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ORIGINAL RESEARCH

The Well-Posedness of Fractional Systems with Affine-Periodic Boundary Conditions

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Abstract This paper is devoted to study the existence and uniqueness of solutions for a class of nonlinear fractional dynamical systems with affine-periodic boundary conditions. We can show that there exists a solution for an α-fractional system via the homotopy invariance of Brouwer degree, where $0 < \alpha \leq 1$. Furthermore, using Gronwall–Bellman inequality, we can prove the uniqueness of the solution if the nonlinearity satisfies the Lipschitz continuity. We apply the main theorem to the fractional kinetic equation and fractional oscillator with constant coefficients subject to affine-periodic boundary conditions. And in appendix, we give the proof of the nonexistence of affine-periodic solution to a given (α, Q, T)-affine-periodic system in the sense of Riemann–Liouville fractional integral and Caputo derivative for $0 < \alpha < 1$.

Keywords Affine-periodic boundary values · Nonlinear fractional dynamical systems · Existence · Uniqueness

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Introduction

The structure of the solution of an differential equation plays an important role in the theory of dynamical systems, such as periodicity, anti-periodicity, harmonic-periodicity, quasi-periodicity and so on. In 2103 [46], Li et al. introduced the concept "affine-periodic" which is a kind of symmetry rather than periodicity, which is the general version of periodicity, anti-periodicity, harmonic-periodicity and quasi-periodicity. There are some natural phenomena presenting affine-periodicity [9,34], such as, spiral wave (or affine-periodic wave), spiral line in geometry, and the orbit of the earth goes round the sun: the orbit is a circle or ellipse exactly in a plane, but in fact, the circle in the plane is only a projection in the space along the time axis. The orbit of the earth when it goes round the sun is cubic rather than just in a plane and the space position is rotating as the time walks a periodic, that is to say, the orbit of the earth goes round the sun presents the "affine-symmetry".

The subject of affine-periodic problem is essential and more and more researchers pay attention to "affine-symmetry" and consider the existence of affine-periodic solutions of different affine-periodic dynamical systems, such as dissipative systems [21,46], discrete dynamical systems [22], nonlinear dynamic equations on time scales [39] and so on [11,13, 40,41].

Our problem is to consider the existence and uniqueness of affine-periodic solutions to fractional affine-periodic systems for fractional-order models are found to be more adequate than integer-order models for some real world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth, involves derivatives of fractional order. And the problem of existence and uniqueness is very essential, which is mentioned by mathematician Hadamard [17], and he believed that mathematical models of physical phenomena should have the properties that (1) a solution exists, (2) the solution is unique, and (3) the solution is stable, that is the solution's behavior changes continuously with the initial conditions, which is called Hadamard well-posedness or well-posedness. The solution is called well-posed if the solution is existing, unique and stable.

As a general of periodicity, anti-periodicity, harmonic-periodicity and quasi-periodicity, there are also many research in the spacial case for fractional dynamical systems. Such as, in [14], Devi developed the generalized monotone method to fractional differential equations with periodic boundary values and obtained some existence results. In [2], Ahmad and Nieto obtained some existence results for a differential equation of fractional order with anti-periodic boundary conditions using Leray–Schauder degree theory. In [43], Wei et al. considered the existence and uniqueness of the solution of the periodic boundary value problem for a fractional differential equation involving a Riemann–Liouville fractional derivative by using the monotone iterative method. In [42], Wang and Bai investigated the existence and uniqueness of solution of the periodic boundary value problem for nonlinear impulsive fractional differential equation involving Riemann–Liouville fractional derivative by using Banach contraction principle. In [12], Chen and Chen considered the anti-periodic boundary value problem for nonlinear fractional differential equation and obtained some existence results by means of the Banach fixed point theorem and Schauder fixed point theorem. In [26], Nieto studied a linear fractional differential equation with a periodic boundary condition and gave the explicit form of the solution and the corresponding Green's function. Meanwhile, some new comparison results are presented by some properties of the Green's function. In [18], Hu et al. considered periodic boundary value problem for fractional differential equation and obtained a new result on the existence of solutions for above fractional boundary value problem using the coincidence degree theory.

Under the help of these fruitful results, we consider the existence and uniqueness of affine-periodic solutions to affine-periodic systems with initial values at first. But, we find that the affine-periodic solutions may not exist in the sense of Riemann–Liouville fractional integral and Caputo fractional derivative for $0 < \alpha < 1$ if the nonlinearity holds affine-symmetry, the proof can be found in Sect. 6.2. Thus, we consider the existence and uniqueness of the solutions to α-fractional systems with affine-periodic boundary conditions, where $0 < \alpha \leq 1$.

The rest of this paper is organized as follows: in Sect. 2, we present some definitions and notations for the fractional calculus and the concept of affine-periodic. Section 3 contains some sufficient conditions for the existence and uniqueness of the solution for affine-periodic boundary problem and the proof of the main results. In Sect. 5, some applications are presented. Section 6 includes some auxiliary proof procedure.

Fractional Calculation and α-Fractional System

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. There are several definitions of the fractional integration of order $\alpha \geq 0$, and not necessarily equivalent to each other, see [23]. Riemann–Liouville and Caputo fractional definitions are the two most used from all the other definitions of fractional calculus which have been introduced recently [10,19,36,38].

Definition 2.1 [19,36,38] A real function $f(t), t > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, +\infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(t) \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2 [19,36,38] The Riemann–Liouville fractional integral operator of order $\beta \geq 0$ of a function $f \in C_\mu$, $\mu \geqslant -1$ is defined as

$$
{}^{RL}_{\ a}\mathrm{I}_t^\beta f(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\beta-1} f(\tau)\mathrm{d}\tau, & \beta > 0, \\ f(t), & \beta = 0, \end{cases}
$$

where $a \in \mathbb{R}$ and the symbol "${}^{RL}\mathrm{I}$" represents the fractional integral in the Riemman–Liouville sense.

Definition 2.3 [19,36,38] The Caputo fractional derivative of order $\beta > 0$ of a function $f \in C_{-1}^n$, $n \in \mathbb{N}$ is defined as

$$
{}^{C}_{a}\mathrm{D}_t^\beta f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta+1-n}} \mathrm{d}\tau, & n-1 < \beta < n, \\ \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \beta = n, \end{cases}
$$

where $a \in \mathbb{R}$ and the symbol "${}^{C}\mathrm{D}$" represents the fractional derivative in the Caputo sense.

With the definitions above, we can obtain the following lemma.

Lemma 2.4 *If* $n-1<\beta\le n$, $n\in\mathbb{N}$, *then*

$$ {}^{RL}_{a}\mathrm{I}^{\beta}_{t}\,{}^{C}_{a}\mathrm{D}^{\beta}_{t} f(\tau) = f(t) - \sum_{k=0}^{n-1} c_k t^k, $$

where $c_k\in\mathbb{R}$, $k=0,1,\cdots,n-1$.

And the proof of Lemma 2.4 can be found in Sect. 6.1.

