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ORIGINAL RESEARCH

On the Cauchy Problem and Solitons for a Class of 1D Boussinesq Systems

José R. Quintero[1] · **Alex M. Montes**[2]

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Abstract In this paper we show the local and global well-posedness for the Cauchy problem associated with a special class of 1D-Boussinesq systems that emerges in the study of the evolution of long water waves with small amplitude in the presence of surface tension. We also show the existence of solitons (finite energy travelling wave solutions) in the case of wave speed $0 < |\omega| < \omega_0$, for some $\omega_0 > 0$.

Keywords Boussinesq system · Well-posedness · Solitary waves · Variational methods

Introduction

In this work we consider the study of the one-dimensional Boussinesq type system

$$\begin{cases} (I - a\mu\partial_x^2)\eta_t + \partial_x^2\Phi - b\mu\partial_x^4\Phi + \epsilon\partial_x(\eta\,(\partial_x\Phi)^p) = 0, \\ (I - c\mu\partial_x^2)\Phi_t + \eta - d\mu\partial_x^2\eta + \frac{\epsilon}{p+1}\,(\partial_x\Phi)^{p+1} = 0, \end{cases} \tag{1}$$

where $\eta = \eta(x,t)$ and $\Phi = \Phi(x,t)$ are real-valued functions, μ and ϵ are small positive parameters, p is a rational number of the form $p = \frac{p_1}{p_2}$ with $(p_1, p_2) = 1$ and p_2 an odd number, and the constants $a \geq 0$, $c \geq 0$, $b > 0$, and $d > 0$ are such that

$$a + c - (b + d) = \frac{1}{3} - \sigma,$$

where σ^{-1} is known as the Bond number. Regarding these models, it can be established that the evolution of long water waves with small amplitude is reduced to studying the solution

✉ José R. Quintero
jose.quintero@correounivalle.edu.co

Alex M. Montes
amontes@unicauca.edu.co

1 Departamento Matemáticas, Universidad del Valle, Cali, Colombia

2 Departamento Matemáticas, Universidad del Cauca, Popayán, Colombia

(η, Φ) of the system (1) in the case of $p = 1$, where ϵ is the amplitude parameter (nonlinearity coefficient), μ is the long-wave parameter (dispersion coefficient) and σ is the inverse of the Bond number (associated with the surface tension). The variable Φ represents the rescale nondimensional velocity potential on the bottom $z = 0$, and the variable η corresponds the rescaled free surface elevation. The model considered in the paper is the 1D version of some Boussinesq system obtained by Quintero and Montes [8] in the case $a = c = \frac{1}{2}, b = \frac{2}{3}, d = \sigma$ (see also Montes [6]) and by Quintero [9] in the case $a = c = 0, b = \frac{1}{6}, d = \sigma - \frac{1}{2}$, which appear when looking at the evolution of long water waves with small amplitude in the presence of surface tension. Results for the two-dimensional version of the Boussinesq system (1), we want to mention [6–9]. For instance, in the cases $a = \frac{1}{2} = c, b = \frac{2}{3}, d = \sigma$ and $a = c = 0, b = \frac{1}{6}, d = \sigma - \frac{1}{2}$, well-posedness for the Cauchy problem for $s \geq 2$ and $p \geq 1$ were obtained by Quintero and Montes in work in revision and by Quintero [10], respectively, and the existence results of solitons (finite energy travelling wave solutions) were obtained by Quintero and Montes [8] and Quintero [9], respectively.

As happens in water wave models, there is a Hamiltonian type structure which is clever to characterize solitary waves as critical points of the action functional and also provides relevant information for the study of the Cauchy problem. In our particular Boussinesq system (1), the Hamiltonian functional $\mathcal{H}$ is defined as

$$\mathcal{H}\begin{pmatrix}\eta\\ \Phi\end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}}\left(\eta^2 + d\mu(\eta_x)^2 + (\Phi_x)^2 + b\mu(\Phi_{xx})^2 + \frac{2\epsilon}{p+1}\eta\,(\Phi_x)^{p+1}\right)dx.$$

and the Hamiltonian type structure is given by

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix} = \mathcal{J}\mathcal{H}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & \left(I - c\mu\partial_x^2\right)^{-1}\\ -\left(I - a\mu\partial_x^2\right)^{-1} & 0\end{pmatrix}.$$

Note that for $a = c$ the operator $\mathcal{J}$ becomes skew symmetric

$$\mathcal{J} = (I - a\mu\partial_x^2)^{-1}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

We see directly that the functional $\mathcal{H}$ is well defined when for t in some interval we have that $\eta(\cdot, t), \Phi_x(\cdot, t) \in H^1(\mathbb{R})$. These conditions already characterize the natural space (energy space) in which we consider the well-posedness of the Cauchy problem and the existence of travelling wave solutions. Another special characteristic on the system (1) is that some well known water wave models as the one-dimensional Benney–Luke equation (see [12–14]) and the Korteweg–de Vries equation emerge from this Boussinesq type system (up to some order with respect to ϵ and μ), making the system (1) very interesting from the physical and numerical view points.

In this paper, we will establish the local well-posedness for the Cauchy problem associated with the system (1) in the space $H^s \times \mathcal{V}^{s+1}$, where $H^s = H^s(\mathbb{R})$ is the usual Sobolev space of order s and $\mathcal{V}^s$ is defined by the norm $\|\psi\|_{\mathcal{V}^s} = \|\psi'\|_{H^{s-1}}$. We also show global well-posedness for the Cauchy problem in the energy space $H^1 \times \mathcal{V}^2$ when the initial date is small enough. We will see as usual that local well-posedness for the Cauchy problem associated with the system (1) follows by the Banach fixed point theorem and appropriate linear and nonlinear estimates using different results as a key ingredient in the case of spatial dimension one:

(a) For $a, c > 0$, we will use a bilinear estimative obtained by Bona and Tzvetkov [2].
(b) For $a = c = 0$, we will use the well known estimates for Kato's commutator used successfully in the KdV model (see works by Kato [3–5]).

On the other hand, global existence for $a = c$ follows from the local existence, the conservation in time of the Hamiltonian, a Sobolev type inequality and the use of energy estimates. Existence of solitons involve the use of the mountain pass theorem and the existence of an appropriate local compact embedding from the space $H^1(\mathbb{R}) \times \mathcal{V}$ to a special $L^q(\mathbb{R})$ type space for $q \geq 2$.

The paper is organized as follows. In "Local Existence", using semigroup estimates and nonlinear estimates, we show a local existence and uniqueness result for the Boussinesq system (1), via a standard fixed point argument. In "Global Existence for $a = c$", from a variational approach which involves the characterization of invariant sets under the flow for the Boussinesq system (1) we obtain the global existence result for initial data small enough, in the case $a = c$. In "Existence of Solitons", we prove the existence of solitons for the system (1) for $0 < |w| < w_0$. We will see that solitons are characterized as critical points of a functional of action. Throughout this work, if not specified, we denote by K a generic constant varying line by line.

Local Existence

In this section we consider the Cauchy problem associated to the system (1) with the initial condition

$$\eta(0, \cdot) = \eta_0, \quad \Phi(0, \cdot) = \Phi_0. \tag{2}$$

The main objective is to show that the Cauchy problem for the system (1) is *locally well-posed*. The notion of well-posedness to be used here is in the sense of Kato: consider an abstract Cauchy problem

$$\frac{du}{dt} = f(u), \quad u(0) = u_0. \tag{3}$$

Suppose that there are two Banach spaces $Y \hookrightarrow X$, with the embedding continuous, such that f is continuous from Y to X. We say that the problem (3) is *locally well-posed* in Y, if for each $u_0 \in Y$ there are a real number $T = T(u_0) > 0$ and a unique function $u \in C([0, T], Y)$ satisfying the integral equation associated to (3), depending continuously on the initial data in the sense that the solution map $u_0 \mapsto u$ is continuous: if $u_n \to u$ en Y and $T' \in (0, T)$, then for n large enough $u_n \in C([0, T'], Y)$ and,

$$\lim_{n\to\infty} \sup_{[0,T']} \|u_n(t) - u(t)\|_Y = 0.$$

We say that the problem is *globally well-posed* in Y, if for every $u_0 \in Y$ the number T can be taken arbitrarily large,. We recall that if E is a Banach space then $C([0, T], E)$ denote the space of continuous functions defined in $[0, T]$ with values in E.

The natural space in which we consider the well-posedness of the Cauchy problem associated with the Boussinesq system (1) is dictated by the definition of the Hamiltonian. Remember that the Hamiltonian is well defined when for t in some interval we have that $\eta(\cdot, t), \Phi_x(\cdot, t) \in H^1(\mathbb{R})$, then we consider the following spaces. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ is defined as the completion of the Schwartz space $\mathcal{S}(\mathbb{R})$ with respect to the norm given by

$$\|f\|^2_{H^s} = \|(1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi,$$

where the Fourier transform of a function w defined on $\mathbb{R}$ is given by

$$(\mathcal{F}w)(\xi) = \widehat{w}(\xi) = \int_{\mathbb{R}} e^{-ix\cdot\xi} w(x)\, dx.$$

The space $\mathcal{V}^s$ denote the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm given by

$$\|f\|^2_{\mathcal{V}^s} = \|f'\|^2_{H^{s-1}}.$$

Note that $\mathcal{V}^s$ is a Hilbert space with inner product

$$(f, g)_{\mathcal{V}^s} = (f', g')_{H^{s-1}}.$$

Moreover,

$$\|f\|^2_{\mathcal{V}^s} = \int_{\mathbb{R}} (1 + |\xi|^2)^{s-1} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi.$$

We will show, under some conditions on a, c, p and s, the local well-posedness for the Boussinesq system (1) with the initial condition (2), in the space $H^s \times \mathcal{V}^{s+1}$. Hereafter, we assume $b, d > 0$.

We note that if we formally derive the second equation of the Boussinesq system (1), we find that the system (1) is transformed in the following system

$$\begin{cases} (I - a\mu\partial_x^2)\eta_t + \partial_x(I - b\mu\partial_x^2)u + \epsilon\partial_x(\eta u^p) = 0, \\ (I - c\mu\partial_x^2)u_t + \partial_x(I - d\mu\partial_x^2)\eta + \frac{\epsilon}{p+1}\partial_x(u^{p+1}) = 0 \end{cases} \tag{4}$$

in the variables $\eta, u = \Phi_x$. For this system, we see that the quantities

$$\int_{\mathbb{R}} u(t, x)\, dx, \quad \int_{\mathbb{R}} \eta(t, x)\, dx$$

are conserved in time for classical solutions and even for mild solutions. So, if we consider the Cauchy problem associated with initial data in an appropriate Sobolev space such that

$$\widehat{u_0}(0) = \int_{\mathbb{R}} u_0(x)\, dx = 0, \tag{5}$$

then we have that

$$\widehat{u}(t, \xi) = \int_{\mathbb{R}} u(t, x)\, dx = 0,$$

for $t \in \mathbb{R}$, as long as the solution exists. Now, it is known that if

$$\dot{H}^r := H^r \cap \{f \in H^r : \hat{f}(0) = 0\},$$

then there is an onto linear map $\partial_x^{-1} : \dot{H}^r \to H^{r+1}$ defined via the Fourier transform by

$$\widehat{\partial_x^{-1}(f)}(\xi) = \frac{\widehat{f}(\xi)}{i\xi}.$$

Moreover, for a given function $u \in \dot{H}^r$, the function $\Phi = \partial_x^{-1} u \in \mathcal{V}^{r+1}$ is such that $u = \Phi_x$. So, by solving the Cauchy problem associated for the system (4) with a initial condition satisfying (5), we are able to solve the Cauchy problem associated with the system (1).

