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ORIGINAL RESEARCH

On General Fractional Differential Inclusions with Nonlocal Integral Boundary Conditions

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Abstract This paper investigates the existence of solutions for fractional differential inclusions involving Caputo fractional derivative of any order together with nonlocal integral boundary conditions. Our study includes the cases when the multivalued map involved in the problem has convex as well as non-convex values. Some standard fixed point theorems for multivalued maps are applied to establish the main results, which are well illustrated with the aid of examples.

Keywords Existence · Fractional differential inclusions · Nonlocal boundary conditions · Fixed point theorems · Multivalued maps

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Introduction

In this paper, we study the following nonlocal integral boundary value problem of Caputo type fractional differential inclusions:

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$$\begin{cases} {}^cD^r_{t_0}x\,(t) \in F\,(t, x\,(t))\,, & t \in J = [t_0, T]\,, \quad n-1 < r < n, \\ x^{(k)}\,(\delta) = x_k + \int\limits_{t_0}^{\delta} g_k(s, x(s))ds, \; k = 0, 1 \ldots, n-1, & \delta \in (t_0, T)\,, \end{cases} \tag{1.1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $g_k : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ${}^cD^r_{t_0}$ denotes the Caputo fractional derivative of order r, $n = [r] + 1$, $[r]$ denotes the integer part of the real number r.

Fractional differential equations and inclusions have been extensively studied by many researchers in the recent years. It has been mainly due to the fact that fractional differential operators appear naturally in a number of disciplines of pure and applied sciences such as biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, circuits theory, control theory, etc., for instance, see [1–3]. The differential inclusions also find decent applications in some areas of physics and control [4]. For some recent work on fractional differential equations and inclusions, we refer the reader to a series of papers [5–15] and the references cited therein. Recently, Ahmad et al. [12] discussed the existence of solutions for a general differential equation of an arbitrary fractional order with nonlocal integral boundary conditions at an interior point of the given finite interval.

In this article, motivated by aforementioned work, we obtain some existence theorems for the inclusion problem (1.1) involving convex as well as nonconvex multivalued maps. These results are based on the nonlinear alternative of Leray–Schauder type, a selection theorem due to Bressan and Colombo, and a fixed point theorem due to Covitz and Nadler. The methods employed to establish the desired results are standard; however their exposition in the framework of problem (1.1) is new and enriches the literature dealing with fractional differential inclusions with nonlocal integral boundary conditions. In passing, we remark that the present work generalizes the problem addressed in [12] to its multivalued case.

This paper is organized as follows. In Sect. 2, we recall some preliminaries about fractional calculus and multivalued mappings analysis. Section 3 contains the main results for the fractional inclusion problem (1.1). We have also discussed some examples to show the applicability of the accomplished work.

Preliminaries

First of all, we fix our terminology and recall some basic ideas of fractional calculus [16], and multivalued analysis (see [17–20]) that we need in the sequel.

Let $C(J, \mathbb{R})$ be the Banach space of all continuous real valued functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in J\}$. By $L^1(J, \mathbb{R})$ we denote the Banach space of all measurable functions $x : J \to \mathbb{R}$ which are Lebesgue integrable endowed with the norm $\|x\|_{L^1} = \int_{t_0}^{T} |x(t)|dt$.

Definition 2.1 The fractional integral of order r with the lower limit zero for a function ϱ is defined as

$$I^r\varrho(t) = \frac{1}{\Gamma(r)}\int_0^t \frac{\varrho(s)}{(t-s)^{1-r}}ds, \quad t > 0, \quad r > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $r > 0,\ n-1 < r < n,\ n \in N$, is defined as

$$D^r_{0+}\varrho(t) = \frac{1}{\Gamma(n-r)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-r-1}\varrho(s)ds,$$

where the function $\varrho(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 The Caputo derivative of order r for a function $\varrho : [0,\infty) \to R$ can be written as

$${}^cD^r\varrho(t) = D^r\left(\varrho(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}\varrho^{(k)}(0)\right), \quad t > 0, \quad n-1 < r < n.$$

Remark 2.4 If $\varrho(t) \in C^n[0,\infty)$, then

$${}^cD^r\varrho(t) = \frac{1}{\Gamma(n-r)}\int_0^t \frac{\varrho^{(n)}(s)}{(t-s)^{r+1-n}}ds = I^{n-r}\varrho^{(n)}(t), \quad t > 0, \quad n-1 < r < n.$$

Definition 2.5 For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Definition 2.6 Let $F : X \to \mathcal{P}(X)$ be a multivalued map.