In the following, we present the definition of α-fractional system, (α,Q,T)-affine-periodic system, (α,Q,T)-affine-periodic solution, and (Q,T)-affine -periodic boundary conditions.

Definition 2.5 [22,39,46] The dynamical system

$$ {}^{C}_{a}\mathrm{D}^{\alpha}_{t}\boldsymbol{x} = \boldsymbol{f}(t,\boldsymbol{x}), \quad (t,\boldsymbol{x})\in(I\subseteq\mathbb{R})\times\mathbb{R}^n \tag{2.1} $$

is called an α-fractional system. If there exists some $T>0$ and $Q\in GL_n(\mathbb{R})$ such that

$$ \boldsymbol{f}(t+T,\boldsymbol{x}) = Q\boldsymbol{f}(t,Q^{-1}\boldsymbol{x}). $$

where $0<\alpha\le 1$, then (2.1) is called an (α,Q,T)-affine-periodic system .

Remark 2.6 An interesting problem is to seek for a solution $\boldsymbol{x}=\boldsymbol{x}(t)$ which can keep the affine-symmetry of $\boldsymbol{f}$, that is, the space variable $\boldsymbol{x}$ has a transformation or rotating Q as the time variable walks a periodic T, which is the concept of (α,Q,T)-affine-periodic solution in Definition 2.7.

Definition 2.7 [22,39,46] A solution $\boldsymbol{x}=\boldsymbol{x}(t)$ of (2.1) is called (α,Q,T)-affine-periodic solution if

$$ \boldsymbol{x}(t+T) = Q\boldsymbol{x}(t), \quad \forall t\in I. $$

Remark 2.8 Obviously, an (α,Q,T)-affine-periodic solution is periodic solution, anti-periodic solution, harmonic-periodic solution or quasi periodic solution when $Q=I$ (the identical matrix), $Q=-I$, $Q^m=I$ for some $m\in\mathbb{N}$ or $Q\in O_n(\mathbb{R})$.

Remark 2.9 It is easy to see that if $\boldsymbol{x}=\boldsymbol{x}(t)$ is an (α,Q,T)-affine-periodic solution of (2.1), then

$$ \boldsymbol{x}(T) = Q\boldsymbol{x}_0, $$

where $\boldsymbol{x}_0=\boldsymbol{x}(0)$ is the initial value.

In this paper, we find that the definition of Riemann–Liouville fractional integral has not shift-invariant, which leads to the (α,Q,T)-affine-periodic solution may not exist for $0<\alpha<1$, the proof can be seen in Sect. 6.2. Thus, we consider the α-fractional system with boundary condition $\boldsymbol{x}(T)=Q\boldsymbol{x}(0)$ which is called "(Q,T)-affine-periodic boundary condition" in the Definition 2.10, where $0<\alpha\le 1$.

Definition 2.10 [22,39,46] The (α,Q,T)-affine-periodic system (2.1) is called satisfying (α,Q,T)-affine-periodic boundary conditions if

$$ \boldsymbol{x}(0)=\boldsymbol{\xi}, \quad \boldsymbol{x}(T)=Q\boldsymbol{\xi}, $$

where $\boldsymbol{\xi}\in\mathbb{R}^n$.

Main Results

Consider the α-fractional system with (Q, T)-affine-periodic boundary conditions

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha \boldsymbol{x} = \boldsymbol{f}(t, \boldsymbol{x}), & (t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^n, \\ \boldsymbol{x}(0) = \boldsymbol{\xi}, & \boldsymbol{x}(T) = Q\boldsymbol{\xi}, \end{cases} \tag{3.1}$$

with the auxiliary α-fractional system with (Q, T)-affine-periodic boundary conditions

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha \boldsymbol{x} = \lambda \boldsymbol{f}(t, \boldsymbol{x}), & (t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^n, \\ \boldsymbol{x}(0) = \boldsymbol{\xi}, & \boldsymbol{x}(T) = Q\boldsymbol{\xi}, \end{cases} \tag{3.2}$$

where $0 < \alpha \leq 1$, $Q \in GL_n(\mathbb{R})$, $0 \leq \lambda \leq 1$. Then we have the following result:

Theorem 3.1 *For the following assumptions:*

A_1 : *For each $\lambda \in (0, 1]$, every possible solution $\boldsymbol{x}(t)$ for system (3.2) satisfies that if $\boldsymbol{x}(t) \in \bar{D}$, then $\boldsymbol{x}(t) \notin \partial D, \forall t \in [0, T]$.*

A_2 : *The Brouwer degree*

$$\deg(\boldsymbol{g}, D \cap \mathrm{Ker}(\boldsymbol{I} - Q), \boldsymbol{0}) \neq 0, \quad \textit{if} \quad \mathrm{Ker}(\boldsymbol{I} - Q) \neq \{\boldsymbol{0}\},$$

where $\boldsymbol{g}(\cdot) = {}^{RL}_{\ 0}\mathrm{I}_T^\alpha \boldsymbol{P}\boldsymbol{f}(\tau, \cdot)$, $\boldsymbol{P} : \mathbb{R}^n \to \mathrm{Ker}(\boldsymbol{I} - Q)$ is an orthogonal projection.

A_3 : *The function $\boldsymbol{f}$ is Lipschitz continuous, that is, there exists a constant $L \in \mathbb{R}^+$ such that*

$$||\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})|| \leq L||\boldsymbol{x} - \boldsymbol{y}||, \quad \forall (t, \boldsymbol{x}), (t, \boldsymbol{y}) \in [0, T] \times \mathbb{R}^n.$$

If the α-fractional system (3.2) holds the assumptions A_1 and A_2, then the α-fractional system (3.1) admits at least a solution. And furthermore, if the assumption A_3 holds as well, then the solution of (3.1) is unique.

Before the proof procedure of Theorem 3.1, some lemmas which will be used in the proof of the main theorem are listed firstly.

Lemma 3.2 [35] *Let $\mathscr{X} = C([0, T])$ denote the set of all continuous functions in $\mathbb{R}^n$ on the interval $[0, T]$, and $\mathscr{X}$ be a linear space which holds*

$$(k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2)(t) = k_1\boldsymbol{x}_1(t) + k_2\boldsymbol{x}_2(t), \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathscr{X}, \quad \forall t \in [0, T].$$

Define $\|\boldsymbol{x}\| = \sup_{t\in[0,T]} \|\boldsymbol{x}(t)\|$, then $(\mathscr{X}, \|\cdot\|)$ is a Banach space.

