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ORIGINAL RESEARCH

Application of Rothe's Method to Some Functional Differential Equations with Dirichlet Boundary Conditions

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Abstract The existence and uniqueness of a strong solution for a class of partial functional differential equations with Dirichlet boundary conditions is established by applying Rothe's method. As an application, we included an example to illustrate the main result.

Keywords Rothe's method · Functional differential equation · Strong solution · Semigroup of bounded linear operators

Mathematics Subject Classification 34K06 · 34K07 · 34K10 · 35R10

Introduction

In this paper, we study the existence and uniqueness of a strong solution of the following partial functional differential equation with Dirichlet boundary conditions

$$\frac{\partial}{\partial t}[w(t,x)+F(t,x,w(t,x))] = \frac{\partial^2}{\partial x^2}[w(t,x)+F(t,x,w(t,x))]+G(t,x,w(t,x)),$$
$$t\in[0,T],\quad x\in[0,1], \tag{1}$$
$$w(t,0)=w(t,1)=0,\quad t\in[-T,T], \tag{2}$$
$$w(t,x)=\Phi(t,x),\quad (t,x)\in[-T,0]\times[0,1], \tag{3}$$

where F, G and Φ are some suitable functions.

Equations of the type (1) in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, are called neutral differential equations. Neu-

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tral differential equations have many applications in various physical situations, for example, in the theory of vibration of masses attached to an elastic bar [1], and in the study of two or more simple oscillatory systems with some interconnections between them [1,2].

The existence of solutions of neutral differential equations has been considered by many authors, for example, Islam and Raffoul [3] studied the existence of periodic solutions of the nonlinear system of neutral differential equations

$$\frac{d}{dt}[u(t) - Q(t, u(t - g(t)))] = A(t)u(t) + G(t, u(t), u(t - g(t))),$$

where $A(t)$ is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements. The functions $Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in their respective arguments. Damak *et al.* [4] studied the existence of weighted pseudo almost periodic solutions of an autonomous neutral functional differential equation

$$\frac{d}{dt}[u(t) - F(t, u(t - r))] = A[u(t) - F(t, u(t - r))] + G(t, u(t), u(t - r)), \quad t \in \mathbb{R},$$

in a Banach space $\mathbb{X}$, where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, and $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is Weighted pseudo almost periodic and $G : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is Stepanov-weighted pseudo almost periodic functions.

The Rothe's method was introduced by Rothe in [5], for solving the following scalar parabolic initial boundary value problem of second order,

$$\begin{aligned} &R(t, x)\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = S(t, x, u), \quad 0 < x < 1, \ t > 0, \\ &u(0, x) = u_0(x), \\ &u(t, 0) = u(t, 1) = 0, \quad t \geq 0, \end{aligned}$$

where R and S are sufficiently smooth functions of t and x in $[0, T] \times (0, 1)$ satisfying certain additional conditions. Here T means an arbitrary finite positive number. His method consist in dividing $[0, T]$ into n number of subintervals $[t^n_{j-1}, t^n_j]$, $t^n_j = jh$, $j = 1, 2, \ldots, n$ with $t^n_0 = 0$ of equal lengths $h(h = \frac{T}{n})$ and replacing the partial derivative $\frac{\partial u}{\partial t}$ of the unknown function u by the difference quotients $\frac{u^n_j - u^n_{j-1}}{h}$. After defining a sequence of polygonal functions

$$U^n(x, t) = u^n_{j-1}(x) + \frac{1}{h}(t - t^n_{j-1})(u^n_j(x) - u^n_{j-1}(x)), \quad t \in [t^n_{j-1}, t^n_j].$$

Rothe has proved the convergence of the sequence $\{U^n\}$ to the unique solution of the problem as $n \to \infty$ using some a priori estimates on $\{U^n\}$. After Rothe this method has been used by many authors, for example, see [6–28].

