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ORIGINAL RESEARCH

Optimal Control Problem for Coupled Time-Fractional Evolution Systems with Control Constraints

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Abstract In this paper, fractional optimal control problem for two dimensional coupled evolution system is investigated. The fractional time derivative is considered in Caputo sense. Constraints on controls are imposed. First, the existence and uniqueness of the state for these systems is proved. Then, the necessary and sufficient optimality conditions for the fractional Dirichlet problems with the quadratic performance functional are derived. Finally we give some examples to illustrate the applicability of our results.

Keywords Optimal control · Coupled time-fractional system · Caputo derivative

Mathematics Subject Classification 46C05 · 49J20 · 93C20

Introduction

This paper deals with fractional optimal control problems for coupled evolution equations fractional dynamic systems. A fractional dynamic system (FDS) is a system whose dynamics is described by fractional differential equations (FDEs), and a fractional optimal control problem (FOCP) is an optimal control problem for a FDS. Evolution equations represent an important class of linear problems and occur in the mathematical description of a large variety of physical problems. The most recent method in the study of free boundary value

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problems arising in filtration, heat conduction and diffusion theory uses a reformulation of these problems as evolution equations.

Integer order optimal control problems for evolution equations have been extensively studied by many authors, for comprehensive treatment of this topic we refer to the classical monograph by Lions [20]. Extensive treatment and various applications of the fractional calculus are discussed in the works of Agrawal et al. [1,2], Ahmad and Ntouyas [3], Bahaa et al. [6–9,12], Mophou [21,22], Debbouche and Nieto [14,15], Wang and Zhou [25], Tang and Ma [24] etc. It has been demonstrated that fractional order differential equations (FODEs) models dynamic systems and processes more accurately than integer order differential equations do, and fractional controllers perform better than integer order controllers.

In this paper, we consider optimal control problem for coupled evolution equations with Caputo derivatives. The novelties of this contribution is we generalize the previous studies in Agrawal et al. [1,2] and Mophou [21,22] for fractional coupled evolution systems which can used to describe many physical, chemical, mathematical and biological models. The existence and uniqueness of solutions for such equations are proved. Fractional optimal control is characterized by the adjoint problem. By using this characterization, particular properties of fractional optimal control are proved.

This paper is organized as follows. In "Preliminaries", we introduce some basic definitions and preliminary results. In "Coupled evolution system with Caputo derivatives", we formulate the fractional Dirichlet problem for evolution equations. In "Optimization theorem and the fractional control problem", we show that our fractional optimal control problem holds and we give the optimality conditions for the optimal control. Some illustrate examples are stated.

Preliminaries

Many definitions have been given of a fractional derivative, which include Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivative. We will formulate the problem in terms of the left and right Caputo fractional derivatives which will be given later.

Let $n \in N^*$ and Ω be a bounded open subset of R^n with a smooth boundary Γ of class C^2. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$.

Definition 2.1 Let $x : [a, b] \to R$ be a continuous function on $[a, b]$ and $\alpha > 0$ be a real number, and $n = [\alpha]$, where $[\alpha]$ denotes the smallest integer greater than or equal to α. The left (left RLFI) and the right (right RLFI) Riemann–Liouville fractional integrals of order α are defined by

$$
{}_aI_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds \quad \text{(left RLFI)},
$$

$$
{}_tI_b^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}x(s)ds \quad \text{(right RLFI)},
$$

where

$$
\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du, \quad {}_aI_t^0x(t) = {}_tI_b^0x(t) = x(t).
$$

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

The left (left RLFD) and the right (right RLFD) Riemann-Liouville fractional derivatives of order α are defined by

$$
\begin{aligned}
{}_aD_t^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}x(s)ds \ \ \text{(left RLFD)}\\
{}_tD_b^{\alpha}x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (s-t)^{n-\alpha-1}x(s)ds \ \ \text{(right RLFD)}
\end{aligned}
$$

where $\alpha \in (n-1, n), n \in N$.

Moreover, The left (left CFD) and the right (right CFD) Caputo fractional derivatives of order α are defined by

$$
{}_a^CD_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds \ \ \text{(left CFD)}
$$

provided that the integral is defined.

$$
{}_t^CD_b^{\alpha}x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (s-t)^{n-\alpha-1}x^{(n)}(s)ds \ \ \text{(right CFD)}
$$

provided that the integral is defined.

The relation between the right RLFD and the right CFD is as follows:

$$
{}_t^CD_b^{\alpha}x(t) = {}_tD_b^{\alpha}x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{(k-\alpha)}.
$$

If x and $x(i), i = 1, ..., n-1$, vanish at $t = a$, then ${}_aD_t^{\alpha}x(t) = {}_a^CD_t^{\alpha}x(t)$, and if they vanish at $t = b$, then ${}_tD_b^{\alpha}x(t) = {}_t^CD_b^{\alpha}x(t)$.

Further, it holds

$$
{}_0^CD_t^{\alpha}\, c = 0, \ \text{where c is a constant,}
$$

and

$$
{}_0^CD_t^{\alpha}\, t^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < [\alpha];\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq [\alpha], \end{cases}
$$

where $\mathbb{N}_0 = 0, 1, 2, \ldots$. We recall that for $\alpha \in \mathbb{N}$ the Caputo differential operator coincides with the usual differential operator of integer order.

Lemma 2.1 *Let $T > 0, u \in C^m([0, T]), p \in (m-1, m), m \in N$ and $v \in C^1([0, T])$. Then for $t \in [0, T]$, the following properties hold*

$$
\begin{gathered}
{}_aD_t^p v(t) = \frac{d}{dt}\,{}_aI_t^{1-p}v(t), \quad m = 1,\\
{}_aD_t^p\,{}_aI_t^p v(t) = v(t);\\
{}_0I_t^p\,{}_0D_t^p u(t) = u(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}u^{(k)}(0);\\
\lim_{t\to 0^+}{}_0^CD_t^p u(t) = \lim_{t\to 0^+}{}_0I_t^p u(t) = 0.
\end{gathered}
$$

Note also that when $T = +\infty$, ${}_0^CD_t^{\alpha}f(t)$ is the Weyl fractional integral of order α of f.