Lemma 3.3 [27] *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset and $\boldsymbol{f} : \bar{\Omega} \to \mathbb{R}^n$ be a continuous mapping. If $\boldsymbol{p} \notin \boldsymbol{f}(\partial\Omega)$, then there exists an integer $\deg(\boldsymbol{f}, \Omega, \boldsymbol{p})$ called the Brouwer degree of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{p}$ in Ω satisfying the following properties:*

Pro1 (Normality) $\deg(\boldsymbol{I}, \Omega, \boldsymbol{p}) = 1$ *if and only if $p \in \Omega$, where $\boldsymbol{I}$ denotes the identity mapping.*

Pro2 (Solvability) *If $\deg(\boldsymbol{f}, \Omega, \boldsymbol{p}) \neq 0$, then $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{p}$ has a solution in Ω.*

Pro3 (Homotopy) *If $\boldsymbol{f}_t(\boldsymbol{x}) = \boldsymbol{H}(t, \boldsymbol{x}) : [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ is continuous and $p \notin \boldsymbol{f}_t(\partial\Omega)$ for all $t \in [0, 1]$, then the $\deg(\boldsymbol{f}_t(\cdot), \Omega, \boldsymbol{p}) = \deg(\boldsymbol{H}(t, \cdot), \Omega, \boldsymbol{p})$ is independent on t.*

Lemma 3.4 [28] *Let $u(t)$ and $f(t)$ be nonnegative, continuous function on $I = [0, \infty)$ for which the inequality holds*

$$u(t) \leq u_0 + \int_0^t f(\tau)u(\tau)\mathrm{d}\tau, \quad t \in I,$$

where u_0 is a nonnegative constant. Then

$$u(t) \leq u_0 \exp^{\int_0^t f(\tau)\mathrm{d}\tau},$$

which is called the Gronwall–Bellman inequality.

Proof of Main Results

In this section, we give the proof of the main Theorem 3.1.

Proof The solution of (3.2) is equivalent to the following integral form

$$\boldsymbol{x}(t) = \boldsymbol{\xi} + \lambda_0^{RL}\mathrm{I}_t^{\alpha} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau))$$

with (Q, T)-affine-periodic boundary condition

$$\boldsymbol{x}(0) = \boldsymbol{\xi}, \quad \boldsymbol{x}(T) = Q\boldsymbol{\xi},$$

which yields

$$(\boldsymbol{I} - Q)\boldsymbol{\xi} + \lambda_0^{RL}\mathrm{I}_T^{\alpha} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) = \boldsymbol{0}. \tag{4.1}$$

Case 1 $\mathrm{Ker}(\boldsymbol{I} - Q) \neq \{\boldsymbol{0}\}$, that is $(\boldsymbol{I} - Q)^{-1}$ does not exists. By a coordinate transform and take Q as the following form

$$Q = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & Q_1 \end{pmatrix}$$

without loss of generality we assume $(\boldsymbol{I} - Q_1)^{-1}$ exists.

Set

$$\mathbb{R}^n = \mathrm{Ker}(\boldsymbol{I} - Q) \oplus \mathrm{Im}(\boldsymbol{I} - Q),$$

then for all $\boldsymbol{\xi} \in \mathbb{R}^n$, there exists $\boldsymbol{\xi}^{\mathrm{Ker}} \in \mathrm{Ker}(\boldsymbol{I} - Q)$ and $\boldsymbol{\xi}^{\perp} \in \mathrm{Im}(\boldsymbol{I} - Q)$ such that

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\mathrm{Ker}} + \boldsymbol{\xi}^{\perp}.$$

Let $\boldsymbol{P}\colon \mathbb{R}^n \to \mathrm{Ker}(\boldsymbol{I} - Q)$ be the orthogonal projection, then (4.1) is equivalent to

$$\begin{pmatrix} 0 & 0 \\ 0 & (\boldsymbol{I} - Q_1) \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}^{\mathrm{Ker}} \\ \boldsymbol{\xi}^{\perp} \end{pmatrix} + \begin{pmatrix} \lambda{}^{RL}_{0}\mathrm{I}_T^{\alpha} \boldsymbol{P}\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \\ \lambda{}^{RL}_{0}\mathrm{I}_T^{\alpha} (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \end{pmatrix} = \boldsymbol{0}.$$

And furthermore

$$\begin{cases} \lambda{}^{RL}_{0}\mathrm{I}_T^{\alpha} \boldsymbol{P}\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) = 0, \\ (\boldsymbol{I} - Q_1)\boldsymbol{\xi}^{\perp} = -\lambda{}^{RL}_{0}\mathrm{I}_T^{\alpha}((\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)). \end{cases}$$

For $\boldsymbol{x} \in \mathscr{X}$ which satisfies $\boldsymbol{x}(t) \in \bar{D}$ for all $t \in [0, T]$, we define an operator

$$\Psi(\boldsymbol{\xi}^{\mathrm{Ker}}, \boldsymbol{x}, \lambda) = \begin{pmatrix} \boldsymbol{\xi}^{\mathrm{Ker}} + {}^{RL}_{0}\mathrm{I}_T^{\alpha} \boldsymbol{P}\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \\ \boldsymbol{\xi}^{\mathrm{Ker}} - \lambda(\boldsymbol{I} - Q_1)^{-1}{}^{RL}_{0}\mathrm{I}_T^{\alpha}((\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) + \lambda{}^{RL}_{0}\mathrm{I}_t^{\alpha} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \end{pmatrix}.$$

We claim that each fixed point of Ψ in $\mathscr{X}$ is a solution of (3.2). In fact, if $\boldsymbol{x}$ is a fixed point of Ψ, then

$$\begin{pmatrix} \boldsymbol{\xi}^{\mathrm{Ker}} \\ \boldsymbol{x}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\xi}^{\mathrm{Ker}} + {}^{RL}_{0}\mathrm{I}_T^{\alpha} P\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \\ \boldsymbol{\xi}^{\mathrm{Ker}} - \lambda(\boldsymbol{I} - Q_1)^{-1}{}^{RL}_{0}\mathrm{I}_T^{\alpha}(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) + \lambda{}^{RL}_{0}\mathrm{I}_t^{\alpha} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \end{pmatrix}.$$

Thus,

$$\begin{cases} {}^{RL}_{0}\mathrm{I}^{\alpha}_{T}\boldsymbol{P}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))=0, \\ \boldsymbol{x}(t)=\boldsymbol{\xi}^{\mathrm{Ker}}-\lambda(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}((\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))+\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)). \end{cases} \tag{4.2}$$

Substituting $t=0$ and $t=T$ into $\boldsymbol{x}(t)$, it gives

$$\begin{aligned} \boldsymbol{\xi}&=\boldsymbol{\xi}^{\mathrm{Ker}}-\lambda(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)),\\ \boldsymbol{x}(T)&=\boldsymbol{\xi}^{\mathrm{Ker}}-\lambda(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))+\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)). \end{aligned}$$

In order to prove $\boldsymbol{x}$ is the solution of (3.2), a necessary condition is

$$\begin{aligned} &Q\boldsymbol{x}(0)=\boldsymbol{x}(T)\\ &\Leftrightarrow \begin{pmatrix}\boldsymbol{I} & 0\\ 0 & Q_1\end{pmatrix}\begin{pmatrix}\boldsymbol{\xi}^{Ker}\\ \boldsymbol{\xi}^{\perp}\end{pmatrix}=\begin{pmatrix}\boldsymbol{\xi}^{\mathrm{Ker}}+\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}\boldsymbol{P}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))\\ -\lambda(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))\\ +\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))\end{pmatrix}\\ &\Leftrightarrow \lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}\boldsymbol{P}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))=0, \end{aligned}$$

which is obvious.