Raheem and Bahuguna [7] have applied Rothe's method to establish the existence and uniqueness of a strong solution for the following delayed cooperation diffusion system with Dirichlet boundary conditions.

$$\begin{cases} \dfrac{\partial w}{\partial t}(t, x) = \dfrac{\partial^2 w}{\partial x^2}(t, x) + kw(t, x)[1 - w(t - \tau, x)] + f(t, x), \\ \qquad\qquad\qquad t \in [t_0, T], \quad x \in [0, \pi], \\ w(t, 0) = w(t, \pi) = 0, \quad t \in [t_0 - \tau, T], \\ w(t, x) = \phi(t, x), \quad (t, x) \in [t_0 - \tau, t_0] \times [0, \pi], \end{cases}$$

where $w(t, x)$ is the density of species at time t and space location x, and k, τ are positive constants, and the maps f, ϕ are defined from $[t_0, T] \times [0, \pi]$ and $[t_0 - \tau, t_0] \times [0, \pi]$ into $L^2[0, \pi]$ respectively.

Merazga and Bouziani [20] have proved the existence and uniqueness of a weak solution for a semilinear heat equation with integral conditions in a non classical function space. Recently in [9], authors have applied Rothe's method to a fractional integral diffusion and established the existence and uniqueness of a strong solution.

By literature, it is clear that Rothe's method can be used for solving many physical, mathematical, biological problems modeled by partial differential equations.

In the present work, we shall use Rothe's method to solve functional differential equations with Dirichlet boundary conditions defined by (1)–(3).

Preliminaries and Main Result

Consider $\mathbb{H} := L^2[0, 1]$, the real Hilbert space of all real valued square integrable functions defined on [0, 1] with the usual inner product and the norm generated by the inner product. Define the linear operator A by

$$D(A) := \{u \in \mathbb{H} : u'' \in \mathbb{H}, u(0) = u(1) = 0\}, \quad Au = -u''.$$

Then, $-A$ is the infinitesimal generator of a C_0-semigroup $T(t),\ t \geq 0$, of contractions in $\mathbb{H}$. Define the maps $f : [0, T] \times \mathbb{H} \to \mathbb{H}$ by

$$f(t, h)(x) = F(t, x, h(x)), \tag{4}$$

and $g : [0, T] \times \mathbb{H} \to \mathbb{H}$ by

$$g(t, h)(x) = G(t, x, h(x)). \tag{5}$$

If we define $u : [-T, T] \to \mathbb{H}$ by $u(t)(x) = w(t, x)$, and $\phi : [-T, 0] \to \mathbb{H}$ by $\phi(t)(x) = \Phi(t, x)$, then (1)–(3) can be rewritten as

$$\frac{d}{dt}[u(t) + f(t, u(t))] + A[u(t) + f(t, u(t))] = g(t, u(t)), \quad t \in [0, T], \tag{6}$$

$$u(t) = \phi(t), \quad t \in [-T, 0]. \tag{7}$$

Lemma 1 (Theorem 1.4.3, [29]) *If $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, then A is m-accretive, that is, $\langle Au, u\rangle \geq 0$ for $u \in D(A)$, and $R(I + \lambda A) = \mathbb{H}$ for any $\lambda > 0$, where I is the identity operator on $\mathbb{H}$, and $R(\cdot)$ is the range of an operator.*

Lemma 2 (Lemma 2.5, [12]) *Let $-A$ be the infinitesimal generator of a C_0-semigroup of contractions. If $Y^n \in D(A)$ for $n \in \mathbb{N}$, $Y^n \to u \in \mathbb{H}$ and $\|AY^n\|$ are bounded, then $u \in D(A)$ and $AY^n \rightharpoonup Au$, where $\rightharpoonup$ denotes the weak convergence in $\mathbb{H}$.*

We consider the following assumptions:

(H1) For each $t \in [0, T]$, the function $S_t : \mathbb{H} \to \mathbb{H}$ defined by $S_t(h) = h + f(t, h)$, $h \in \mathbb{H}$, is bijective.

(H2) f is continuous, and there exists $0 < K < 1$ such that

$$\|f(t_1, u_1) - f(t_2, u_2)\| \leq K\|u_1 - u_2\|, \quad \forall t_1, t_2 \in [0, T],\ \forall u_1, u_2 \in \mathbb{H}.$$

(H3) There exists $L_g > 0$ such that

$$\|g(t_1, u_1) - g(t_2, u_2)\| \leq L_g(|t_1 - t_2| + \|u_1 - u_2\|),\ \forall t_1, t_2 \in [0, T],\ \forall u_1, u_2 \in \mathbb{H}.$$

(H4) ϕ is continuous function.

Denote by $C([a,b];\mathbb{H})$ the space of all continuous functions from the interval $[a,b]$ into $\mathbb{H}$.