An important tool is the integration by parts formula for Caputo fractional derivatives, which is stated in the following lemma.

Lemma 2.2 [1,2]. *Let $\alpha \in (0,1)$, and $x, y : [a,b] \to R$ be two functions of class C^1. Then the following integration by parts formula holds:*

$$\int_a^b y(t)\,{}_a^C D_t x(t)dt = [{}_t I_b^{1-\alpha} y(t)x(t)]_a^b + \int_a^b x(t)\,{}_t D_b^\alpha y(t)dt.$$

Lemma 2.3 (Fractional Green's formula [23]). *Let $0 < \alpha \leq 1$. Then for any $\phi \in C^\infty(\overline{Q})$ we have*

$$\begin{aligned}\int_0^T \int_\Omega ({}_0^C D_t^\alpha y(x,t) + \mathcal{A}y(x,t))\phi(x,t)dxdt &= \int_\Omega \phi(x,T)\,{}_0^C I_t^{1-\alpha} y(x,T)dx \\ &- \int_\Omega \phi(x,0)\,{}_0^C I_t^{1-\alpha} y(x,0^+)dx + \int_0^T \int_{\partial\Omega} y \frac{\partial \phi}{\partial \nu_\mathcal{A}} d\Gamma dt \\ &- \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu_\mathcal{A}} \phi d\Gamma dt + \int_0^T \int_\Omega y(x,t)(-{}_0^C D_t^\alpha \phi(x,t) \\ &+ \mathcal{A}^* \phi(x,t))dxdt.\end{aligned}$$

where $\mathcal{A}$ is a given operator which is defined by (3.6) below and

$$\frac{\partial y}{\partial \nu_\mathcal{A}} = \sum_{i,j=1}^n a_{ij} \frac{\partial y}{\partial x_j} cos(n, x_j) \quad on\, \Gamma,$$

$cos(n, x_j)$ is the i-th direction cosine of n,n being the normal at Γ exterior to Ω.

We also introduce the space

$$\mathcal{W}(0,T) := \{y : y \in L^2(0,T; H_0^1(\Omega)),\ {}_0^C D_t^\alpha y(t) \in L^2(0,T; H^{-1}(\Omega))\}$$

in which a solution of a fractional differential systems is contained. The spaces considered in this paper are assumed to be real.

Lemma 2.4 *Let $0 < \alpha < 1$, $\mathbb{X}$ be a Banach space and $f \in \mathcal{C}([0,T], \mathbb{X})$. Then for all, $t_1, t_2 \in [0,T]$*

$$||{}_0 I_t^\alpha f(t_1) - {}_0 I_t^\alpha f(t_2)||_\mathbb{X} \leq \frac{||f||_{L^\infty((0,T);\mathbb{X})}}{\Gamma(\alpha+1)} |t_1 - t_2|^\alpha.$$

Remark 2.5 Since $\mathcal{C}([0,T], \mathbb{X}) \subset L^\infty((0,T); \mathbb{X}) \subset L^2((0,T); \mathbb{X})$ because $[0,T]$ is a bounded subset of $\mathbb{R}$, Lemma 2.4 holds for $f \in L^2((0,T); \mathbb{X})$ and we have that ${}_0 I_t^\alpha f \in \mathcal{C}([0,T], \mathbb{X}) \subset L^2((0,T); \mathbb{X})$.

Coupled evolution system with Caputo derivatives

For $y_{10}, y_{20} \in H_0^1(\Omega)$ and $f_1, f_2 \in L^2(0,T; H^{-1}(\Omega))$, let us consider the fractional problem for coupled evolution system:

Find

$$y = \{y_1, y_2\} \in \mathcal{W}(0,T) \times \mathcal{W}(0,T)$$

such that

$$ {}_0^C D_t^\alpha y_1(t) + \mathcal{A} y_1(t) + y_1(t) - y_2(t) = f_1(t), \quad a.e.\, t \in]0, T[, \tag{3.1} $$

$$ {}_0^C D_t^\alpha y_2(t) + \mathcal{A} y_2(t) + y_2(y) + y_1(t) = f_2(t), \quad a.e.\, t \in]0, T[, \tag{3.2} $$

$$ y_1(x, 0) = y_{0,1}(x), \quad x \in \Omega, \tag{3.3} $$

$$ y_2(x, 0) = y_{0,2}(x), \quad x \in \Omega, \tag{3.4} $$

$$ y_1(x, t) = 0, \quad y_2(x, t) = 0\,, x \in \Gamma, t \in (0, T), \tag{3.5} $$

here ${}_0^C D_t^\alpha y(t)$ is the Caputo fractional derivatives of $y : [0, t] \to H^{-1}(\Omega)$, where Ω has the same properties as in "Introduction". The monotone operator $\mathcal{A}$ in the state equations (3.1), (3.2) is a second order elliptic operator given by

$$ \mathcal{A} y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right) + a_0(x) y, \tag{3.6} $$

where $a_{ij}, i, j = 1, 2, \ldots, n$, be given function on Ω with the properties

$$ a_0(x), a_{ij}(x) \in L^\infty(\Omega) \quad \text{(with real values)}, $$

$$ a_0(x) \geq \beta > 0, \quad \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \beta(\xi_1^2 + \cdots + \xi_n^2), \quad \forall \xi \in R^n, $$

almost everywhere on Ω. The operator $\mathcal{A} \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega) \right)$ satisfying for some $\omega > 0$, and real κ, the coercivity condition

$$ (\mathcal{A} y, y)_{L^2(\Omega)} + \kappa |y|^2_{L^2(\Omega)} \geq \omega ||y||^2_{H_0^1(\Omega)} \ \ \forall y \in H_0^1(\Omega) \tag{3.7} $$