- (i) F is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$.
- (ii) F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$.
- (iii) F is an upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$.
- (iv) F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$.
- (v) F has a fixed point if there is $x \in X$ such that $x \in F(x)$.
- (vi) If F is completely continuous with nonempty compact values, then F is u.s.c if and only if F has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

The fixed point set of the multivalued operator F will be denoted by $FixF$.

Definition 2.7 A multivalued map $F : J \to \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, F(t)) = \inf\{|y-z| : z \in F(t)\}$$

is measurable.

Definition 2.8 A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if: (i) $t \to F(t,x)$ is measurable for each $x \in \mathbb{R}$, (ii) $x \to F(t,x)$ is upper semi-continuous for almost all $t \in J$. Further a Carathéodory function F is called L^1-Carathéodory if for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t)$$

for all $\|x\| \le \alpha$ and for a.e. $t \in J$.

Definition 2.9 Let Y be a Banach space, Z a nonempty closed subset of Y. The multivalued operator $F : Z \to \mathcal{P}(Y)$ is said to be lower semi-continuous (l.s.c.) if the set $\{z \in Z : F(z) \cap B \neq \phi\}$ is open for any open set B in Y.

Definition 2.10 Let A be a subset of $J \times \mathbb{R}$. A is said to be $\mathcal{L} \otimes \mathcal{B}$-measurable if A belongs to the σ-algebra generated by all sets of the form $L \times B$, where L is Lebesgue measurable in J and B is Borel measurable in $\mathbb{R}$.

Definition 2.11 A subset A of $L^1(J, \mathbb{R})$ is decomposable if for all $u, v \in A$ and measurable sets $I \subset J$, the function $u\chi_I + v\chi_{J-I} \in A$, where χ_I stands for the characteristic function of I.

Definition 2.12 If $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with nonempty compact values and $u \in C(J, \mathbb{R})$, then the set of selections of $F(\cdot, \cdot)$, denoted by $S_{F,u}$, is of lower semi-continuous type if

$$S_{F,u} = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, u(t)) \quad \text{for a.e. t} \in J\}$$

is lower semi-continuous with nonempty closed and decomposable values.

Definition 2.13 Let (X, d) be a metric space associated with the metric d. The Pompeiu–Hausdorff distance of the closed subsets $A,\ B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},$$

where $d^*(A, B) = \sup\{d(a, B) : a \in A\}$, and $d(x, B) = \inf_{y \in B} d(x, y)$.

Definition 2.14 A multivalued operator F on X with nonempty values in X is called:

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_H(F(x), F(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X,$$

(b) A contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemmas will be used in what follows.

Lemma 2.15 [18] *Let X be a Banach space. Let $F : J \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let H be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Theta \circ S_F : C(J, X) \to \mathcal{P}_{cp,c}(C(J, X)),$$
$$x \to (H \circ S_F)(x) = H(S_{F,x})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.16 [21] *Let Y be a separable metric space and let $F : Y \to \mathcal{P}(L^1(J, \mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $F(\cdot)$ has a continuous selection, i.e., there exists a continuous mapping (single valued) $f : Y \to L^1(J, \mathbb{R})$ such that $f(y) \in F(y)$ for every $y \in Y$.*

We conclude this section by stating the following fixed point theorems needed for the forthcoming analysis.

Theorem 2.17 (Nonlinear alternative of Leray–Schauder type [22]) *Let X be a Banach space, $\mathcal{X}$ be a closed convex subset of X, $\mathcal{U}$ be an open subset of $\mathcal{X}$ with $0 \in \mathcal{U}$. Suppose that $F : \overline{\mathcal{U}} \to P_{cp,c}(\mathcal{X})$ is an upper semicontinuous compact map. Then either F has a fixed point in $\overline{\mathcal{U}}$ or there are $\S \in \partial\mathcal{U}$ and $\lambda \in (0, 1)$ such that $\S \in \lambda F(\S)$.*

