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Global Asymptotic Stability of a Generalized SEIRS Epidemic Model

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Abstract A disease transmission model of SEIRS type with a wide class of nonlinear incidence functions in a constant population is proposed. This model has two equilibria: a disease-free equilibrium and an endemic equilibrium. To our knowledge, the global asymptotic stability of the endemic equilibrium is not investigated in the general case. In the present paper, we use a geometric approach to offer a useful solution of this problem.

Keywords SEIRS epidemic model · Nonlinear incidence function · Global asymptotic stability · Geometric approach · Compound matrix

Introduction

In most communicable diseases such as cholera, pertussis, influenza and malaria, it has been observed that recovered individuals can return to the susceptible period (temporary immunity). This observation has been described by SIRS and SEIRS models (see [17–19, 23]). The above model utilize an ordinary differential system with four equations to represent the flow of individuals between Susceptible (S), Exposed (E), Infectious (I) and Recovered (R) compartments. The major contribution of this representation is the simultaneous modeling of the latent period (the period during which the individual is said to be exposed but not infectious) and the temporary immunity of the recovered individuals. In the present paper,

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we propose the following SEIRS model with a wide class of nonlinear incidence function (the number of new exposed per unit time) in a constant population :

$$
\begin{aligned}
\frac{dS}{dt} &= A - dS - f(S, I) + \delta R, \\
\frac{dE}{dt} &= f(S, I) - (d + \sigma)E, \\
\frac{dI}{dt} &= \sigma E - (d + \gamma)I, \\
\frac{dR}{dt} &= \gamma I - (d + \delta)R.
\end{aligned}
\tag{1.1}
$$

The initial conditions are

$$
S(0) > 0, \quad E(0) \geq 0, \quad I(0) > 0, \quad R(0) \geq 0 \tag{1.2}
$$

Here $A = dN$ is the recruitment rate, where $N = S + E + I + R$ is the total number of population, S is the number of susceptible individuals, I is the number of infectious individuals, E is the number of exposed individuals, R is the number of recovered individuals, d is the natural death, $f(S, I)$, is the incidence function, γ is the recovery rate of the infectious individuals, σ is the rate at which exposed individuals become infectious and δ is the rate of loss of immunity.

A fundamental problem in epidemic model is to determine the incidence functions because they has played an important role in the study of the transmission of the infection. All studies on epidemiological modeling have suggested that this disease transmission shall have a nonlinear incidence (see [2,3,24,39]). In our proposition, the incidence $f(S, I)$ is a continuously differentiable function on $]0, +\infty[\times]0, +\infty[$ satisfying [12]:

$(H_0):$ $f(0, I) = f(S, 0) = 0$;

$(H_1):$ f is a strictly monotone increasing function of S, for any fixed $I > 0$, and f is a strictly monotone increasing function of I, for any fixed $S > 0$;

$(H_2):$ $\phi(S, I) = \frac{f(S,I)}{I}$ is a bounded and monotone decreasing function of $I > 0$, for any fixed $S \geq 0$, and $K(S) = \lim_{I \to 0+} \phi(S, I)$ is a continuous and monotone increasing function on $S \geq 0$.

This proposition includes the following forms:

1. The saturated incidence $\frac{\beta SI}{\alpha+S+I}$ proposed in [1], where β and α are the positive constants;
2. The bilinear incidence βSI introduced in [14,20,37,42,43];
3. The standard incidence $\frac{\beta SI}{N}$ advanced in [11,15];
4. The saturated incidence $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ proposed in [1,5–7,20,22,38,40,41], where α_1 and α_2 are the positive constants.

However, hypotheses (H_1) and (H_2) cannot be applied generally in the following cases:

1. The first one is $\frac{\beta S^q I^p}{1+\alpha I^s}$ [16,19,26,31,35], where $\alpha \geq 0$, p, q and s are the positive constants;
2. The second one is $\beta I(1 + \nu I^q)S$ [2,10,29], where $\nu \geq 0$ and $0 < q \leq 1$.