Lemma 3.1 *Let $f_1, f_2 \in L^2(Q)$ and $y_1, y_2 \in L^2((0, T); H_0^1(\Omega))$ be such that ${}_0^C D_t^\alpha y_1(t), {}_0^C D_t^\alpha y_2(t) \in L^2(Q)$ and ${}_0^C D_t^\alpha y_1(t) + \mathcal{A} y_1(t) + y_1(t) - y_2(t) = f_1(t)$, ${}_0^C D_t^\alpha y_2(t) + \mathcal{A} y_2(t) + y_2(t) + y_1(t) = f_2(t)$. Then we have*

(i) $y_1|_\Sigma,\ y_2|_\Sigma$ *exists and belongs $L^2((0, T); H^{-1}(\Gamma))$,*
(ii) $y_1(0),\ y_2(0)$ *belongs to $L^2(\Omega)$.*

Proof Since $a_{ij} \in C^1(\overline{\Omega})$ for $1 \leq i, j \leq n$, we have (i). On the other hand, in view of Lemma 2.4, ${}_0I_t^\alpha({}_0^C D_t^\alpha y_1(t)), {}_0I_t^\alpha({}_0^C D_t^\alpha y_2(t)) \in L^2(\Omega)$ because ${}_0^C D_t^\alpha y_1(t), {}_0^C D_t^\alpha y_2(t) \in L^2(Q)$. Hence, $y_1(0), y_2(0)$ exists and belongs to $L^2(\Omega)$ since ${}_0I_t^\alpha({}_0^C D_t^\alpha y_1(t)) = y_1(t) - y_1(0)$, ${}_0I_t^\alpha({}_0^C D_t^\alpha y_2(t)) = y_2(t) - y_2(0)$ and $y_1(t), y_2(t) \in L^2(\Omega)$. □

For the operator A in (3.6) we define the bilinear form as follows:

Definition 3.1 For each $t \in]0, T[$, $y = (y_1, y_2)$ and $\phi = (\phi_1, \phi_2)$ we define a family of bilinear forms $\pi(t; y, \phi)$ on $(H_0^1(\Omega))^2$ by:

$$ \pi(t; y, \phi) = \Big(\mathcal{A} y_1 + y_1 - y_2, \phi_1 \Big)_{L^2(\Omega)} + \Big(\mathcal{A} y_2 + y_2 + y_1, \phi_2 \Big)_{L^2(\Omega)}, \quad y, \phi \in (H_0^1(\Omega))^2. \tag{3.8} $$

Then Eq. (3.8) can be written as

$$ \pi(t; y, \phi) = \Big(\mathcal{A} y_1 + y_1 - y_2, \phi_1 \Big)_{L^2(\Omega)} + \Big(\mathcal{A} y_2 + y_2 + y_1, \phi_2 \Big)_{L^2(\Omega)} $$

$$= \left(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y_1}{\partial x_j}\right)+a_0(x)y_1+y_1-y_2,\phi_1(x)\right)_{L^2(\Omega)}$$

$$+\left(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y_2}{\partial x_j}\right)+a_0(x)y_2+y_2+y_1,\phi_2(x)\right)_{L^2(\Omega)}$$

$$= \int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y_1(x)\frac{\partial}{\partial x_j}\phi_1(x)dx+\int_\Omega a_0(x)y_1(x)\phi_1(x)dx.$$

$$+\int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y_2(x)\frac{\partial}{\partial x_j}\phi_2(x)dx+\int_\Omega a_0(x)y_2(x)\phi_2(x)dx$$

$$+\int_\Omega[y_1\phi_1+y_2\phi_2-y_2\phi_1+y_1\phi_2]dx \tag{3.9}$$

Lemma 3.2 *The bilinear form $\pi(t;y,\phi)$ in (3.9) is coercive on $(H_0^1(\Omega))^2$ that is for $y=(y_1,y_2)$ we have*

$$\pi(t;y,y)\geq\lambda\,||y||^2_{(H_0^1(\Omega))^2},\quad \lambda>0. \tag{3.10}$$

Proof It is well known that the ellipticity of $\mathcal{A}$ is sufficient for the coerciveness of $\pi(t;y,\phi)$ on $(H_0^1(\Omega))^2$. Since we have

$$\pi(t;y,\phi)=\int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y_1(x)\frac{\partial}{\partial x_j}\phi_1(x)dx+\int_\Omega a_0(x)y_1(x)\phi_1(x)dx$$

$$+\int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y_2(x)\frac{\partial}{\partial x_j}\phi_2(x)dx+\int_\Omega a_0(x)y_2(x)\phi_2(x)dx$$

$$+\int_\Omega[y_1\phi_1+y_2\phi_2-y_2\phi_1+y_1\phi_2]dx,$$

then we get

$$\pi(t;y,y)=\int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y_1(x)\frac{\partial}{\partial x_j}y_1(x)dx+\int_\Omega a_0(x)y_1(x)y_1(x)dx$$

$$+\int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_i}y_2(x)\frac{\partial}{\partial x_j}y_2(x)dx+\int_\Omega a_0(x)y_2(x)y_2(x)dx$$

$$+\int_\Omega[y_1y_1+y_2y_2-y_2y_1+y_1y_2]dx$$

$$\geq\beta\sum_{i,j=1}^{n}a_{ij}||\frac{\partial}{\partial x_i}y_1(x)||^2_{L^2(\Omega)}+\beta||y_1(x)||^2_{L^2(\Omega)}+||y_1(x)||^2_{L^2(\Omega)}$$

$$+\beta\sum_{i,j=1}^{n}a_{ij}||\frac{\partial}{\partial x_i}y_2(x)||^2_{L^2(\Omega)}+\beta||y_2(x)||^2_{L^2(\Omega)}+||y_2(x)||^2_{L^2(\Omega)}$$

$$\geq \lambda_1 ||y_1||^2_{H_0^1(\Omega)} + \lambda_1 ||y_2||^2_{H_0^1(\Omega)}$$
$$\geq \lambda ||y||^2_{(H_0^1(\Omega))^2}, \ \lambda = \max(\lambda_1, \lambda_2) > 0.$$

□

Also we assume that $\forall y, \phi \in (H_0^1(\Omega)^2$ the function $t \to \pi(t; y, \phi)$ is continuously differentiable in $]0, T[$ and the bilinear form $\pi(t; y, \phi)$ is symmetric,

$$\pi(t; y, \phi) = \pi(t; \phi, y) \qquad \forall y, \phi \in (H_0^1(\Omega))^2. \tag{3.11}$$

Then (3.1)–(3.5) constitute a fractional Dirichlet coupled problem. First by using the Lax–Milgram lemma, we prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (3.1)–(3.5).