Theorem 2.18 (Covitz and Nadler [23]) *Let (X, d) be a complete metric space. If $F : X \to \mathcal{P}_{cl}(X)$ is a contraction, then F has a fixed point.*

Existence Results

To define the solution for problem (1.1), we consider its linear variant given by

$$\begin{cases} {}^cD^r_{t_0}x(t) = \widetilde{f}(t), & t \in J, \\ x^{(k)}(\delta) = x_k + \int\limits_{t_0}^{\delta} g_k(s)ds, & k = 0, 1, \ldots, n-1, \quad \delta \in J, \end{cases} \tag{3.1}$$

where $\widetilde{f} \in C(J, \mathbb{R})$.

Lemma 3.1 [12] *The fractional nonlocal boundary value problem* (3.1) *is equivalent to the integral equation*

$$x(t) = I^r \widetilde{f}(t) + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} g_k(s)ds - I^{r-k} \widetilde{f}(\delta) \right), \quad t \in J. \tag{3.2}$$

Next, we formulate the hypotheses for proving the existence of solutions for problem (1.1).

(A) $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has convex values;
(B) There exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in C(J, \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le p(t)\psi(|x|),$$

for each $(t, x) \in J \times \mathbb{R}$;
(C) There exist continuous nondecreasing functions $\psi_k : [0, \infty) \to (0, \infty)$ and functions $p_k \in C(J, \mathbb{R}^+)$ such that

$$|g_k(t,x)| \le p_k(t)\psi_k(|x|), \quad k = 0, 1, \ldots, n-1$$

for each $(t, x) \in J \times \mathbb{R}$; and
(D) There exists a number $M > 0$ such that

$$\frac{M}{\gamma_1 \psi(M) \|p\| + \gamma_2} > 1,$$

where

$$\gamma_1 = \left\{ \frac{2}{\Gamma(r+1)} + \sum_{k=1}^{n-1} \frac{1}{k!\Gamma(r-k+1)} \right\} (T - t_0)^r$$

and

$$\gamma_2 = \sum_{k=0}^{n-1} \frac{|T-t_0|^k}{k!} \left(|x_k| + \delta - t_0\right) \psi_k(\|\eta\|) \|p_k\|.$$

Theorem 3.2 *Assume that the conditions* (*A*)–(*D*) *hold. Then the fractional differential inclusion problem* (1.1) *has at least one solution on* J.

Proof Using Theorem 3.1, define an operator $\Omega_F : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ by

$$\Omega_F(x) = \left\{ h \in C(J,\mathbb{R}) : h(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} f(s)\,ds \right.$$
$$\left. + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s,x(s))ds - \frac{(\delta-s)^{r-k-1} f(s)}{\Gamma(r-k)} \right) ds \right) \right\}$$

for $f \in S_{F,x}$. We show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. We complete the proof in several steps.

Step I $\Omega_F(x)$ is convex for each $x \in C(J,\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit its proof.

Step II We show that $\Omega_F(x)$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$. For a positive number η, let $B_\eta = \{x \in C(J,\mathbb{R}) : \|x\| \le \eta\}$ be a bounded set in $C(J,\mathbb{R})$. Then, for each $h \in \Omega_F(x),\ x \in B_\eta$, there exists $f \in S_{F,x}$ such that

$$h(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} f(s)\,ds$$
$$+ \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s,x(s))ds - \frac{(\delta-s)^{r-k-1} f(s)}{\Gamma(r-k)} \right) ds \right).$$

Then, for $t \in J$, we have

$$|h(t)| \le \int_{t_0}^{t} \frac{(t-s)^{r-1}\,|f(s)|}{\Gamma(r)} ds$$
$$+ \sum_{k=0}^{n-1} \frac{|t-\delta|^k}{k!} \left(|x_k| + \int_{t_0}^{\delta} \left(|g_k(s,x(s))|\,ds + \frac{(\delta-s)^{r-k-1}\,|f(s)|}{\Gamma(r-k)} \right) ds \right)$$
$$\le \psi(\|x\|)\,\|p\| \left\{ \frac{1}{\Gamma(r+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(r-k+1)} \right\} (T-t_0)^r$$
$$+ \sum_{k=0}^{n-1} \frac{|T-t_0|^k}{k!} \left(|x_k| + \delta - t_0\right) \psi_k(\|x\|)\,\|p_k\|.$$