Many authors have studied the dynamical behavior of System (1.1) with various incidence functions (see [8,23,28,28,30,33] and references therein). Much attention has been paid to the analysis of the global stability of the disease-free equilibrium and the endemic equilibrium of this system. Liu et al. [30] considered an SEIRS epidemic model with $f(S, I) = \beta S^p I^q$. They showed that the qualitative behavior is not affected by changing q from unity, but

is affected significantly by changing p from unity: For $0 < p < 1$ the disease free equilibrium is always unstable and the endemic equilibrium is asymptotically stable and for $p > 1$ they proved that System (1.1) has three equilibria due to a saddle-node bifurcation of the positive equilibrium at the threshold. In [33], Li and Muldowney have studied System (1.1) with bilinear incidence function which takes the form $f(S, I) = \beta SI$. Using a geometric approach to global stability, they established that when the rate of loss of immunity δ is close to unity, the unique endemic equilibrium is globally asymptotically stable. A geometric approach was also found in [32] to determine the global stability of SEIRS model with $f(S, I) = \beta Sg(I)$. The authors established that the unique endemic equilibrium is globally asymptotically stable if $g(I)$ satisfies $|Ig'(I)| \leq g(I)$. Later, Cheng and Yang [8], proposed to complete the study of the global stability in [32] for all parameters μ, σ, γ and δ,, they relaxed the constraint $|Ig'(I)| \leq g(I)$ by the following condition $|Ig'(I)|(1 + \tau) \leq g(I)$ and they presented the complete resolution of the global stability issue of the bilinear SEIRS model for the full range of μ, σ, γ and δ, in the parameter space. In [28] Li et al. considered System (1.1) with bilinear incidence. they applied a geometric approach to global stability problems to show that when the basic reproduction number is greater than one, then there exists a $\bar{\delta} > 0$ such that the unique endemic equilibrium of (1.1) is globally asymptotically stable when $\delta \leq \bar{\delta}$. In [23] Khan et al. presented the System (1.1) with $f(S, I) = \frac{\beta SI}{\Psi(I)}$ where Ψ is a positive function with $\Psi(0) = 1$ and $\Psi' \geq 0$. The authors showed the global stability of the endemic equilibrium by a geometric approach.

In the present paper, a SEIRS epidemic model with a wide class of nonlinear incidence (called here a generalized SEIRS epidemic model) is proposed. By a geometric approach, the global asymptotic stability of the unique endemic equilibrium is proved when the basic reproduction number is greater than unity.

The rest of the paper is organized as follows: In "Basic Properties of the Generalized SEIRS Model" section, the basic proprieties of the generalized SEIRS epidemic model (1.1) are investigated. In "Local Asymptotic Stability of the Endemic Equilibrium" section, the local asymptotic stability of the endemic equilibrium is established. In "Global Asymptotic Stability of the Endemic Equilibrium" section, the global asymptotic stability of the endemic equilibrium are discussed. In "Discussion" section, a brief discussion is presented to conclude this paper.

Basic Properties of the Generalized SEIRS Model

In this section, we state the basic proprieties of the generalized SEIRS epidemic model (1.1). The total population size N satisfies the equation $N = S + E + I + R$, which reduces the system (1.1) to the following system:

$$
\begin{aligned}
\frac{dS}{dt} &= A + \delta N - (d + \delta)S - f(S, I) - \delta E - \delta I, \\
\frac{dE}{dt} &= f(S, I) - (d + \sigma)E, \\
\frac{dI}{dt} &= \sigma E - (d + \gamma)I.
\end{aligned}
\tag{2.1}
$$

Therefore, the biologically feasible region for the system (2.2)

$$T = \{(S, E, I) \in \mathbb{R}_+^3 : S + E + I \leq N\}$$

is bounded and positively invariant.

We define the basic reproduction number of system (2.2) by

$$R_0 = \frac{\sigma \frac{\partial f(N,0)}{\partial I}}{(d + \gamma)(d + \sigma)}.$$

In our first basic result, we show the existence and the uniqueness of the equilibrium points.

Proposition 2.1 *On the one hand, the system* (2.2) *always has a disease-free equilibrium* $P_0 = (N, 0, 0)$ *provided that hypothesis* (H_0) *is satisfied. On the other hand, under the hypotheses* (H_1) *and* (H_2), *if* $R_0 > 1$, *then system* (2.2) *also admits a unique endemic equilibrium:* $P_* = (S^*, E^*, I^*)$.

Proof It is easy to verify that if $f(S, 0) = 0$, the vector $(N, 0, 0)$ always is an equilibrium of system (2.2).