Definition 1 A function $u \in C([-T,T];\mathbb{H})$ is called a strong solution of the problem (6)–(7) if $u(t)=\phi(t)$ for $t\in[-T,0]$, $u(t)+f(t,u(t))\in D(A)$ for $t\in[0,T]$, $u(t)$ is Lipschitzian on $[0,T]$ and satisfies the Eq. (6) a.e. on $[0,T]$.

Theorem 1 *Suppose that the conditions (H1)–(H4) are satisfied. Then* (6)–(7) *has a strong solution u on* $[-T,\tilde{T}]$, $0<\tilde{T}<T$, *which can be continued uniquely either on the whole interval* $[-T,T]$ *or on the maximal interval of existence* $[-T,t_{\max})$, $0<t_{\max}\le T$, *such that u is a strong solution of* (6)–(7) *on every subinterval* $[-T,\tilde{T}]$, $0<\tilde{T}<t_{\max}$.

Discretization Scheme and a priori Estimates

Fix $R>0$ and choose t_0 such that $0<t_0\le T$ and $t_0M_0\le R$, where

$$M_0 := L_g\left(T+\frac{R}{1-K}\right)+g(0,\phi(0))+\|A(\phi(0)+f(0,\phi(0)))\|. \tag{8}$$

For each $n\in\mathbb{N}$, let $t_0^n=0$, $h_n=\frac{t_0}{n}$, and $t_j^n=jh_n$, for $j=1,2,\ldots,n$. Let $u_0^n=\phi(0)$ for all $n\in\mathbb{N}$, and define each of $\{u_j^n\}_{j=1}^n$ successively as the unique solution of the following equation

$$\frac{1}{h_n}[u+f(t_j^n,u)-u_{j-1}^n-f(t_{j-1}^n,u_{j-1}^n)]+A(u+f(t_j^n,u))=g(t_j^n,u_{j-1}^n). \tag{9}$$

The existence of a unique u_j^n satisfying (9) is obtained by using Lemma 1 and the assumption (H1). We define the sequence $\{U^n\}$ by

$$U^n(t)=\begin{cases}\phi(t), \quad t\in[-T,0],\\ u_{j-1}^n+f(t_{j-1}^n,u_{j-1}^n)+\frac{1}{h_n}(t-t_{j-1}^n)(u_j^n+f(t_j^n,u_j^n)-u_{j-1}^n\\ \qquad -f(t_{j-1}^n,u_{j-1}^n)), \quad t\in(t_{j-1}^n,t_j^n].\end{cases} \tag{10}$$

Lemma 3 *For* $n\in\mathbb{N}$ *and* $j=1,2,\ldots,n$,

$$\|u_j^n+f(t_j^n,u_j^n)-\phi(0)-f(0,\phi(0))\|\le R.$$

Proof From (9) for $j=1$, we have

$$\frac{1}{h_n}[u_1^n+f(t_1^n,u_1^n)-u_0^n-f(t_0^n,u_0^n)]+A(u_1^n+f(t_1^n,u_1^n))=g(t_1^n,u_0^n).$$

Then,

$$\begin{aligned}&\left\langle\frac{1}{h_n}[u_1^n+f(t_1^n,u_1^n)-u_0^n-f(t_0^n,u_0^n)],u_1^n+f(t_1^n,u_1^n)-u_0^n-f(t_0^n,u_0^n)\right\rangle\\ &\quad+\langle A(u_1^n+f(t_1^n,u_1^n)-u_0^n-f(t_0^n,u_0^n)),u_1^n+f(t_1^n,u_1^n)-u_0^n-f(t_0^n,u_0^n)\rangle\\ &=\langle g(t_1^n,u_0^n),u_1^n+f(t_1^n,u_1^n)-u_0^n-f(t_0^n,u_0^n)\rangle\\ &\quad-\langle A(u_0^n+f(t_0^n,u_0^n)),u_1^n+f(t_1^n,u_1^n)-u_0^n-f(t_0^n,u_0^n)\rangle.\end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned}
&\frac{1}{h_n}\|u_1^n + f(t_1^n, u_1^n) - u_0^n - f(t_0^n, u_0^n)\| \\
&\le \|g(t_1^n, u_0^n)\| + \|A(u_0^n + f(t_0^n, u_0^n))\| \\
&\le \|g(t_1^n, u_0^n) - g(0, \phi(0))\| + \|g(0, \phi(0))\| + \|A(\phi(0) + f(0, \phi(0))\| \\
&\le L_g|t_1^n| + \|g(0, \phi(0))\| + \|A(\phi(0) + f(0, \phi(0))\| \\
&\le M_0.
\end{aligned} \tag{11}$$

