\frac{\partial \tilde{I}_{r,1}(h, u_0)}{\partial h} = j_r \bar{A}_r\, h^{j_r-1} + \frac{1}{4r-2} + O(h). \tag{32}$$

On the other hand, by (26), we have

$$\frac{\partial \tilde{I}_{r,1}(h,u_0)}{\partial h}=\frac{1}{2}\int_0^1\frac{u^{2r}\,du}{\sqrt{h+u^4}}=\frac{1}{2}h^{j_r-1}\int_0^{h^{-\frac{1}{4}}}\frac{v^{2r}\,dv}{\sqrt{1+v^4}}. \tag{33}$$

- For $r=0$, comparing (32) and (33) gives

$$\bar{A}_0=\frac{1}{2\,j_r}\lim_{h\to 0}\frac{\frac{\partial \tilde{I}_{r1}}{\partial h}}{h^{j_r-1}}=\frac{2}{3}\int_0^{\infty}\frac{dv}{\sqrt{1+v^4}}=1.236049785$$

- For $r=1$, by using $\frac{v^2}{\sqrt{1+v^4}}=1-\frac{1}{\sqrt{1+v^4}(v^2+\sqrt{1+v^4})}$, it follows from (33) that

$$\frac{\partial \tilde{I}_{r,1}(h,u_0)}{\partial h}=\frac{1}{2}-\frac{1}{2}h^{\frac{1}{4}}\int_0^{h^{-\frac{1}{4}}}\frac{dv}{\sqrt{1+v^4}(v^2+\sqrt{1+v^4})}.$$

Comparing the above with (32) we obtain

$$\bar{A}_1=-\frac{2}{5}\int_0^{\infty}\frac{dv}{\sqrt{1+v^4}(v^2+\sqrt{1+v^4})}=-0.3388852337.$$

Proposition 2.4 *For the functions $M^*(h,\delta)$ given in (24), we have the following expansion:*

$$M^*(h,\delta)=c_0^*+c_1^*\,h^{\frac{3}{4}}+c_2^*\,h+c_3^*\,h^{\frac{5}{4}}+c_4^*\,h^{\frac{7}{4}}+O(h^2), \tag{34}$$

where

$$\begin{aligned} c_0^* &= M^*(0,\delta)=\oint_{L_0} q\,dx-\ p\,dy|_{\varepsilon=0}=c_0+\tilde{c}_0,\\ c_1^* &= 2\tilde{A}_0\,\tilde{r}_{01},\quad c_3^*=2\tilde{A}_1\tilde{r}_{11},\quad c_4^*=2\tilde{A}_0\left(-\frac{1}{7}\,\tilde{r}_{21}+\frac{6}{7}\,\tilde{r}_{03}\right),\\ c_2^* &= \oint_{L_0}\left[(p_x+q_y)|_{\varepsilon=0}-a_0\right]dt+O_1(c_1^*). \end{aligned} \tag{35}$$

Proof By (24), (29), (30), and (31) for $0<h\ll 1$ we have

$$M^*(h,\delta)=\varphi^*(h,\delta)+2\,h^{\frac{3}{4}}\left[\tilde{A}_0\,\tilde{I}_{10}^*(h)+\tilde{A}_1\,h^{\frac{1}{2}}\,\tilde{I}_{11}^*(h)\right], \tag{36}$$

where

$$\tilde{I}_{10}^*(h)=\sum_{\substack{k=2m\\ m\geq 0,\ j\geq 1\ odd}}\tilde{r}_{k,j}\tilde{\alpha}_{k,j}\tilde{\beta}_k h^{m+[\frac{j}{2}]}=\tilde{r}_{01}\tilde{\alpha}_{01}\tilde{\beta}_0+(\tilde{r}_{21}\tilde{\alpha}_{21}\tilde{\beta}_2+\tilde{r}_{03}\tilde{\alpha}_{03}\tilde{\beta}_0)\,h+O(h^2),$$

$$\tilde{I}_{11}^*(h)=\sum_{\substack{k=2m+1\\ m\geq 0,\ j\geq 1\ odd}}\tilde{r}_{k,j}\tilde{\alpha}_{k,j}\tilde{\beta}_k h^{m+[\frac{j}{2}]}=\tilde{r}_{11}\tilde{\alpha}_{11}\tilde{\beta}_1+O(h),$$

with

$$\tilde{\alpha}_{01}=\tilde{\beta}_0=\tilde{\beta}_1=1,\quad \tilde{\alpha}_{11}=\tilde{\alpha}_{21}=1,\quad \tilde{\beta}_2=-\frac{1}{7},\quad \tilde{\alpha}_{03}=\frac{6}{7}.$$

Therefore, we can obtain the given expansion for $M^*(h,\delta)$ by inserting the above into (34) with $c_0^*=M^*(0,\delta)=c_0+\tilde{c}_0$, and

$$c_1^*=2\tilde{A}_0\,\tilde{r}_{01},\quad c_3^*=2\tilde{A}_1\tilde{r}_{11},\quad c_4^*=2\tilde{A}_0\left(-\frac{1}{7}\,\tilde{r}_{21}+\frac{6}{7}\,\tilde{r}_{03}\right),$$

To calculate $\tilde{r}_{ij}$, as before assume that $x = \psi^{-1}(u) = \tau_0 u + \tau_1 u^3 + \tau_2 u^5 + O(u^7)$ and $\frac{1}{\psi'(x)} = \lambda_0 + \lambda_1 x^2 + \lambda_2 x^4 + O(x^6)$. Then we observe that

$$\begin{aligned}
\tilde{r}_{01} &= 2\sqrt{2}\,\lambda_0\,(a_{10}+b_{01}),\\
\tilde{r}_{11} &= 2\sqrt{2}\,\tau_0^2\,(\lambda_0\,(3\,a_{30}+b_{21})+\lambda_1\,(a_{10}+b_{01})),\\
\tilde{r}_{21} &= 2\sqrt{2}\left(\lambda_0\left(2\,(3\,a_{30}+b_{21})\,\tau_0\tau_1+(5\,a_{50}+b_{41})\,\tau_0{}^4\right)+\lambda_1\tau_0{}^4\,(3\,a_{30}+b_{21})\right.\\
&\quad\left.+\left(\lambda_2\tau_0{}^4+2\,\lambda_1\tau_0\tau_1\right)(a_{10}+b_{01})\right),\\
\tilde{r}_{03} &= \frac{4}{3}\sqrt{2}\,\lambda_0\,(a_{12}+3\,b_{03}).
\end{aligned} \tag{37}$$

To prove the formula of c_3^* in (35) see [5]. □

Remark 2.5 Under the conditions of Propositions 2.2 and 2.4, it is easy to see that

$$c_1^* = \frac{2\,\tilde{A}_0}{\bar{A}_0}\,c_1,\quad c_3^* = \frac{2\,\tilde{A}_1}{\bar{A}_2}\,c_4,\quad c_4^* = -\frac{2\,\tilde{A}_0}{\bar{A}_0}\,c_5.$$