Now, we will focus in the local well posedness for the the Cauchy problem associated with the system (4). Note that by defining the operators $A = I - a\mu\partial_x^2$, $B = I - b\mu\partial_x^2$, $C = I - c\mu\partial_x^2$ and $D = I - d\mu\partial_x^2$ via the Fourier transform as

$$\widehat{Af} = (1 + a\mu\xi^2)\widehat{f}, \quad \widehat{Bf} = (1 + b\mu\xi^2)\widehat{f}, \quad \widehat{Cf} = (1 + c\mu\xi^2)\widehat{f}, \quad \widehat{Df} = (1 + d\mu\xi^2)\widehat{f},$$

we see that the system (4) can be written as

$$\begin{pmatrix}\eta\\ u\end{pmatrix}_t + M\begin{pmatrix}\eta\\ u\end{pmatrix} + F\begin{pmatrix}\eta\\ u\end{pmatrix} = 0, \tag{6}$$

where M is a linear operator and F corresponds to the nonlinear part,

$$M = \begin{pmatrix} 0 & \partial_x A^{-1}B\\ \partial_x C^{-1}D & 0\end{pmatrix}, \quad F\begin{pmatrix}\eta\\ u\end{pmatrix} = \epsilon\begin{pmatrix}\partial_x A^{-1}(\eta u^p)\\ \frac{1}{p+1}\partial_x C^{-1}(u^{p+1})\end{pmatrix}.$$

In order to consider the Cauchy problem associated with the first order equation (6), we need to describe the semigroup $S(t)$ associated with the linear problem

$$\begin{pmatrix}\eta\\ u\end{pmatrix}_t + M\begin{pmatrix}\eta\\ u\end{pmatrix} = 0. \tag{7}$$

If we consider the Sobolev type space $Y^s = H^s \times H^s$ with norm given by

$$\|(\eta, u)\|^2_{Y^s} = \|\eta\|^2_{H^s} + \|u\|^2_{H^s}.$$

Then the unique solution of the linear problem (7) with the initial condition

$$(\eta(0,\cdot), u(0,\cdot)) = (\eta_0, u_0) \in Y^s, \tag{8}$$

is given by

$$(\eta(t), u(t)) = S(t)(\eta_0, u_0),$$

where $S(t)$ is defined as

$$S(t) = \begin{pmatrix}\mathcal{F}^{-1} & 0\\ 0 & \mathcal{F}^{-1}\end{pmatrix}\begin{pmatrix}\cos(\xi\Lambda(\xi)t) & -i\frac{\varphi_1(\xi)}{\Lambda(\xi)}\sin(\xi\Lambda(\xi)t)\\ -i\frac{\varphi_2(\xi)}{\Lambda(\xi)}\sin(\xi\Lambda(\xi)t) & \cos(\xi\Lambda(\xi)t)\end{pmatrix}\begin{pmatrix}\mathcal{F} & 0\\ 0 & \mathcal{F}\end{pmatrix},$$

and the functions $\varphi_i, \Lambda\colon \mathbb{R} \longrightarrow \mathbb{R}$ are given by

$$\varphi_1(\xi) = \sqrt{\frac{1+b\mu\xi^2}{1+a\mu\xi^2}}, \quad \varphi_2(\xi) = \sqrt{\frac{1+d\mu\xi^2}{1+c\mu\xi^2}}, \quad \Lambda^2(\xi) = \frac{(1+b\mu\xi^2)(1+d\mu\xi^2)}{(1+a\mu\xi^2)(1+c\mu\xi^2)}.$$

It is convenient to set

$$Q(t)(\widehat{\eta}, \widehat{u}) = \big(Q_1(t), Q_2(t)\big)(\widehat{\eta}, \widehat{u}),$$

where

$$Q_1(t)(\widehat{\eta}, \widehat{u})(\xi) = \cos(\xi\Lambda(\xi)t)\,\widehat{\eta}(\xi) - i\frac{\varphi_1(\xi)}{\Lambda(\xi)}\sin(\xi\Lambda(\xi)t)\widehat{u}(\xi),$$

$$Q_2(t)(\widehat{\eta}, \widehat{u})(\xi) = -i\frac{\varphi_2(\xi)}{\Lambda(\xi)}\sin(\xi\Lambda(\xi)t)\,\widehat{\eta}(\xi) + \cos(\xi\Lambda(\xi)t)\,\widehat{u}(\xi).$$

Then we have that

$$S(t)(\eta, \Phi) = (\mathcal{F}^{-1}(Q_1(t)(\widehat{\eta}, \widehat{\Phi})), \mathcal{F}^{-1}(Q_2(t)(\widehat{\eta}, \widehat{\Phi}))).$$

On the other hand, it is known that the Duhamel's principle implies that if (η, Φ) is a solution of (6) with the initial condition (8), then this solution satisfies the integral equation

$$\begin{pmatrix}\eta\\ \Phi\end{pmatrix}(t) = S(t)\begin{pmatrix}\eta_0\\ u_0\end{pmatrix} - \int_0^t S(t-\tau)\,F\begin{pmatrix}\eta\\ u\end{pmatrix}(\tau)\,d\tau. \tag{9}$$

Hereafter, we refer a couple $(\eta, u) \in C([0, T], Y^s)$ satisfying the integral equation (9) as a mild solution for the Cauchy problem associated with the system (7) with initial condition (8). Now, we will establish the existence of mild solutions. For this, we use some linear and nonlinear estimates. Let us start with the following result.

Lemma 2.1 *Suppose $s \in \mathbb{R}$. Then for all $t \in \mathbb{R}$, $S(t)$ is a bounded linear operator from Y^s into Y^s. Moreover, there exists $K_1 > 0$ such that for all $t \in \mathbb{R}$,*

$$\|S(t)(\eta, u)\|_{Y^s} \leq K_1 \|(\eta, u)\|_{Y^s}.$$

Proof First note that there is a constant $\beta > 0$ such that $0 < \frac{\varphi_i}{\Lambda} \leq \beta$. Then we have that

$$\begin{aligned}
\|\mathcal{F}^{-1}(Q_1(t)(\widehat{\eta}, \widehat{u}))\|^2_{H^s} &\leq \int_{\mathbb{R}} (1+\xi^2)^s \, |\cos(\xi\Lambda(\xi)t)|^2 \, |\widehat{\eta}(\xi)|^2 d\xi \\
&\quad + \int_{\mathbb{R}} (1+\xi^2)^s \frac{\varphi_1^2(\xi)}{\Lambda^2(\xi)} |\sin(\xi\Lambda(\xi)t)|^2 |\widehat{u}(\xi)|^2 d\xi \\
&\leq \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{\eta}(\xi)|^2 d\xi + \beta^2 \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{u}(\xi)|^2 d\xi \\
&\leq K(\beta)(\|\eta\|^2_{H^s} + \|u\|^2_{H^s}).
\end{aligned}$$

In a similar fashion, we see that

$$\begin{aligned}
\|\mathcal{F}^{-1}(Q_2(t)(\widehat{\eta}, \widehat{u}))\|^2_{H^s} &= \int_{\mathbb{R}} (1+\xi^2)^s \frac{\varphi_2^2(\xi)}{\Lambda^2(\xi)} |\sin(\xi\Lambda(\xi)t)|^2 |\widehat{\eta}(\xi)|^2 \, d\xi \\
&\quad + \int_{\mathbb{R}} (1+|\xi|^2)^s |\cos(\xi\Lambda(\xi)t)|^2 |\widehat{u}(\xi)|^2 d\xi \\
&\leq \beta^2 \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{\eta}(\xi)|^2 + \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{u}(\xi)|^2 d\xi \\
&\leq K(\beta)(\|\eta\|^2_{H^s} + \|u\|^2_{H^s}).
\end{aligned}$$

Then we obtain that

$$\begin{aligned}
\|S(t)(\eta, u)\|^2_{Y^s} &= \|\mathcal{F}^{-1}(Q_1(t)(\widehat{\eta}, \widehat{u}))\|^2_{H^s} + \|\mathcal{F}^{-1}(Q_2(t)(\widehat{\eta}, \widehat{u})\|^2_{H^s} \\
&\leq K\|(\eta, u)\|^2_{Y^s},
\end{aligned}$$

and $S(t)$ have the required property.

Next, we want to perform the estimates for nonlinear terms of system (6) (Lemma 2.4), which will follow by an estimate obtained by J. Bona and N. Tzvetkov (see Lemma 1 in [2]) in the case $a, c > 0$ and the well known estimates for the commutator of Kato in the case $a = c = 0$. First, note that for $r > 0$, the Fourier multiplier $\psi_r(\xi) = \frac{\xi}{1+r\xi^2}$ is associated with the operator $R^{-1}\partial_x$ with $R = I - r\partial_x^2$, since we have that

$$\widehat{R^{-1}\partial_x u}(\xi) = \psi_r(\xi)\widehat{u}(\xi).$$

Lemma 2.2 (Bona and Tzvetkov [2]) *Let $r > 0$, $s \geq 0$ and $u, v \in H^s(\mathbb{R})$. Then there exists a constant $K(r) > 0$ such that*

$$\|R^{-1}(\partial_x)uv\|_{H^s(\mathbb{R})} \leq K(r)\|u\|_{H^s(\mathbb{R})}\|v\|_{H^s(\mathbb{R})}.$$

Now, let $J = \left(I - \partial_x^2\right)^{1/2}$ be the operator defined by

$$\widehat{Jf} = (1 + |\xi|^2)^{1/2}\widehat{f},$$

and let [,] be the commutator defined by

$$[J^s, u]v = J^s(uv) - uJ^s v.$$

Lemma 2.3 (Kato [3–5]) *Suppose $s > \frac{3}{2}$, $t > \frac{1}{2}$ and $u \in H^s(\mathbb{R})$, $w \in H^{s-1}(\mathbb{R})$. Then there exists a constant $K > 0$ such that*

(1) $\|[J^s, u]w\|_{L^2(\mathbb{R})} \le K\|u\|_{H^s(\mathbb{R})}\|w\|_{H^{s-1}(\mathbb{R})}$.
(2) $\|u\partial_x w\|_{L^2(\mathbb{R})} \le K\|\partial_x u\|_{H^t(\mathbb{R})}\|w\|_{L^2(\mathbb{R})}$.

