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ORIGINAL RESEARCH

Positive Solutions for a System of Fractional Differential Equations with Nonlocal Integral Boundary Conditions

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Abstract In this paper, we discuss by means of a fixed point theorem, the existence of positive solutions of a system of nonlinear Caputo fractional differential equations with integral boundary conditions. An example is given to illustrate the main results.

Keywords Integral condition · Fractional Caputo derivative · Positive solution · Fixed point theorem

Mathematics Subject Classification 34B10 · 26A33 · 34B15

Introduction

Fractional calculus still gaining increasing attention even it is an old mathematical topic, this is due to their multidisciplinary applications such in viscoelasticity, electrochemistry, electromagnetism, rheology…[1]. It is shown that the mean advantages of fractional order derivatives and integrals is their ability to describe memory and hereditary properties of different materials, for more details see [1–3]. Let us note that there exists abundant literature concerning the relation between fractional derivatives and fractional powers of operators [4,5], fractional partial differential equations [6] and boundary value problems involving fractional calculus [7–13].

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The study of systems of fractional differential equations has also known an increasing interest since the behavior of many physical systems can be properly described by fractional-order systems, for example, dielectric polarization, electromagnetic waves, viscoelastic systems, diffusion waves [14]. For some recent results on systems of fractional differential equations and their applications, we refer to [14,15].

In this work, we consider the following system of fractional differential equations with integral boundary conditions:

$$(S): \begin{cases} {}^{c}D_{0+}^{q}u(t) = g(t)f(u(t)), \quad 0 < t < 1, \\ u'(0) = 0, \quad Eu(0) - Bu(1) = \int_0^1 h(u(s))\,ds, \end{cases}$$

where $f, h : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, 1] \to \mathbb{R}$ are given functions, $u : [0, 1] \to \mathbb{R}^n$ is the unknown function, ${}^{c}D_{0+}^{q}$ denotes the Caputo's fractional derivative, $1 < q < 2$. Denote $u = (u_1, u_2, \ldots, u_n)^T$,

$$f(u) = (f_1(u_1, u_2, \ldots, u_n), \ldots, f_n(u_1, u_2, \ldots, u_n))^T,$$
$$h(u) = (h_1(u_1, u_2, \ldots, u_n), \ldots, h_n(u_1, u_2, \ldots, u_n))^T.$$

Denote by E the identity matrix in $M_n(\mathbb{R})$ (The vector space of real matrices with n rows and n columns) and $B = diag(\lambda_1, \ldots \lambda_n) \in M_n(\mathbb{R})$, $0 < \lambda_i < 1$, $\forall i \in \{1, \ldots, n\}$.

Similar boundary value problem for systems of nonlinear fractional differential equations has been widely studied by many authors, but most of them obtained only some existence results without any information about the positivity of solutions, see [1,5,8,16–20].

Boundary value problem with integral boundary conditions is a mathematical model for of various phenomena of physics, ecology, biology, chemistry, etc. Integral conditions come up when values of the function on the boundary is connected to values inside the domain or when direct measurements on the boundary are not possible, see [13,21–25]

Many methods are used to investigate the existence of solutions for boundary value problems, one can cite fixed point theory, the upper and lower solution method, the variational method…We refer the reader to [7–12,22–31] for recent developments in this area. On the other hand, fixed point theory is a very powerful mathematical tool in the study of boundary value problems where the existence, uniqueness, positivity and stability knowledge are needed.

This paper is organized as follows: We state in Sect. 2 some background materials and preliminaries. Main results and their proofs are exposed in Sect. 3, which we achieved by an example illustrating the obtained results.

Background Materials and Preliminaries

We recall some basic definitions of the fractional integrals and derivatives. For more details on fractional calculus, see Kilbas et al. [2].

Definition 1 The Riemann-Liouville fractional integral of order α of a function g is defined by

$$I_{a+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}}ds,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}dt$ is the Gamma function, $\alpha > 0$.

Definition 2 Let $q \geq 0, n = [q] + 1$. If $g \in C^n[a, b]$, then the Caputo fractional derivative of order q of g defined by

$$^{c}D^{q}_{a+}g(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{q-n+1}} ds,$$

exists almost everywhere on $[a, b]$ ($[q]$ is the entire part of q).