Moreover, we have

$$\boldsymbol{\xi}^{\perp}=-\lambda(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)).$$

So

$$\begin{aligned} \boldsymbol{x}(t)&=\boldsymbol{\xi}^{\mathrm{Ker}}-\lambda(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))+\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))\\ &=\boldsymbol{\xi}^{\mathrm{Ker}}+\boldsymbol{\xi}^{\perp}+\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))\\ &=\boldsymbol{\xi}+\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)). \end{aligned}$$

It means that the fixed point $\boldsymbol{x}$ of Ψ is a solution of (3.2).

Provided that there exists a fix point of operator Ψ, we have proved that the fixed point is the solution of (3.2). So we need to illustrate the existence of fixed points of Ψ.

Set

$$\boldsymbol{f}(t,\boldsymbol{x})=(f_1(t,\boldsymbol{x}),f_2(t,\boldsymbol{x}),\cdots,f_n(t,\boldsymbol{x})),$$

where f_i is continuous since $\boldsymbol{f}$ is continuous, then there exists constants m_i and M_i such that

$$0\le m_i\le\frac{\max_{t\in[0,T]}|f_i(t,\boldsymbol{x}(t))|}{\Gamma(\alpha+1)}\le M_i,\quad i=1,2,\cdots,n.$$

Set

$$M=\max_{1\le i\le n}M_i,\quad m=\max_{1\le i\le n}m_i,$$

and

$$\mathscr{X}_\lambda=\left\{x\in\mathscr{X}:\frac{\|\boldsymbol{x}(t)-\boldsymbol{x}(s)\|}{|t^\alpha-s^\alpha|}\le\lambda\sqrt{n}(M-m),\quad\forall t\ne s\right\},$$

then it is easy to define a retraction $\sigma_\lambda:\mathscr{X}\to\mathscr{X}_\lambda$.

Define a homotopy

$$\begin{aligned} &\boldsymbol{H}(\boldsymbol{\xi}^{\mathrm{Ker}},\boldsymbol{x},\lambda)\\ &=\begin{pmatrix}\boldsymbol{\xi}^{\mathrm{Ker}}+{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}P\boldsymbol{f}(\tau,\sigma_\lambda\circ\boldsymbol{x}(\tau))\\ \sigma_\lambda\circ\boldsymbol{\xi}^{\mathrm{Ker}}-\lambda(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\sigma_\lambda\circ\boldsymbol{x}(\tau))+\lambda{}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\sigma_\lambda\circ\boldsymbol{x}(\tau))\end{pmatrix}, \end{aligned}$$

where

$$(\boldsymbol{\xi}^{\mathrm{Ker}}, \boldsymbol{x}, \lambda) \in (D \cap \mathrm{Ker}(\boldsymbol{I} - Q)) \times \tilde{D} \times [0, 1], \quad \tilde{D} = \{\boldsymbol{x} \in \mathscr{X} | \boldsymbol{x}(t) \in D, \quad \forall\, t \in [0, T]\}.$$

We claim that

$$\mathbf{0} \notin (\boldsymbol{I} - \boldsymbol{H})(\partial((D \cap \mathrm{Ker}(\boldsymbol{I} - Q)) \times \tilde{D}) \times [0, 1]),$$

which is proved by contradiction. Suppose there exists $(\bar{\boldsymbol{\xi}}^{\mathrm{Ker}}, \bar{\boldsymbol{x}}, \bar{\lambda}) \in (\partial((D \cap \mathrm{Ker}(\boldsymbol{I} - Q)) \times \tilde{D}) \times [0, 1])$ such that

$$(\boldsymbol{I} - \boldsymbol{H})(\bar{\boldsymbol{\xi}}^{\mathrm{Ker}}, \bar{\boldsymbol{x}}, \bar{\lambda}) = 0.$$

As $\bar{\boldsymbol{\xi}}^{\mathrm{Ker}} \in \partial\,(D \cap \mathrm{Ker}(\boldsymbol{I} - Q)) \subset \partial D$ is a contradictory to the assumption A_1, we know that $\bar{\boldsymbol{\xi}}^{\mathrm{Ker}} \notin \partial\,(D \cap \mathrm{Ker}(\boldsymbol{I} - Q)) \subset \partial D$. In other words, $\bar{\boldsymbol{x}} \in \partial \tilde{D}$. We discuss it in two cases:

(i) If $\bar{\lambda} = 0$, then

$$\mathscr{X}_0 = \left\{\boldsymbol{x} \in \mathscr{X} \Big| \frac{||\boldsymbol{x}(t) - \boldsymbol{x}(s)||}{|t^\alpha - s^\alpha|} \le 0\right\}.$$

Hence $\boldsymbol{x}(t) = \boldsymbol{p} \in \mathscr{X}_0$ and $\sigma_0 \circ \boldsymbol{x}(t) = \boldsymbol{p}$ for all $t \in [0, T]$. Since $(\boldsymbol{I} - \boldsymbol{H})(\bar{\boldsymbol{\xi}}^{\mathrm{Ker}}, \bar{\boldsymbol{x}}, \bar{\lambda}) = 0$, we have

$$\begin{pmatrix} \bar{\boldsymbol{\xi}}^{\mathrm{Ker}} \\ \bar{\boldsymbol{x}}(t) \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{\xi}}^{\mathrm{Ker}} + {}^{RL}_{0}\mathrm{I}^{\alpha}_{T} \boldsymbol{P} \boldsymbol{f}(\tau, \boldsymbol{p}) \\ \sigma_0 \circ \bar{\boldsymbol{\xi}}^{\mathrm{Ker}}, \end{pmatrix},$$

thus $g(\boldsymbol{p}) = 0$. Notice that $\bar{\boldsymbol{x}} \in \partial \tilde{D}$, hence there exists $t_0 \in [0, T]$ such that $\boldsymbol{p} = \boldsymbol{x}(t_0) \in \partial D$. It is a contradictory to the assumption A_2, because the Brouwer degree $\deg(\boldsymbol{g}, D, \mathbf{0}) \neq 0$.