Lemma 3.3 [21,22] (Fractional Green's formula for evolution systems). *Let $y = (y_1, y_2)$ be the solution of system (3.1)–(3.5). Then for any $\phi = (\phi_1, \phi_2) \in (C^\infty(\overline{Q}))^2$ such that $\phi(x, T) = (\phi_1, \phi_2)(x, T) = 0$ in Ω and $\phi = (\phi_1, \phi_2) = 0$ on Σ, we have for each $i = 1, 2$*

$$\begin{aligned}
&\int_0^T \int_\Omega ({}_0^C D_t^\alpha y_i(x,t) + \mathcal{A} y_i(x,t))\phi_i(x,t)dxdt \\
&= -\int_\Omega \phi_i(x,0) y_i(0) dx + \int_0^T \int_{\partial\Omega} y_i \frac{\partial \phi_i}{\partial \nu} d\Gamma dt \\
&\quad - \int_0^T \int_{\partial\Omega} \frac{\partial y_i}{\partial \nu} \phi_i d\Gamma dt + \int_0^T \int_\Omega y_i(x,t)(-{}_0^C D_t^\alpha \phi_i(x,t) \\
&\quad + \mathcal{A}^* \phi_i(x,t)) dxdt.
\end{aligned}$$

Lemma 3.4 *If (3.10) and (3.11) hold, then the problem (3.1)–(3.5) admits a unique solution $y(t) = (y_1(t), y_2(t)) \in (\mathcal{W}(0,T))^2$.*

Proof From the coerciveness condition (3.10) and using the Lax–Milgram lemma, there exists a unique element $y(t) = (y_1(t), y_2(t)) \in (H_0^1(\Omega))^2$ such that

$$({}_0^C D_t^\alpha y(t), \phi)_{(L^2(Q))^2} + \pi(t; y, \phi) = L(\phi) \quad \text{for all} \quad \phi = (\phi_1, \phi_2) \in (H_0^1(\Omega))^2, \tag{3.12}$$

where $L(\phi)$ is a continuous linear form on $(H_0^1(\Omega))^2$ and takes the form

$$\begin{aligned}
L(\phi) &= \int_Q [f_1\phi_1 + f_2\phi_2] dxdt + \int_\Omega [y_{0,1}\phi_1(x,0) + y_{0,2}\phi_2(x,0)]dx, \\
f &= (f_1, f_2) \in (L^2(Q))^2, \ y_0 = (y_{0,1}, y_{0,2}) \in (L^2(\Omega))^2.
\end{aligned} \tag{3.13}$$

Then Eq. (3.12) equivalents to there exists a unique solution $y(t) = (y_1(t), y_2(t)) \in (H_0^1(\Omega))^2$ for

$$\begin{aligned}
&\left({}_0^C D_t^\alpha y_1(t) + \mathcal{A} y_1(t) + y_1 + y_2, \phi_1(x)\right)_{L^2(Q)} \\
&\quad + \left({}_0^C D_t^\alpha y_2(t) + \mathcal{A} y_2(t) + y_2 - y_1, \phi_2(x)\right)_{L^2(Q)} = L(\phi).
\end{aligned} \tag{3.14}$$

Then Eq. (3.14) is equivalent to the fractional evolution equations

$$ {}_0^C D_t^{\alpha} y_1(t) + \mathcal{A}y_1(t) + y_1 + y_2 = f_1, \tag{3.15} $$
$$ {}_0^C D_t^{\alpha} y_2(t) + \mathcal{A}y_2(t) + y_2 - y_1 = f_2, \tag{3.16} $$

"tested" against $\phi_1(x)$, $\phi_2(x)$ respectively.

Let us multiply both sides in (3.16), (3.17) by $\phi_1(x)$, $\phi_2(x)$ respectively and applying Green's formula (Lemma 3.3), we have

$$ \int_Q ({}_0^C D_t^{\alpha} y_1(t) + \mathcal{A}y_1(t) + y_1 + y_2)\phi_1(x)dxdt $$
$$ = \int_Q f_1\phi_1 dxdt \quad \text{for all } \phi_1(x) \in H_0^1(\Omega), \tag{3.17} $$
$$ \int_Q ({}_0^C D_t^{\alpha} y_2(t) + \mathcal{A}y_2(t) + y_2 - y_1)\phi_2(x)dxdt $$
$$ = \int_Q f_2\phi_2 dxdt \quad \text{for all } \phi_2(x) \in H_0^1(\Omega) \tag{3.18} $$

applying Green's formula (Lemma 3.3), we have

$$ \begin{aligned} &-\int_\Omega \phi_1(x,0)y_1(x,0)dx + \int_0^T\int_{\partial\Omega} y_1\frac{\partial\phi_1}{\partial\nu}d\Gamma dt - \int_0^T\int_{\partial\Omega}\frac{\partial y_1}{\partial\nu}\phi d\Gamma dt \\ &\quad + \int_0^T\int_\Omega y_1(x,t)(-{}_0^C D_t^{\alpha}\phi_1(x,t) + \mathcal{A}^*\phi_1(x,t))dxdt + \int_Q (y_1+y_2)\phi_1 dxdt \\ &= \int_Q f_1\phi_1(x)dxdt - \int_\Omega \phi_2(x,0)y_2(x,0)dx + \int_0^T\int_{\partial\Omega} y_2\frac{\partial\phi_1}{\partial\nu}d\Gamma dt \\ &\quad - \int_0^T\int_{\partial\Omega}\frac{\partial y_2}{\partial\nu}\phi d\Gamma dt + \int_0^T\int_\Omega y_2(x,t)(-{}_0^C D_t^{\alpha}\phi_2(x,t) + \mathcal{A}^*\phi_2(x,t))dxdt \\ &\quad + \int_Q (y_2 - y_1)\phi_2 dxdt = \int_Q f_2\phi_2(x)dxdt \end{aligned} $$

whence comparing with (3.12), (3.13), we get

$$ \int_\Omega \phi_1(x,0)y_1(x,0)dx - \int_0^T\int_{\partial\Omega} y_1\frac{\partial\phi_1}{\partial\nu}d\Gamma dt = \int_\Omega y_{0,1}\phi_1(x,0)dx. $$
$$ \int_\Omega \phi_2(x,0)y_2(x,0)dx - \int_0^T\int_{\partial\Omega} y_2\frac{\partial\phi_2}{\partial\nu}d\Gamma dt = \int_\Omega y_{0,2}\phi_2(x,0)dx. $$