Asymptotic Expansion of the Melnikov Function Near the Centers

In this section, we calculate the expansions of $M_1(h,\delta)$ and $M_2(h,\delta)$ near the centers C_1 and C_2, respectively. First, we calculate the expansion of $M_1(h,\delta)$. By introducing the transformation $(x,y)=(X-x^*,Y)$, we shift $C_1(x^*,0)$ to the origin. Then (rewriting again X as x and Y as y) we get

$$\begin{aligned}
\dot{x} &= y+\varepsilon\,\bar{p}(x,y,\delta),\\
\dot{y} &= -2\,(x+x^*)+\frac{(x+x^*+\alpha)}{\sqrt{(x+x^*+\alpha)^2+\beta^2}}+\frac{(x+x^*-\alpha)}{\sqrt{(x+x^*-\alpha)^2+\beta^2}}+\varepsilon\bar{q}(x,y,\delta),
\end{aligned} \tag{38}$$

where

$$\bar{p}(x,y,\delta)=p(x+x^*,y,\delta)=\sum_{i+j\ge 0}\bar{a}_{ij}\,x^i\,y^j,\quad \bar{q}(x,y,\delta)=q(x+x^*,y,\delta)=\sum_{i+j\ge 0}\bar{b}_{ij}\,x^i\,y^j.$$

For $\varepsilon=0$ the Hamiltonian function of system (38) is

$$\bar{H}(x,y)=\frac{y^2}{2}+(x+x^*)^2-\sqrt{(x+x^*+\alpha)^2+\beta^2}-\sqrt{(x+x^*-\alpha)^2+\beta^2}+c=\frac{y^2}{2}+\bar{H}_0(x),$$

where $c=-x^{*2}+\sqrt{(x^*+\alpha)^2+\beta^2}+\sqrt{(x^*-\alpha)^2+\beta^2}$ is a constant. Recall that $\alpha=\beta^{\frac{2}{3}}\sqrt{1-\beta^{\frac{2}{3}}}$ and $\beta\in(0,\frac{8}{5\sqrt{5}})$.

The Hamiltonian system $(38)|_{\varepsilon=0}$ has a family of periodic orbits $\Gamma_h : \bar{H}(x,y)=h$ for $h>0$ small, surrounding the origin. So, we have

$$M_1(h,\delta)=\oint_{\Gamma_h}\bar{q}\,dx-\bar{p}\,dy=\iint_{\bar{H}\le h}(\bar{p}_x+\bar{q}_y)\,dx\,dy=\oint_{\Gamma_h}\hat{q}(x,y,\delta)\,dx,$$

where

$$\hat{q}(x,y,\delta)=\bar{q}(x,y,\delta)-\bar{q}(x,0,\delta)+\int_0^y\bar{p}_x(x,u,\delta)\,du$$

verifies $\hat{q}_y = \bar{p}_x + \bar{q}_y$ and $\hat{q}(x,0,\delta) = 0$. If $\bar{p}_x(x,y,\delta) + \bar{q}_y(x,y,\delta) = \sum_{i+j\geq 0} c_{ij}\, x^i\, y^j$, then $\hat{q}(x,y,\delta) = y\sum_{i+j\geq 0}\hat{b}_{ij}\, x^i\, y^j = \sum_{j\geq 1}\hat{q}_j(x)\, y^j$, where

$$\hat{q}_{j+1}(x) = \frac{1}{(j+1)!}\frac{\partial^j}{\partial y^j}\left(\bar{p}_x + \bar{q}_y\right)\Big|_{\varepsilon=y=0} = \sum_{i\geq 1}\hat{b}_{ij}\, x^i, \quad j \geq 0.$$

Note that the equation $\bar{H}(x,y) = h$ has two C^ω solutions $y^\pm = \pm\sqrt{2}w(1 + O(|x,w|))$, where $w = \sqrt{h - \bar{H}_0(x)}$. Let $\zeta_1(h) < 0$ and $\zeta_2(h) > 0$ be the solutions of the equation $\bar{H}_0(x) = h$. Then

$$M_1(h,\delta) = \oint_{\Gamma_h}\hat{q}\, dx = \int_{\zeta_1}^{\zeta_2}(\hat{q}(x,y^+,\delta) - \hat{q}(x,y^-,\delta))\, dx = \sum_{j\geq 0}\int_{\zeta_1}^{\zeta_2} q_j^*(x)\, w^{2j+1}\, dx,$$

where $q_j^* = 2^{3j+\frac{3}{2}}\hat{q}_{2j+1}$. Let $u^2 = \bar{H}_0(x)$. Then, by introducing the variable $u = \psi(x) = sgn(x)(\bar{H}_0(x))^{\frac{1}{2}}$, we obtain

$$M_1(h,\delta) = \sum_{j\geq 0}\int_{-\sqrt{h}}^{\sqrt{h}}\check{q}_j(u)\, w^{2j+1}\, du = \sum_{j\geq 0}\int_0^{\sqrt{h}}\left(\check{q}_j(u) + \check{q}_j(-u)\right)\, w^{2j+1}\, du = \sum_{i+j\geq 0}\bar{r}_{ij}\,\bar{I}_{ij},$$

where $w = \sqrt{h-u^2}$, $\check{q}_j(u) = \frac{q_j^*(x)}{\psi'(x)}\Big|_{x=\psi^{-1}(u)}$, $\check{q}_j(u) + \check{q}_j(-u) = \sum_{j\geq 0}\bar{r}_{ij}\, u^{2i}$ and $\bar{I}_{ij} = \int_0^{\sqrt{h}} u^{2i}\, w^{2j+1}\, du$. By introducing $v = \frac{u}{\sqrt{h}}$, we get

$$\bar{I}_{ij} = h^{i+j+1}\int_0^1 v^{2i}\left(1-v^2\right)^j\sqrt{1-v^2}\, dv = h^{i+j+1}\, J_{ij}.$$

Therefore,

$$M_1(h,\delta) = h\sum_{i+j\geq 0}\bar{r}_{ij}\, J_{ij}\, h^{i+j} = h\sum_{k\geq 0} b_k(\delta)\, h^k, \qquad where \quad b_k(\delta) = \sum_{i+j=k}\bar{r}_{ij}\, J_{ij}. \quad (39)$$