Thus,

$$\|h\| \le \psi(\|\eta\|)\,\|p\| \left\{ \frac{2}{\Gamma(r+1)} + \sum_{k=1}^{n-1} \frac{1}{k!\Gamma(r-k+1)} \right\} (T-t_0)^r$$
$$+ \sum_{k=0}^{n-1} \frac{|T-t_0|^k}{k!} \left(|x_k| + \delta - t_0\right) \psi_k(\|\eta\|)\,\|p_k\|.$$

Step III We show that Ω_F maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$, and $x \in B_\eta$. In view of the hypothesis (C), for each $h \in \Omega_F(x)$, we obtain

$$
\begin{aligned}
&|h(t_2) - h(t_1)| \\
&= \Bigg| \int_{t_0}^{t_1} \frac{(t_2-s)^{r-1} - (t_1 - s)^{r-1}}{\Gamma(r)} f(s)\, ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{r-1} f(s)}{\Gamma(r)} ds \\
&\quad + \sum_{k=0}^{n-1} \left(\frac{(t_2 - \delta)^k}{k!} - \frac{(t_1 - \delta)^k}{k!} \right) \\
&\quad \times \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x(s)) ds - \frac{(\delta - s)^{r-k-1} f(s)}{\Gamma(r-k)} \right) ds \right) \Bigg| \\
&\leq \psi(\|x\|) \left(\int_{t_0}^{t_1} \frac{\left|(t_2-s)^{r-1} - (t_1 - s)^{r-1}\right| p(s)}{\Gamma(r)} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{r-1} p(s)}{\Gamma(r)} ds \right) \\
&\quad + \sum_{k=0}^{n-1} \left| \frac{(t_2 - \delta)^k}{k!} - \frac{(t_1 - \delta)^k}{k!} \right| \\
&\quad \times \left(|x_k| + \psi_k(\|x\|) \int_{t_0}^{\delta} p_k(s) ds + \psi(\|x\|) \int_{t_0}^{\delta} \frac{(\delta - s)^{r-k-1} p(s)}{\Gamma(r-k)} ds \right).
\end{aligned}
$$

The right hand side of the above inequality tends to zero independently of $x \in B_\eta$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, it follows by the Arzelá-Ascoli Theorem that $\Omega_F : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

Step IV We show that Ω_F has a closed graph. Let $x_n \to x_*$, $h_n \in \Omega_F(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(x_*)$. Associated with $h_n \in \Omega_F(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in J$,

$$
\begin{aligned}
h_n(t) = &\int_{t_0}^{t} \frac{(t - s)^{r-1} f_n(s)}{\Gamma(r)} ds + \sum_{k=0}^{n-1} \frac{(t - \delta)^k}{k!} \\
&\times \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x_n(s)) ds - \frac{(\delta - s)^{r-k-1} f_n(s)}{\Gamma(r-k)} \right) ds \right).
\end{aligned}
$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in J$,

$$
\begin{aligned}
h_*(t) = &\int_{t_0}^{t} \frac{(t - s)^{r-1} f_*(s)}{\Gamma(r)} ds + \sum_{k=0}^{n-1} \frac{(t - \delta)^k}{k!} \\
&\times \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x_*(s)) ds - \frac{(\delta - s)^{r-k-1} f_*(s)}{\Gamma(r-k)} \right) ds \right).
\end{aligned} \tag{3.3}
$$

Let us consider the continuous linear operator $\Theta : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ given by

$$f \to \Theta(f)(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1} f(s)}{\Gamma(r)} ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x(s)) ds - \frac{(\delta - s)^{r-k-1} f(s)}{\Gamma(r-k)} \right) ds \right).$$

Observe that

$$|h_n(t) - h_*(t)| \le \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} |f_n(s) - f_*(s)| ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \int_{t_0}^{\delta} \frac{(\delta-s)^{r-k-1}}{\Gamma(r-k)} |f_n(s) - f_*(s)| ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \int_{t_0}^{\delta} |g_k(s, x_n(s)) - g_k(s, x_*(s))| ds.$$

Thus, it follows by Lemma 2.15 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$; since $x_n \to x_*$, we have h_* satisfying (3.3) for some $f_* \in S_{F,x_*}$.