From this we deduce the initial conditions

$$ y_1(x,0) = y_{0,1}, \quad x \in \Omega $$
$$ y_2(x,0) = y_{0,2}, \quad x \in \Omega $$

which completes the proof. □

Optimization theorem and the fractional control problem

For a control $u = (u_1, u_2) \in (L^2(Q))^2$, the state $y(u) = (y_1(u), y_1(u))$ of the system is given by the fractional variation coupled systems:

$$ {}_0^C D_t^{\alpha} y_1(u) + \mathcal{A}y_1(u) + y_1(u) - y_2(u) = f_1(t) + u_1, \text{ in } Q, $$

$$a.e.\, t \in]0, T[, \;\; f_1 \in L^2(Q), \tag{4.1}$$
$$ {}_0^C D_t^\alpha y_2(u) + \mathcal{A} y_2(u) + y_2(u) + y_1(u) = f_2(t) + u_2, \text{ in } Q,$$
$$a.e.\, t \in]0, T[, \;\; f_2 \in L^2(Q), \tag{4.2}$$
$$y_1(x, 0; u) = y_{0,1}(x) \in L^2(\Omega), \quad x \in \Omega, \tag{4.3}$$
$$y_2(x, 0; u) = y_{0,2}(x) \in L^2(\Omega), \quad x \in \Omega, \tag{4.4}$$
$$y_1(x, t) = 0, \quad y_2(x, t) = 0\,, x \in \Gamma, t \in (0, T), \tag{4.5}$$

The observation equations are given by

$$z_i(u) = y_i(u), \text{ for each } i = 1, 2. \tag{4.6}$$

The cost function $J(v)$ for $v = \{v_1, v_2\}$ is given by

$$J(v) = \int_Q [(y_1(v) - z_{d,1})^2 + (y_2(v) - z_{d,2})^2] dxdt + (Nv, v)_{(L^2(Q))^2}$$

where $z_d = \{z_{d,1}, z_{d,2}\}$ is a given element in $(L^2(Q))^2$ and $N = \{N_1, N_2\} \in \mathcal{L}(L^2(Q), L^2(Q))$ is hermitian positive definite operator:

$$(Nu_i, u_i) \geq c||u_i||^2_{L^2(Q)}, \; c > 0, \text{ for each } i = 1, 2. \tag{4.7}$$

Control constraints We define U_{ad}(set of admissible controls) is closed, convex subset of $U = L^2(Q) \times L^2(Q)$.

Control problem We want to minimize J over U_{ad} i.e. find $u = \{u_1, u_2\}$ such that

$$J(u) = \inf_{v=\{v_1, v_2\} \in U_{ad}} J(v). \tag{4.8}$$

Under the given considerations we have the following theorem:

Theorem 4.1 *The problem* (4.8) *admits a unique solution given by* (4.1)–(4.5) *and the optimality condition*

$$\int_Q [p_1(v_1 - u_1) + p_2(v_2 - u_2)]\, dxdt + (Nu, v - u)_U \geq 0, \quad \forall v \in U_{ad}, \; u \in U_{ad} \tag{4.9}$$

where $p(u) = \{p_1(u), p_2(u)\}$ *is the adjoint state.*

Proof Since the control $u \in U_{ad}$ is optimal if and only if

$$J'(u)(v - u) \geq 0 \quad \text{for all } v \in U_{ad}$$

The above condition, when explicitly calculated for this case, gives

$$(y_1(u) - z_{d,1}, y_1(v) - y_1(u))_{L^2(Q)} + (y_2(u) - z_{d,2}, y_2(v) \\ - y_2(u))_{L^2(Q)} + (Nu, v - u)_U \geq 0$$

i.e.

$$\int_Q [y_1(u) - z_{d,1})(y_1(v) - y_1(u)) + (y_2(u) - z_{d,2})(y_2(v) - y_2(u))] dxdt \\ + (Nu, v - u)_{(L^2(Q))^2} \geq 0. \tag{4.10}$$

For the control $u \in (L^2(Q))^2$ the adjoint state $p(u) = \{p_1(u), p_2(u)\} \in (L^2(Q))^2$ is defined by

$$- {}_0^C D_t^\alpha p_1(u) + \mathcal{A}^* p_1(u) + p_1(u) + p_2(u) = y_1(u) - z_{d,1}, \quad \text{in } Q, \tag{4.11}$$

$$-{}_0^C D_t^\alpha p_2(u) + \mathcal{A}^* p_2(u) + p_2(u) - p_1(u) = y_2(u) - z_{d,2}, \quad \text{in } Q, \tag{4.12}$$

$$p_1(u) = 0, \quad p_2(u) = 0 \quad \text{on } \Sigma, \tag{4.13}$$

$$p_1(x, T; u) = 0, \quad p_2(x, T; u) = 0 \quad \text{in } \Omega, \tag{4.14}$$

where $\mathcal{A}^*$ is the adjoint operator for the operator $\mathcal{A}$, which given by

$$\mathcal{A}^* p = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial p}{\partial x_i}\right) + a_0(x)p.$$