Therefore,

$$\|u_1^n + f(t_1^n, u_1^n) - \phi(0) - f(0, \phi(0))\| \le h_n M_0 \le t_0 M_0 \le R. \tag{12}$$

Hence,

$$\begin{aligned}
(1-K)\|u_1^n - \phi(0)\| &\le \|u_1^n - \phi(0)\| - \|f(t_1^n, u_1^n) - f(0, \phi(0))\| \\
&\le \|u_1^n + f(t_1^n, u_1^n) - \phi(0) - f(0, \phi(0))\| \\
&\le R.
\end{aligned}$$

Therefore, $\|u_1^n - \phi(0)\| \le \frac{R}{1-K}$.

Assume that

$$\|u_i^n + f(t_i^n, u_i^n) - \phi(0) - f(0, \phi(0))\| \le R \quad \text{for } i = 1, 2, \ldots, j-1. \tag{13}$$

Then, $\|u_i^n - \phi(0)\| \le \frac{R}{1-K}$ for $i = 1, 2, \ldots, j-1$. Now, From (9) for $i = j$, $2 \le j \le n$, we have

$$\frac{1}{h_n}[u_j^n + f(t_j^n, u_j^n) - u_{j-1}^n - f(t_{j-1}^n, u_{j-1}^n)] + A(u_j^n + f(t_j^n, u_j^n)) = g(t_j^n, u_{j-1}^n).$$

Then,

$$\begin{aligned}
&\left\langle \frac{1}{h_n}[u_j^n + f(t_j^n, u_j^n) - u_{j-1}^n - f(t_{j-1}^n, u_{j-1}^n)], u_j^n + f(t_j^n, u_j^n) - u_0^n - f(t_0^n, u_0^n) \right\rangle \\
&\quad + \langle A(u_j^n + f(t_j^n, u_j^n) - u_0^n - f(t_0^n, u_0^n)), u_j^n + f(t_j^n, u_j^n) - u_0^n - f(t_0^n, u_0^n)\rangle \\
&= \langle g(t_j^n, u_{j-1}^n), u_j^n + f(t_j^n, u_j^n) - u_0^n - f(t_0^n, u_0^n)\rangle \\
&\quad - \langle A(u_0^n + f(t_0^n, u_0^n)), u_j^n + f(t_j^n, u_j^n) - u_0^n - f(t_0^n, u_0^n)\rangle.
\end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned}
&\frac{1}{h_n}\|u_j^n + f(t_j^n, u_j^n) - u_{j-1}^n - f(t_{j-1}^n, u_{j-1}^n)\| \\
&\le \|g(t_j^n, u_{j-1}^n)\| + \|A(u_0^n + f(t_0^n, u_0^n))\| \\
&\le \|g(t_j^n, u_{j-1}^n) - g(0, \phi(0))\| + \|g(0, \phi(0))\| + \|A(\phi(0) + f(0, \phi(0))\| \\
&\le L_g(|t_j^n| + \|u_{j-1}^n - \phi(0)\|) + +\|g(0, \phi(0))\| + \|A(\phi(0) + f(0, \phi(0))\| \\
&\le L_g\left(|t_j^n| + \frac{R}{1-K}\right) + \|g(0, \phi(0))\| + \|A(\phi(0) + f(0, \phi(0))\| \\
&\le M_0.
\end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned}
&\frac{1}{h_n}\|u_j^n + f(t_j^n, u_j^n) - u_0^n - f(t_0^n, u_0^n)\| \\
&\quad \le \frac{1}{h_n}\|u_{j-1}^n + f(t_{j-1}^n, u_{j-1}^n) - u_0^n - f(t_0^n, u_0^n)\| \\
&\qquad + \frac{1}{h_n}\|u_j^n + f(t_j^n, u_j^n) - u_{j-1}^n - f(t_{j-1}^n, u_{j-1}^n)\| \\
&\quad \le \frac{1}{h_n}\|u_{j-1}^n + f(t_{j-1}^n, u_{j-1}^n) - u_0^n - f(t_0^n, u_0^n)\| + M_0
\end{aligned}$$