For computing $\{b_k\}$, note that, by Taylor expansion, we obtain $x = \psi^{-1}(u) = \nu_1\, u + \nu_2\, u^2 + \nu_3\, u^3 + O(u^4)$, $\frac{1}{\psi'(x)} = \gamma_0 + \gamma_1\, x + \gamma_2\, x^2 + O(x^3)$, where

$$\begin{aligned}
\nu_1 = \Big(& 6x^{*2}\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}} + 4\alpha^2x^{*2} + 4\beta^2x^{*2} + 4x^{*4}\\
&+\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}}\alpha^2 + 2\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}}x^*\alpha + \sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}}\beta^2\\
&-3x^{*2}\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}} + \sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\alpha^2 - 2\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}x^*\alpha\\
&+\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\beta^2 - 3x^{*2}\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\\
&-\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}} + \alpha^2 - \beta^2 - x^{*2}\Big)^{-\frac{1}{2}}\\
&\times\Big(\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}}\alpha^2 + 2\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}}x^*\alpha + \sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}}\beta^2\\
&+x^{*2}\sqrt{\alpha^2-2\alpha x^*+\beta^2+x^{*2}} + \sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\alpha^2 - 2\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}x^*\alpha\\
&+\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\beta^2 + x^{*2}\sqrt{\alpha^2+2\alpha x^*+\beta^2+x^{*2}}\Big)^{\frac{1}{2}}
\end{aligned}$$

$$\gamma_0 = \Big(\sqrt{2}\sqrt{\alpha^4+2\alpha^2\beta^2-2\alpha^2x^{*2}+\beta^4+2\beta^2x^{*2}+x^{*4}}\,\Big(-8\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}x^{*4}$$

$$+4\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha^2\beta^2-9\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\beta^2x^{*2}$$
$$+2\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\beta^4+6\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\alpha\,\beta^2x^*$$
$$+18\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}x^{*4}-8\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\alpha^3x^*$$
$$-\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\alpha^2\beta^2+8\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\alpha^2x^{*2}-\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\beta^4$$
$$+8\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\alpha\,x^{*3}-20\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha^2x^{*2}$$
$$+20\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\beta^2x^{*2}+8\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha^3x^*$$
$$-\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha^2\beta^2+8\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha^2x^{*2}-\beta^4\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}$$
$$-6\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha\,\beta^2x^*-8\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha\,x^{*3}-9\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\beta^2x^{*2}$$
$$+2\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\alpha^4\Big)^{\frac{1}{2}}\Bigg)$$
$$\times\Bigg(\sqrt[4]{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\sqrt[4]{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\left(2\alpha^4+4\alpha^2\beta^2-8\alpha^2x^{*2}+2\beta^4+8\beta^2x^{*2}+6x^{*4}\right.$$
$$+2\alpha\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}x^*-\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}\beta^2-2x^{*2}\sqrt{\alpha^2+2x^*\alpha+\beta^2+x^{*2}}$$
$$\left.-2\alpha\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}x^*-\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\beta^2-2x^{*2}\sqrt{\alpha^2-2x^*\alpha+\beta^2+x^{*2}}\right)^{-1},$$

where $\alpha=\beta^{\frac{2}{3}}\sqrt{1-\beta^{\frac{2}{3}}}$ and the other coefficients have long terms that can be easily calculated by using the Taylor expansion. Also,

$$q_0^*(x)=2\sqrt{2}\hat{q}_1=2\sqrt{2}(\bar{p}_x+\bar{q}_y)|_{\varepsilon=y=0}=2\sqrt{2}\sum_{i=1}^{\infty}\left(\bar{b}_{i1}+(i+1)\bar{a}_{i+1,0}\right)x^i,$$
$$q_1^*(x)=16\sqrt{2}\hat{q}_3=\frac{8}{3}\sqrt{2}(\bar{p}_{xyy}+\bar{q}_{yyy})|_{\varepsilon=y=0}=\frac{16}{3}\sqrt{2}\sum_{i=1}^{\infty}\left(3\bar{b}_{i3}+(i+1)\bar{a}_{i+1,2}\right)x^i.$$

Thus,

$$\bar{r}_{00}=4\sqrt{2}\,\gamma_0\left(\bar{a}_{10}+\bar{b}_{01}\right),\quad \bar{r}_{01}=\frac{32}{3}\sqrt{2}\,\gamma_0\left(\bar{a}_{12}+3\,\bar{b}_{03}\right),$$
$$\bar{r}_{10}=4\sqrt{2}\left[\gamma_0\left(\nu_2\left(2\,\bar{a}_{20}+\bar{b}_{11}\right)+{\nu_1}^2\left(3\,\bar{a}_{30}+\bar{b}_{21}\right)\right)+\gamma_1{\nu_1}^2\left(2\,\bar{a}_{20}+\bar{b}_{11}\right)\right.$$
$$\left.+\left(\gamma_2{\nu_1}^2+\gamma_1\nu_2\right)\left(\bar{a}_{10}+\bar{b}_{01}\right)\right].$$

Therefore,

$$b_0(\delta)=\bar{r}_{00}\,J_{00},\quad b_1(\delta)=\bar{r}_{10}\,J_{10}+\bar{r}_{01}\,J_{01},\tag{40}$$

where

$$J_{00}=\int_0^1\sqrt{1-v^2}\,dv=\frac{\pi}{4},\quad J_{10}=\int_0^1 v^2\sqrt{1-v^2}\,dv=\frac{\pi}{16},\quad J_{01}=\int_0^1(1-v^2)\sqrt{1-v^2}\,dv=\frac{3\pi}{16}.$$

Now, we calculate the expansion of $M_2(h,\delta)$. First, by introducing the transformation $(x,y)=(X+x^*,Y)$, we shift $C_2(-x^*,0)$ to the origin. Then (rewriting again X as x and Y as y) we get

$$\dot{x}=y+\varepsilon\hat{p}(x,y,\delta),\tag{41}$$
$$\dot{y}=-2\,(x-x^*)+\frac{(x-x^*+\alpha)}{\sqrt{(x-x^*+\alpha)^2+\beta^2}}+\frac{(x-x^*-\alpha)}{\sqrt{(x-x^*-\alpha)^2+\beta^2}}+\varepsilon\hat{q}(x,y,\delta),$$

where

$$\hat{p}(x, y, \delta) = p(x-x^*, y, \delta) = \sum_{i+j\geq 0} \hat{a}_{ij}\, x^i\, y^j, \qquad \hat{q}(x, y, \delta) = q(x-x^*, y, \delta) = \sum_{i+j\geq 0} \hat{b}_{ij}\, x^i\, y^j.$$

For $\varepsilon = 0$ the Hamiltonian function of system (41) is

$$\hat{H}(x, y) = \frac{y^2}{2} + (x - x^*)^2 - \sqrt{(x - x^* + \alpha)^2 + \beta^2} - \sqrt{(x - x^* - \alpha)^2 + \beta^2} + c = \frac{y^2}{2} + \bar{H}_0(x),$$

where $c = -x^{*2} + \sqrt{(-x^* + \alpha)^2 + \beta^2} + \sqrt{(-x^* - \alpha)^2 + \beta^2}$ is a constant.