We now will establish the nonlinear estimates.

Lemma 2.4 *Suppose a, c, p and s are such that*

(i) $a, c > 0,\ p = 1, s \ge 0$, *or*
(ii) $a, c > 0,\ p > 1,\ s > \frac{1}{2}$, *or*
(iii) $a = c = 0,\ p \ge 1,\ s > \frac{3}{2}$.

Then there are constants $K_2, K_3 > 0$ such that

$$\|F(\eta, u)\|_{Y^s} \le K_2\|(\eta, u)\|_{Y^s}^{p+1}, \tag{10}$$

$$\|F(\eta, u) - F(\eta_1, u_1)\|_{Y^s} \le K_3\|(\eta, u) - (\eta_1, u_1)\|_{Y^s}\left(\|(\eta, u)\|_{Y^s} + \|(\eta_1, u_1)\|_{Y^s}\right)^p. \tag{11}$$

Proof We write $F = \epsilon\left(F_1, \frac{1}{p+1}F_2\right)$ where

$$F_1(\eta, u) = A^{-1}\partial_x(\eta u^p), \quad F_2(\eta, u) = C^{-1}\partial_x(u^{p+1}).$$

First we assume that $a, c > 0$, $p = 1$ and $s \ge 0$. Note that the Lemma 2.2 holds for $\psi_a(\xi) = \frac{\xi}{1+a\xi^2}$. Then we have that

$$\begin{aligned}\|F_1(\eta, u)\|_{H^s} &= \|A^{-1}\partial_x\,(\eta u)\,\|_{H^s}\\ &\le K(a)\|\eta\|_{H^s}\|u\|_{H^s}\\ &\le K(a)(\|\eta\|_{H^s}^2 + \|u\|_{H^s}^2)\\ &= K(a)\|(\eta, u)\|_{Y^s}^2.\end{aligned}$$

Similarly we have that

$$\|F_2(\eta, u)\|_{H^s} = \|C^{-1}\partial_x(u^2)\|_{H^s} \le K(c)\|u\|_{H^s}^2 \le K(c)\|(\eta, u)\|_{Y^s}^2.$$

In other words, we have established estimate (1). Now we prove estimate (2). In fact,

$$\begin{aligned}\|F_1(\eta, u) - F_1(\eta_1, u_1)\|_{H^s} &\le \|A^{-1}\partial_x\,(\eta u - \eta_1 u_1)\,\|_{H^s}\\ &\le \|A^{-1}\partial_x(\eta(u - u_1))\|_{H^s} + \|A^{-1}\partial_x(\eta - \eta_1)u_1\|_{H^s}\\ &\le K(a)\,(\|\eta\|_{H^s}\|u - u_1\|_{H^s} + \|\eta - \eta_1\|_{H^s}\|u_1\|_{H^s})\\ &\le K(a)\,(\|\eta\|_{H^s} + \|u_1\|_{H^s})\,(\|\eta - \eta_1\|_{H^s} + \|u - u_1\|_{H^s})\\ &\le K(a)\,(\|(\eta, u)\|_{Y^s} + \|(\eta_1, u_1)\|_{Y^s})\,\|(\eta, u) - (\eta_1, u_1)\|_{Y^s}.\end{aligned}$$

In a similar fashion we have that

$$\begin{aligned}\|F_2(\eta,u)-F_2(\eta_1,u_1)\|_{H^s}&=\|C^{-1}\partial_x(u^2-u_1^2)\|_{H^s}\\&=\|C^{-1}\partial_x(u+u_1)(u-u_1)\|_{H^s}\\&\le K(c)\|u+u_1\|_{H^s}\|u-u_1\|_{H^s}\\&\le K(c)\left(\|(\eta,u)\|_{Y^s}+\|(\eta_1,u_1)\|_{Y^s}\right)\|(\eta,u)-(\eta_1,u_1)\|_{Y^s}.\end{aligned}$$

Then we conclude that

$$\begin{aligned}\|F(\eta,u)-F(\eta_1,u_1)\|_{Y^s}\le K(a,c)(\|F_1(\eta,u)-F_1(\eta_1,u_1)\|_{H^s}+\|F_2(\eta,u)\\-F_2(\eta_1,u_1)\|_{H^s})\le K(a,c)(\|(\eta,u)\|_{Y^s}\\+\|(\eta_1,u_1)\|_{Y^s})\|(\eta,u)-(\eta_1,u_1)\|_{Y^s}.\end{aligned}$$

Now we suppose that $s>\frac{1}{2}$ and $p>1$. Using the Lemma 2.2 and that $H^s(\mathbb{R})$ is an algebra we obtain that

$$\begin{aligned}\|F_1(\eta,u)\|_{H^s}&=\|A^{-1}\partial_x(\eta u^p)\|_{H^s}\\&\le K(a)\|\eta u^p\|_{H^s}\\&\le K(a)\|\eta\|_{H^s}\|u\|_{H^s}^p\\&\le K(a)\|(\eta,u)\|_{Y^s}^{p+1},\end{aligned}$$

and also that

$$\|F_2(\eta,u)\|_{H^s}=\|C^{-1}\partial_x(u^{p+1})\|_{H^s}\le K(c)\|u\|_{H^s}^{p+1}\le K(c)\|(\eta,u)\|_{Y^s}^{p+1}.$$

Moreover, we see that

$$\begin{aligned}\|F_1(\eta,u)-F_1(\eta_1,u_1)\|_{H^s}&\le\|A^{-1}\partial_x(\eta(u^p-u_1^p))\|_{H^s}+\|A^{-1}\partial_x((\eta-\eta_1)u_1^p)\|_{H^s}\\&\le K(a)\|\eta\|_{H^s}\|u^p-u_1^p\|_{H^s}+\|\eta-\eta_1\|_{H^s}\|u_1\|_{H^s}^p.\end{aligned}$$

But a simple calculation shows that

$$\|u^p-u_1^p\|_{H^s}\le K(p)\|u-u_1\|_{H^s}\left(\|u\|_{H^s}+\|u_1\|_{H^s}\right)^{p-1}.$$

Then we have that

$$\|F_1(\eta,u)-F_1(\eta_1,u_1)\|_{Y^s}\le K(a,c,p)\|(\eta,u)-(\eta_1,u_1)\|_{Y^s}\left(\|(\eta,u)\|_{Y^s}+\|(\eta_1,u_1)\|_{Y^s}\right)^p.$$

In a similar fashion we obtain the same estimate for $\|F_2(\eta,u)-F_2(\eta_1,u_1)\|_{Y^s}$ and then (1) and (2) hold.

We assume now that $a=c=0$, $p\ge 1$ and $s>\frac{3}{2}$. First we will that if $v,\partial_x w\in H^s$ then there exists $K>0$ such that

$$\|v\partial_x w\|_{H^s}\le K\|v\|_{H^s}\|w\|_{H^s}. \tag{12}$$

In fact, from Lemma 2.3 we see that

$$\begin{aligned}\|v\partial_x w\|_{H^s}&=\|J^s(v\partial_x w)\|_{L^2}\\&\le\|\,[J^s,v]\partial_x w\,\|_{L^2}+\|v\partial_x J^s w\|_{L^2}\\&\le K(\|v\|_{H^s}\|\partial_x w\|_{H^{s-1}}+\|\partial_x v\|_{H^{s-1}}\|J^s w\|_{L^2})\\&\le K\|v\|_{H^s}\|w\|_{H^s}.\end{aligned}$$

Then, using (12) and that $H^s(\mathbb{R})$ is an algebra we have that

$$\begin{aligned}\|F_1(\eta,u)\|_{H^s} &\le K(p)(\|\partial_x \eta u^p\|_{H^s} + \|\eta u^{p-1}\partial_x u\|_{H^s})\\ &\le K(p)\|\eta\|_{H^s}\|u\|^p_{H^s}\\ &\le K(p)\|(\eta,u)\|^{p+1}_{Y^s},\end{aligned}$$

and also that

$$\|F_2(\eta,u)\|_{H^s} \le K(p)\|u^p\partial_x u\|_{H^s} \le K(p)\|u\|^{p+1}_{H^s} \le K(p)\|(\eta,u)\|^{p+1}_{Y^s}.$$

Thus, we conclude that there exists $K > 0$ such that

$$\|F(\eta,u)\|_{Y^s} \le K\|(\eta,u)\|^{p+1}_{Y^s}.$$

In a similar way we obtain the part (2) and then the theorem follows.

Next, we establish the local well-posedness for the system (4) in the space $Y^s = H^s \times H^s$. For this we will show the existence of a mild solution for the integral equation (9) for a, c, p and s as in Lemma 2.4, using the Banach fixed point theorem. Moreover, if a, c, p and s are as in Lemma 2.4, with $s > \frac{1}{2}$ in the case $a, c > 0$ and $p = 1$, we already have classical solutions.

Theorem 2.1 *Let a, c, p and s be as in Lemma 2.4. Then for all $(\eta_0, u_0) \in Y^s$ there exists a time $T > 0$ which depends only on $\|(\eta_0,u_0)\|_{Y^s}$ such that the problem (4) with initial condition (8) has a unique solution (η, u) satisfying that $(\eta,u) \in C([0,T], Y^s)$. Moreover, as in Lemma 2.4 with $s > \frac{1}{2}$ for $a, c > 0$, $p = 1$, we have*

$$(\eta,u) \in C([0,T], Y^s) \cap C^1([0,T], Y^{s-1}).$$

On the other hand, for all $0 < T' < T$ there exists a neighborhood $\mathbb{V}$ of (η_0,u_0) in Y^s such that the correspondence $(\tilde{\eta}_0,\tilde{u}_0) \longrightarrow (\tilde{\eta}(\cdot),\tilde{u}(\cdot))$, that associates to $(\tilde{\eta}_0,\tilde{u}_0)$ the solution $(\tilde{\eta}(\cdot),\tilde{u}(\cdot))$ of the problem (4) with initial condition $(\tilde{\eta}_0,\tilde{u}_0)$ is a Lipschitz mapping from $\mathbb{V}$ in $C([0,T'], Y^s)$.