Lemma 3 *For $q > 0$, $g \in C(0, 1)$, the homogenous fractional differential equation $^{c}D^{q}_{a+}g(t) = 0$ has a solution*

$$g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, and $n = [q] + 1$.

Lemma 4 *Let $p, q \geq 0$, $f \in L_1[a, b]$. Then $I^p_{0+} I^q_{0+} f(t) = I^{p+q}_{0+} f(t) = I^q_{0+} I^p_{0+} f(t)$ and $^{c}D^{q}_{a+} I^q_{0+} f(t) = f(t)$, for all $t \in [a, b]$.*

Let X be the Banach space of all functions $u \in C^n[0, 1] = C[0, 1] \times \cdots \times C[0, 1]$ with the norm $||.||$ defined by $||u|| = \sum_{i=1}^{i=n} \max_{t\in[0,1]} |u_i(t)|$.

The system (S) can be turned to the following system of n equations with nonlocal conditions:

$$(S_i) : \begin{cases} ^{c}D^{q}_{0+}u_i(t) = g(t) f_i(u(t)), & 0 < t < 1, \\ u_i'(0) = 0, \quad u_i(0) - \lambda_i u(1) = \int_0^1 h_i(u(s))\, ds, & i \in \{1, \ldots, n\}. \end{cases}$$

Lemma 5 *If $0 < \lambda_i < 1$, $\forall i \in \{1, \ldots, n\}$, then the linear nonhomogeneous problem*

$$(S_i) : \begin{cases} ^{c}D^{q}_{0+}u_i(t) = y(t), & 0 < t < 1, \\ u_i'(0) = 0, \quad u_i(0) - \lambda_i u_i(1) = \int_0^1 z(s)ds, \end{cases}$$

has the following solution

$$u_i(t) = \frac{1}{\Gamma(q)} \int_0^1 G_i(t, s)\, y(s)ds + \frac{1}{1-\lambda_i} \int_0^1 z(s)\, ds,$$

where

$$G_i(t, s) = \begin{cases} (t-s)^{q-1} + \frac{\lambda_i}{1-\lambda_i}(1-s)^{q-1}, & s \leq t, \\ \frac{\lambda_i}{1-\lambda_i}(1-s)^{q-1}, & t \leq s. \end{cases}$$

Define the integral operator $T : X \to X$ by $Tu = (T_1 u, T_2 u, \ldots, T_n u)$ where

$$T_i u(t) = \frac{1}{\Gamma(q)} \int_0^1 g(s) G_i(t, s)\, f_i(u(s))\, ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s))\, ds,$$

then the system (S) is equivalent to a system of integral equations:

$$Tu(t) = \frac{1}{\Gamma(q)} \int_0^1 g(s) G(t, s)\, f(u(s))\, ds + \int_0^1 H_\lambda(u(s))\, ds,$$

where the matrix $G(t, s) = diag(G_1(t, s), \ldots, G_n(t, s))$ and

$$\int_0^1 H_\lambda(u(s))\, ds = \left(\frac{1}{1-\lambda_1} \int_0^1 h_1(u(s))\, ds, \ldots, \frac{1}{1-\lambda_n} \int_0^1 h_n(u(s))\, ds \right)^T.$$

Lemma 6 *The function $u \in X$ is solution of the system (S) if and only if $T_i u(t) = u(t)$, for all $t \in [0,1]$, $\forall i \in \{1,\ldots,n\}$.*

Consequently, existence of solutions for the system (S) can be turned into a fixed point problem in X for the operator T.

Define a positive solution as:

Definition 7 A function u is called positive solution of the system (S) if $u_i(t) \geq 0, \forall t \in [0,1]$, $\forall i \in \{1,\ldots,n\}$ and it satisfies the boundary conditions in (S_i).

Main Results

Let us state the properties of the Green functions:

Lemma 8 *For all $t, s \in [0,1]$ and $\forall i \in \{1,\ldots,n\}$, we have the following:*

(i) $G_i(t,s) \in C([0,1] \times [0,1])$, $G_i(t,s) \geq 0$.
(ii) $\lambda_i \gamma_i(s) \leq G_i(t,s) \leq \gamma_i(s)$, *where* $\gamma_i(s) = \frac{(1-s)^{q-1}}{1-\lambda_i}$.