Thus, we get

$$\frac{1}{h_n}\|u_j^n + f(t_j^n, u_j^n) - u_0^n - f(t_0^n, u_0^n)\| \le jM_0$$

Therefore,

$$\|u_j^n + f(t_j^n, u_j^n) - \phi(0) - f(0, \phi(0))\| \le jh_n M_0 \le t_0 M_0 \le R. \tag{15}$$

Hence proved. □

Corollary 1 *For $n \in \mathbb{N}$ and $j = 1, 2, \ldots, n$,*

$$\frac{1}{h_n}\|u_j^n + f(t_j^n, u_j^n) - u_{j-1}^n - f(t_{j-1}^n, u_{j-1}^n)\| \le M_0.$$

Proof Proof is obvious by Lemma 3 and from the Eqs. (13) and (14). □

Now, we defined a sequence $\{Y^n\}$ of step functions from $[-h_n, t_0]$ into $\mathbb{H}$ by

$$Y^n(t) = \begin{cases} \phi(0), & t \in [-h_n, 0], \\ u_j^n + f(t_j^n, u_j^n), & t \in (t_{j-1}^n, t_j^n]. \end{cases} \tag{16}$$

Remark 1 From Corollary 1, it is clear that the functions $U^n(t)$ are Lipschitz continuous on $[0, t_0]$, and the sequence $U^n(t) - Y^n(t) \to 0$ in $\mathbb{H}$ as $n \to \infty$ uniformly on $[0, t_0]$. Moreover, $Y^n(t) \in D(A)$ for $t \in [0, t_0]$ and the sequences $\{U^n(t)\}$, $\{Y^n(t)\}$ and $\{AY^n(t)\}$ are bounded uniformly in $n \in \mathbb{N}$ and $t \in [0, t_0]$.

If we suppose that

$$g^n(t) = g(t_j^n, u_{j-1}^n), \quad t \in (t_{j-1}^n, t_j^n], \quad 1 \le j \le n, \tag{17}$$

then (9) can be written as

$$\frac{d^-}{dt}U^n(t) + AY^n(t) = g^n(t), \quad t \in (0, t_0], \tag{18}$$

where $\frac{d^-}{dt}$ denotes the left derivative in $(0, t_0]$. Also, for $t \in (0, t_0]$, we have

$$\int_0^t AY^n(s)ds = \phi(0) - U^n(t) + \int_0^t g^n(s)ds. \tag{19}$$

Lemma 4 *There exists a function $u \in C([-T, t_0]; \mathbb{H})$ such that $U^n(t) \to u(t) + f(t, u(t))$ in $C([-T, t_0]; \mathbb{H})$ as $n \to \infty$. Moreover, u is Lipschitz continuous on $[0, t_0]$.*

Proof From (18) and using Lemma 1, for $t \in (0, t_0]$, we have

$$\left\langle \frac{d^-}{dt} U^n(t) - \frac{d^-}{dt} U^k(t), Y^n(t) - Y^k(t) \right\rangle \leq \langle g^n(t) - g^k(t), Y^n(t) - Y^k(t) \rangle.$$

Using above inequality, we get

$$\begin{aligned} &\frac{1}{2} \frac{d^-}{dt} \|U^n(t) - U^k(t)\|^2 \\ &= \left\langle \frac{d^-}{dt} (U^n(t) - U^k(t)), U^n(t) - U^k(t) \right\rangle \\ &\leq \left\langle \frac{d^-}{dt} (U^n(t) - U^k(t)) - g^n(t) + g^k(t), U^n(t) - U^k(t) - Y^n(t) + Y^k(t) \right\rangle \\ &+ \langle g^n(t) - g^k(t), U^n(t) - U^k(t) \rangle. \end{aligned} \tag{20}$$

For $t \in (t^n_{j-1}, t^n_j]$ and $t \in (t^k_{l-1}, t^k_l]$, $1 \leq j \leq n$, $1 \leq l \leq k$, we have

$$\begin{aligned} \|u^n_{j-1} - u^k_{l-1}\| \leq{}& \|U^n(t) - U^k(t)\| + \|f(t^n_{j-1}, u^n_{j-1}) - f(t^k_{l-1}, u^k_{l-1})\| \\ &+ \|\frac{1}{h_n}(t - t^n_{j-1})(u^n_j + f(t^n_j, u^n_j) - u^n_{j-1} - f(t^n_{j-1}, u^n_{j-1}))\| \\ &+ \|\frac{1}{h_k}(t - t^k_{l-1})(u^k_l + f(t^k_l, u^k_l) - u^k_{l-1} - f(t^k_{l-1}, u^k_{l-1}))\|. \end{aligned}$$