Proof Given $T > 0$ we define the space $X^s(T) = C([0,T], Y^s)$, equipped with the norm defined by

$$\|U\|_{X^s(T)} = \max_{t\in[0,T]} \|U(\cdot,t)\|_{Y^s}.$$

It is easy to see that $X^s(T)$ is a Banach space. Let $B_R(T)$ be the closed ball of radius R centered at the origin in $X^s(T)$, i.e.

$$B_R(T) = \{U \in X^s(T) \colon \|U\|_{X^s(T)} \le R\}.$$

For fixed $U_0 = (\eta_0, u_0) \in Y$, we define the map

$$\Psi(U(t)) = S(t)U_0 - \int_0^t S(t-\tau)F(U(\tau))\,d\tau,$$

where $U = (\eta, u) \in X(T)$. We will show that the correspondence $U(t) \mapsto \Psi(U(t))$ maps $B_R(T)$ into itself and is a contraction if R and T are well chosen. In fact, if $t \in [0,T]$ and $U \in B_R(T)$, then using Lemma 2.1 and statement (1) of Lemma 2.4 we have that

$$\begin{aligned}\|\Psi(U(t))\|_{Y^s} &\le K_1\left(\|U_0\|_{Y^s} + K_2\int_0^t \|U(\tau)\|^{p+1}_{Y^s}d\tau\right)\\ &\le K_1(\|U_0\|_{Y^s} + K_2R^{p+1}T).\end{aligned}$$

Choosing $R = 2K_1\|U_0\|_{Y^s}$ and $T > 0$ such that

$$(2K_1)^{p+1}K_2\|U_0\|_{Y^s}^p T \leq 1,$$

we obtain that

$$\|\Psi(U(t))\|_{Y^s} \leq K_1 \|U_0\|_{Y^s}(1 + (2K_1)^{p+1}C_2\|U_0\|_{Y^s}^p T) \leq 2 K_1\|U_0\|_{Y^s} = R.$$

So that Ψ maps $B_R(T)$ to itself. Let us prove that Ψ is a contraction. If $U, V \in B_R(T)$, then by the definition of Ψ we have that

$$\Psi(U(t)) - \Psi(V(t)) = -\int_0^t S(t-\tau)[F(U(\tau)) - F(V(\tau))]d\tau.$$

Then using the statement (2) of Lemma 2.4 we see that for $t \in [0, T]$,

$$\begin{aligned}\|\Psi(U(t)) - \Psi(V(t))\|_{Y^s} &\leq K_1K_3\int_0^t (\|U(\tau)\|_{Y^s} + \|V(\tau)\|_{Y^s})^p \|U(\tau) - V(\tau)\|_{Y^s}\, d\tau\\ &\leq K_1K_3(2R)^pT\|U - V\|_{X^s(T)}\\ &\leq 4^pK_1^{p+1}K_3\|U_0\|_{Y^s}^p T\|U - V\|_{X^s(T)}.\end{aligned}$$

We choose T enough small so that (2) holds and

$$\alpha = 4^pK_1^{p+1}K_3\|U_0\|_{Y^s}^p T \leq \frac{1}{2}.$$

So, we conclude that

$$\|\Psi(U) - \Psi(V)\|_{X^s(T)} \leq \alpha\|U - V\|_{X^s(T)}.$$

Therefore Ψ is a contraction. Thus, there exists a unique fixed point of Ψ in $B_R(T)$, which is a solution of the integral equation (9). Now, if $(\eta(t), u(t)) \in C([0, T], Y^s)$ is a integral or mild solution, obviously $(\eta(0), u(0)) = (\eta_0, u_0)$.

Now assume that a, c, p and s are as in Lemma 2.4, with $s > \frac{1}{2}$ in the case $a, c > 0$ and $p = 1$. We define the function $H \in C([0, T] : Y^s)$ by $H(t) = F(\eta(t), u(t))$. From Lemma 2.4, we have that $H \in L^1([0, T]\colon Y^s)$ since from inequality (10) for $s > \frac{1}{2}$,

$$\|F(\eta(t), u(t))\|_{C_b(\mathbb{R})} \leq K_3\|F(\eta(t), u(t))\|_{Y^s} \leq K_3K_2\|(\eta(t), u(t))\|_{Y^s}^{p+1},$$

where $C_b(\mathbb{R})$ denotes the space of bounded continuous functions defined on $\mathbb{R}$. From this fact and the smoothness properties of the semigroup S, we conclude that the function defined on $[0, T]$ by

$$W(t) = \int_0^t S(t-\tau)F(\eta(\tau), u(\tau))\, d\tau$$

is such that $W \in C([0, T]\colon Y^s)$. On the other hand, we also have that

$$\begin{aligned}\frac{1}{h}(W(t+h) - W(t)) &= \frac{1}{h}\int_t^{t+h} S(t-\tau+h)F(\eta(\tau), u(\tau))\, d\tau\\ &\quad + \left(\frac{S(h) - I}{h}\right)\int_0^t S(t-\tau)F(\eta(\tau), u(\tau))\, d\tau. \qquad (13)\end{aligned}$$

Taking limit as $h \to 0$ and using the continuity of H, we have that

$$W'(t) = F(\eta(t), u(t)) - M(\eta(t), u(t)),$$

and so $U(t) = S(t)U_0 - W(t)$ is such that $U \in C([0,T]: Y^s) \cap C^1([0,T]: Y^{s-1})$ is a local classical solution of (7). In other words, $(\eta(t), u(t))$ is a local classical solution for the Cauchy problem associated with the system (6) and initial condition (8). The uniqueness and continuous dependence of the solution are obtained by standard arguments.

Our main result in this section related with the existence and uniqueness of mild and classical solutions for the Cauchy problem associated with (1) is a direct consequence of the Theorem 2.1. For system (1), we have existence of integral or mild solutions for a, c, p and s as in Lemma 2.4, and for a, c, p and s as in Lemma 2.4, with $s > \frac{1}{2}$ in the case $a, c > 0$ and $p = 1$, we already have classical solutions.

Theorem 2.2 *Let a, c, p and s be as in Lemma 2.4. Then for all $(\eta_0, \Phi_0) \in H^s \times \mathcal{V}^{s+1}$ there exists a time $T > 0$ which depends only on $\|(\eta_0, \Phi_0)\|_{H^s \times \mathcal{V}^{s+1}}$ such that the Cauchy problem associated with the Boussinesq system (1) and the initial condition (η_0, Φ_0) has a unique solution (η, Φ) satisfying that $(\eta, u) \in C([0,T], Y^s) \cap C^1([0,T], Y^{s-1})$. Moreover, as in Lemma 2.4 with $s > \frac{1}{2}$ for $a, c > 0$, $p = 1$, we have*

$$(\eta, \Phi) \in C([0,T], H^s \times \mathcal{V}^{s+1}) \cap C^1([0,T], H^{s-1} \times \mathcal{V}^s).$$

Moreover, for all $0 < T' < T$ there exists a neighborhood $\mathbb{V}$ of (η_0, Φ_0) in $H^s \times \mathcal{V}^{s+1}$ such that the correspondence $(\tilde{\eta}_0, \tilde{\Phi}_0) \longrightarrow (\tilde{\eta}(\cdot), \tilde{\Phi}(\cdot))$, that associates to $(\tilde{\eta}_0, \tilde{\Phi}_0)$ the solution $(\tilde{\eta}(\cdot), \tilde{\Phi}(\cdot))$ of the problem (1) with initial condition $(\tilde{\eta}_0, \tilde{\Phi}_0)$ is a Lipschitz mapping from $\mathbb{V}$ in $C([0,T'], H^s \times \mathcal{V}^{s+1})$.

Proof By hypothesis, if $u_0 = \partial_x \Phi_0$, then we have that $(\eta_0, u_0) \in Y^s$, and also that $\widehat{u_0}(0) = 0$. Now, from Theorem 2.1, there exist $T = T(\|(\eta_0, u_0)\|_{Y^s}) > 0$ and a unique solution (η, u) of the problem (4) with initial condition (η_0, Φ_0) satisfying that

$$(\eta, u) \in C([0,T], Y^s) \cap C^1([0,T], Y^{s-1}),$$

and also that

$$\widehat{u_0}(0) = \int_{\mathbb{R}} u_0(x)\,dx = \int_{\mathbb{R}} u(t,x)\,dx = \widehat{u}(t,\xi) = 0.$$

So, the couple (η, Φ) where $\Phi(t,x) = \partial_x^{-1} u(t,x)$ is a mild solution of the Cauchy problem (1) with initial condition (η_0, Φ_0) satisfying that

$$(\eta, \Phi) \in C([0,T], H^s \times \mathcal{V}^{s+1}) \cap C^1([0,T], H^{s-1} \times \mathcal{V}^s).$$

The last part follows by noting that if $\Phi(t,x) = \partial_x^{-1} u(t,x)$, then we have that

$$\|(\eta(t,\cdot), u(t,\cdot))\|_{Y^s} = \|(\eta(t,\cdot), \Phi(t,\cdot))\|_{H^s \times \mathcal{V}^{s+1}}.$$

Global Existence for $a = c$

In this section for $a = c$ we will establish that any local solution in time of the system (1) can be extended for any $t > 0$. The result will depends strongly on fact that the Hamiltonian $\mathcal{H}$ is conserved in time on classical and mild solutions. Before we go further, for solutions of the system (1) a direct computation shows that

$$\partial_t \mathcal{H}(\eta(t,x), \Phi(t,x)) = \mu(c-a) \int_{\mathbb{R}} \partial_x^2 \Phi_t \eta_t \,dx,$$

meaning that the Hamiltonian $\mathcal{H}$ is conserved in time on classical and mild solutions if and only if $a = c$. Now note that

$$\begin{aligned}\mathcal{H}(\eta,\Phi) &= \frac{1}{2}\int_{\mathbb{R}}\left(\eta^2 + d\mu\,(\eta_x)^2 + (\Phi_x)^2 + b\mu\,(\Phi_{xx})^2 + \frac{2\epsilon}{p+1}\eta\,(\Phi_x)^{p+1}\right)dx \\ &= \frac{1}{2}\left(\mathcal{E}(\eta,\Phi) + G(\eta,\Phi)\right), \end{aligned} \tag{14}$$

where functional $\mathcal{E}$ (energy) and G are given by

$$\begin{aligned}\mathcal{E}(\eta,\Phi) &= \int_{\mathbb{R}}(\eta^2 + d\mu\,(\eta_x)^2 + (\Phi_x)^2 + b\mu\,(\Phi_{xx})^2)dx, \\ G(\eta,\Phi) &= \frac{2\epsilon}{p+1}\int_{\mathbb{R}}\eta\,(\Phi_x)^{p+1}\,dx.\end{aligned}$$

We will see that the global well-posedness follows by using a variational approach and the fact that the energy $\sqrt{\mathcal{E}}$ is a norm in the space $H^1(\mathbb{R}) \times \mathcal{V}^2$, since for some constant $K(b,d,\mu) > 1$,

$$K(b,d,\mu)^{-1}\|(\eta,\Phi)\|^2_{H^1\times\mathcal{V}^2} \le \mathcal{E}(\eta,\Phi) \le K(b,d,\mu)\|(\eta,\Phi)\|^2_{H^1\times\mathcal{V}^2}. \tag{15}$$