Step V We discuss *a priori bounds* on solutions. Let x be a solution of (1.1). Then there exists $f \in L^1(J, \mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in J$, we have

$$x(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} f(s) ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x_*(s)) ds - \frac{(\delta - s)^{r-k-1} f(s)}{\Gamma(r-k)} \right) ds \right).$$

As in Step II, we find that

$$|x(t)| \le \psi(\|x\|) \|p\| \left\{ \frac{2}{\Gamma(r+1)} + \sum_{k=1}^{n-1} \frac{1}{k!\Gamma(r-k+1)} \right\} (T - t_0)^r + \sum_{k=0}^{n-1} \frac{|T - t_0|^k}{k!} (|x_k| + \delta - t_0) \psi_k(\eta) \|p_k\|,$$

which, after taking norm for $t \in J$, implies that

$$\frac{\|x\|}{\gamma_1 \psi(\|x\|) \|p\| + \gamma_2} \le 1.$$

In view of assumption (D), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J, \mathbb{R}) : \|x\| < M + 1\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda\Omega_F(x)$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray–Schauder type, we deduce that Ω_F has a fixed point $x \in \overline{U}$ which is a solution of problem (1.1). This completes the proof. □

The next result deals with the case when F is not necessarily convex valued and its proof relies on the nonlinear alternative of Leray–Schauder type together with the selection theorem due to Bressan and Colombo [21] for lower semi-continuous maps with decomposable values. For that, we consider the following assumption instead of hypothesis (A):

(E) Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a nonempty compact-valued multivalued map such that (a) $(t, x) \to F(t, x)$ is $L \otimes B$ measurable, and (b) $x \to F(t, x)$ is lower semi-continuous for each $t \in J$.

Theorem 3.3 *Assume that the hypotheses* (B), (C), (D), *and* (E) *hold. Then the fractional differential inclusion problem* (1.1) *has at least one solution on* J.

Proof In view of the hypotheses (B) and (E), we deduce that F is of $l.s.c.$ type. Then, by Lemma 2.16, there exists a continuous function $f : C(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ such that $f(x) \in F(x)$ for all $x \in C(J,\mathbb{R})$. Next, we consider the problem

$$\begin{cases} {}^cD^r_{t_0}x(t) = f(x(t)), & t \in J, \\ x^{(k)}(\delta) = x_k + \int_{t_0}^{\delta} g_k(s, x(s))ds, & k = 0, 1, \ldots, n-1, \quad \delta \in J. \end{cases} \tag{3.4}$$

Observe that if $x \in C(J,\mathbb{R})$ is a solution of (3.4), then x is a solution to problem (1.1). In order to transform problem (3.4) into a fixed point problem, we define an operator Υ as

$$\Upsilon x(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} f(x(s))\,ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x(s))ds - \frac{(\delta-s)^{r-k-1} f(x(s))}{\Gamma(r-k)} \right) ds \right).$$

As in the preceding result, one can show that the operator Υ is completely continuous. The rest of the proof is similar to that of Theorem 3.2, so we omit it. This completes the proof. □

In our last result, we discuss the existence of solutions for problem (1.1) with a nonconvex valued map by means of a fixed point theorem for multivalued maps due to Covitz and Nadler [23]. In the sequel, we need the following assumptions:

(F) Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be such that $F(\cdot, x) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;

(G) $d_H(F(t, x), F(t, y)) \le z(t)|x-y|$ for almost all $t \in J$ and $x, y \in \mathbb{R}$ with $z \in C(J,\mathbb{R}^+)$ and $d(0, F(t, 0)) \le z(t)$ for almost all $t \in J$;

(H) There exist functions $p_k \in C(J,\mathbb{R}^+)$ such that

$$|g_k(t, x) - g_k(t, y)| \le p_k(t)|x - y|,$$

for $t \in J$, $k = 0, 1, \ldots, n-1$ and $x, y \in \mathbb{R}$.