Now, multiplying the Eqs. (4.11), (4.12) by $(y_1(v) - y_1(u))$, $(y_2(v) - y_2(u))$ respectively and applying Green's formula, we obtain

$$\begin{aligned}
\int_Q (y_1(u) - z_{d,1})(y_1(v) - y_1(u))dxdt &= \int_Q [-{}_0^C D_t^\alpha p_1(u) \\
&\quad + \mathcal{A}^* p_1(u) + p_1(u) + p_2(u)](y_1(v) - y_1(u))\,dxdt \\
&= -\int_\Omega p_1(x,0)(y_1(v; x, 0) - y_1(u; x, 0^+))dx \\
&\quad + \int_\Sigma p_1(u)(\frac{\partial y_1(v)}{\partial \nu_{\mathcal{A}}} - \frac{\partial y_1(u)}{\partial \nu_{\mathcal{A}}})\,d\Sigma \\
&\quad - \int_\Sigma \frac{\partial p_1(u)}{\partial \nu_{\mathcal{A}}}(y_1(v) - y_1(u))\,d\Sigma \\
&\quad + \int_Q [p_1(u)({}_0^C D_t^\alpha + \mathcal{A} + 1) + p_2(u)](y_1(v) - y_1(u))\,dxdt,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\int_Q (y_2(u) - z_{d,2})(y_2(v) - y_2(u))dxdt &= \int_Q [-{}_0^C D_t^\alpha p_2(u) \\
&\quad + \mathcal{A}^* p_2(u) + p_2(u) - p_1(u)](y_2(v) - y_2(u))\,dxdt \\
&= -\int_\Omega p_2(x,0)(y_2(v; x, 0) - y_2(u; x, 0^+))dx \\
&\quad + \int_\Sigma p_2(u)(\frac{\partial y_2(v)}{\partial \nu_{\mathcal{A}}} - \frac{\partial y_2(u)}{\partial \nu_{\mathcal{A}}})\,d\Sigma \\
&\quad - \int_\Sigma \frac{\partial p_2(u)}{\partial \nu_{\mathcal{A}}}(y_2(v) - y_2(u))\,d\Sigma \\
&\quad + \int_Q [p_2(u)({}_0^C D_t^\alpha + \mathcal{A} + 1) - p_1(u))](y_2(v) - y_2(u))\,dxdt.
\end{aligned} \tag{4.16}$$

Using (4.1)–(4.5), (4.13) and (4.14), we have

$$\begin{aligned}
&\int_Q [p_1(u)({}_0^C D_t^\alpha + \mathcal{A} + 1) + p_2(u)](y_1(v) - y_1(u))\,dxdt = \int_Q p_1(u)(v_1 - u_1)\,dxdt \\
&\int_Q [p_2(u)({}_0^C D_t^\alpha + \mathcal{A} + 1) - p_1(u))](y_2(v) - y_2(u))\,dxdt = \int_Q p_2(u)(v_2 - u_2)\,dxdt \\
&y_1(u)|_\Sigma = 0,\ y_2(u)|_\Sigma = 0 \quad p_1(u)|_\Sigma = 0,\ p_2(u)|_\Sigma = 0, \\
&y_1(v; x, 0) - y_1(u; x, 0) = y_{0,1}(x) - y_{0,1}(x) = 0 \\
&y_2(v; x, 0) - y_2(u; x, 0) = y_{0,2}(x) - y_{0,2}(x) = 0
\end{aligned}$$

Then (4.15) becomes

$$\int_Q (y_1(u) - z_{d,1})(y_1(v) - y_1(u))dxdt = \int_Q p_1(u)(v_1 - u_1)dxdt,$$

and (4.16) becomes

$$\int_Q (y_2(u) - z_{d,2})(y_2(v) - y_2(u))dxdt = \int_Q p_2(u)(v_2 - u_2)dxdt,$$

and hence (4.10) is equivalent to

$$\int_Q p_1(u)(v_1 - u_1)\, dxdt + \int_Q p_2(u)(v_2 - u_2)\, dxdt + (Nu, v - u)_{(L^2(Q))^2} \geq 0$$

which can be written as:

$$\int_Q [(p_1(u) + N_1 u_1)(v_1 - u_1) + (p_2(u) + N_2 u_2)(v_2 - u_2)\, dxdt \geq 0$$

which completes the proof. □

Example 4.1 see [16] and [17]. We consider an example of an evolution equation which is analogous to that considered in section 2 but with Neumann boundary condition and boundary control.

In this example we consider the space

$$\mathcal{W}(0, T) := \{y : y \in L^2(0, T; H^1(\Omega)),\ {}_0^C D_t^\alpha y(t) \in L^2(0, T; (H^1(\Omega))^{'})\}$$

in which a solution of a fractional differential systems is contained. Let $y(u) = \{y_1(u), y_2(u)\} \in \mathcal{W}(0, T)$ be the state of the system which is given by,

$${}_0^C D_t^\alpha y_1(u) + \mathcal{A}y_1(u) + y_1(u) - y_2(u) = f_1(t), \text{ in } Q, \quad a.e.\, t \in]0, T[,\ f_1 \in L^2(Q), \tag{4.17}$$

$${}_0^C D_t^\alpha y_2(u) + \mathcal{A}y_2(u) + y_2(u) + y_1(u) = f_2(t), \text{ in } Q, \quad a.e.\, t \in]0, T[,\ f_2 \in L^2(Q), \tag{4.18}$$

$$y_1(x, 0; u) = y_{0,1}(x) \in L^2(\Omega), \quad x \in \Omega, \tag{4.19}$$

$$y_2(x, 0; u) = y_{0,2}(x) \in L^2(\Omega), \quad x \in \Omega, \tag{4.20}$$

$$\frac{\partial y_1(x, t)}{\partial \nu_A}|_\Sigma = u_1, \quad \frac{\partial y_2(x, t)}{\partial \nu_A}|_\Sigma = u_2\,, x \in \Gamma, t \in (0, T). \tag{4.21}$$