(ii) If $\bar{\lambda} \in (0, 1]$, since $(\boldsymbol{I} - \boldsymbol{H})(\bar{\boldsymbol{\xi}}^{\mathrm{Ker}}, \bar{\boldsymbol{x}}, \bar{\lambda}) = 0$, we have

$$\begin{pmatrix} \bar{\boldsymbol{\xi}}^{\mathrm{Ker}} \\ \bar{\boldsymbol{x}}(t) \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{\xi}}^{Ker} + {}^{RL}_{0}\mathrm{I}^{\alpha}_{T} \boldsymbol{P} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \\ \sigma_{\bar{\lambda}} \circ \bar{\boldsymbol{\xi}}^{\mathrm{Ker}} - \bar{\lambda}(\boldsymbol{I} - Q_1)^{-1} {}^{RL}_{0}\mathrm{I}^{\alpha}_{T} (\boldsymbol{I} - \boldsymbol{P}) \boldsymbol{f}(\tau, \sigma_{\bar{\lambda}} \circ \bar{\boldsymbol{x}}(\tau)) + \bar{\lambda}\, {}^{RL}_{0}\mathrm{I}^{\alpha}_{t} \boldsymbol{f}(\tau, \sigma_{\bar{\lambda}} \circ \bar{\boldsymbol{x}}(\tau)) \end{pmatrix},$$

thus

$${}^{RL}_{0}\mathrm{I}^{\alpha}_{T} \boldsymbol{P} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) = \mathbf{0},$$

and

$$\bar{\boldsymbol{x}}(t) = \sigma_{\bar{\lambda}} \circ \bar{\boldsymbol{\xi}}^{\mathrm{Ker}} - \bar{\lambda}(\boldsymbol{I} - Q_1)^{-1} {}^{RL}_{0}\mathrm{I}^{\alpha}_{T} (\boldsymbol{I} - \boldsymbol{P}) \boldsymbol{f}(\tau, \sigma_{\bar{\lambda}} \circ \bar{\boldsymbol{x}}(\tau)) + \bar{\lambda}\, {}^{RL}_{0}\mathrm{I}^{\alpha}_{t} \boldsymbol{f}(\tau, \sigma_{\bar{\lambda}} \circ \bar{\boldsymbol{x}}(\tau)). \quad (4.3)$$

Notice that

$$\begin{aligned}\frac{\|\bar{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(s)\|}{|t^\alpha - s^\alpha|} &= \frac{\|\bar{\lambda}{}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\sigma_{\bar{\lambda}}\circ\bar{\boldsymbol{x}}(\tau)) - \bar{\lambda}{}^{RL}_{0}\mathrm{I}^{\alpha}_{s}\boldsymbol{f}(\tau,\sigma_{\bar{\lambda}}\circ\bar{\boldsymbol{x}}(\tau))\|}{|t^\alpha - s^\alpha|}\\
&= \frac{\bar{\lambda}}{\Gamma(\alpha)|t^\alpha - s^\alpha|}\bigg\| \int_0^t (t-\tau)^{\alpha-1}\boldsymbol{f}(\tau,\sigma_{\bar{\lambda}}\circ\bar{\boldsymbol{x}}(\tau)\mathrm{d}\tau\\
&\quad - \int_0^s (s-\tau)^{\alpha-1}\boldsymbol{f}(\tau,\sigma_{\bar{\lambda}}\circ\bar{\boldsymbol{x}}(\tau))\mathrm{d}\tau\bigg\|\\
&= \frac{\bar{\lambda}}{\Gamma(\alpha)|t^\alpha - s^\alpha|}\Bigg(\sum_{i=1}^{n}\bigg| \int_0^t (t-\tau)^{\alpha-1} f_i(t,\boldsymbol{x})\mathrm{d}\tau\\
&\quad - \int_0^s (s-\tau)^{\alpha-1} f_i(t,\boldsymbol{x})\mathrm{d}\tau\bigg|^2\Bigg)^{\frac{1}{2}}\\
&\le \frac{\bar{\lambda}}{\Gamma(\alpha)|t^\alpha - s^\alpha|}\Bigg(\sum_{i=1}^{n}(M_i - m_i)^2\bigg| \int_0^t (t-\tau)^{\alpha-1} d\tau\\
&\quad - \int_0^s (s-\tau)^{\alpha-1}\mathrm{d}\tau\bigg|^2\Bigg)^{\frac{1}{2}}\\
&\le \frac{\bar{\lambda}\sqrt{n}(M-m)\Gamma(\alpha+1)}{\Gamma(\alpha)|t^\alpha - s^\alpha|}\bigg| \int_0^t (t-\tau)^{\alpha-1}\mathrm{d}\tau - \int_0^s (s-\tau)^{\alpha-1}\mathrm{d}\tau\bigg|\\
&\le \bar{\lambda}\sqrt{n}(M-m),\end{aligned}$$

which means $\bar{\boldsymbol{x}} \in \mathscr{X}_{\bar{\lambda}}$, thus $\sigma_{\bar{\lambda}} \circ \bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}$. Now we rewrite (4.3) as

$$\bar{\boldsymbol{x}}(t) = \bar{\boldsymbol{\xi}}^{\mathrm{Ker}} - \bar{\lambda}(\boldsymbol{I}-Q_1)^{-1}{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}(\tau,\bar{\boldsymbol{x}}(\tau)) + \bar{\lambda}{}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\bar{\boldsymbol{x}}(\tau)). \tag{4.4}$$

And similar to the discussion in (4.2), we can prove that $\bar{\boldsymbol{x}}(t)$ is a solution of (3.2). By assumption A_1, we know that $\bar{\boldsymbol{x}}(t) \notin \partial D$ for all $t \in [0,T]$, which is a contradictory to $\bar{\boldsymbol{x}} \in \partial\tilde{D}$. Thus we have

$$0 \notin (\boldsymbol{I}-\boldsymbol{H})(\partial((D\cap Ker(\boldsymbol{I}-Q))\times\tilde{D})\times[0,1]).$$

Therefore, by the homotopy invariance and the theory of the Brouwer degree, we have

$$\begin{aligned}&\deg((\boldsymbol{I}-\boldsymbol{H})(\boldsymbol{\xi}^{\mathrm{Ker}},\cdot,1),(D\cap\mathrm{Ker}(\boldsymbol{I}-Q))\times\tilde{D},\boldsymbol{0})\\
&= \deg((\boldsymbol{I}-\boldsymbol{H})(\boldsymbol{\xi}^{\mathrm{Ker}},\cdot,0),(D\cap\mathrm{Ker}(\boldsymbol{I}-Q))\times\tilde{D},\boldsymbol{0})\\
&= \deg(\boldsymbol{g},D\cap\mathrm{Ker}(\boldsymbol{I}-Q),\boldsymbol{0}) \neq 0,\end{aligned}$$

which means that there exists $\boldsymbol{x}^* \in \tilde{D}$ such that

$$\begin{pmatrix}\boldsymbol{x}^{*\mathrm{Ker}}\\ \boldsymbol{x}^*(t)\end{pmatrix} = \boldsymbol{H}(\boldsymbol{x}^{*\mathrm{Ker}},\boldsymbol{x}^*,1)$$

and $\boldsymbol{x}^* \in \mathscr{X}_1$ then $\boldsymbol{H}(\boldsymbol{x}^{*\mathrm{Ker}},\boldsymbol{x}^*,1) = \Phi(\boldsymbol{x}^{*\mathrm{Ker}},\boldsymbol{x}^*,1)$, and $\boldsymbol{x}^*$ is a fixed point of Ψ in $\mathscr{X}$, thus $\boldsymbol{x}^*(t)$ is a solution of (3.2).
Case 2 If $\mathrm{Ker}(\boldsymbol{I}-Q) = \{\boldsymbol{0}\}$, that is $(\boldsymbol{I}-Q)^{-1}$ exists, then

$$\boldsymbol{\xi} = -(\boldsymbol{I}-Q)^{-1}\lambda\,{}^{RL}_{0}\mathrm{I}^{\alpha}_{T}\boldsymbol{P}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)).$$