The Hamiltonian system $(41)|_{\varepsilon=0}$ has a continuous family of periodic orbits γ_h : $\hat{H}(x, y) = h$ for $h > 0$ small, surrounding the origin. So, similar to M_1, we have

$$M_2(h, \delta) = \oint_{\gamma_h} \hat{q}\, dx - \hat{p}\, dy = \iint_{\hat{H}\leq h} (\hat{p}_x + \hat{q}_y)\, dx\, dy = \oint_{\gamma_h} \check{q}(x, y, \delta)\, dx,$$

where

$$\check{q}(x, y, \delta) = \hat{q}(x, y, \delta) - \hat{q}(x, 0, \delta) + \int_0^y \hat{p}_x(x, u, \delta)\, du$$

verifies $\check{q}_y = \hat{p}_x + \hat{q}_y$ and $\check{q}(x, 0, \delta) = 0$. If $\hat{p}_x(x, y, \delta) + \hat{q}_y(x, y, \delta) = \sum_{i+j\geq 0} \hat{c}_{ij}\, x^i\, y^j$, then $\check{q}(x, y, \delta) = y \sum_{i+j\geq 0} \breve{b}_{ij}\, x^i\, y^j = \sum_{j\geq 1} \check{q}_j(x)\, y^j$, where

$$\check{q}_{j+1}(x) = \frac{1}{(j+1)!} \frac{\partial^j}{\partial y^j} (\hat{p}_x + \hat{q}_y)\Big|_{\varepsilon=y=0} = \sum_{i\geq 1} \breve{b}_{ij}\, x^i, \quad j \geq 0.$$

Note that the equation $\hat{H}(x, y) = h$ has two C^ω solutions $y^\pm = \pm\sqrt{2}w(1 + O(|x, w|))$, where $w = \sqrt{h - \hat{H}_0(x)}$. Let $\bar{\zeta}_1(h) < 0$ and $\bar{\zeta}_2(h) > 0$ be the solutions of the equation $\hat{H}_0(x) = h$. Then

$$M_2(h, \delta) = \oint_{\gamma_h} \check{q}\, dx = \int_{\bar{\zeta}_1}^{\bar{\zeta}_2} (\check{q}(x, y^+, \delta) - \check{q}(x, y^-, \delta))\, dx = \sum_{j\geq 0} \int_{\bar{\zeta}_1}^{\bar{\zeta}_2} \check{q}_j(x)\, w^{2j+1}\, dx,$$

where $\check{q}_j = 2^{3j+\frac{3}{2}} \check{q}_{2j+1}$. Let $u^2 = \hat{H}_0(x)$. Then, by introducing the variable $u = \rho(x) = sgn(x)(\hat{H}_0(x))^{\frac{1}{2}}$, we obtain

$$M_2(h, \delta) = \sum_{j\geq 0} \int_{-\sqrt{h}}^{\sqrt{h}} q_j^*(u)\, w^{2j+1}\, du = \sum_{j\geq 0} \int_0^{\sqrt{h}} (q_j^*(u) + q_j^*(-u))\, w^{2j+1}\, du = \sum_{i+j\geq 0} \hat{r}_{ij} \hat{I}_{ij},$$

where $w = \sqrt{h - u^2}$, $q_j^*(u) = \frac{\check{q}_j(x)}{\rho'(x)}\Big|_{x=\rho^{-1}(u)}$, $q_j^*(u) + q_j^*(-u) = \sum_{j\geq 0} \hat{r}_{ij}\, u^{2i}$ and $\hat{I}_{ij} = \int_0^{\sqrt{h}} u^{2i}\, w^{2j+1}\, du$. By introducing $v = \frac{u}{\sqrt{h}}$, we get,

$$\hat{I}_{ij} = \bar{I}_{ij} = h^{i+j+1} \int_0^1 v^{2i}\, (1 - v^2)^j \sqrt{1 - v^2}\, dv = h^{i+j+1}\, J_{ij}.$$

Thus,

$$M_2(h, \delta) = h \sum_{i+j\geq 0} \hat{r}_{ij}\, J_{ij}\, h^{i+j} = h \sum_{k\geq 0} \bar{b}_k(\delta)\, h^k, \quad where\ \bar{b}_k(\delta) = \sum_{i+j=k} \hat{r}_{ij}\, J_{ij}. \quad (42)$$

For computing $\{\bar{b}_k\}$, we see that $x = \rho^{-1}(u) = \nu_1\, u - \nu_2\, u^2 + \nu_3\, u^3 - \nu_4\, u^4 + O(u^5)$ and $\frac{1}{\rho'(x)} = \gamma_0 - \gamma_1\, x + \gamma_2\, x^2 + O(x^3)$, because of symmetry. Then

$$\begin{aligned}\hat{r}_{00} &= 4\sqrt{2}\,\gamma_0\left(\hat{a}_{10} + \hat{b}_{01}\right), \quad \hat{r}_{01} = \frac{32}{3}\sqrt{2}\,\gamma_0\left(\hat{a}_{12} + 3\,\hat{b}_{03}\right),\\ \hat{r}_{10} &= 4\sqrt{2}\Big[\gamma_0\left(-\nu_2\left(2\,\hat{a}_{20} + \hat{b}_{11}\right) + \nu_1{}^2\left(3\,\hat{a}_{30} + \hat{b}_{21}\right)\right)\\ &\quad -\gamma_1\nu_1{}^2\left(2\,\hat{a}_{20} + \hat{b}_{11}\right) + \left(\gamma_2\nu_1{}^2 + \gamma_1\nu_2\right)\left(\hat{a}_{10} + \hat{b}_{01}\right)\Big].\end{aligned}$$

Therefore,

$$\bar{b}_0(\delta) = \hat{r}_{00}\, J_{00}, \quad \bar{b}_1(\delta) = \hat{r}_{10}\, J_{10} + \hat{r}_{01}\, J_{01}. \tag{43}$$

Limit Cycle Bifurcation

In this section, by using the first nonvanishing coefficients of the expansions obtained in the previous sections, we discuss about the number of limit cycles which can be generated from system (6).

Let $L_0 = \bar{L}_0 \cup \tilde{L}_0$ be a double homoclinic loop defined by $H(x, y) = 0$. Assume that $H(x^*, 0) = h_{c_1}$ and $H(-x^*, 0) = h_{c_2}$. Consider the expantions of M, $\widetilde{M}$, M^*, M_1 and M_2, then we have the following theorems.

Theorem 2.6 *Under the above conditions, If there exists some $\delta_0 \in \mathbb{R}^m$ such that*

$$\begin{aligned}&c_0(\delta_0) = \tilde{c}_0(\delta_0) = c_1(\delta_0) = c_2(\delta_0) = c_3(\delta_0) = \tilde{c}_3(\delta_0) = 0, \quad c_4(\delta_0) \neq 0,\\ &b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{k_1-1}(\delta_0) = 0, \quad b_{k_1}(\delta_0) \neq 0,\\ &\bar{b}_0(\delta_0) = \bar{b}_1(\delta_0) = \cdots = \bar{b}_{k_2-1}(\delta_0) = 0, \quad \bar{b}_{k_2}(\delta_0) \neq 0,\end{aligned}$$

and

$$rank\,\frac{\partial(c_0, \tilde{c}_0, c_1, c_2, c_3, \tilde{c}_3, b_0, b_1, \ldots, b_{k_1-1}, \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{k_2-1})}{\partial(\delta_1, \ldots, \delta_k)} = 6 + k_1 + k_2, \tag{44}$$

then (6) can have $8 + k_1 + k_2 + \frac{1-sgn(M(h_2,\delta_0)M_1(h_1,\delta_0))}{2} + \frac{1-sgn(\widetilde{M}(h_4,\delta_0)M_2(h_3,\delta_0))}{2}$ limit cycles for some (ε, δ) near $(0, \delta_0)$ from which 8 limit cycles are near the double homoclinic loop, k_1 limit cycles are near the center C_1, k_2 limit cycles are near the center C_2, $\frac{1-sgn(M_1(h_1,\delta_0)M(h_2,\delta_0))}{2}$ limit cycle is located between C_1 and $\bar{L}_0$ and $\frac{1-sgn(\widetilde{M}(h_4,\delta_0)M_2(h_3,\delta_0))}{2}$ limit cycle is located between C_2 and $\tilde{L}_0$, where $h_1 = h_{c_1} + \varepsilon_1$, $h_2 = 0 - \varepsilon_2$, $h_3 = h_{c_2} + \varepsilon_3$, $h_4 = 0 - \varepsilon_4$ with ε_1, ε_2, ε_3 and ε_4 are positive and very small.