Next, we prove the existence of the unique endemic equilibrium. If (S, E, I) is an equilibrium of System (2.2), then we have the following system:

$$
\begin{aligned}
A + \delta N - (d + \delta)S - f(S, I) - \delta E - \delta I &= 0, \\
f(S, I) - (d + \sigma)E &= 0, \\
\sigma E - (d + \gamma)I &= 0.
\end{aligned}
\tag{2.2}
$$

If $I \neq 0$, we have

$$
\begin{aligned}
S &= \frac{\sigma(A + \delta N) - (d + \sigma + \delta)(d + \gamma)I - \sigma\delta I}{\sigma(d + \delta)}, \\
E &= \frac{d + \gamma}{\sigma}I, \\
f(S, I) &= \frac{(d + \gamma)(d + \sigma)}{\sigma}I.
\end{aligned}
\tag{2.3}
$$

We consider the following function

$$h(I) := \phi\left(\frac{\sigma(A + \delta N) - (d + \sigma + \delta)(d + \gamma)I - \sigma\delta I}{\sigma(d + \delta)}, I\right) - \frac{(d + \gamma)(d + \delta)}{\sigma}.$$

By (H_1) and (H_2), we have h is continuous and strictly monotone decreasing function of $I > 0$, and for $R_0 > 1$,

$$\lim_{I \to 0^+} h(I) = \frac{(d + \gamma)(d + \delta)}{\sigma}(R_0 - 1) > 0,$$

and

$$h\left(\frac{\sigma(A + \delta N)}{\sigma\delta + (d + \gamma)(d + \sigma + \delta)}\right) = -\left(\frac{(d + \gamma)(d + \delta) + \sigma\delta}{\sigma}\right) < 0.$$

Hence, there exist a unique endemic equilibrium $P_* = (S^*, E^*, I^*)$. This complete the proof.   □

Next, we discuss the global asymptotic stability of the disease-free equilibrium P_0.

Theorem 2.2   (i) *If $R_0 \leq 1$, then the disease-free equilibrium P_0 is globally asymptotically stable.*

(ii) *If $R_0 > 1$, then the disease free equilibrium P_0 is unstable.*

Proof Set $V = \sigma E + (d + \sigma)I$, then

$$V' = (d + \gamma)(d + \sigma)I \left(\frac{\sigma \phi(S, I)}{(d + \gamma)(d + \sigma)} - 1 \right) \leq (d + \gamma)(d + \sigma)I(R_0 - 1),$$

if $R_0 \leq 1$, then $V' \leq 0$. Furthermore, $V' = 0$ if and only if $I = 0$. Therefore the largest compact invariant set in $\{(S, E, I) \in T / V' = 0\}$, when $R_0 \leq 1$, is the singleton $\{P_0\}$. LaSalle's Invariance Principle implies that P_0 is globally asymptotically stable in T.   □

Local Asymptotic Stability of the Endemic Equilibrium

In this section, we discuss the local and the global asymptotic stability of the endemic equilibrium of the generalized SEIRS epidemic model (1.1).

In order to prove our main results, we will need to demonstrate the following results:

Lemma 3.1 *The following statements are logically equivalent:*

 (i) *The function $\phi(S, I) = \frac{f(S,I)}{I}$ is monotone of $I > 0$, for any fixed $S \geq 0$.*
 (ii) *$\frac{f(S,I)}{I} - \frac{\partial f(S,I)}{\partial I}$ has a constant sign.*

Proof Consider the function $\phi(S, I) = \frac{f(S,I)}{I}$. Then, the partial derivative of ϕ, with respect to I is given by

$$\frac{\partial \phi(S, I)}{\partial I} = \frac{I \frac{\partial f(S,I)}{\partial I} - f(S, I)}{I^2},$$

and a simple calculation, leads to

$$\frac{f(S, I)}{I} - \frac{\partial f(S, I)}{\partial I} = -I . \frac{\partial \phi(S, I)}{\partial I}.$$

This immediately allows to demonstrate the proposed equivalence.   □

The following corollary is an immediate consequence of Lemma 3.1 .

Corollary 3.2 *The following statements are logically equivalent:*

 (i) *$\phi(S, I) = \frac{f(S,I)}{I}$ is a monotone decreasing function of $I > 0$, for any fixed $S \geq 0$.*
 (ii) *$\frac{f(S,I)}{I} - \frac{\partial f(S,I)}{\partial I} \geq 0$.*

Now let us start to discuss the local asymptotic stability of the endemic equilibrium P_*.

Theorem 3.3 *Suppose the hypotheses (H_1) and (H_2) hold.*

If $R_0 > 1$, then the endemic equilibrium P_* is locally asymptotically stable.

Proof Let $x = S - S^*$, $y = E - E^*$ and $z = I - I^*$.