Consider the homotopy

$$H(x,\lambda) = -(\boldsymbol{I}-Q)^{-1}\lambda\ {}^{RL}_{0}\mathrm{I}^{\alpha}_{T}\boldsymbol{P}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)) + \lambda\ {}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)).$$

Similar to the proof when $\mathrm{Ker}(\boldsymbol{I}-\boldsymbol{Q}) \neq \{0\}$, we have $\boldsymbol{0} \notin (\boldsymbol{I}-\boldsymbol{H})(\partial\tilde{D}\times[0,1])$. Hence

$$\deg(\boldsymbol{I}-\boldsymbol{H}(\cdot,1),\tilde{D},\boldsymbol{0}) = \deg(\boldsymbol{I}-\boldsymbol{H}(\cdot,0),\tilde{D},\boldsymbol{0}) = \deg(\boldsymbol{I},\tilde{D},\boldsymbol{0}) = 1,$$

which means that there exists $\boldsymbol{x}^*(t)\in D$ for all $t\in\mathbb{R}^1$ such that

$$\boldsymbol{x}^*(t) = \boldsymbol{x}^*(0) + \lambda\ {}^{RL}_{0}\mathrm{I}^{\alpha}_{t}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau)).$$

Therefore $\boldsymbol{x}^*(t)$ is a solution of (3.2).

Since we know the solution exists, then the uniqueness of the solution with boundary values can be described by the uniqueness of the solution with initial value. Let us consider the α-fractional system with initial value as follows

$$\begin{cases} {}^{C}_{0}\mathrm{D}^{\alpha}_{t}\boldsymbol{x} = \boldsymbol{f}(t,\boldsymbol{x}),\\ \boldsymbol{x}(0) = \boldsymbol{\xi}, \end{cases}$$

which is equivalent to the following integral formal

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))\mathrm{d}\tau.$$

If there is another solution $\boldsymbol{y}(t)$, and thanks to the assumption A_3, then

$$\begin{aligned} \|\boldsymbol{x}(t)-\boldsymbol{y}(t)\| &= \left\| \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\boldsymbol{f}(\tau,\boldsymbol{x}(\tau))\,\mathrm{d}\tau \right.\\ &\quad \left. - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\boldsymbol{f}(\tau,\boldsymbol{y}(\tau))\,\mathrm{d}\tau \right\|\\ &= \frac{1}{\Gamma(\alpha)}\left\| \int_0^t (t-\tau)^{\alpha-1}\left(\boldsymbol{f}(\tau,\boldsymbol{x}(\tau) - \boldsymbol{f}(\tau,\boldsymbol{y}(\tau))\,\mathrm{d}\tau \right\|\\ &\le \frac{L}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|\boldsymbol{x}(\tau)-\boldsymbol{y}(\tau)\|\mathrm{d}\tau. \end{aligned} \tag{4.5}$$

Setting $h(t) = \|\boldsymbol{x}(t)-\boldsymbol{y}(t)\|$, then $h(0)=0$ and (4.5) is equivalent to

$$h(t) \le \frac{L}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)\mathrm{d}\tau.$$

According to the Lemma 3.4, we have

$$h(t) \le h(0)\cdot \exp^{\frac{Lt^{\alpha}}{\Gamma(\alpha+1)}} \le h(0)\cdot\exp^{\frac{LT^{\alpha}}{\Gamma(\alpha+1)}} = 0,$$

which indicates that $\boldsymbol{x}=\boldsymbol{y}$. That shows the uniqueness of the solution. □

Some Applications

In this section, we will apply the main theorem to the fractional relaxor kinetic equations and fractional harmonic oscillator equations. It shows the existence and uniqueness of the solutions under the affine-periodic boundary conditions.

Fractional Relaxor Kinetic Equation in $\mathbb{R}^n$

It is a well known fact that many fundamental laws of physics can be formulated as generalized fractional kinetic equations [1,16,29,30] of the form

$$ {}_0^C\mathrm{D}_t^{\alpha}\boldsymbol{x} = -\boldsymbol{A}\boldsymbol{x} \tag{5.1}$$

with affine-periodic boundary condition

$$ \boldsymbol{x}(0) = \boldsymbol{\xi}, \quad \boldsymbol{x}(T) = Q\boldsymbol{\xi}, \tag{5.2}$$

where $0 < \alpha \leq 1, \boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, n \in \mathbb{N}, 0 < t \leq T$, and $\boldsymbol{A} = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}, d_i > 0, i = 1, 2, \ldots, n$.

When the index $\alpha = 1$ and $n = 1$, this equation represents a relaxation process described by the solutions $x(t) = x(0)e^{-t/\tau}$ with a characteristic time scale $\tau = A^{-1}$ for the exponential decay. When $\alpha = 1$ and $n \in \mathbb{N}$, Eq. (5.2) could be Maxwell's equations or Schrödinger's equation if $\boldsymbol{A}$ is limited to linear operators, or it could be Newton's law of motion or Einstein's equations for geodesics if $\boldsymbol{A}$ may also be a nonlinear operator [30].

In this subsection, we consider α-order kinetic equation in $\mathbb{R}^n$, where $0 < \alpha \leq 1$. We focus on the fractional relaxor, i.e. $0 < \alpha \leq 1$. The fractional kinetic equation has different limit cases which have been discussed intensively during the last decade. One case is related to the so-called Levý process and Levý flights [20,24,33], another to the problem of fractal time and fractal Brownian motion [31]. Both limit cases can be matched to the Montroll–Weiss equation and to the continuous time random walk [8,25,32,44,45]. All of these cases can be united by the fractional kinetic equation. The fractional kinetic equation being written in some generalized form can cover all described cases, including non-Markovian case, and the corresponding generalized solutions can be obtained. Sometimes, the well-known integral equations, like the Benjamin–Ono equation for the internal ocean waves, can be rewritten in a form with fractional derivatives.

Theorem 5.1 *For any* $0 < \alpha \leq 1$, *if* $Q = \mathrm{diag}\{E_\alpha((-d_1T)^\alpha), E_\alpha((-d_2T)^\alpha), \ldots, E_\alpha((-d_nT)^\alpha)\}$, *then the problem* (5.1)–(5.2) *admits a unique solution.*

Proof Set the open set

$$ D := \prod_{i=1}^{n} D_i \subset \mathbb{R}^n, \quad \text{where } D_i = (-m_i - \delta_0, \ \ m_i + \delta_0), $$
$$ m_i = \max\{|\xi_i|E_\alpha((-d_iT)^\alpha), |\xi_i|\}, \quad \delta_0 \in \mathbb{R}^+. $$