Now we state the assumptions that will be used to prove the existence of positive solutions:

$$\begin{aligned}
&(\mathbf{K}_1) \ : \ 0 < \lambda_i < 1,\ f_i \in C\left(\mathbb{R}_+^n, \mathbb{R}_+\right), \quad \forall i \in \{1,\ldots,n\}, \\
&\quad g \in L^1([0,1], \mathbb{R}_+), \ \int_0^1 (1-s)^{q-1} g(s) ds > 0. \\
&(\mathbf{K}_2) \ : \ h_i \in C\left(\mathbb{R}_+^n, \mathbb{R}_+\right), h_i(v_1,\ldots,v_n) \leq l(v_1 + \cdots + v_n), \\
&\quad \forall i \in \{1,\ldots,n\},\ l < \left(\sum_{i=1}^n \frac{1}{1-\lambda_i}\right)^{-1}.
\end{aligned}$$

Lemma 9 *Assume that (K_1) and (K_2) hold. Then the solution u of the system (S) satisfies*

$$\min_{t\in[0,1]} \sum_{i=1}^n u_i(t) \geq \left(\min_{1\leq i\leq n} \lambda_i\right) \|u\|.$$

Proof For all $i \in \{1,\ldots,n\}$ we have

$$u_i(t) = \frac{1}{\Gamma(q)} \int_0^1 G_i(t,s) g(s) f_i(u(s)) ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s)) ds.$$

By Lemma 8 we obtain

$$u_i(t) \leq \frac{1}{\Gamma(q)} \int_0^1 \gamma_i(s) g(s) f_i(u(s)) ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s)) ds. \tag{3.1}$$

Taking the supremum over $[0,1]$, then summing the obtained inequalities according to i from 1 to n, we get

$$\|u\| \leq \frac{1}{\Gamma(q)} \sum_{i=1}^n \left(\int_0^1 \gamma_i(s) g(s) f_i(u(s)) ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s)) ds\right). \tag{3.2}$$

On the other hand, taking into account the left hand side of inequalities in (ii) of Lemma 8 and the fact that $0 < \lambda_i < 1$, it yields

$$\begin{aligned} u_i(t) &\geq \frac{\lambda_i}{\Gamma(q)} \int_0^1 \gamma_i(s)\, g(s) f_i(u(s))\, ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s))\, ds \\ &\geq \frac{\lambda_i}{\Gamma(q)} \left(\int_0^1 \gamma_i(s)\, g(s) f_i(u(s))\, ds + \frac{1}{(1-\lambda_i)} \int_0^1 h_i(u(s))\, ds \right) \\ &\geq \left(\min_{1\leq i\leq n} \lambda_i \right) \frac{1}{\Gamma(q)} \left(\int_0^1 \gamma_i(s)\, g(s) f_i(u(s))\, ds + \frac{1}{(1-\lambda_i)} \int_0^1 h_i(u(s))\, ds \right). \end{aligned}$$

Summing the n obtained inequalities, taking the minimum over t on [0, 1], then applying (3.2) we get the desired result. The proof is complete. □

Let us introduce the following notations

$$f_0^i = \lim_{\|u\|\to 0^+} \frac{f_i(u)}{\|u\|}, \quad f_\infty^i = \lim_{\|u\|\to +\infty} \frac{f_i(u)}{\|u\|}, \quad f_0 = \sum_{i=0}^n f_0^i,\ f_\infty = \sum_{i=0}^n f_\infty^i.$$

The case $f_0 = 0$ and $f_\infty = +\infty$ is called superlinear case and the case $f_0 = +\infty$ and $f_\infty = 0$ is called sublinear case. Let K be the cone:

$$\begin{aligned} K = \Big\{ & u = (u_1, \ldots, u_2) \in X, u_i(t) \geq 0, \quad \forall t \in [0,1], \quad \forall i \in \{1,\ldots,n\}, \\ & \min_{t\in[0,1]} \sum_{i=1}^n u_i(t) \geq \left(\min_{1\leq i\leq n} \lambda_i \right) \|u\| \Big\}. \end{aligned}$$

Lemma 10 *The map $T : X \to X$ is completely continuous and $TK \subset K$.*

Proof By Ascoli-Arzela theorem we prove that T_i is completely continuous mapping for all $i = 1, \ldots, n$, then T is completely continuous mapping. To prove that $TK \subset K$, let us take $u \in K$, then

$$T_i u(t) = \frac{1}{\Gamma(q)} \int_0^1 G_i(t,s)\, g(s) f_i(u(s))\, ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s))\, ds,$$

arguing as in the proof of Lemma 9, we obtain

$$\min_{t\in[0,1]} \sum_{i=1}^n Tu(t) \geq \left(\min_{1\leq i\leq n} \lambda_i \right) \|Tu\|,$$

thus $TK \subset K$. □

Now we are ready to state the main result.