By Corollary 1 and by the assumption (H2), we get

$$\|u^n_{j-1} - u^k_{l-1}\|\| \leq \|U^n(t) - U^k(t)\| + K\|u^n_{j-1} - u^k_{l-1}\| + (h_n + h_k)M_0, \tag{21}$$

that is,

$$\|u^n_{j-1} - u^k_{l-1}\|\| \leq \frac{1}{1-K}(\|U^n(t) - U^k(t)\| + (h_n + h_k)M_0). \tag{22}$$

Now,

$$\begin{aligned} \|g^n(t) - g^k(t)\| &= \|g(t^n_j, u^n_{j-1}) - g(t^k_l, u^k_{l-1})\| \\ &\leq L_g(|t^n_j - t^k_l| + \|u^n_{j-1} - u^k_{l-1}\|) \\ &\leq \epsilon_{nk}(t) + \frac{L_g}{1-K}\|U^n(t) - U^k(t)\|, \end{aligned}$$

where

$$\epsilon_{nk}(t) = L_g\left(|t^n_j - t^k_l| + \frac{M_0(h_n + h_k)}{1-K}\right).$$

Clearly, $\epsilon_{nk}(t) \to 0$ as $n, k \to \infty$ uniformly on $[0, t_0]$. This implies that for a.e. $t \in [0, t_0]$, we have

$$\frac{d^-}{dt}\|U^n(t) - U^k(t)\|^2 \leq M_0(\epsilon^1_{nk} + \|U^n(t) - U^k(t)\|^2),$$

where ϵ^1_{nk} is a sequence of numbers such that $\epsilon^1_{nk} \to 0$ as $n, k \to \infty$. Notice that $U^n = \phi$ on $[-T, 0]$ for all n. Hence, we obtain

$$\|U^n(t) - U^k(t)\|^2 \leq M_0\left(T\epsilon^1_{nk} + \int_0^t \|U^n(s) - U^k(s)\|^2 ds\right).$$

Using Gronwall's inequality, we conclude that there exists $v \in C([-T, t_0]; \mathbb{H})$ such that $U^n \to v$ in $C([-T, t_0]; \mathbb{H})$ as $n \to \infty$. By the assumption (H1), for each $t \in [0, t_0]$ there exists $u(t)$ such that $v(t) = u(t) + f(t, u(t))$, and for each $t \in [-T, 0]$, we define $u(t) = v(t)$. By the assumption (H2), it is clear that $u(t)$ is continuous. It is easy to see that $v = \phi$ on $[-T, 0]$, therefore $u(t) = \phi(t)$ on $[-T, 0]$. Since v is Lipschitz continuous, by the assumption (H2) u is also Lipschitz continuous on $[0, t_0]$. Hence proved. □

Proof of Theorem 1 From Remark 1 it follows that $Y^n(t) \to u(t) + f(t, u(t))$ as $n \to \infty$, and $u(t) + f(t, u(t)) \in \mathbb{H}$ for $t \in [0, t_0]$. Since $\|AY^n\|$ are bounded, by Lemma 2 it is clear that $AY^n(t) \rightharpoonup A(u(t) + f(t, u(t)))$. For $t \in (t^n_{j-1}, t^n_j]$, we have

$$\begin{aligned}\|u^n_{j-1} - u(t)\| &\le \|u^n_j + f(t^n_j, u^n_j) - u(t) - f(t, u(t))\| \\ &\quad + \|u^n_j + f(t^n_j, u^n_j) - u^n_{j-1} - f(t^n_{j-1}, u^n_{j-1})\| \\ &\quad + \|f(t, u(t)) - f(t^n_{j-1}, u^n_{j-1})\| \\ &\le \|u^n_j + f(t^n_j, u^n_j) - u(t) - f(t, u(t))\| \\ &\quad + M_0 h_n + K\|u^n_{j-1} - u(t)\|,\end{aligned}$$

that is,

$$\|u^n_{j-1} - u(t)\| \le \frac{1}{1-K}(\|u^n_j + f(t^n_j, u^n_j) - u(t) - f(t, u(t))\| + M_0 h_n).$$