A key ingredient in our analysis depends upon the variational characterization of the number δ_0 defined by

$$\begin{aligned}\delta_0 &= \inf\left\{\sup_{\lambda\ge 0}\mathcal{H}(\lambda(\eta,\Phi))\colon (\eta,\Phi)\in H^1\times\mathcal{V}^2\backslash\{0\}\right\} \\ &= \inf\left\{\sup_{\lambda\ge 0}\mathcal{H}(\lambda(\eta,\Phi))\colon (\eta,\Phi)\in H^1\times\mathcal{V}^2,\ G(\eta,\Phi)<0\right\}.\end{aligned}$$

Note that for $G(\eta,\Phi)\ge 0$, we have that $\sup_{\lambda\ge 0}\mathcal{H}(\lambda(\eta,\Phi)) = \infty$. It is straightforward to see that

$$\delta_0 = \frac{p}{2(p+2)}\left(\frac{2}{p+2}\right)^{\frac{2}{p}} K_p^{-\frac{p+2}{p}}, \tag{16}$$

where K_p is defined as

$$K_p = \sup\left\{\frac{G^{\frac{2}{p+2}}(\eta,\Phi)}{\mathcal{E}(\eta,\Phi)}\colon (\eta,\Phi)\in H^1\times\mathcal{V}^2\backslash\{0\}\right\}. \tag{17}$$

In fact, from the Young inequality and that the embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous for $q \ge 2$, we see that there is $K_1 = K_1(\epsilon, p) > 0$ such that for all $(\eta,\Phi)\in H^1(\mathbb{R})\times\mathcal{V}^2$,

$$|G(\eta,\Phi)| \le K_1(\|\eta\|^{p+2}_{H^1} + \|\Phi\|^{p+2}_{\mathcal{V}^2}) \le K_1\|(\eta,\Phi)\|^{\frac{p+2}{2}}_{H^1\times\mathcal{V}^2}.$$

Thus, from (15), we obtain that

$$|G(\eta,\Phi)|^{\frac{2}{p+2}} \le K_1(\epsilon,p)K(b,d,\mu)\mathcal{E}(\eta,\Phi),$$

meaning that K_p is finite. Now, for $\lambda \ge 0$, we define the function

$$V(\lambda) = \mathcal{H}(\lambda(\eta,\Phi)) = \frac{\lambda^2}{2}\mathcal{E}(\eta,\Phi) + \frac{\lambda^{p+2}}{2}G(\eta,\Phi).$$

Then we have that $V'(\lambda) = 0$ if and only if $\lambda_0 = 0$ or $\lambda_1^p G(\eta, \Phi) = -\frac{2}{p+2}\mathcal{E}(\eta, \Phi)$. Since $V(0) = 0$, then

$$\sup_{\lambda \geq 0} V(\lambda) = V(\lambda_1) = \frac{p}{2(p+2)} \left(\frac{2}{p+2}\right)^{\frac{2}{p}} \left(\frac{G^{\frac{2}{p+2}}(\eta, \Phi)}{\mathcal{E}(\eta, \Phi)}\right)^{-\frac{p+2}{p}}.$$

This formula implies the desired equality (16). We note that the constant K_p establishes a Sobolev type inequality, since we have that

$$|G(\eta, \Phi)|^{\frac{1}{p+2}} \leq K_p^{\frac{1}{2}} \sqrt{\mathcal{E}(\eta, \Phi)} \leq K(b, d, \mu)^{\frac{1}{2}} K_p^{\frac{1}{2}} \|(\eta, \Phi)\|_{H^1 \times \mathcal{V}^2}. \tag{18}$$

Before we go further, we consider the auxiliary functional $\mathcal{H}_1(U) = \mathcal{H}'(U)(U)$, which has can be expressed as

$$\mathcal{H}_1(\eta, \Phi) = \mathcal{E}(\eta, \Phi) + \frac{p+2}{2} G(\eta, \Phi). \tag{19}$$

In particular, we have that

$$\mathcal{H}(\eta, \Phi) = \frac{p}{2(p+2)} \mathcal{E}(\eta, \Phi) + \frac{1}{p+2} \mathcal{H}_1(\eta, \Phi). \tag{20}$$

We have the following result related with invariance of quantities under the flow of solutions for the Cauchy problem associated with the system (1).

Lemma 3.1 *Let* (η, Φ) *be a local solution of* (1) *with initial condition* $(\eta_0, \Phi_0) \in H^1(\mathbb{R}) \times \mathcal{V}^2$ *on* $[0, T_0)$ *such that* $\mathcal{H}(\eta_0, \Phi_0) < \delta_0$ *and* $\mathcal{H}_1(\eta_0, \Phi_0) > 0$. *Then for* $t \in [0, T_0)$ *we have that* $\mathcal{H}(\eta(t), \Phi(t)) < \delta_0$, $\mathcal{H}_1(\eta(t), \Phi(t)) > 0$ *and*

$$e(t) = \sup_{r \in [0,t]} \mathcal{E}(\eta(r), \Phi(r)) < \frac{2(p+2)}{p} \delta_0.$$

Proof First we observe that the Hamiltonian $\mathcal{H}$ is conserved in time on solutions. In fact, after integration by parts, we obtain that

$$\begin{aligned} \frac{d}{dt} \mathcal{H}(\eta(t), \Phi(t)) &= \int_{\mathbb{R}} \left(\left(\eta - d\mu \partial_x^2 \eta + \frac{\epsilon}{p+1} (\partial_x \Phi)^{p+1} \right) \eta_t \right. \\ &\quad \left. + (-\partial_x^2 \Phi + b\mu \partial_x^4 \Phi - \epsilon \partial_x [\eta\, (\partial_x \Phi)^p]) \Phi_t \right) dx \\ &= \int_{\mathbb{R}} ((I - a\mu \partial_x^2) \eta_t \Phi_t - (I - a\mu \partial_x^2) \Phi_t \eta_t)\, dx \\ &= 0, \end{aligned}$$

since the operator $I - a\mu\partial_x^2$ is self adjoint on $L^2(\mathbb{R})$. In other words, we have on classical solutions that

$$\mathcal{H}(\eta(t), \Phi(t)) = \mathcal{H}(\eta_0, \Phi_0) < \delta_0, \tag{21}$$

as long as the solution exist for $0 \leq t < T_0$.

Now, assume that there is $t_2 \in (0, T_0)$ such that $\mathcal{H}_1(\eta(t_2), \Phi(t_2)) < 0$, then by continuity, there is $0 < t_1 < t_2$ such that

$$\mathcal{H}_1(\eta(t_1), \Phi(t_1)) = 0, \quad (\eta(t_1), \nabla\Phi(t_1)) \neq 0. \tag{22}$$

Then, from (20), we have that

$$0 < \mathcal{E}(\eta(t_1), \Phi(t_1)) = \frac{2(p+2)}{p}\mathcal{H}(\eta(t_1), \Phi(t_1)) - \frac{2}{p}\mathcal{H}_1(\eta(t_1), \Phi(t_1)) < \frac{2(p+2)}{p}\delta_0. \tag{23}$$

But from the Sobolev type inequality (18) we conclude that

$$\begin{aligned} |G(\eta(t_1), \quad \Phi(t_1))| &\leq K_p^{\frac{p+2}{2}} \left[\mathcal{E}(\eta(t_1), \Phi(t_1))\right]^{\frac{p}{2}} \mathcal{E}(\eta(t_1), \Phi(t_1)) \\ &< K_p^{\frac{p+2}{2}} \left[\frac{2(p+2)}{p}\delta_0\right]^{\frac{p}{2}} \mathcal{E}(\eta(t_1), \Phi(t_1)) \\ &< \left(\frac{2}{p+2}\right)\mathcal{E}(\eta(t_1), \Phi(t_1)), \end{aligned}$$

which implies, by using (19), that we already have $\mathcal{H}_1(\eta(t_1), \Phi(t_1)) > 0$, but this is a contradiction. In other words, we have shown that $\mathcal{H}_1(\eta(t), \Phi(t)) > 0$, since the case $\mathcal{H}_1(\eta(t_2), \Phi(t_2)) = 0$ also provides a contradiction.

Now, as a consequence of invariance of the Hamiltonian given by (21), we have for $t \in [0, T_0)$ and the Sobolev type inequality (18) that

$$\mathcal{E}(\eta(t), \Phi(t)) \leq 2\mathcal{H}(\eta(0), \Phi(0)) + |G(\eta(t), \Phi(t))| \leq 2\delta_0 + K_p^{\frac{p+2}{2}} (\mathcal{E}(\eta(t), \Phi(t)))^{\frac{p+2}{2}}. \tag{24}$$

Then from this inequality we conclude that

$$e(t) \leq 2\delta_0 + K_p^{\frac{p+2}{2}} (e(t))^{\frac{p+2}{2}}.$$

Now consider the function f defined for $x > 0$ as $f(x) = x - K_p^{\frac{p+2}{2}} x^{\frac{p+2}{2}} - 2\delta_0$. Note that $f(0) = -2\delta_0 < 0$ and that there is a unique $x_0 > 0$ such that $f'(x_0) = 0$. In fact,

$$f'(x_0) = 0 \Leftrightarrow x_0 = \left(\frac{2}{p+2}\right)^{\frac{2}{p}} K_p^{-\frac{p+2}{p}} = \frac{2(p+2)}{p}\delta_0,$$

and so, we also have that $f(x_0) = 0$ and that $f(x) < 0$ for $x \neq x_0$. We want to show that in fact $e(t) < x_0$ for $t \in [0, T_0)$. So, assume that for some $0 < t_1 < T_0$ we have that $\mathcal{E}(\eta(t_1), \Phi(t_1)) \geq x_0$. Then from Eq. (20) we have that

$$\begin{aligned} \delta_0 > \mathcal{H}(\eta(t_1), \Phi(t_1)) &= \frac{p}{2(p+2)}\mathcal{E}(\eta(t_1), \Phi(t_1)) + \frac{1}{p+2}\mathcal{H}_1(\eta(t_1), \Phi(t_1)) \\ &\geq \delta_0 + \frac{1}{p+2}\mathcal{H}_1(\eta(t_1), \Phi(t_1)), \end{aligned}$$

meaning that $\mathcal{H}_1(\eta(t_1), \Phi(t_1)) \leq 0$, but we have that $\mathcal{H}_1(\eta(t), \Phi(t)) > 0$ para $t \in [0, T_0)$. In other words, $\mathcal{E}(\eta(t), \Phi(t)) < x_0$, and so $e(t) \leq x_0$, for $t \in [0, T_0)$ as claimed.

The proof of the existence of global solutions for the system (1) is based on the Lemma 3.1.