It is easy that the exact solution of (5.1) and (5.2) is

$$ \boldsymbol{x}(t) = (\xi_1 E_\alpha((-d_1t)^\alpha), \xi_2 E_\alpha((-d_2t)^\alpha), \ldots, \xi_n E_\alpha((-d_nt)^\alpha)) \in \bar{D}, $$

and $\boldsymbol{x}(t) \notin \partial D$ for all $t \in [0, T]$. Since $\mathrm{Ker}(\boldsymbol{I} - Q) = \{\boldsymbol{0}\}$ and by the proof of the Theorem 3.1 when $\mathrm{Ker}(\boldsymbol{I} - Q) = \{\boldsymbol{0}\}$, there admits a solution of the problem (5.1) and (5.2). And

$$ \|(-\boldsymbol{A}\boldsymbol{x}) - (-\boldsymbol{A}\boldsymbol{y})\| = \|\boldsymbol{A}\|\|\boldsymbol{x} - \boldsymbol{y}\| \leq L||\boldsymbol{x} - \boldsymbol{y}||, $$

where $L = \sqrt{n}\max_{1\leq i\leq n} d_i$, thus the solution is unique. □

Fractional Harmonic Oscillators

The harmonic oscillator, one of the simplest mechanical systems, whose motion is governed by a second order linear differential equation with constant coefficients

$$my^{''} + \mu y^{'} + ky = 0, \tag{5.3}$$

which describes the displacement (elongation) of a body of mass m, in time t, from the equilibrium position, subject to Hooke's Law, $-ky(t)$, a damping force $-\mu y'$, where the prime $' = \frac{d}{dt}$ and μ, k are positive constants. A lucid treatment of the various aspects of the dynamics of the simple harmonic oscillator can be found in the Feynmann Lectures on Physics [15].

The purpose of this subsection is to present a generalization of the classical harmonic oscillator based on the methods of fractional calculus, into what will be referred to hereafter as the fractional oscillator. Consider fractional homogeneous linear vibration equation with constant coefficients as

$$m \ {}_0^C\mathrm{D}_t^{2\alpha} y + \mu \ {}_0^C\mathrm{D}_t^{\alpha} y + ky = 0, \quad 1/2 < \alpha \leq 1. \tag{5.4}$$

Here we investigate a generalized damped harmonic oscillator, where the first derivative in the damping term has been replaced by a derivative of arbitrary order α, where $1/2 < \alpha \leq 1$. The "oscillator" part is also described by a fractional 2α-order derivative. This equation was used to describe the properties of viscoelastic materials by Bagley and Torvik [37]. Their work shows that constitutive equations containing fractional derivatives are effective in describing the frequency-dependent behavior of viscoelastic polymers [7] and that the fractional calculus leads to well-posed problems for the motion of structures containing elastic and viscoelastic components. A particular virtue of constitutive relations containing fractional derivatives is that they lead to a casual response at zero time [6], thereby having a distinct advantage over convolution methods employing a structural damping model in that they may be safely employed to predict transient response.

Set

$$2n = \frac{\mu}{m}, \quad \omega^2 = \frac{k}{m}, \quad x_1 = y, \quad x_2 = {}_0^C\mathrm{D}_t^{\alpha} x_1,$$

then (5.4) is equivalent to the α-order linear equations with constant coefficients

$$\begin{cases} {}_0^C\mathrm{D}_t^{\alpha} x_1 = x_2, \\ {}_0^C\mathrm{D}_t^{\alpha} x_2 = -\omega^2 x_1 - 2n x_2, \end{cases}$$

which can be written in compact form

$${}_0^C\mathrm{D}_t^{\alpha} \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}, \quad 1/2 < \alpha \leq 1, \tag{5.5}$$

where

$$\boldsymbol{x} = (x_1, x_2)^T, \quad \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2n \end{pmatrix},$$

and furthermore we consider Eq. (5.5) with affine-periodic boundary conditions

$$\boldsymbol{x}(0) = \boldsymbol{\xi}, \quad \boldsymbol{x}(T) = Q\boldsymbol{\xi}, \tag{5.6}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2, \quad T > 0.$

Theorem 5.2 *For any* $1/2 < \alpha \le 1$, *if*

$$Q = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2 E_\alpha(\lambda_1 T^\alpha) - \lambda_1 E_\alpha(\lambda_2 T^\alpha) & -E_\alpha(\lambda_1 T^\alpha) + E_\alpha(\lambda_2 T^\alpha) \\ \lambda_1\lambda_2 E_\alpha(\lambda_1 T^\alpha) - \lambda_1\lambda_2 E_\alpha(\lambda_2 T^\alpha) & -\lambda_1 E_\alpha(\lambda_1 T^\alpha) + \lambda_2 E_\alpha(\lambda_2 T^\alpha) \end{pmatrix},$$

then there exists a unique solution of system (5.5)–(5.6), *where* λ_1 *and* λ_2 *are two different eigenvalues of* $\boldsymbol{A}$ *when* $n \neq \omega > 0$.

Proof Since $n \neq \omega > 0$, then the matrix $\boldsymbol{A}$ admits two different eigenvalues λ_1 and λ_2, and there exists a non-degenerate linear transformation $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{z}$ such that (5.5) can be transformed into

$${}_0^C\mathrm{D}_t^\alpha \boldsymbol{z} = \boldsymbol{B}\boldsymbol{z}, \quad 1/2 < \alpha \le 1, \quad \boldsymbol{B} = \boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We discuss the above equation in two different cases:
Case 1 If $n > \omega$, then $\lambda_1 = -n - \sqrt{n^2 - \omega^2}$, $\lambda_2 = -n + \sqrt{n^2 - \omega^2}$ and

$$\boldsymbol{R} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$

It is easy to get

$$\boldsymbol{z} = \left(C_1 E_\alpha(\lambda_1 t^\alpha), C_2 E_\alpha(\lambda_2 t^\alpha)\right)^T,$$

and thus

$$\boldsymbol{x} = \boldsymbol{R}\boldsymbol{z} = (C_1 E_\alpha(\lambda_1 t^\alpha) + C_2 E_\alpha(\lambda_2 t^\alpha), \lambda_1 C_1 E_\alpha(\lambda_1 t^\alpha) + \lambda_2 C_2 E_\alpha(\lambda_2 t^\alpha))^T.$$

Obviously, $\boldsymbol{x}$ is continuous on $[0, T]$. Then there exists an open set $D \subset \mathbb{R}^2$ such that $\boldsymbol{x}(t) \notin \partial D$ for all $t \in [0, T]$ if $\boldsymbol{x}(t) \in \bar{D}$.
Case 2 If $n < \omega$, then $\lambda_1 = -n - \mathrm{i}\sqrt{\omega^2 - n^2}$, $\lambda_2 = -n + \mathrm{i}\sqrt{\omega^2 - n^2}$. By the same way as in Case 1, there exists an open set $D \subset \mathbb{R}^2$ such that $\boldsymbol{x}(t) \notin \partial D$ for all $t \in [0, T]$ if $\boldsymbol{x}(t) \in \bar{D}$.

After simple calculation, we can get

$$\det(\boldsymbol{I} - \boldsymbol{Q}) = (E_\alpha(\lambda_1 T^\alpha) - 1)(E_\alpha(\lambda_2 T^\alpha) - 1) \neq 0,$$

thus $\mathrm{Ker}(\boldsymbol{I} - \boldsymbol{Q}) = \{\boldsymbol{0}\}$. Hence, there exists a solution of problem (5.5)-(5.6).