Now, for $t \in (t^n_{j-1}, t^n_j]$, we have

$$\begin{aligned}\|g^n(t) - g(t, u(t))\| &= \|g(t^n_j, u^n_{j-1}) - g(t, u(t))\| \\ &\le L_g(|t^n_j - t| + \|u^n_{j-1} - u(t)\|) \\ &\le \frac{L_g}{1-K}\|u^n_j + f(t^n_j, u^n_j) - u(t) - f(t, u(t))\| \\ &\quad + \left(1 + \frac{M_0}{1-K}\right) L_g h_n.\end{aligned}$$

Therefore, $\|g^n(t) - g(t, u(t))\| \to 0$ as $n \to \infty$ uniformly on $[0, t_0]$. From (19), for every $v \in \mathbb{H}$, we have

$$\int_0^t \langle AY^n(s), v\rangle ds = \langle \phi(0), v\rangle - \langle U^n(t), v\rangle + \int_0^t \langle g^n(s), v\rangle ds.$$

By Lemma 4 and the bounded convergence theorem, we get as $n \to \infty$,

$$\begin{aligned}\int_0^t \langle A(u(s) + f(s, u(s))), v\rangle ds &= \langle \phi(0), v\rangle - \langle (u(t) + f(t, u(t))), v\rangle \\ &\quad + \int_0^t \langle g(s, u(s)), v\rangle ds. \end{aligned} \tag{23}$$

Since $A(u(t) + f(t, u(t)))$ is Bochner integrable on $[0, t_0]$, from (23) we obtain

$$\frac{d}{dt}(u(t) + f(t, u(t))) + A(u(t) + f(t, u(t))) = g(t, u(t)) \text{ a.e. } t \in [0, t_0]. \tag{24}$$

Now we prove the uniqueness of a function $u \in C([-T, t_0]; \mathbb{H})$ such that $u(t) + f(t, u(t))$ is differentiable a.e. on $[0, t_0]$ with $u(t) + f(t, u(t)) \in D(A)$ a.e. on $[0, t_0]$ and $u = \phi$ on $[-T, 0]$ satisfying (24). Suppose there exist two strong solutions $u_1, u_2 \in C([-T, t_0]; \mathbb{H})$ of (24) with

$u_1 = u_2 = \phi$ on $[-T, 0]$. Let $u(t) + f(t, u(t)) = u_1(t) + f(t, u_1(t)) - u_2(t) - f(t, u_2(t))$ on $[0, t_0]$. Then

$$\begin{aligned}\|u_1(t) - u_2(t)\| &\le \|u_1(t) + f(t, u_1(t)) - u_2(t) - f(t, u_2(t))\| \\ &\quad + \|f(t, u_1(t)) - f(t, u_2(t))\| \\ &\le \|u(t) + f(t, u(t))\| + K\|u_1(t) - u_2(t)\|.\end{aligned}$$

Therefore, $\|u_1(t) - u_2(t)\| \le \frac{1}{1-K}\|u(t) + f(t, u(t))\|$ on $[0, t_0]$. Now, from (24) and using Lemma 1, we have

$$\begin{aligned}\frac{d}{dt}\|u(t) + f(t, u(t))\|^2 &= 2\langle\frac{d}{dt}(u(t) + f(t, u(t))), u(t) + f(t, u(t))\rangle \\ &\le 2\langle g(t, u_1(t)) - g(t, u_2(t)), u(t) + f(t, u(t))\rangle \\ &\le 2\|g(t, u_1(t)) - g(t, u_2(t))\|\|u(t) + f(t, u(t))\| \\ &\le 2L_g(1 + K)\|u_1(t) - u_2(t)\|^2 \\ &\le \frac{2L_g(1 + K)}{(1 - K)^2}\|u(t) + f(t, u(t))\|^2.\end{aligned}$$

Since $u(t) + f(t, u(t)) = 0$ on $[-T, 0]$, therefore we obtain

$$\|u(t) + f(t, u(t))\|^2 \le \frac{2L_g(1 + K)}{(1 - K)^2}\int_0^t \|u(s) + f(s, u(s))\|^2 ds.$$