Then by linearizing system (2.2) around P_*, we have

$$\begin{aligned}
\frac{dx}{dt} &= -\left(d + \frac{\partial f(S^*, I^*)}{\partial S} + \delta \right) x(t) - \left(\frac{\partial f(S^*, I^*)}{\partial I} + \delta \right) z(t) - \delta y(t), \\
\frac{dy}{dt} &= -(d + \sigma)y(t) + \frac{\partial f(S^*, I^*)}{\partial S} x(t) + \frac{\partial f(S^*, I^*)}{\partial I} z(t), \quad\quad (3.1) \\
\frac{dz}{dt} &= \sigma y(t) - (d + \gamma)z(t)
\end{aligned}$$

The characteristic equation associated to system (3.2) is given by

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0, \tag{3.2}$$

where

$$a = 3d + \gamma + \sigma + \delta + \frac{\partial f(S^*, I^*)}{\partial S},$$

$$b = (d + \gamma)(d + \sigma) - \sigma \frac{\partial f(S^*, I^*)}{\partial I} + (2d + \gamma + \sigma)\left(d + \delta + \frac{\partial f(S^*, I^*)}{\partial S}\right)$$

$$+ \delta \frac{\partial f(S^*, I^*)}{\partial S},$$

$$c = (d + \delta)\left[(d + \gamma)(d + \sigma) - \sigma \frac{\partial f(S^*, I^*)}{\partial I}\right] + \frac{\partial f(S^*, I^*)}{\partial S}\left[(d + \delta + \sigma)(d + \gamma) + \sigma\delta\right].$$

Firstly, from hypothesis (H_1), we have $\frac{\partial f(S^*, I^*)}{\partial S} \geq 0$, which implies that $a > 0$. Secondly, by using the second and the third equations in system (2.2), We find that

$$(d + \gamma)(d + \sigma) - \sigma \frac{\partial f(S^*, I^*)}{\partial I} = \sigma\left(\frac{f(S^*, I^*)}{I^*} - \frac{\partial f(S^*, I^*)}{\partial I}\right).$$

Hence, the Corollary 3.2 implies that $b > 0$, $c > 0$, and

$$ab - c = \sigma(2d + \gamma + \sigma)\left(\frac{f(S^*, I^*)}{I^*} - \frac{\partial f(S^*, I^*)}{\partial I}\right)$$

$$+ \left(3d + \gamma + \sigma + \delta + \frac{\partial f(S^*, I^*)}{\partial S}\right)$$

$$\times \left[(2d + \gamma + \sigma)\left(d + \delta + \frac{\partial f(S^*, I^*)}{\partial S}\right) + \delta \frac{\partial f(S^*, I^*)}{\partial S}\right]$$

$$- \frac{\partial f(S^*, I^*)}{\partial S}\left[(d + \delta + \sigma)(d + \gamma) + \sigma\delta\right] > 0.$$

So, by the Routh–Hurwitz criterion, we obtain the local stability of P_* for $R_0 > 1$. This concludes the proof of Theorem 3.3. □

Global Asymptotic Stability of the Endemic Equilibrium

In order to study the global asymptotic stability of the endemic equilibrium P^*, we use the geometrical approach which is developed in the papers of Smith [36] and Li and Muldowney [28]. We obtain simple sufficient conditions that P_* is globally asymptotically stable when $R_0 > 1$.

To show the existence of a compact set in the interior of T that is absorbing for (2.2) is equivalent to proving that (2.2) is uniformly persistent, which means that there exists a constant $c > 0$ such that every solution (S, E, I) of (2.2) with the initial data $(S(0), E(0), I(0))$ in the interior of T satisfies

$$\liminf_{t \to \infty} S(t) \geq c, \quad \liminf_{t \to \infty} E(t) \geq c, \quad \liminf_{t \to \infty} I(t) \geq c \tag{4.1}$$

Here c is independent of initial data in T, see [28]. We can prove the following result.

Proposition 4.1 *The system* (2.2) *is uniformly persistent if and only if* $R_0 > 1$.

Proof By Theorem 4.3 in [13], we can see that uniform persistence of system (2.2) is equivalent to instability of the disease-free equilibrium $P_0 = (\frac{A}{d}, 0, 0)$. Combine the local stability analysis for this equilibrium in Theorem 3.3 and Theorem 4.3 in [13], we know that system (2.2) is uniformly persistent if and only if $R_0 > 1$.   □

We state our main result in the following theorem.