The control $u = \{u_1, u_2\}$ is taken in $L^2(\Sigma) \times L^2(\Sigma)$:

$$u = \{u_1, u_2\} \in U = L^2(\Sigma) \times L^2(\Sigma).$$

Problem (4.17)–(4.21) admits a unique solution. To see this we apply Theorem (1.2) [20], with

$$V = H^1(\Omega) \times H^1(\Omega), \quad \phi = \{\phi_1, \phi_2\} \in V,$$

$$\pi(t; y, \phi) = \pi(y, \phi) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_i} y_1(x) \frac{\partial}{\partial x_j} \phi_1(x) dx + \int_\Omega a_0(x) y_1(x) \phi_1(x) dx$$

$$+\int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial}{\partial x_i}y_2(x)\frac{\partial}{\partial x_j}\phi_2(x)dx + \int_\Omega a_0(x)y_2(x)\phi_2(x)dx$$
$$+\int_\Omega [y_1\phi_1 + y_2\phi_2 - y_2\phi_1 + y_1\phi_2]dx,$$

$$L(\phi) = (f,\phi) = \int_\Omega [f_1(x,t)\phi_1(x) + f_2(x,t)\phi_2(x)]dx + \int_\Gamma [u_1(t)\phi_1(x) + u_2(t)\phi_2(x)]d\Gamma.$$

Let us consider the case where we have partial observation of the final state

$$z(v) = y_1(x,T;v), \tag{4.22}$$

and the cost function $J(v)$ for $v = \{v_1, v_2\}$ is given by

$$J(v) = \int_\Omega (y_1(x,T;v) - z_d)^2 dx + (Nv, v)_{(L^2(\Sigma))^2}, \quad z_d \in L^2(\Omega).$$

where $N = \{N_1, N_2\} \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma))$ is hermitian positive definite operator:

$$(Nu, u) \geq c||u||^2_{L^2(\Sigma)}, \; c > 0 \tag{4.23}$$

Control constraints We define U_{ad}(set of admissible controls) is closed, convex subset of $U = L^2(\Sigma) \times L^2(\Sigma)$.

Control problem We want to minimize J over U_{ad} i.e. find $u = \{u_1, u_2\}$ such that

$$J(u) = \inf_{v=\{v_1,v_2\}\in U_{ad}} J(v). \tag{4.24}$$

The adjoint state is given by

$$-{}_0^C D_t^\alpha p_1(u) + \mathcal{A}^* p_1(u) + p_1(u) + p_2(u) = 0, \quad \text{in } Q, \tag{4.25}$$
$$-{}_0^C D_t^\alpha p_2(u) + \mathcal{A}^* p_2(u) + p_2(u) - p_1(u) = 0, \quad \text{in } Q, \tag{4.26}$$
$$\frac{\partial p_1(u)}{\partial \nu_{A^*}} = 0, \quad \frac{\partial p_2(u)}{\partial \nu_{A^*}} = 0 \quad \text{on } \Sigma, \tag{4.27}$$
$$p_1(x,T;u) = y_1(u) - z_d, \quad \text{in } \Omega, \quad p_2(x,T;u) = 0 \quad \text{in } \Omega. \tag{4.28}$$

The optimality condition is

$$\int_\Sigma [p_1(u)(v_1 - u_1) + p_2(u)(v_2 - u_2)]\, d\Sigma + (Nu, v - u)_{(L^2(\Sigma))^2} \geq 0,$$
$$\forall v \in U_{ad},\; u \in U_{ad}. \tag{4.29}$$

Example 4.2 In the case of no constraint on the control ($U = U_{ad}$) and $N = \{N_1, N_2\}$ is a diagonal matrix of operators. Then (4.29) reduces to

$$p_1 + N_1 u_1 = 0 \quad \text{on } \Sigma, \quad p_2 + N_2 u_2 = 0 \quad \text{on } \Sigma,$$

which equivalent to

$$u_1 = -N_1^{-1}(p_1(u)|_{\Sigma})), \quad u_2 = -N_2^{-1}(p_2(u)|_\Sigma). \tag{4.30}$$

The fractional optimal control is obtained by the simultaneous solving (4.17)–(4.21) and (4.25)–(4.28) (where we eliminate u_1, u_2 with the aid of (4.30)) and then utilizing (4.30).

Example 4.3 If we take

$$\mathcal{U}_{ad} = \left\{ u_i | u_i \in L^2(\Sigma), u_i \geq 0 \quad \text{almost everywhere on} \quad \Sigma, i = 1, 2 \right\},$$

and $N = \nu \times$Identity, (4.29) gives

$$\begin{aligned} &u_1 \geq 0, \quad p_1(u) + \nu_1 u_1 \geq 0, \quad u_1(p_1(u) + \nu_1 u_1) = 0 \quad \text{on } \Sigma, \\ &u_2 \geq 0, \quad p_2(u) + \nu_2 u_2 \geq 0, \quad u_2(p_2(u) + \nu_2 u_2) = 0 \quad \text{on } \Sigma. \end{aligned}$$

The fractional optimal control is obtained by the solution of the fractional problem

$$\begin{aligned} &{}_0^C D_t^\alpha y_1(u) + \mathcal{A} y_1(u) + y_1(u) - y_2(u) = f_1(t), \text{ in } Q, \quad a.e. \, t \in]0, T[, \; f_1 \in L^2(Q), \\ &{}_0^C D_t^\alpha y_2(u) + \mathcal{A} y_2(u) + y_2(u) + y_1(u) = f_2(t), \text{ in } Q, \quad a.e. \, t \in]0, T[, \; f_2 \in L^2(Q), \\ &-{}_0^C D_t^\alpha p_1(u) + \mathcal{A}^* p_1(u) + p_1(u) + p_2(u) = 0, \quad \text{in } Q, \\ &-{}_0^C D_t^\alpha p_2(u) + \mathcal{A}^* p_2(u) + p_2(u) - p_1(u) = 0, \quad \text{in } Q, \\ &y_1(x, 0; u) = y_{0,1}(x) \in L^2(\Omega), \quad x \in \Omega, \\ &y_2(x, 0; u) = y_{0,2}(x) \in L^2(\Omega), \quad x \in \Omega, \\ &p_1(x, T; u) = y_1(u) - z_d, \quad \text{in } \Omega, \\ &p_2(x, T; u) = 0 \quad \text{in } \Omega. \\ &\frac{\partial y_1(x,t)}{\partial \nu_A}|_\Sigma \geq 0, \quad \frac{\partial y_2(x,t)}{\partial \nu_A}|_\Sigma \geq 0, x \in \Gamma, t \in (0, T). \\ &\frac{\partial p_1(u)}{\partial \nu_{A^*}} = 0, \quad \frac{\partial p_2(u)}{\partial \nu_{A^*}} = 0 \quad \text{on } \Sigma, \\ &p_1 + \nu_1 \frac{\partial y_1}{\partial \nu_A} \geq 0, \quad \frac{\partial y_1}{\partial \nu_A}[p_1 + \nu_1 \frac{\partial y_1}{\partial \nu_A}] = 0 \quad \text{on } \Sigma, \\ &p_2 + \nu_2 \frac{\partial y_2}{\partial \nu_A} \geq 0, \quad \frac{\partial y_2}{\partial \nu_A}[p_2 + \nu_2 \frac{\partial y_2}{\partial \nu_A}] = 0 \quad \text{on } \Sigma, \end{aligned}$$