Finally, since $\boldsymbol{A}$ is a bounded linear operator, there exists a positive constant L such that

$$\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{y}\| \le L\|\boldsymbol{x} - \boldsymbol{y}\|,$$

which indicates the uniqueness of the solution. □

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Appendix

Proof of Lemma 2.4

Proof *Case 1* If $\beta = n$, then

$$
\begin{aligned}
{}^{RL}_{a}\mathrm{I}^{\beta}_{t}\,{}^{C}_{a}\mathrm{D}^{\beta}_{t} f(\tau) &= \underbrace{\int_a^t \cdots \int_a^t}_{n-times} f^{(n)}(\tau)\mathrm{d}\tau = \underbrace{\int_a^t \cdots \int_a^t}_{(n-1)-times} (f^{(n-1)}(\tau) - a_1^{(0)})\mathrm{d}\tau \\
&= \underbrace{\int_a^t \cdots \int_a^t}_{(n-2)-times} (f^{(n-2)}(\tau) - a_2^{(0)} - a_2^{(1)}t)\mathrm{d}\tau \\
&= \cdots \\
&= \int_a^t (f^{'}(\tau) - a_{n-1}^{(0)} - a_{n-1}^{(1)}t - \cdots - a_{n-1}^{(n-2)}t^{n-2})\mathrm{d}\tau \\
&= f(t) - a_n^{(0)} - a_n^{(1)}t - \cdots - a_n^{(n-1)}t^{n-1} \\
&\triangleq f(t) - \sum_{k=0}^{n-1} c_k t^k,
\end{aligned}
$$

where $c_k = a_{n-1}^{(k)} \in \mathbb{R}^1$.
Case 2 If $n-1 < \beta < n$, then

$$
{}^{RL}_{a}\mathrm{I}^{\beta}_{t}\,{}^{C}_{a}\mathrm{D}^{\beta}_{t} f(\tau) = {}^{RL}_{a}\mathrm{I}^{\beta}_{t}\,{}^{RL}_{a}\mathrm{I}^{n-\beta}_{t}\,{}^{C}_{a}\mathrm{D}^{n}_{t} f(\tau) = {}^{RL}_{a}\mathrm{I}^{n}_{t}\,{}^{C}_{a}\mathrm{D}^{n}_{t} f(\tau),
$$

and by the proof of Case 1, we have

$$
{}^{RL}_{a}\mathrm{I}^{\beta}_{t}\,{}^{C}_{a}\mathrm{D}^{\beta}_{t} f(\tau) = f(t) - \sum_{k=0}^{n-1} c_k t^k.
$$

□

A Proof of the Nonexistence of (α, Q, T)-Affine-Periodic Solution for a Given (α, Q, T)-Affine-Periodic System when $0 < \alpha < 1$

Proof Consider the (α, Q, T)-affine-periodic system

$$
{}^{C}_{a}\mathrm{D}^{\alpha}_{t}\boldsymbol{x} = \boldsymbol{f}(t, \boldsymbol{x}), \quad (t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^n, \tag{6.1}
$$

where $0 < \alpha < 1$, $Q \in GL_n(\mathbb{R})$, $T > 0$, $a \in \mathbb{R}$.

If $\boldsymbol{x} = \boldsymbol{x}(t)$ is an (α, Q, t)-affine-periodic solution of (6.1), then $\boldsymbol{x}(t+T) = Q\boldsymbol{x}(t)$, $\forall t \in [0, T]$.

Set

$$
\boldsymbol{y}(t) \triangleq \boldsymbol{x}(t) = Q^{-1}\boldsymbol{x}(t+T),
$$

then y should satisfy ${}_a^C\mathrm{D}_t^\alpha \boldsymbol{y} = \boldsymbol{f}(t, \boldsymbol{y})$. But in fact, it follows the definitions of Riemann–Liouville fractional integral and Caputo fractional derivative that

$$\begin{aligned}
{}_a^C\mathrm{D}_t^\alpha \boldsymbol{y}(t) &= {}_a^{RL}\mathrm{I}_t^{1-\alpha}\,{}_a^C\mathrm{D}_t^1 \boldsymbol{y}(t)\\
&= {}_a^{RL}\mathrm{I}_t^{1-\alpha} \boldsymbol{y}^{'}(t)\\
&= \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha}\boldsymbol{y}^{'}(\tau)\mathrm{d}\tau\\
&= \frac{Q^{-1}}{\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha}\boldsymbol{x}^{'}(\tau+T)\mathrm{d}\tau\\
&= \frac{Q^{-1}}{\Gamma(1-\alpha)}\int_{a+T}^{t+T} (t+T-\tau)^{-\alpha}\boldsymbol{x}^{'}(\tau)\mathrm{d}\tau\\
&= \frac{Q^{-1}}{\Gamma(1-\alpha)}\int_a^{t+T} (t+T-\tau)^{-\alpha}\boldsymbol{x}^{'}(\tau)\mathrm{d}\tau\\
&\quad - \frac{Q^{-1}}{\Gamma(1-\alpha)}\int_a^{a+T} (t+T-\tau)^{-\alpha}\boldsymbol{x}^{'}(\tau)\mathrm{d}\tau\\
&= Q^{-1}f(t+T, \boldsymbol{x}(t+T)) - \frac{Q^{-1}}{\Gamma(1-\alpha)}\int_a^{a+T} (t+T-\tau)^{-\alpha}\boldsymbol{x}^{'}(\tau)\mathrm{d}\tau\\
&= f(t, Q^{-1}\boldsymbol{x}(t+T)) - \frac{Q^{-1}}{\Gamma(1-\alpha)}\int_a^{a+T} (t+T-\tau)^{-\alpha}\boldsymbol{x}^{'}(\tau)\mathrm{d}\tau\\
&= f(t, \boldsymbol{y}(t)) - \frac{Q^{-1}}{\Gamma(1-\alpha)}\int_a^{a+T} (t+T-\tau)^{-\alpha}\boldsymbol{y}^{'}(\tau)\mathrm{d}\tau.
\end{aligned}$$

When $a = -\infty$,

$${}_a^C\mathrm{D}_t^\alpha \boldsymbol{y} = f(t, \boldsymbol{y}),$$

which is reasonable and this case is also in the sense of Weyl fractional integral, but we don't know what is the placement in infinity. When $a > -\infty$, that is to say, the lower limit of integral is finite or there is a "truncation" destroys the shift-invariant, which shows the nonexistence of the (α, Q, T)-affine-periodic solution for a given (α, Q, T)-affine-periodic system. And some similar results about periodicity or quasi-periodicity for fractional integrals and derivatives of periodic functions can be found in Nieto et al.'s work, see [3–5]. ◻

Remark 6.1 The above conclusion shows that in order to obtain the existence and uniqueness of affine-periodic solutions for fractional affine-periodic dynamical systems, the fractional differential operators must keep shift-invariant apart from Weyl fractional differential operators. Thus, one direction for the future research is to modify the definition of fractional derivative or to seek "quasi-periodic solutions". This is our ongoing work and will be reported elsewhere.

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