Proof Since $c_4(\delta_0) \neq 0$, in the same way as in Theorem 3.1 in [5], we can conclude that 8 limit cycles occur near the double homoclinic loop L_0. By (44), we know that $b_0, b_1, \ldots, b_{k_1-1}, \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{k_2-1}$ can be taken as free parameters. Now, we change the sign of these parameter to obtain the zeros of $M_j(h, \delta)$ for $j = 1, 2$. If

$$b_{j-1}b_j < 0, \quad j = 1, \ldots, k_1, \quad 0 < |b_0| \ll |b_1| \ll \ldots \ll |b_{k_1}|,$$

then we can find k_1 limit cycles are near the center C_1. If

$$\bar{b}_{j-1}\bar{b}_j < 0, \quad j = 1, \ldots, k_2, \quad 0 < |\bar{b}_0| \ll |\bar{b}_1| \ll \ldots \ll |\bar{b}_{k_2}|,$$

then we can find k_2 limit cycles are near the center C_2.

It is clear that if there exists $h_1 = h_{c_1} + \varepsilon_1$ and $h_2 = 0 - \varepsilon_2$ with ε_1 and ε_2 positive and very small such that $M_1(h_1, \delta_0).M(h_2, \delta_0)) < 0$, then we have $\frac{1-sgn(M_1(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ limit cycle is located between C_1 and $\bar{L}_0$. Similarly, for $h_3 = h_{c_2} + \varepsilon_3$ and $h_4 = 0 - \varepsilon_4$ with ε_3 and ε_4 positive and very small, we have $\frac{1-sgn(\widetilde{M}(h_4,\delta_0)M_2(h_3,\delta_0))}{2}$ limit cycle is located between C_2 and $\tilde{L}_0$. □

The next two theorems can be proved similarly.

Theorem 2.7 *Under the conditions of Theorem 2.6, if there exists some $\delta_0 \in \mathbb{R}^m$ such that*

$$
\begin{aligned}
&c_0(\delta_0) = \tilde{c}_0(\delta_0) = c_1(\delta_0) = c_2(\delta_0) = c_3(\delta_0) = \tilde{c}_3(\delta_0) = c_4(\delta_0) = 0, \quad c_5(\delta_0) \neq 0,\\
&b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{k_1-1}(\delta_0) = 0, \quad b_{k_1}(\delta_0) \neq 0,\\
&\bar{b}_0(\delta_0) = \bar{b}_1(\delta_0) = \cdots = \bar{b}_{k_2-1}(\delta_0) = 0, \quad \bar{b}_{k_2}(\delta_0) \neq 0,
\end{aligned}
$$

and

$$
rank \frac{\partial(c_0, \tilde{c}_0, c_1, c_2, c_3, \tilde{c}_3, c_4, b_0, b_1, \ldots, b_{k_1-1}, \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{k_2-1})}{\partial(\delta_1, \ldots, \delta_k)} = 7 + k_1 + k_2,
$$

then (6) can have $10 + k_1 + k_2 + \frac{1-sgn(M(h_2,\delta_0)M_1(h_1,\delta_0))}{2} + \frac{1-sgn(\widetilde{M}(h_4,\delta_0)M_2(h_3,\delta_0))}{2}$ limit cycles for some (ε, δ) near $(0, \delta_0)$ from which 10 limit cycles are near the double homoclinic loop, k_1 limit cycles are near the center C_1, k_2 limit cycles are near the center C_2, $\frac{1-sgn(M(h_2,\delta_0)M_1(h_1,\delta_0))}{2}$ limit cycle is located between C_1 and $\bar{L}_0$ and $\frac{1-sgn(\widetilde{M}(h_4,\delta_0)M_2(h_3,\delta_0))}{2}$ limit cycle is located between C_2 and $\tilde{L}_0$, where $h_1 = h_{c_1} + \varepsilon_1$, $h_2 = 0 - \varepsilon_2$, $h_3 = h_{c_2} + \varepsilon_3$, $h_4 = 0 - \varepsilon_4$ with ε_1, ε_2, ε_3 and ε_4 are positive and very small.

Theorem 2.8 *Under the conditions of Theorem 2.6, if there exists some $\delta_0 \in \mathbb{R}^m$ such that $M^*(h_0, \delta) \neq 0$ for some h_0 and,*

$$
\begin{aligned}
&c_0(\delta_0) = \tilde{c}_0(\delta_0) = c_1(\delta_0) = c_2(\delta_0) = c_3(\delta_0) = \tilde{c}_3(\delta_0) = c_4(\delta_0) = c_5(\delta_0) = 0, \quad c_6(\delta_0) \neq 0,\\
&b_0(\delta_0) = b_1(\delta_0) = \cdots = b_{k_1-1}(\delta_0) = 0, \quad b_{k_1}(\delta_0) \neq 0,\\
&\bar{b}_0(\delta_0) = \bar{b}_1(\delta_0) = \cdots = \bar{b}_{k_2-1}(\delta_0) = 0, \quad \bar{b}_{k_2}(\delta_0) \neq 0,
\end{aligned}
$$

and,

$$
rank \frac{\partial(c_0, \tilde{c}_0, c_1, c_2, c_3, \tilde{c}_3, c_4, b_0, b_1, \ldots, b_{k_1-1}, \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{k_2-1})}{\partial(\delta_1, \ldots, \delta_k)} = 8 + k_1 + k_2,
$$

then (6) can have $12 + k_1 + k_2 + \frac{1-sgn(M(h_2,\delta_0)M_1(h_1,\delta_0))}{2} + \frac{1-sgn(\widetilde{M}(h_4,\delta_0)M_2(h_3,\delta_0))}{2}$ limit cycles for some (ε, δ) near $(0, \delta_0)$ from which 12 limit cycles are near the double homoclinic loop, k_1 limit cycles are near the center C_1, k_2 limit cycles are near the center C_2, $\frac{1-sgn(M(h_2,\delta_0)M_1(h_1,\delta_0))}{2}$ limit cycle is located between C_1 and $\bar{L}_0$ and $\frac{1-sgn(\widetilde{M}(h_4,\delta_0)M_2(h_3,\delta_0))}{2}$ limit cycle is located between C_2 and $\tilde{L}_0$, where $h_1 = h_{c_1} + \varepsilon_1$, $h_2 = 0 - \varepsilon_2$, $h_3 = h_{c_2} + \varepsilon_3$, $h_4 = 0 - \varepsilon_4$ with ε_1, ε_2, ε_3 and ε_4 are positive and very small.