Using Gronwall's inequality, we conclude that $u(t) + f(t, u(t)) = 0$ on $[-T, t_0]$. Therefore, by the assumption (H2), we get $u_1 = u_2$ on $[-T, t_0]$. Now, we prove the continuation of the solution u on $[-T, T]$. Suppose $t_0 < T$, then consider

$$\begin{cases}\frac{d}{dt}[v(t) + f(t, v(t))] + A[v(t) + f(t, v(t))] = \tilde{g}(t, v(t)), & 0 \le t \le T - t_0, \\ v(t) = \phi(t), \quad t \in [-T, 0],\end{cases} \tag{25}$$

where $\tilde{g}(t, v(t)) = g(t+t_0, v(t)), \quad 0 \le t \le T-t_0$. Since $\tilde{g}$ satisfies the assumption (H3), we can proceed as before and prove the existence of a unique solution $v \in C([-T - t_0, t_1]; \mathbb{H})$, $0 < t_1 \le T - t_0$, such that v is Lipschitz continuous on $[0, t_1]$, $v(t) \in D(A)$ for $t \in [0, t_1]$ and v satisfies the following equation

$$\begin{cases}\frac{d}{dt}[v(t) + f(t, v(t))] + A[v(t) + f(t, v(t))] = \tilde{g}(t, v(t)), & \text{a.e } t \in [0, t_1], \\ v(t) = \phi(t), \quad t \in [-T, 0].\end{cases} \tag{26}$$

Then the function

$$\tilde{u}(t) = \begin{cases}u(t), & t \in [-T, t_0], \\ v(t - t_0), & t \in [t_0, t_0 + t_1],\end{cases} \tag{27}$$

is Lipschitz continuous on $[0, t_0 + t_1]$, $\tilde{u}(t) \in D(A)$ for $t \in [0, t_0 + t_1]$ and satisfies a.e. on $[0, t_0 + t_1]$. By continuing in this way, we can prove the existence on $[-T, T]$ or on the maximal interval of existence $[-T, t_{\max})$, $0 < t_{\max} \le T$ such that u is a strong solution on every interval $[-T, \tilde{T}]$, $0 < \tilde{T} < t_{\max}$. Hence proved. □

Application

Consider the following differential equation

$$\begin{cases} \dfrac{d}{dt}((1+\lambda)u(t)+\lambda\sin u(t)) + A((1+\lambda)u(t)+\lambda\sin u(t)) = \mu(t+\cos u(t)), \;\; t\in[0,T], \\ u(t)=\phi(t), \quad t\in[-T,0], \end{cases} \tag{28}$$

in the Hilbert space $\mathbb{H} := L^2[0,1]$, where $-A$ is the infinitesimal generator of a C_0-semigroup $T(t),\;\; t \geq 0$, of contractions in $\mathbb{H}$, T and μ are positive real numbers, λ is sufficiently small positive real number lies in the interval $(0, \frac{1}{2})$, and $\phi : [-T,0] \to \mathbb{H}$ is any continuous function. Define the maps $f : [0,T]\times\mathbb{H} \to \mathbb{H}$ by

$$f(t,h) = \lambda(h+\sin h), \tag{29}$$

and $g : [0,T]\times\mathbb{H} \to \mathbb{H}$ by

$$g(t,h) = \mu(t+\cos h). \tag{30}$$

Then, for each $t\in[0,T]$, the function $S_t : \mathbb{H}\to\mathbb{H}$ defined by

$$S_t(h) = h + f(t,h) = (1+\lambda)h + \lambda\sin h,$$

$h\in\mathbb{H}$, is bijective. Now, for $t_1, t_2 \in [0,T]$, and $h_1, h_2 \in \mathbb{H}$, consider

$$\begin{aligned} \|f(t_1,h_1) - f(t_2,h_2)\| &= \lambda\|h_1 + \sin h_1 - h_2 - \sin h_2\| \\ &\leq 2\lambda\|h_1 - h_2\|, \end{aligned}$$

and

$$\begin{aligned} \|g(t_1,h_1) - g(t_2,h_2)\| &= \mu\|t_1 + \cos h_1 - t_2 - \cos h_2\| \\ &\leq \mu(|t_1 - t_2| + \|h_1 - h_2\|). \end{aligned}$$

Thus, the functions f and g satisfy the assumptions (H1)–(H3). Therefore, by Theorem 1, the problem (28) has a strong solution.

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