Theorem 4.2 *Assume that* $R_0 > 1$. *Then there exists* $\overline{\delta} > 0$ *such that the unique endemic equilibrium* P_* *is globally asymptotically stable when* $\delta \leq \overline{\delta}$.

Proof By Proposition 4.1, when $R_0 > 1$, there exists a compact set K in the interior of T that is absorbing for (2.2). The proof of the theorem consists of choosing a suitable vector norm in $\mathbb{R}^3$ and a 3×3 matrix-valued function $A(x)$ so that

$$\overline{q}_2 := \limsup_{t \to \infty} \sup_{x_0 \in K} \frac{1}{t} \int_0^t \overline{\mu}_1(B(x(s, x_0))) ds < 0 \tag{4.2}$$

where $B = A_g A^{-1} + A J^{[2]} A^{-1}$, $x = (S, E, I)$ and $g(x)$ denote the vector field of (2.2), i.e. $\frac{dx(t)}{dt} = g(x)$. The Jacobian matrix $J = \frac{\partial g}{\partial x}$ associated with a general solution $x(t)$ of (2.2) is given by:

$$J = \begin{pmatrix} -d - \delta - \frac{\partial f(S,I)}{\partial S} & -\delta & -\frac{\partial f(S,I)}{\partial S} - \delta \\ \frac{\partial f(S,I)}{\partial S} & -d - \sigma & \frac{\partial f(S,I)}{\partial I} \\ 0 & \sigma & -d - \gamma \end{pmatrix}$$

The second additive compound matrix $J^{[2]}$ of the Jacobian matrix J is given by

$$J^{[2]} = \begin{pmatrix} -2d - \delta - \sigma - \frac{\partial f(S,I)}{\partial S} & \frac{\partial f(S,I)}{\partial I} & \frac{\partial f(S,I)}{\partial I} + \delta \\ \sigma & -2d - \gamma - \delta - \frac{\partial f(S,I)}{\partial S} & -\delta \\ 0 & \frac{\partial f(S,I)}{\partial S} & -2d - \gamma - \sigma \end{pmatrix}$$

Set the function $A(x) = A(S, E, I) = diag\{1, \frac{E}{I}, \frac{E}{I}\}$. Then,

$$A_g A^{-1} = diag\left\{0, \frac{E'}{E} - \frac{I'}{I}, \frac{E'}{E} - \frac{I'}{I}\right\},$$

where the matrix A_g is obtained by replacing each entry a_{ij} of $A(x)$ by its derivative in the direction of g. The matrix $B = A_g A^{-1} + A J^{[2]} A^{-1}$ can be written in the following block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{4.3}$$

where $B_{11} = -2d - \delta - \sigma - \frac{\partial f(S,I)}{\partial S}$,

$$B_{12} = \frac{I}{E} \cdot \left(\frac{\partial f(S,I)}{\partial I} \frac{\partial f(S,I)}{\partial I} + \delta\right), \qquad B_{21} = \begin{pmatrix} \frac{\sigma E}{I} \\ 0 \end{pmatrix}$$

$$B_{22} = \begin{pmatrix} \frac{E'}{E} - \frac{I'}{I} - 2d - \delta - \gamma - \frac{\partial f(S,I)}{\partial S} & -\delta \\ \frac{\partial f(S,I)}{\partial S} & \frac{E'}{E} - \frac{I'}{I} - 2d - \sigma - \gamma \end{pmatrix}$$

Let (u, v, w) denote the vectors in $\mathbb{R}^3$, we select a norm in $\mathbb{R}^3$ as

$$|(u, v, w)| = \max\{|u|, |v| + |w|\},$$

and let $\overline{\mu}_1$ denote the Lozinskiǐ measure with respect to this norm. Using the method of estimating $\overline{\mu}_1$ in ([34]), we have

$$\overline{\mu}_1(B) \leq \sup(g_1, g_2) \tag{4.4}$$

where

$$g_1 = \overline{\mu}_1(B_{11}) + |B_{12}| \tag{4.5}$$
$$g_2 = \overline{\mu}_1(B_{22}) + |B_{21}|, \tag{4.6}$$

$|B_{12}|, |B_{21}|$ are matrix norms with respect to the l_1 vector norm. We have

$$\overline{\mu}_1(B_{11}) = -2d - \delta - \sigma - \frac{\partial f(S, I)}{\partial S}, \tag{4.7}$$

$$|B_{21}| = \frac{\sigma E}{I}, \tag{4.8}$$

$$|B_{12}| = \left(\frac{\partial f(S, I)}{\partial I} + \delta \right) \frac{I}{E}. \tag{4.9}$$