Theorem 3.1 *Assume $a = c \geq 0$ and $p \geq 1$. Let $(\eta_0, \Phi_0) \in H^1(\mathbb{R}) \times \mathcal{V}^2$ be such that $\mathcal{H}(\eta_0, \Phi_0) < \delta_0$ and $\mathcal{H}_1(\eta_0, \Phi_0) > 0$. Then there exists a unique global solution $(\eta, \Phi) \in C([0, \infty), H^1(\mathbb{R}) \times \mathcal{V}^2)$ of the Boussinesq system (1) satisfying the initial condition*

$$(\eta(0, \cdot), \Phi(0, \cdot)) = (\eta_0, \Phi_0).$$

Proof First we assume $a = c > 0$. Then, if $(\eta_0, \Phi_0) \in H^1(\mathbb{R}) \times \mathcal{V}^2$, by the local existence result, there is a maximal existence time $T_0 > 0$ and a unique solution $(\eta, \Phi) \in C([0, T_0), H^1(\mathbb{R}) \times \mathcal{V}^2)$ of the Cauchy problem associated with the system (1) with initial condition $(\eta(0, \cdot), \Phi(0, \cdot)) = (\eta_0, \Phi_0)$. From the conservation in time of the Hamiltonian and the hypothesis we see that

$$\mathcal{H}(\eta(t), \quad \Phi(t)) = \frac{p}{2(p+2)}\mathcal{E}\,(\eta(t), \quad \Phi(t)) + \frac{1}{p+2}\mathcal{H}_1(\eta(t), \Phi(t)) = \mathcal{H}(\eta_0, \Phi_0) < \delta_0.$$

Hence, using Lemma 3.1 we have that $\mathcal{H}_1(\eta(t), \Phi(t)) > 0$. Then we also have that

$$\mathcal{E}(\eta(t), \quad \Phi(t)) \leq \frac{2(p+2)}{p}\mathcal{H}(\eta_0, \Phi_0) < \frac{2(p+2)}{p}\delta_0.$$

But from (15) we obtain that for $t \in [0, T_0)$,

$$\|(\eta(t), \Phi(t))\|^2_{H^1 \times \mathcal{V}^2} \leq K(b, d, \mu)\mathcal{E}(\eta(t), \Phi(t)) < \frac{2(p+2)}{p}K(b, d, \mu)\delta_0.$$

This fact implies that the solution (η, Φ) is bounded in time on the space $H^1(\mathbb{R}) \times \mathcal{V}^2$ and that for any finite $T_0 < \infty$ we are able to conclude that

$$\lim_{t \to T_0^-} \|(\eta(t), \Phi(t))\|^2_{H^1 \times \mathcal{V}^2} < \infty.$$

In other words, we have that (η, Φ) can be extended in time.

Now, we assume $a = c = 0$. Let $s_0 > \frac{3}{2}$ be fixed, then by density there exists $(\eta_{0,k}\Phi_{0,k}) \in H^{s_0} \times \mathcal{V}^{s_0+1}$ such that

$$(\eta_{0,k}, \Phi_{0,k}) \to (\eta_0, \Phi_0) \quad \text{in } H^1 \times \mathcal{V}^2, \quad \text{as } k \to \infty.$$

From the local existence result, for each $k \in \mathbb{Z}^+$ there is $T_{0,k} > 0$ and a unique solution (η_k, Φ_k) of the Cauchy problem for the Boussinesq system (1) with initial condition $(\eta_k(0, \cdot), \Phi_k(0, \cdot)) = (\eta_{0,k}, \Phi_{0,k})$. On the other hand, there exists $k_0 \in \mathbb{Z}^+$ such that $\mathcal{H}(\eta_{0,k}, \Phi_{0,k}) < \delta_0$ and $\mathcal{H}_1(\eta_{0,k}, \Phi_{0,k}) > 0$ for $k \geq k_0$. Now, for $k \geq k_0$ we have that

$$\mathcal{H}(\eta_k, \Phi_k) = \frac{p}{2(p+2)}\mathcal{E}\,(\eta_k, \Phi_k) + \frac{1}{p+2}\mathcal{H}_1(\eta_k, \Phi_k) = \mathcal{H}(\eta_{0,k}, \Phi_{0,k}) < \delta_0.$$

From Lemma 3.1 we have that $\mathcal{H}_1(\eta_k, \Phi_k) > 0$ for $k \geq k_0$. Then we also have that

$$\mathcal{E}(\eta_k, \Phi_k) \leq \frac{2(p+2)}{p}\mathcal{H}(\eta_{0,k}, \Phi_{0,k}) < \frac{2(p+2)}{p}\delta_0.$$

But from (15) we obtain for $k \geq k_0$ and $t \in [0, T_0)$ that

$$\|(\eta_k, \Phi_k)\|^2_{H^1 \times \mathcal{V}^2} \leq K\mathcal{E}(\eta_k, \Phi_k) < \frac{2(p+2)}{p}K\delta_0.$$

This fact implies that $\{(\eta_k, \Phi_k)\}_k$ is bounded sequence in the space $H^1(\mathbb{R}) \times \mathcal{V}^2$ and that for any finite $T_0 < \infty$ and $k \geq k_0$ we are able to conclude that

$$\lim_{t \to T_0^-} \|(\eta_k, \Phi_k)\|^2_{H^1 \times \mathcal{V}^2} < \infty.$$

In other words, for $k \geq k_0$ we have that (η_k, Φ_k) can be extended in time. Since $\{(\eta_k, \Phi_k)\}_k$ is bounded sequence in $H^1(\mathbb{R}) \times \mathcal{V}^2$, then there is a subsequence, denoted the same, and $(\eta, \Phi) \in H^1(\mathbb{R}) \times \mathcal{V}^2$ such that

$$(\eta_k, \Phi_k) \rightharpoonup (\eta, \Phi) \quad \text{(weakly) in } H^1(\mathbb{R}) \times \mathcal{V}^2, \quad \text{as } k \to \infty.$$

It is no hard to prove that $(\eta, \Phi) \in C([0, \infty),\ H^1(\mathbb{R}) \times \mathcal{V}^2)$ is a weak solution of the Cauchy problem for the system (1) satisfying $(\eta(0, \cdot), \Phi(0, \cdot)) = (\eta_0, \Phi_0)$.

As a consequence of the previous result, we are able to establish that the Cauchy problem associated with the Boussinesq system (1) has global solution in time for initial data $(\eta_0, \Phi_0) \in H^1(\mathbb{R}) \times \mathcal{V}^2$ small enough such that $(\eta_0, \Phi_0) \neq 0$.

Theorem 3.2 *Let $p \geq 1$. Then there exists $\delta > 0$ such that for any $(\eta_0, \Phi_0) \in H^1(\mathbb{R}) \times \mathcal{V}^2$ with $\|(\eta_0, \Phi_0)\|_{H^1 \times \mathcal{V}^2} \leq \delta$, the Cauchy problem (1)–(2) has a unique global solution*

$$(\eta, \Phi) \in C([0, \infty), H^1(\mathbb{R}) \times \mathcal{V}^2) \cap C^1([0, \infty), L^2(\mathbb{R}) \times H^1(\mathbb{R})).$$

Proof If $G(\eta_0, \Phi_0) \geq 0$, then using (19) we have directly that

$$\mathcal{H}_1(\eta_0, \Phi_0) = \mathcal{E}(\eta_0, \Phi_0) + \frac{p+2}{2} G(\eta_0, \Phi_0) > 0.$$

Now, If $G(\eta_0, \Phi_0) < 0$, then we see from (15) that

$$\begin{aligned} \mathcal{H}_1(\eta_0, \Phi_0) &= \mathcal{E}(\eta_0, \Phi_0) + \frac{p+2}{2} G(\eta_0, \Phi_0) \\ &\geq K^{-1}(b, d, \mu) \left(\|(\eta_0, \Phi_0)\|^2_{H^1 \times \mathcal{V}^2} + \frac{p+2}{2} K(b, d, \mu) G(\eta_0, \Phi_0) \right). \end{aligned}$$

Thus, for $\|(\eta_0, \Phi_0)\|^2_{H^1 \times \mathcal{V}^2}$ sufficiently small we would have $\mathcal{H}_1(\eta_0, \Phi_0) > 0$, since

$$G(\eta_0, \Phi_0) = O(\mathcal{E}(\eta_0, \Phi_0)^{\frac{p+2}{2}}) = O(\|(\eta_0, \Phi_0)\|^{p+2}_{H^1 \times \mathcal{V}^2}).$$

From (15), (14) and (18) we see that there exists $K_1(b, d, \mu, \epsilon, p) > 0$ such that

$$\mathcal{H}(\eta_0, \Phi_0) \leq K_1(b, d, \mu, \epsilon, p)(1 + \|(\eta_0, \Phi_0)\|^p_{H^1 \times \mathcal{V}^2}) \|(\eta_0, \Phi_0)\|^2_{H^1 \times \mathcal{V}^2},$$

and from (15) we have that

$$\mathcal{E}(\eta_0, \Phi_0) \leq K(b, d, \mu) \|(\eta_0, \Phi_0)\|^2_{H^1 \times \mathcal{V}^2}.$$

Hence, we choose $\delta > 0$ in a such way that

$$K_1(b, d, \mu, \epsilon, p)(1 + \delta^p)\delta^2 < \delta_0 \quad \text{and} \quad K(b, d, \mu)\delta^2 < \frac{2(p+2)}{p} \delta_0.$$

Let $(\eta_0, \Phi_0) \in H^1 \times \mathcal{V}^2$ be such that $\|(\eta_0, \Phi_0)\|_{H^1 \times \mathcal{V}^2} \leq \delta$, then we see that $\mathcal{H}(\eta_0, \Phi_0) < \delta_0$. Moreover, from the Sobolev type inequality (18) we obtain that

$$\begin{aligned} |G(\eta_0, \Phi_0)| &\leq K_p^{\frac{p+2}{2}} (\mathcal{E}(\eta_0, \Phi_0))^{\frac{p}{2}} \mathcal{E}(\eta_0, \Phi_0) \\ &< K_p^{\frac{p+2}{2}} \left(\frac{2(p+2)}{p} \delta_0 \right)^{\frac{p}{2}} \mathcal{E}(\eta_0, \Phi_0) \\ &< \frac{p+2}{2} \mathcal{E}(\eta_0, \Phi_0). \end{aligned}$$

Then from (19) we have that $\mathcal{H}_1(\eta_0, \Phi_0) > 0$ and the conclusion follows from the previous lemma.

Existence of Solitons

In this section we will establish the existence of finite energy travelling wave solutions or solitons for the 1D-Boussinesq system with $a = c \geq 0, b, d > 0$ and wave speed ω satisfying $0 < |\omega| < \omega_0$, where $\omega_0 = \min\left\{1, \frac{d}{a}, \frac{b}{a}\right\}$ for $a \neq 0$ and $\omega_0 = 1$ for $a = 0$. We will see that the solitary waves are characterized as critical points of some functional, for which the existence of critical points follows as a consequence of the mountain pass theorem without the Palais–Smale condition and the existence of a local compact embedding result (see Lemma 4.1).