Theorem 11 *Assume that (K_1)–(K_2) hold. Then the fractional boundary value problem (S) has at least one positive solution in superlinear as well as sublinear case.*

We recall the well-known Guo-Krasnosel'skii fixed point theorem on cone.

Theorem 12 [32,33] *Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let*

$$\mathcal{A} : K \cap \left(\overline{\Omega_2} \backslash \Omega_1 \right) \to K,$$

be a completely continuous operator such that

(i) $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right)$.

Proof of Theorem 11 Let us consider the superlinear case. From the definition of the limit $f_0 = 0$, we deduce that $f_0^i = 0$, $\forall i = 1, \ldots, n$, then for any $\varepsilon > 0$, there exists $R^i > 0$, such that if $0 < \|u\| \leq R^i$, then $f_i(u) \leq \varepsilon \|u\|$, $\forall i = 1, \ldots, n$. Let $R_1 = \min\left\{R^i, i = 1, \ldots, n, \right\}$ $\Omega_1 = \{u \in E, \|u\| < R_1\}$, so, for any $u \in K \cap \partial\Omega_1$, we get by using $(K_1) - (K_2)$ and Lemma 8

$$\begin{aligned} T_i u(t) &= \frac{1}{\Gamma(q)} \int_0^1 G_i(t,s)\, g(s) f_i(u(s)) ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s)) ds \\ &\leq \frac{\varepsilon \|u\|}{\Gamma(q)} \int_0^1 \gamma_i(s) g(s) ds + \frac{l\,\|u\|}{1-\lambda_i}, \end{aligned}$$

then by taking the maximum over $t \in [0, 1]$ it follows

$$\max_{0\leq t\leq 1} T_i u(t) \leq \left(\frac{\varepsilon}{\Gamma(q)} \int_0^1 \gamma_i(s) g(s) ds + \frac{l}{1-\lambda_i}\right) \|u\|. \tag{3.3}$$

Summing the n obtained inequalities in (3.3) it yields

$$\|Tu\| \leq \|u\| \left(\frac{\varepsilon}{\Gamma(q)} \sum_{i=1}^n \int_0^1 \gamma_i(s) g(s) ds + \sum_{i=1}^n \frac{l}{1-\lambda_i}\right).$$

In view of assumptions (K_1), one can choose ε such that

$$\varepsilon \leq \frac{\Gamma(q)\left(1 - l\sum_{i=1}^n \frac{1}{1-\lambda_i}\right)}{\sum_{i=1}^n \int_0^1 \gamma_i(s) g(s) ds} = \frac{\Gamma(q)\left(1 - l\sum_{i=1}^n \frac{1}{1-\lambda_i}\right)}{\left(\sum_{i=1}^n \frac{1}{1-\lambda_i}\right)\left(\int_0^1 (1-s)^{q-1} g(s) ds\right)},$$

thus $||Tu|| \leq ||u||$, for $u \in K \cap \partial\Omega_1$.

Next, since $f_\infty = \infty$, then $f_\infty^i = \infty$, $\forall i = 1, \ldots, n$. For any $M > 0$, there exists $\xi^i > 0$, such that $f_i(u) \geq M\|u\|$ for $\|u\| \geq \zeta^i$. Let $R = \max\left\{\zeta^i, i = 1, \ldots, n, \right\}$, choose $R_2 > \max\left\{R_1, \frac{R}{(\min_{1\leq i\leq n} \lambda_i)}\right\}$ and denote by $\Omega_2 = \{u \in E : ||u|| < R_2\}$. Let $u \in K \cap \partial\Omega_2$ then for all $t \in [0, 1]$ we have

$$\|u\| = \sum_{i=1}^n \max_{0\leq t\leq 1} |u_i(t)| \geq \sum_{i=1}^n \min_{[0,1]} u_i(t) \geq \left(\min_{1\leq i\leq n} \lambda_i\right) \|u\| = \left(\min_{1\leq i\leq n} \lambda_i\right) R_2 > R.$$