hence

$$u_1 = \frac{\partial y_1}{\partial \nu_A}|_\Sigma, \quad u_2 = \frac{\partial y_2}{\partial \nu_A}|_\Sigma.$$

Example 4.4 We can generalize our results to n dimensional coupled fractional system as follows. The state of the system is given, for each $i = 1, 2, \ldots, n$, by

$${}_0^C D_t^\alpha y_i(u) + \mathcal{A} y_i(u) + \sum_{j=1}^{n} b_{ij} y_j(u) = f_i(t), \text{ in } Q, \quad a.e. \, t \in]0, T[, \; f_1 \in L^2(Q), \tag{4.31}$$

$$y_i(x, 0; u) = y_{0,i}(x) \in L^2(\Omega), \quad x \in \Omega, \tag{4.32}$$

$$\frac{\partial y_i(x,t)}{\partial \nu_A}|_\Sigma = u_i, \quad x \in \Gamma, t \in (0, T), \tag{4.33}$$

$$b_{ij} = \begin{cases} 1, & i \geq j; \\ -1, & i < j. \end{cases} \tag{4.34}$$

The control $u = \{u_1, u_2, \ldots, u_n\}$ is taken in $(L^2(\Sigma))^n$:

$$u = \{u_1, u_2, \ldots, u_n\} \in U = (L^2(\Sigma))^n.$$

Problem (4.31)–(4.33) admits a unique solution. To see this, we use the method developed in [20]:

$$V = (H^1(\Omega))^n, \quad \phi = \{\phi_1, \phi_2, \ldots, \phi_n\} \in V,$$

$$\pi(t; y, \phi) = \pi(y, \phi) = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} y_i(x) \frac{\partial}{\partial x_j} \phi_i(x) dx + \int_\Omega a_0(x) \sum_{j=1}^{n} y_j(x)\phi_j dx + \sum_{i=1}^{n} \int_\Omega \sum_{j=1}^{n} b_{i,j} y_j(x)\phi_i(x) dx,$$

$$L(\phi) = (f, \phi) = \int_\Omega \sum_{i=1}^{n} f_i(x, t)\phi_i(x) dx + \int_\Gamma \sum_{i=1}^{n} u_i(t)\phi_i(x) d\Gamma.$$

Let us consider the case where we have partial observation of the final state

$$z(v) = y_1(x, T; v), \tag{4.35}$$

and the cost function $J(v)$ for $v = \{v_1, v_2\}$ is given by

$$J(v) = \int_\Omega (y_1(x, T; v) - z_d)^2 dx + (Nv, v)_{(L^2(\Sigma))^n}, \quad z_d \in L^2(\Omega).$$

where $N = \{N_1, N_2, \ldots, N_n\} \in \mathcal{L}((L^2(\Sigma))^n, (L^2(\Sigma))^n)$ is hermitian positive definite operator:

$$(Nu, u) \geq c||u||^n_{L^2(\Sigma)}, \ c > 0 \tag{4.36}$$

Control constraints We define U_{ad}(set of admissible controls) is closed, convex subset of $U = (L^2(\Sigma))^n$.

Control problem We want to minimize J over U_{ad} i.e. find $u = \{u_1, u_2, \ldots, u_n\}$ such that

$$J(u) = \inf_{v=\{v_1,v_2,\ldots,v_n\}\in U_{ad}} J(v). \tag{4.37}$$

The adjoint state is given by

$$-{}_0^C D_t^\alpha p_i(u) + \mathcal{A}^* p_i(u) + \sum_{j=1}^{n} b_{ji} y_j(u) = 0, \quad \text{in } Q, \tag{4.38}$$

$$\frac{\partial p_i(u)}{\partial \nu_{A^*}} = 0, \quad \text{on } \Sigma, \tag{4.39}$$

$$p_1(x, T; u) = y_1(u) - z_d, \quad \text{in } \Omega, \quad p_k(x, T; u) = 0, \ k = 2, 3, \ldots, n \quad \text{in } \Omega, \tag{4.40}$$

where b_{ji} are the transpose of b_{ij}. The optimality condition is

$$\int_\Sigma \sum_{i=1}^{n} p_i(u)(v_i - u_i)\, d\Sigma + (Nu, v - u)_{(L^2(\Sigma))^n} \geq 0, \quad \forall v \in U_{ad}, \ u \in U_{ad}. \tag{4.41}$$

Remark 4.2 If we take $\alpha = 1$ in the previews sections we obtain the classical results in the optimal control with integer derivatives.

Conclusions

In this paper we considered optimal control problem for coupled evolution systems with Caputo derivatives. The analytical results were given in terms of Euler-Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems for coupled parabolic systems. The optimization problem presented in this paper constitutes a generalization of the optimal control problem of evolution equations with Dirichlet boundary conditions considered in [20] to systems with Caputo time derivatives. Also the main results of the paper contain necessary and sufficient conditions of optimality for coupled evolution equations that give characterization of fractional optimal control (Theorem 4.1).

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