By a solitary wave solution we shall mean a solution (η, Φ) of (1) of the form

$$\eta(t,x) = \frac{1}{\epsilon} u\left(\frac{x-\omega t}{\sqrt{\mu}}\right), \quad \Phi(t,x) = \frac{\mu}{\epsilon} v\left(\frac{x-\omega t}{\sqrt{\mu}}\right).$$

Then we have that the travelling wave profile (u, v) should satisfy the system

$$\begin{cases} bv'''' - v'' + \omega(u' - au''') - [u(v')^p]' = 0, \\ u - du'' - \omega(v' - cv''') + \frac{1}{p+1}(v')^{p+1} = 0. \end{cases} \tag{25}$$

Next, we define the appropriate spaces. The usual space $H^1(U),\ U \subset \mathbb{R}$, is the Hilbert space defined as the closure of $C^\infty(U)$ with respect to the norm

$$\|\phi\|^2_{H^1(U)} = \int_U (\phi^2 + (\phi')^2) dx.$$

We denote by $\mathcal{V}$ the closure of $C_0^\infty(\mathbb{R})$ with respect to the norm given by

$$\|\psi\|^2_{\mathcal{V}} = \int_{\mathbb{R}} ((\psi')^2 + (\psi'')^2) dx = \|\psi'\|^2_{H^1(\mathbb{R})}.$$

Note that $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is a Hilbert space with inner product

$$(\phi, \psi)_{\mathcal{V}} = (\phi', \psi')_{H^1(\mathbb{R})}.$$

Also we define the Hilbert space $\mathcal{X} = H^1(\mathbb{R}) \times \mathcal{V}$ with respect to norm

$$\|(\phi, \psi)\|^2_{\mathcal{X}} = \|\phi\|^2_{H^1(\mathbb{R})} + \|\psi\|^2_{\mathcal{V}}.$$

The existence of solitons for the system (1) is a consequence of a variational approach which apply a minimax type result, since solutions (u, v) of the system (25) are critical points of the functional J_ω given by

$$J_\omega = I_\omega(u, v) + G(u, v),$$

where the functionals I_ω and G are defined on the space $\mathcal{X}$ by

$$\begin{aligned} I_\omega(u, v) &= \int_{\mathbb{R}} [u^2 + d(u')^2 + (v')^2 + b(v'')^2 - 2\omega uv' - \omega\,(a+c)\,u'v'']dx \\ &= \int_{\mathbb{R}} [u^2 + d(u')^2 + (v')^2 + b(v'')^2 - 2\omega(uv' + au'v'')]dx, \\ G(u, v) &= \frac{2}{p+1} \int_{\mathbb{R}} u(v')^{p+1} dx. \end{aligned}$$

First we have that $I_\omega, G, J_\omega \in C^2(\mathcal{X}, \mathbb{R})$ and its derivatives in (u, v) in the direction of (z, w) are given by

$$\langle I'_\omega(u, v), (z, w)\rangle = 2\int_{\mathbb{R}} (uz + du'z' + v'w' + bv''w'')dx$$
$$- 2\omega \int_{\mathbb{R}} (uw' + v'z + a(u'w'' + v''z'))dx$$
$$\langle G'(u, v), (z, w)\rangle = \frac{2}{p+1}\int_{\mathbb{R}} ((v')^{p+1}z + (p+1)u(v')^p w')dx.$$

As a consequence of this, after integration by parts, we conclude that

$$J'_\omega(u, v) = 2\begin{pmatrix} u - du'' - \omega(v' - av''') + \frac{1}{p+1}(v')^{p+1} \\ bv'''' - v'' + \omega(u' - au''') - [u(v')^p]' \end{pmatrix},$$

meaning that critical points of the functional J_ω satisfy the travelling wave Eq. (25). Hereafter, we will say that weak solutions for (25) are critical points of the functional J_ω. In particular, we have that

$$\left\langle J'_\omega(u, v), (u, v)\right\rangle = 2I_\omega(u, v) + (p+2)G(u, v)$$
$$= 2J_\omega(u, v) + pG(u, v). \tag{26}$$

Thus on any critical point (u, v), we have that

$$J_\omega(u, v) = \frac{p}{p+2} I_\omega(u, v), \tag{27}$$
$$J_\omega(u, v) = -\frac{p}{2} G(u, v), \tag{28}$$
$$I_\omega(u, v) = -\frac{p+2}{2} G(u, v). \tag{29}$$

One can see easily that the functional G is well-defined on $\mathcal{X}$. Note that $u, v' \in H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for all $q \geq 2$, therefore by applying Young's inequality we obtain that

$$|G(u, v)| \leq K(\|u\|^{p+2}_{L^{p+2}(\mathbb{R})} + \|v'\|^{p+2}_{L^{p+2}(\mathbb{R})}) \leq K\|(u, v)\|^{p+2}_{\mathcal{X}}. \tag{30}$$

Moreover, for $0 < |\omega| < \omega_0$ there are some positive constants $K_1(a, b, d, \omega) < K_2(a, b, d, \omega)$ such that

$$K_1\|(u, v)\|^2_{\mathcal{X}} \leq I_\omega(u, v) \leq K_2\|(u, v)\|^2_{\mathcal{X}}. \tag{31}$$

In fact, using the definition of I_ω and Young inequality we obtain that

$$I_\omega(u, v) \leq \int_{\mathbb{R}} [(1 + |\omega|)u^2 + (1 + |\omega|)(v')^2 + (b + |\omega|a)(v'')^2 + (d + |\omega|a)(u')^2]dx$$
$$\leq \max(1 + |\omega|, b + |\omega|a, d + |\omega a|)\,\|(u, v)\|^2_{\mathcal{X}}.$$

Additionally,

$$I_\omega(u, v) \geq \int_{\mathbb{R}} [(1 - |\omega|)u^2 + (1 - |\omega|)(v')^2 + (b - |\omega|a)(v'')^2 + (d - |\omega|a)(u')^2]dx$$
$$\geq \min(1 - |\omega|, b - |\omega|a, d - |\omega|a)\,\|(u, v)\|^2_{\mathcal{X}},$$

showing that the inequality (31) holds.

Our approach to show the existence of a non trivial critical point for J_ω is to use the mountain pass theorem without the Palais–Smale condition (see Ambrosetti et. al. [1], Willem [15]) to build a Palais–Smale sequence for J_ω for a minimax value and use a local embedding result to obtain a critical point for J_ω as a weak limit of such Palais–Smale sequence.

Theorem 4.1 *Let X be a Hilbert space, $\varphi \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\|_X > r$ and*

$$\beta = \inf_{\|u\|_X = r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

Then, given $n \in \mathbb{N}$, there is $u_n \in X$ such that

$$\varphi(u_n) \to \delta, \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{in } X', \tag{PS}$$

where

$$\delta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1], X) \colon \gamma(0) = 0, \ \ \gamma(1) = e\}.$$

Before we go further, we establish an important result for our analysis, which is related with the characterization of "vanishing sequences" in $\mathcal{X}$. Define ϱ on $\mathcal{X}$ as

$$\varrho(u, v) = u^2 + (v')^2,$$

and for $\zeta \in \mathbb{R}$ and $r > 0$ we will denote by $B_r(\zeta)$ the ball in $\mathbb{R}$ of center ζ and radius r.

Theorem 4.2 *Let $q \geq 2$. If $\{(u_n, v_n)\}_n$ is a bounded sequence in $\mathcal{X}$ and there is a positive constant $r > 0$ such that*

$$\lim_{n\to\infty} \sup_{\zeta \in \mathbb{R}} \int_{B_r(\zeta)} \varrho(u_n, v_n)\, dx = 0. \tag{32}$$

Then we have that

$$\lim_{n\to\infty} \|v_n\|_{\mathcal{M}^q(\mathbb{R})} = \lim_{n\to\infty} \|u_n\|_{L^q(\mathbb{R})} = 0.$$

Proof First suppose that $\{w_n\}_n$ is a bounded sequence in $H^1(\mathbb{R})$ and assume there is a positive constant $r > 0$ such that

$$\lim_{n\to\infty} \sup_{\zeta \in \mathbb{R}} \int_{B_r(\zeta)} w_n^2\, dx = 0. \tag{33}$$

We will see that $\lim_{n\to\infty} \|w_n\|_{L^q(\mathbb{R})} = 0$. In fact, let $\{w_n\}_n$ be a bounded sequence in $H^1(\mathbb{R})$ satisfying the limit (33). Then we have for $q \geq 2$ that

$$\|w_n\|^q_{L^q(B_r(\zeta))} \leq \|w_n\|_{L^2(B_r(\zeta))} \|w_n\|^{q-1}_{L^{2(q-1)}(B_r(\zeta))} \leq \|w_n\|_{L^2(B_r(\zeta))} \|w_n\|^{q-1}_{H^1(\mathbb{R})}.$$

Covering $\mathbb{R}$ by a countable number of balls of radius r in a such way that every point in $\mathbb{R}$ is contained in at most two balls $B_r(\zeta)$, we obtain that

$$\|w_n\|^q_{L^q(\mathbb{R})} \leq 2 \sup_{\zeta \in \mathbb{R}} \|w_n\|_{L^2(B_r(\zeta))} \|w_n\|^{q-1}_{H^1(\mathbb{R})}.$$

We conclude using the hypothesis and that $\{w_n\}_n$ is a bounded sequence in $H^1(\mathbb{R})$ that

$$\lim_{n\to\infty} \|w_n\|_{L^q(\mathbb{R})} = 0.$$

Now suppose that $\{(u_n, v_n)\}_n$ is a bounded sequence in $\mathcal{X}$ and that it satisfies (32). Then $u_n, v_n' \in H^1(\mathbb{R})$. Hence, for w_n being defined as either u_n or v_n' we see that w_n satisfies in each case the condition (33). By the previous observation, we conclude for $q \geq 2$ that $\lim_{n\to\infty} \|w_n\|_{L^q(\mathbb{R})} = 0$. In other words, we have for $q \geq 2$ that

$$\lim_{n\to\infty} \|u_n\|_{L^q(\mathbb{R})} = \lim_{n\to\infty} \|v\|_{\mathcal{M}^q(\mathbb{R})} = 0.$$

Now, we want to verify the mountain pass theorem hypotheses given in Theorem 4.1 and to build a Palais–Smale sequence for J_ω.