Application

In this section, we provide an example as an application of our main results. Let $\beta = \frac{1}{\sqrt{8}}$, then $\alpha = \beta^{\frac{2}{3}}\sqrt{1-\beta^{\frac{2}{3}}} = \frac{1}{\sqrt{8}}$ and system (4) becomes,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -2\,x + \frac{1}{4}\,\frac{2\,\sqrt{2}+8\,x}{\sqrt{2\,x\sqrt{2}+4\,x^2+1}} + \frac{1}{4}\,\frac{-2\,\sqrt{2}+8\,x}{\sqrt{-2\,x\sqrt{2}+4\,x^2+1}}, \end{aligned} \tag{45}$$

with the Hamiltonian function,

$$H(x,y) = \frac{1}{2}\,y^2 + x^2 - \frac{1}{2}\sqrt{2\,x\sqrt{2}+4\,x^2+1} - \frac{1}{2}\sqrt{-2\,x\sqrt{2}+4\,x^2+1} + 1. \tag{46}$$

We Consider the following perturbation,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -2\,x + \frac{1}{2}\,\frac{\sqrt{2}+4\,x}{\sqrt{2\,x\sqrt{2}+4\,x^2+1}} + \frac{1}{2}\,\frac{-\sqrt{2}+4\,x}{\sqrt{-2\,x\sqrt{2}+4\,x^2+1}} + \varepsilon f(x,\delta)y, \end{aligned} \tag{47}$$

where $f(x,\delta) = a_0 + a_1\,x + \cdots + a_7\,x^7 + a_8\,x^8$ and $\delta = (a_0, a_1, \ldots, a_8) \in \mathbb{R}^9$.
We have the following theorem.

Theorem 3.1 *System (47) can have 13 limit cycles.*

Proof System (45) has a nilpotent saddle at $A(0,0)$, two centers at $C_1(x^*,0)$ and $C_2(-x^*,0)$ with $x^* = 0.9013700925$, and a double homoclinic loop $L_0 = \bar{L}_0 \cup \tilde{L}_0$ passing through the nilpotent saddle A, defined by $H(x,y) = 0$, where $\bar{L}_0 = L_0|_{x\geq 0}$ and $\tilde{L}_0 = L_0|_{x\leq 0}$. Note that, by (46), we have that

$$\begin{aligned} \bar{L}_0 : y^2 &= -\,2\,x^2 + \sqrt{2\,x\sqrt{2}+4\,x^2+1} + \sqrt{-2\,x\sqrt{2}+4\,x^2+1} - 2, \quad 0 \leq x \leq \sqrt[4]{3}, \\ \tilde{L}_0 : y^2 &= -\,2\,x^2 + \sqrt{2\,x\sqrt{2}+4\,x^2+1} + \sqrt{-2\,x\sqrt{2}+4\,x^2+1} - 2, \quad -\sqrt[4]{3} \leq x \leq 0. \end{aligned}$$

From Proposition 2.2, we know that

$$c_0(\delta) = M(0,\delta) = \oint_{\bar{L}_0} f(x)y\,dx = \sum_{i=0}^{8} a_i\,I_i, \quad \tilde{c}_0(\delta) = \tilde{M}(0,\delta) = \oint_{\tilde{L}_0} f(x)y\,dx = \sum_{i=0}^{8} a_i\,\tilde{I}_i,$$

$$c_3(\delta) = \oint_{\bar{L}_0} \sum_{i=2}^{n} a_i x^i\,dt = \sum_{i=2}^{8} a_i\,J_i, \quad \tilde{c}_3(\delta) = \oint_{\tilde{L}_0} \sum_{i=2}^{n} a_i x^i\,dt = \sum_{i=2}^{8} a_i\,\tilde{J}_i,$$

where

$$\begin{aligned} I_i &= \oint_{\bar{L}_0} x^i\,y\,dx = 2\int_0^{\sqrt[4]{3}} x^i\sqrt{-\,2\,x^2 + \sqrt{2\,x\sqrt{2}+4\,x^2+1} + \sqrt{-2\,x\sqrt{2}+4\,x^2+1} - 2}\,dx, \\ \tilde{I}_i &= \oint_{\tilde{L}_0} x^i\,y\,dx = 2\int_{-\sqrt[4]{3}}^{0} x^i\sqrt{-\,2\,x^2 + \sqrt{2\,x\sqrt{2}+4\,x^2+1} + \sqrt{-2\,x\sqrt{2}+4\,x^2+1} - 2}\,dx, \end{aligned}$$

$$J_i = \oint_{\bar{L}_0} x^i\,dt = 2\int_0^{\sqrt[4]{3}} \frac{x^i}{\sqrt{-2\,x^2 + \sqrt{2\,x\sqrt{2}+4\,x^2+1} + \sqrt{-2\,x\sqrt{2}+4\,x^2+1} - 2}}\,dx,$$
$$\tilde{J}_i = \oint_{\tilde{L}_0} x^i\,dt = 2\int_{-\sqrt[4]{3}}^{0} \frac{x^i}{\sqrt{-2\,x^2 + \sqrt{2\,x\sqrt{2}+4\,x^2+1} + \sqrt{-2\,x\sqrt{2}+4\,x^2+1} - 2}}\,dx.$$

Using Maple we find that

$$\begin{aligned} &I_0 = \tilde{I}_0 = 0.8520193230, \quad I_1 = -\tilde{I}_1 = 0.6927396352, \quad I_2 = \tilde{I}_2 = 0.6267093568,\\ &I_3 = -\tilde{I}_3 = 0.6080411464, \quad I_4 = \tilde{I}_4 = 0.6195116802, \quad I_5 = -\tilde{I}_5 = 0.6542775182,\\ &I_6 = \tilde{I}_6 = 0.7101531116, \quad I_7 = -\tilde{I}_7 = 0.7875161356, \quad I_8 = \tilde{I}_8 = 0.8884959544, \end{aligned}$$

and

$$\begin{aligned} &J_2 = \tilde{J}_2 = 4.774680819, \quad J_3 = -\tilde{J}_3 = 4.715807320, \quad J_4 = \tilde{J}_4 = 5.253456626,\\ &J_5 = -\tilde{J}_5 = 6.124821684 \quad J_6 = \tilde{J}_6 = 7.321105054, \quad J_7 = -\tilde{J}_7 = 8.892929226,\\ &J_8 = \tilde{J}_8 = 10.92543392. \end{aligned}$$