To calculate $\overline{\mu}_1(B_{22})$, we add the absolute value of the off-diagonal one in each column of B_{22}, and then take the maximum of two sums (see [9]), we obtain

$$\overline{\mu}_1(B_{22}) = \frac{E'}{E} - \frac{I'}{I} - 2d - \delta - \gamma - \frac{\partial f(S, I)}{\partial S} \tag{4.10}$$

$$g_1 = -2d - \delta - \sigma - \frac{\partial f(S, I)}{\partial S} + \left(\frac{\partial f(S, I)}{\partial I} + \delta \right) \frac{I}{E}, \tag{4.11}$$

$$g_2 = \frac{\sigma E}{I} + \frac{E'}{E} - \frac{I'}{I} - 2d - \gamma + \max\{-\delta; -\sigma + \delta\} \tag{4.12}$$

Rewriting the second and the third equations in (2.2), we obtain respectively,

$$\frac{E'}{E} = \frac{f(S, I)}{E} - (d + \sigma) \tag{4.13}$$

$$\frac{I'}{I} = \frac{\sigma E}{I} - (d + \gamma) \tag{4.14}$$

Substituting (4.13) into (4.11) and (4.14) into (4.12), respectively, we have

$$g_1 = \frac{E'}{E} - d - \delta - \frac{\partial f(S, I)}{\partial S} - \frac{I}{E} \left[\left(\frac{f(S, I)}{I} - \frac{\partial f(S, I)}{\partial I} \right) - \delta \right]. \tag{4.15}$$

$$g_2 = \frac{E'}{E} - d + \max\{-\delta; -\sigma + \delta\} \tag{4.16}$$

By the inequality in Corollary 3.2, we have

$$g_1 \leq \frac{E'}{E} - d - \delta - \frac{\partial f(S, I)}{\partial S} + \delta \frac{I}{E} \tag{4.17}$$

If $2\delta \leq \sigma$, then

$$g_2 \leq \frac{E'}{E} - d - \delta \tag{4.18}$$

$$\overline{\mu}_1(B) \leq \frac{E'}{E} - d - \delta + \max \left\{ 0, \delta \frac{I}{E} - \frac{\partial f(S, I)}{\partial S} \right\} \tag{4.19}$$

and by the uniform persistence in Proposition 4.1, there exists $c > 0$ and $t_0 > 0$ such that $t > t_0$ implies $c \leq E(t) \leq \frac{A}{d}$ and $c \leq I(t) \leq \frac{A}{d}$, for all $(S(0), E(0), I(0)) \in K$.

Set

$$\delta_1 := \inf \left\{ \frac{cd}{A} \frac{\partial f(S, I)}{\partial S}, c \leq S, I \leq \frac{A}{d} \right\},$$

Therefore, there exists $\overline{\delta} = \min\{\delta_1; \frac{\sigma}{2}\}$ such that if $\delta \leq \overline{\delta}$, then

$$\overline{\mu}_1(B) \leq \frac{E'}{E} - d - \delta. \tag{4.20}$$

We thus have for $\delta \leq \overline{\delta}$,

$$\frac{1}{t} \int_0^t \overline{\mu}_1(B)dt < \frac{1}{t} \int_0^{t_0} \overline{\mu}_1(B)dt + \frac{1}{t} \log \frac{E(t)}{E(t_0)} - (d + \delta)\frac{t - t_0}{t} < \frac{-(d + \delta)}{2}, \tag{4.21}$$

for all $(S(0), E(0), I(0)) \in K$, which implies

$$\overline{q}_2 < 0 \tag{4.22}$$

This concludes the proof.   □

Discussion

In this paper we analyzed global asymptotic stability of of the generalized SEIRS epidemic model (1.1). The incidence function $f(S, I)$ employed in this paper can be applied generally for a wide class of incidence functions such as the bilinear incidence, the standard incidence and the saturated incidence, which have appeared in many studies in the literature.

The dynamics of the generalized SEIRS epidemic model demonstrates the threshold phenomenon as follows : the disease-free equilibrium is globally asymptotically stable when the basic reproduction number, R_0 is less than unity and when R_0 is greater than unity, there is an unique endemic equilibrium, which is globally asymptotically stable provided that the rate of loss of immunity δ is less than a critical value $\overline{\delta}$. The above result is obtained by the geometric approach.

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