Theorem 3.4 *Assume that the conditions* (*F*), (*G*) *and* (*H*) *hold. Then the fractional differential inclusion problem* (1.1) *has at least one solution on* J *if*

$$\gamma_1 \|z\| + \gamma_3 < 1,$$

where

$$\gamma_3 = \sum_{k=0}^{n-1} \frac{(T-t_0)^k \|p_k\|}{k!}.$$

Proof Observe that the set $S_{F,x}$ is nonempty for each $x \in C(J,\mathbb{R})$ by assumption (F), so F has a measurable selection (see Theorem 3.6 in [17]). Now we show that the operator Ω_F satisfies the assumptions of Theorem 2.18. To show that $\Omega_F(x) \in \mathcal{P}_{cl}((C(J,\mathbb{R}))$ for each $x \in C(J,\mathbb{R})$, let $(u_n)_{n\geq 0} \in \Omega_F(x)$ be such that $u_n \to u$ in $C(J,\mathbb{R})$. Then $u \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in J$, we have

$$u_n(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} v_n(s)ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x_n(s))ds - \frac{(\delta-s)^{r-k-1} v_n(s)}{\Gamma(r-k)} \right) ds \right).$$

Since F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J$,

$$u_n(t) \to u(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} v(s)ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x(s))ds - \frac{(\delta-s)^{r-k-1} v(s)}{\Gamma(r-k)} \right) ds \right).$$

Thus, $u \in \Omega_F(x)$. Next we show that there exists $\tau < 1$ such that

$$d_H(\Omega_F(x), \Omega_F(y)) \leq \tau \|x-y\|,$$

for each $x, y \in C(J,\mathbb{R})$. Let $x, y \in C(J,\mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in J$, we have

$$h_1(t) = \int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} v_1(s)ds + \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, x(s))ds - \frac{(\delta-s)^{r-k-1} v_1(s)}{\Gamma(r-k)} \right) ds \right).$$

By hypothesis (G), we have

$$d_H(F(t,x), F(t,y)) \leq z(t)|x(t)-y(t)|.$$

So, there exists $w_* \in F(t, y(t))$ such that

$$|v_1(t) - w_*| \leq z(t)|x(t) - y(t)|, \ t \in J.$$

Define the multivalued map $V : J \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w_* \in \mathbb{R} : |v_1(t) - w_*| \leq z(t)\,|x(t) - y(t)|\}.$$

Since $V(t) \cap F(t, y(t))$ is measurable (see Proposition 3.4 in [17]), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, y(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq z(t)|x(t) - y(t)|$. Let us define

$$\begin{aligned} h_2(t) = &\int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} v_2(s)ds \\ &+ \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \left(x_k + \int_{t_0}^{\delta} \left(g_k(s, y(s))ds - \frac{(\delta - s)^{r-k-1} v_2(s)}{\Gamma(r-k)} \right) ds \right). \end{aligned}$$

Thus, for each $t \in J$, it follows that

$$\begin{aligned} |h_1(t) - h_2(t)| \leq &\int_{t_0}^{t} \frac{(t-s)^{r-1}}{\Gamma(r)} \, |v_1(s) - v_2(s)|\, ds \\ &+ \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \int_{t_0}^{\delta} \frac{(\delta - s)^{r-k-1}}{\Gamma(r-k)} \, |v_1(s) - v_2(s)|\, ds \\ &+ \sum_{k=0}^{n-1} \frac{(t-\delta)^k}{k!} \int_{t_0}^{\delta} |g_k(s, x(s)) - g_k(s, y(s))|\, ds \\ \leq &\left\{ \left\{ \frac{(T-t_0)^r}{\Gamma(r+1)} + \sum_{k=0}^{n-1} \frac{(T-t_0)^r}{k!\Gamma(r-k+1)} \right\} \|z\| \right. \\ &\left. + \sum_{k=0}^{n-1} \frac{(T-t_0)^k \, \|p_k\|}{k!} \right\} \|x - y\| \,. \end{aligned}$$

Hence

$$\|h_1 - h_2\| \leq (\gamma_1 \, \|z\| + \gamma_3) \, \|x - y\|.$$

Analogously, interchanging the roles of x and y, we obtain

$$\begin{aligned} d_H(\Omega_F(x), \Omega_F(y)) &\leq (\gamma_1 \, \|z\| + \gamma_3) \, \|x - y\| \\ &\leq \tau \, \|x - y\|, \end{aligned}$$

where $\tau < 1$. Thus it follows by Theorem 2.18 that the operator Ω_F has a fixed point x which is a solution of problem (1.1). This completes the proof. □

Remark 3.5 We obtain the existence results for an initial value problem of general fractional differential inclusions with initial conditions: $x^{(k)}(t_0) = b_k,\ k = 0, 1, 2, \ldots, n-1,$ by taking $\theta = t_0$ in the results of this paper, while the results for general fractional differential inclusions with classical nonlinear integral conditions:

$$x^{(k)}(T) = b_k + \int_{t_0}^{T} g_k(s, x(s))ds, \quad k = 0, 1, 2, \ldots, n-1,$$

follow by fixing $\theta = T$ in the obtained results.