Consequently, we obtain that

$$\begin{aligned} c_0 = {} & 0.8520193230\,a_0 + 0.6927396352\,a_1 + 0.6267093568\,a_2 + 0.6080411464\,a_3 + 0.6195116802\,a_4\\ & + 0.6542775182\,a_5 + 0.7101531116\,a_6 + 0.7875161356\,a_7 + 0.8884959544\,a_8,\\ \tilde{c}_0 = {} & 0.8520193230\,a_0 - 0.6927396352\,a_1 + 0.6267093568\,a_2 - 0.6080411464\,a_3 + 0.6195116802\,a_4\\ & - 0.6542775182\,a_5 + 0.7101531116\,a_6 - 0.7875161356\,a_7 + 0.8884959544\,a_8,\\ c_3 = {} & 4.774680819\,a_2 + 4.715807320\,a_3 + 5.253456626\,a_4 + 6.124821684\,a_5 + 7.321105054\,a_6\\ & + 8.892929226\,a_7 + 10.92543392\,a_8,\\ \tilde{c}_3 = {} & 4.774680819\,a_2 - 4.715807320\,a_3 + 5.253456626\,a_4 - 6.124821684\,a_5 + 7.321105054\,a_6\\ & - 8.892929226\,a_7 + 10.92543392\,a_8. \end{aligned}$$

Furthermore, from Proposition 2.2, we get that,

$$\begin{aligned} &c_1 = -2.233794126\,a_0, \quad c_2 = -\frac{1}{6}\sqrt{3}\,a_1, \quad c_4 = 0.3750381616\,a_0 + 0.5000508821\,a_2,\\ &c_5 = 1.650968872\,a_0 + 0.2659278720\,a_2 + 0.2127422976\,a_4,\\ &c_6 = 0.4570689630\,a_1 + 0.7216878361\,a_3 + 0.4811252243\,a_5. \end{aligned}$$

Finally, we calculate the coefficients b_j, $j = 0, 1, \cdots$, in (39). First, by introducing the transformation $(x, y) = (X - x^*, Y)$, we shift $C_1(x^*, 0)$ to the origin. Then (rewriting again X as x and Y as y) we get,

$$\begin{aligned} \dot{x} &= y,\\ \dot{y} &= -2\,(x + x^*) + \frac{1}{4}\,\frac{2\sqrt{2} + 8\,(x + x^*)}{\sqrt{2\,(x + x^*)\sqrt{2} + 4\,(x + x^*)^2 + 1}}\\ &\quad + \frac{1}{4}\,\frac{-2\sqrt{2} + 8\,(x + x^*)}{\sqrt{-2\,(x + x^*)\sqrt{2} + 4\,(x + x^*)^2 + 1}} + \varepsilon\,\hat{f}(x)y, \end{aligned} \tag{48}$$

where

$$\hat{f}(x) = f(x + x^*, \delta) = \hat{a}_0 + \hat{a}_1\,x + \cdots + \hat{a}_8\,x^8.$$

For $\varepsilon = 0$ the Hamiltonian function of system (48) is,

$$\bar{H}(x, y) = \frac{1}{2} y^2 + (x + x^*)^2 - \frac{1}{2}\sqrt{2(x + x^*)\sqrt{2} + 4(x + x^*)^2 + 1} - \frac{1}{2}\sqrt{-2(x + x^*)\sqrt{2} + 4(x + x^*)^2 + 1} + 1.1433077145.$$

By the formulas of b_j, in (40) for $j = 0, 1$, we obtain

$$b_0 = 1.157557250\sqrt{2}\pi\,(a_0 + 0.9013700925\,a_1 + 0.8124680437\,a_2 + 0.7323343957\,a_3 + 0.6601043220\,a_4 + 0.5949982938\,a_5 + 0.5363136671\,a_6 + 0.4834170997\,a_7 + 0.4357377159\,a_8),$$
$$b_1 = \pi\,(1.783836705\,a_0 - 0.4279155516\,a_1 + 1.031864517\,a_2 + 5.139643557\,a_3 + 11.06971485\,a_4 + 18.16201114\,a_5 + 25.89465002\,a_6 + 33.86056122\,a_7 + 41.74763006\,a_8).$$

By introducing the transformation $(x, y) = (X + x^*, Y)$, we shift $C_2(-x^*, 0)$ to the origin. Then (rewriting again X as x and Y as y) we get,

$$\begin{aligned} \dot{x} &= y,\\ \dot{y} &= -2(x - x^*) + \frac{1}{4}\frac{2\sqrt{2} + 8(x - x^*)}{\sqrt{2(x - x^*)\sqrt{2} + 4(x - x^*)^2 + 1}}\\ &\quad + \frac{1}{4}\frac{-2\sqrt{2} + 8(x - x^*)}{\sqrt{-2(x - x^*)\sqrt{2} + 4(x - x^*)^2 + 1}} + \varepsilon \bar{f}(x) y, \end{aligned} \tag{49}$$

where,

$$\bar{f}(x) = f(x - x^*, \delta) = \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_8 x^8.$$

For $\varepsilon = 0$ the Hamiltonian function of system (49) is,

$$\bar{H}(x, y) = \frac{1}{2} y^2 + (x - x^*)^2 - \frac{1}{2}\sqrt{2(x - x^*)\sqrt{2} + 4(x - x^*)^2 + 1} - \frac{1}{2}\sqrt{-2(x - x^*)\sqrt{2} + 4(x - x^*)^2 + 1} + 1.1433077145.$$

By the formulas of b_j, in (40) for $j = 0, 1$, we obtain

$$\bar{b}_0 = 1.157557248\sqrt{2}\pi\,(a_0 - 0.9013700925\,a_1 + 0.8124680437\,a_2 - 0.7323343957\,a_3 + 0.6601043220\,a_4 - 0.5949982938\,a_5 + 0.5363136671\,a_6 - 0.4834170997\,a_7 + 0.4357377159\,a_8),$$
$$\bar{b}_1 = \pi\,(-0.1663960129\,a_1 + 0.3889846550\,a_2 - 1.060341107\,a_3 + 2.041024002\,a_4 - 3.219541028\,a_5 + 4.507717919\,a_6 - 5.836755818\,a_7 + 7.153881753\,a_8 + 0.5653947622\,a_0).$$

(i) We can find $\delta_0 = (0, 0, a_2^*, a_3^*, a_4^*, a_5^*, a_6^*, a_7^*, a_8)$ with

$$a_2^* = -1.462202842\,a_8,\quad a_3^* = -0.4461010105\,a_8,\quad a_4^* = 3.733915692\,a_8,$$
$$a_5^* = 0.7592286090\,a_8,\quad a_6^* = -3.218072410\,a_8,\quad a_7^* = -0.2863413586\,a_8,$$

such that $(c_0, \tilde{c}_0, c_1, c_2, c_3, \tilde{c}_3, b_0, \bar{b}_0)(\delta_0) = (0, 0, 0, 0, 0, 0, 0, 0)$, and

$$c_4(\delta_0) = -0.7311758210\,a_8,\quad b_1(\delta_0) = 0.041997177\,\pi\,a_8,\quad \bar{b}_1(\delta_0) = -0.600087669\,\pi\,a_8.$$