Theorem 4.3 *Let* $0 < |\omega| < \omega_0$. *Then*

(1) *There exists* $\rho > 0$ *small enough such that* $\beta(\omega) \colon = \inf_{\|z\|_{\mathcal{X}}=\rho} J_\omega(z) > 0$.
(2) *There is* $e \in \mathcal{X}$ *with* $\|e\|_{\mathcal{X}} \geq \rho$ *such that* $J_\omega(e) \leq 0$.
(3) *If* $\delta(\omega)$ *is defined as*

$$\delta(\omega) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\omega(\gamma(t)), \quad \Gamma = \{\gamma \in C([0,1], \mathcal{X}) |\, \gamma(0) = 0,\ \gamma(1) = e\},$$

then $\delta(\omega) \geq \beta(\omega)$ *and there is a sequence* $(U_n)_n \in \mathcal{X}$ *such that*

$$J_\omega(U_n) \to \delta, \quad J'_\omega(U_n) \to 0 \ \text{ in } \mathcal{X}'.$$

Proof From inequalities (30)–(31), we have for any $(u, v) \in \mathcal{X}$ that

$$\begin{aligned} J_\omega(u, v) &\geq K_1\|(u, v)\|^2_{\mathcal{X}} - K_2\|(u, v)\|^{p+2}_{\mathcal{X}} \\ &\geq (K_1 - K_2\|(u, v)\|^p_{\mathcal{X}})\|(u, v)\|^2_{\mathcal{X}}. \end{aligned}$$

Then for $\rho > 0$ small enough such that

$$K_1 - \rho^p K_2 > 0,$$

we conclude for $\rho = \|(u, v)\|_{\mathcal{X}}$ that

$$J_\omega(u, v) \geq (K_1 - \rho^p K_2)\rho^2 := \alpha > 0.$$

In particular, we also have that

$$\beta(\omega) = \inf_{\|z\|_{\mathcal{X}}=\rho} J_\omega(z) \geq \alpha > 0. \tag{34}$$

Now, it is not hard to prove that there exist $u_0, v_0 \in C_0^\infty(\mathbb{R})$ such that $G(u_0, v_0) < 0$. Then for any $t \in \mathbb{R}$ we have that

$$\begin{aligned} J_\omega(tu_0, tv_0) &= t^2 I_\omega(u_0, v_0) + t^{p+2}G(u_0, v_0) \\ &= t^2(I_\omega(u_0, v_0) + t^p G(u_0, v_0)). \end{aligned}$$

As a consequence of this, we have that

$$\lim_{t\to\infty} J_\omega(tu_0, tv_0) = -\infty.$$

So, there is $t_0 > 0$ such that $e = t_0(u_0, v_0) \in \mathcal{X}$ satisfies that $t_0\|(u_0, v_0)\|_{\mathcal{X}} = \|e\|_{\mathcal{X}} > \rho$ and that $J_\omega(e) \leq J_\omega(0) = 0$. The third part follows by applying Theorem 4.1.

Now we are in position to establish the main result in this section, in which we use the existence of a local embedding result obtained by J. Quintero in the case two dimensional (see [11]). First of all, we know for $q \geq 2$ that the embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous and the embedding $H^1(\mathbb{R}) \hookrightarrow L^q_{loc}(\mathbb{R})$ is compact. Now, if we set for $q \geq 2$ and $Q \subset \mathbb{R}$ the Banach space

$$\mathcal{M}^q(Q) : \overline{\langle C_0^\infty(Q), \|\cdot\|_{(q)} \rangle}, \quad \|\psi\|^q_{(q)} = \|\psi'\|^q_{L^q(Q)}.$$

Then the following embedding result holds (see [11]).

Lemma 4.1 *For $q \geq 2$ we have that*

(1) *The embedding $\mathcal{V} \hookrightarrow \mathcal{M}^q(\mathbb{R})$ is continuous and the embedding $\mathcal{V} \hookrightarrow \mathcal{M}^q_{loc}(\mathbb{R})$ is compact.*
(2) *The embedding $\mathcal{X} \hookrightarrow L^q(\mathbb{R}) \times \mathcal{M}^q(\mathbb{R})$ is continuous and the embedding $\mathcal{X} \hookrightarrow L^q_{loc}(\mathbb{R}) \times \mathcal{M}^q_{loc}(\mathbb{R})$ is compact.*

Using previous local embedding, we have the following existence result.

Theorem 4.4 *Let $0 < |\omega| < \omega_0$. Then the system (25) has a nontrivial solution in $\mathcal{X}$.*

Proof We will see that $\delta(\omega)$ is in fact a critical value of J_ω. Let $\{(u_n, v_n)\} \subset \mathcal{X}$ be the sequence given by previous lemma. First note from (34) that $\delta(\omega) \geq \beta(\omega) \geq \alpha$. Using the definition of J_ω and (26) we have that

$$I_\omega(u_n, v_n) = \frac{p+2}{p} J_\omega(u_n, v_n) - \frac{1}{p}\langle J'_\omega(u_n, v_n), (u_n, v_n)\rangle.$$

But from (31) we conclude for n large enough that

$$K_1\|(u_n, v_n)\|^2_{\mathcal{X}} \leq I_\omega(u_n, v_n) \leq \frac{p+2}{p}(\delta(\omega) + 1) + \|(u_n, v_n)\|_{\mathcal{X}}.$$

Then we have shown that $\{(u_n, v_n)\}_n$ is a bounded sequence in $\mathcal{X}$. We claim that

$$\alpha^* = \overline{\lim_{n\to\infty}} \sup_{\zeta\in\mathbb{R}} \int_{B_1(\zeta)} \varrho(u_n, v_n)\, dx > 0.$$

If we suppose that

$$\overline{\lim_{n\to\infty}} \sup_{\zeta\in\mathbb{R}} \int_{B_1(\zeta)} \varrho(u_n, v_n)\, dx = 0.$$

Hence from Lemma 4.2 we conclude for $q \geq 2$ that

$$\lim_{n\to\infty} \|u_n\|_{L^q(\mathbb{R})} = 0, \quad \lim_{n\to\infty} \|v\|_{\mathcal{M}^q(\mathbb{R})} = 0.$$

Now, we have from (34), (26) and (30) that

$$\begin{aligned} 0 < \alpha \leq \delta(\omega) &= J_\omega(u_n, v_n) - \frac{1}{2}\langle J'_\omega(u_n, v_n), (u_n, v_n)\rangle + o(1) \\ &= \frac{p}{2} G(u_n, v_n) + o(1) \\ &\leq C[\|u_n\|^{p+2}_{L^{p+2}(\mathbb{R})} + \|v_n\|^{p+2}_{\mathcal{M}^{p+2}(\mathbb{R})}] + o(1) \\ &\leq o(1). \end{aligned}$$

But this is a contradiction. Thus, there is a subsequence of $\{(u_n, v_n)\}_n$, denoted the same, and a sequence $\zeta_n \in \mathbb{R}$ such that

$$\int_{B_1(\zeta_n)} \varrho(u_n, v_n)\, dx \geq \frac{\alpha^*}{2}. \tag{35}$$

Now we define the sequence $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x+\zeta_n), v_n(x+\zeta_n))$. For this sequence we also have that

$$\|(\tilde{u}_n, \tilde{v}_n)\|_{\mathcal{X}} = \|(u_n, v_n)\|_{\mathcal{X}}, \quad J_\omega(\tilde{u}_n, \tilde{v}_n) \to d, \quad J'_\omega(\tilde{u}_n, \tilde{v}_n) \to 0 \ \text{ in } \mathcal{X}'.$$

Then $\{(\tilde{u}_n, \tilde{v}_n)\}_n$ is a bounded sequence in $\mathcal{X}$. Thus, for some subsequence of $\{(\tilde{u}_n, \tilde{v}_n)\}_n$, denoted the same, and for some $(u, v) \in \mathcal{X}$ we have that

$$(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v), \quad \text{as } n \to \infty \quad (\text{weakly in } \mathcal{X}).$$

Since the embedding $\mathcal{X} \hookrightarrow L^q_{loc}(\mathbb{R}) \times \mathcal{M}^q_{loc}(\mathbb{R})$ is locally compact for $q \geq 2$ we see that

$$(\tilde{u}_n, \tilde{v}_n) \to (u, v) \quad \text{in } L^q_{loc}(\mathbb{R}) \times \mathcal{M}^q_{loc}(\mathbb{R}).$$

Then $(u, v) \neq 0$ because using (35) we have that

$$\int_{B_1(0)} \varrho(u, v)\, dxdy = \lim_{n\to\infty} \int_{B_1(0)} \varrho(\tilde{u}_n, \tilde{v}_n)\, dxdy \geq \frac{\alpha^*}{2}.$$

Moreover, if $(z, w) \in C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$ with $supp\, z, w \subset \Omega$ we have that

$$\begin{aligned}\langle I'_\omega(u, v), (z, w)\rangle &= 2\int_\Omega (uz + du'z' + v'w' + bv''w'')dx \\ &\quad - 2\omega \int_\Omega (uw' + v'z + a(u'w'' + v''z'))dx \\ &= 2\lim_{n\to\infty} \int_\Omega (\tilde{u}_n z + d\tilde{u}'_n z' + \tilde{v}'_n w' + b\tilde{v}''_n w'')dx \\ &\quad - 2\omega \lim_{n\to\infty} \int_\Omega (\tilde{u}_n w' + \tilde{v}'_n z + a(\tilde{u}'_n w'' + \tilde{v}''_n z'))dx \\ &= \lim_{n\to\infty} \langle I'_\omega(\tilde{u}_n, \tilde{v}_n), (z, w)\rangle.\end{aligned}$$

Now noting that the sequences $\{(\tilde{v}'_n)^{p+1}\}_n$ and $\{\tilde{u}_n \left(\tilde{v}'_n\right)^p\}_n$ are bounded in $L^2(\mathbb{R})$, then (taking a subsequence, if necessary), we have that

$$(\tilde{v}'_n)^{p+1} \rightharpoonup (v')^{p+1}, \quad \tilde{u}_n \left(\tilde{v}'_n\right)^p \rightharpoonup u \left(v'\right)^p \quad \text{in } L^2(\mathbb{R}).$$

As a consequence of this, we have that

$$\int_\Omega \left(\tilde{v}'_n\right)^{p+1} z dx \to \int_\Omega \left(v'\right)^{p+1} z\, dx, \quad \int_\Omega \tilde{u}_n \left(\tilde{v}'_n\right)^p w' dx \to \int_\Omega u \left(v'\right)^p w' dx.$$

In other words, we have shown that

$$\left\langle G'(u, v), (z, w)\right\rangle = \lim_{n\to\infty} \langle G'(\tilde{u}_n, \tilde{v}_n), (z, w)\rangle$$

and also that

$$\langle J'_\omega(u, v), (z, w)\rangle = \lim_{n\to\infty} \langle J'_\omega(\tilde{u}_n, \tilde{v}_n), (z, w)\rangle = 0.$$

Now, let $(z, w) \in \mathcal{X}$. By density, there is $(z_k, w_k) \in C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$ such that $(z_k, w_k) \to (z, w)$ in $\mathcal{X}$. Then

$$\begin{aligned}|\langle J'_\omega(u, v), (z, w)\rangle| &\leq |\langle J'_\omega(u, v), (z - z_k, w - w_k)\rangle| + |\langle J'_c(u, v), (z_k, w_k)\rangle| \leq \|J'_\omega(u, v) \\ &\quad \times \|_{\mathcal{X}'} \|(z - z_k, w - w_k)\|_{\mathcal{X}} + |\langle J'_\omega(u, v), (z - z_k, w - w_k)\rangle| \to 0.\end{aligned}$$

Thus we have already established that $J'_\omega(u, v) = 0$. In other words, (u, v) is a nontrivial solution for the system (25).

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