Example 3.6 Consider the following fractional differential inclusion problem

$$\begin{cases} {}^cD_0^{5.5} x\,(t) \in F\,(t, x\,(t))\,, & t \in [0, 1], \\ x^{(k)}\left(\frac{1}{2}\right) = 1 + \int_0^{1/2} ke^{-x(t)}dt, & k = 0, 1, \ldots, 5, \end{cases} \tag{3.5}$$

where $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t, x) = \left\{ y \in \mathbb{R} : 0 \leq y \leq \frac{\sqrt[3]{t}\,|x|}{3\,(1 + |x|)} \right\}.$$

Observe that

$$t \to F(t, x) = \left\{ y \in \mathbb{R} : 0 \leq y \leq \frac{\sqrt[3]{t}\,|x|}{3\,(1 + |x|)} \right\}$$

is measurable for each $x \in \mathbb{R}$, since both the lower and upper functions are measurable on $[0, 1] \times \mathbb{R}$. Moreover, the mapping $x \to F(t, x)$ is upper semi-continuous for all $t \in J$. Thus F is a Carathéodory and clearly has convex values satisfying

$$\|F(t, x)\| \leq p(t)\psi(\|x\|) \quad \text{for each } (t, x) \in [0, 1] \times \mathbb{R},$$

where $p(t) = \sqrt[3]{t}$, and $\psi(x) = 1/3$. If we let $g_k(t, x(t)) = ke^{-x(t)}$, $k = 0, 1, 2, \ldots, 5$, then

$$|g_k(t, x)| \leq p_k(t),$$

where $p_k = k$, and $\psi_k(x) = 1$, for $k = 0, 1, 2, \ldots, 5$. In a straightforward manner, we find that

$$\gamma_1 = \frac{2}{\Gamma(6.5)} + \frac{1}{\Gamma(5.5)} + \frac{1}{\Gamma(4.5)} + \frac{1}{\Gamma(3.5)} + \frac{1}{\Gamma(2.5)} + \frac{1}{\Gamma(1.5)} = 2.3$$
$$\gamma_2 = \frac{3}{2}\left(1 + \frac{2}{2} + \frac{3}{6} + \frac{4}{24} + \frac{5}{120}\right) = 0.209.$$

Therefore, choosing M such that

$$M > \frac{1}{4}(2.3) + 0.209 = 0.784,$$

we conclude that there exists a solution of problem (3.5) on [0, 1] by Theorem 3.2.

Next, for $F(t, x) = \left[0, \frac{\sqrt[3]{t}\,|x|}{2\,(1 + |x|)}\right]$, we deduce that

$$\begin{aligned} \sup\{|y| : y \in F(t, x)\} &\leq \frac{\sqrt[3]{t}\,|x|}{3\,(1 + |x|)} \\ &\leq \frac{1}{3} \quad \text{for each } (t, x) \in [0, 1] \times \mathbb{R}, \end{aligned}$$

and

$$d_H(F(t,x), F(t,y)) = d_H\left(\left[0, \frac{\sqrt[3]{t}\,|x|}{3\,(1+|x|)}\right], \left[0, \frac{\sqrt[3]{t}\,|y|}{3\,(1+|y|)}\right]\right)$$
$$\leq \frac{\sqrt[3]{t}}{3}|x-y|.$$

Here $z(t) = \dfrac{\sqrt[3]{t}}{3}$, with $\|z\| \approx 0.33$, and

$$\gamma_1 \|z\| + \gamma_3 \approx 2.3\,(0.33) + 0.14 < 1.$$

The compactness of F together with the above calculations lead to the existence of solution of the problem (3.5) by Theorem 3.4.

Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

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