Hence, for $h_1 = -1.143307714 + \varepsilon_1$, $h_2 = 0 + \varepsilon_2$, $h_3 = -1.143307714 + \varepsilon_3$, $h_4 = 0 + \varepsilon_4$ with ε_1, ε_2, ε_3 and ε_4 positive and sufficiently small, we have

$$M_1(h_1, \delta_0) = b_1(\delta_0)h_1^2 + O\left(h_1^3\right) > 0, \quad M(h_2, \delta_0) = c_4(\delta_0)|h_2|^{\frac{5}{4}} + O\left(|h_2|^{\frac{7}{4}}\right) < 0,$$
$$M_2(h_3, \delta_0) = \bar{b}_1(\delta_0)\, h_3^2 + O\left(h_3^2\right) < 0, \quad \widetilde{M}(h_4, \delta_0) = c_4(\delta_0)|h_4|^{\frac{5}{4}} + O\left(|h_4|^{\frac{7}{4}}\right) < 0.$$

Therefore, $\frac{1-sgn(M_1(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ and $\frac{1-sgn(M_2(h_3,\delta_0)\widetilde{M}(h_4,\delta_0))}{2} = 0$. Also, an easy computation shows that

$$rank \frac{\partial(c_0(\delta), \tilde{c}_0(\delta), c_1(\delta), c_2(\delta), c_3(\delta), \tilde{c}_3(\delta), b_0(\delta), \bar{b}_0(\delta))}{\partial(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)}\Big|_{\delta=\delta_0} = 8.$$

Thus, by Theorem 2.6 there exists some $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ near δ_0 such that system (47) has 11 limit cycles, from which eight limit cycles are near the double homoclinic loop L_0, one limit cycle is near the center C_1, one limit cycle is near the center C_2 and one limit cycle lies between C_1 and $\bar{L}_0$.

(ii) We can find $\delta_0 = (0, 0, 0, a_3^*, a_4^*, a_5^*, a_6^*, a_7^*, a_8)$ with

$$a_3^* = -2.130496054\, a_8, \quad a_4^* = 1.55819359\, a_8, \quad a_5^* = 3.625935646\, a_8,$$
$$a_6^* = -2.610444224\, a_8, \quad a_7^* = -1.367513456\, a_8,$$

such that $(c_0, \tilde{c}_0, c_1, c_2, c_3, \tilde{c}_3, c_4, b_0)(\delta_0) = (0, 0, 0, 0, 0, 0, 0, 0)$, and

$$c_5(\delta_0) = 0.3314936861\, a_8, \quad b_1(\delta_0) = -0.00063048\, \pi\, a_8, \quad \bar{b}_0(\delta_0) = 0.1483763082\, \sqrt{2}\, \pi\, a_8.$$

Thus, for $h_1 = -1.143307714+\varepsilon_1, h_2 = 0+\varepsilon_2, h_3 = -1.143307714+\varepsilon_3, h_4 = 0+\varepsilon_4$ with ε_1, ε_2, ε_3 and ε_4 positive and sufficiently small, we have

$$M_1(h_1, \delta_0) = b_1(\delta_0)h_1^2 + O\left(h_1^3\right) < 0, \quad M(h_2, \delta_0) = c_5(\delta_0)|h_2|^{\frac{7}{4}} + O\left(|h_2|^2 \ln|h_2|\right) > 0,$$
$$M_2(h_3, \delta_0) = \bar{b}_0(\delta_0)\, h_3 + O\left(h_3^2\right) < 0, \quad \widetilde{M}(h_4, \delta_0) = c_5(\delta_0)|h_4|^{\frac{7}{4}} + O\left(|h_4|^2 \ln|h_4|\right) > 0.$$

Hence, $\frac{1-sgn(M_1(h_1,\delta_0)M(h_2,\delta_0))}{2} = 1$ and $\frac{1-sgn(M_2(h_3,\delta_0)\widetilde{M}(h_4,\delta_0))}{2} = 1$. Also, an easy computation shows that

$$rank \frac{\partial(c_0(\delta), \tilde{c}_0(\delta), c_1(\delta), c_2(\delta), c_3(\delta), \tilde{c}_3(\delta), c_4(\delta), b_0(\delta))}{\partial(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)}\Big|_{\delta=\delta_0} = 8.$$

Therefore, by Theorem 2.7, there exists some $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ near δ_0 such that system (47) has 13 limit cycles, from which ten limit cycles are near the double homoclinic loop L_0, one limit cycle is near the center C_1, one limit cycle lies between C_1 and $\bar{L}_0$ and one limit cycle lies between C_2 and $\tilde{L}_0$.

(iii) We can find $\delta_0 = (0, 0, 0, a_3^*, a_4^*, a_5^*, a_6^*, a_7^*, a_8)$ with

$$a_3^* = 2.14389217\, a_8, \quad a_4^* = 1.558193598\, a_8, \quad a_5^* = -3.648734787\, a_8,$$
$$a_6^* = -2.610444224\, a_8, \quad a_7^* = 1.376112101\, a_8,$$

such that $(c_0, \tilde{c}_0, c_1, c_2, c_3, \tilde{c}_3, c_4, \tilde{b}_0)(\delta_0) = (0, 0, 0, 0, 0, 0, 0, 0)$, and

$$c_5(\delta_0) = 0.3314936861\, a_8, \quad b_0(\delta_0) = -10.45484998\, \sqrt{2}\, \pi\, a_8, \quad \bar{b}_1(\delta_0) = 0.009010108\, \pi\, a_8.$$

In consequently, for $h_1 = -1.143307714 + \varepsilon_1, h_2 = 0 + \varepsilon_2, h_3 = -1.143307714 + \varepsilon_3$, $h_4 = 0 + \varepsilon_4$ with ε_1, ε_2, ε_3 and ε_4 positive and sufficiently small, we have

$$M_1(h_1, \delta_0) = b_0(\delta_0)h_1 + O\left(h_1^2\right) > 0, \quad M(h_2, \delta_0) = c_5(\delta_0)|h_2|^{\frac{7}{4}} + O\left(|h_2|^2 \ln |h_2|\right) > 0,$$
$$M_2(h_3, \delta_0) = \bar{b}_1(\delta_0)\, h_3^2 + O\left(h_3^3\right) > 0, \quad \widetilde{M}(h_4, \delta_0) = c_5(\delta_0)|h_4|^{\frac{7}{4}} + O\left(|h_4|^2 \ln |h_4|\right) > 0.$$

Then $\frac{1-sgn(M_1(h_1,\delta_0)M(h_2,\delta_0))}{2} = 0$ and $\frac{1-sgn(M_2(h_3,\delta_0)\widetilde{M}(h_4,\delta_0))}{2} = 0$. Also, an easy computation shows that,

$$rank \left.\frac{\partial(c_0(\delta), \tilde{c}_0(\delta), c_1(\delta), c_2(\delta), c_3(\delta), \tilde{c}_3(\delta), c_4(\delta), \tilde{b}_0(\delta))}{\partial(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)}\right|_{\delta=\delta_0} = 8.$$

Therefore, by Theorem 2.7, there exists some $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ near δ_0 such that system (47) has 11 limit cycles, from which ten limit cycles are near the double homoclinic loop L_0 and one limit cycle is near the center C_2. □

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