Using estimates in Lemma 8 and the fact that $u \in K$, we obtain for all $t \in [0, 1]$

$$\begin{aligned} T_i u(t) &\geq \|u\| \frac{\lambda_i M}{\Gamma(q)} \int_0^1 \gamma_i(s) g(s) ds + \frac{1}{1-\lambda_i} \int_0^1 h_i(u(s)) ds \\ &\geq \|u\| \frac{\lambda_i M}{\Gamma(q)} \int_0^1 \gamma_i(s) g(s) ds, \end{aligned}$$

thus

$$\|Tu\| \geq \|u\| \frac{M}{\Gamma(q)} \sum_{i=1}^n \left(\lambda_i \int_0^1 \gamma_i(s) g(s) ds\right). \tag{3.4}$$

Thanks to assumptions (K_1) one can choose M such that

$$M \geq \frac{\Gamma(q)}{\sum_{i=1}^{n} \lambda_i \int_0^1 \gamma_i(s) g(s) ds} = \frac{\Gamma(q)}{\left(\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i}\right)\left(\int_0^1 (1-s)^{q-1} g(s) ds\right)},$$

then (3.4) implies

$$||Tu|| \geq ||u||, \ \forall u \in K \cap \partial\Omega_2.$$

Now we can apply the first statement of Guo-Krasnosel'skii fixed point theorem on cone to conclude that T has a fixed point in $K \cap \left(\overline{\Omega_2} \diagdown \Omega_1\right)$ such that $R \leq ||u|| \leq R_2$. Applying similar techniques as above, we prove the sublinear case. The proof is complete. □

We conclude with an explicit numerical example:

Example 13 Consider the following two-dimensional fractional order system

$$(S_i): \begin{cases} {}^{c}D_{0^+}^{\frac{3}{2}} u_1(t) = (1-t)\mathrm{e}^{-u_1-u_2}, \\ {}^{c}D_{0^+}^{\frac{3}{2}} u_2(t) = (1-t)\mathrm{e}^{-u_2} \quad 0 < t < 1, \\ u_1'(0) = 0, \quad u_1(0) - \frac{1}{2} u_1(1) = \int_0^1 \left(\frac{u_1(s)+u_2(s)}{20}\right) ds \\ u_2'(0) = 0, \quad u_2(0) - \frac{1}{4} u_2(1) = \int_0^1 \frac{\mathrm{e}^{-2}(u_1(s)+u_2(s))}{1+u_1^2(s)+u_2^2(s)} ds. \end{cases}$$

We have $q = \frac{3}{2}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, $g(t) = (1-t) \in L^1([0,1], \mathbb{R}_+)$, $\int_0^1 (1-s)^{q-1} g(s) ds = \frac{2}{5} > 0$, $f_1(u) = \mathrm{e}^{-u_1-u_2}$, $f_2(u) = \mathrm{e}^{-u_2}$, $f_i \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$, $h_1(u) = \frac{1}{20}(u_1+u_2)$, $h_2(u) = \frac{\mathrm{e}^{-2}(u_1+u_2)}{1+u_1^2+u_2^2} \leq \mathrm{e}^{-2}(u_1+u_2)$, $h_i \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$, one can choose $l = \mathrm{e}^{-2} = 0.135\,34$, then $l < \left(\sum_{i=1}^{2} \frac{1}{1-\lambda_i}\right)^{-1} = 0, 3$. Since (K_1) and (K_2) are satisfied and $f_0^i = \infty$, $f_\infty^i = 0$, then Theorem 11 implies that there exists at least one positive solution in the cone K.

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References

1. Podlubny, I.: Fractional Differential Equations Mathematics in Sciences and Engineering. Academic Press, New York (1999)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
3. Lakshmikantham, V., Vatsala, A.S.: Basic theory of fractional differential equations. Nonlinear Anal. **69**, 2677–2682 (2008)
4. Ashyralyev, A.: A note on fractional derivatives and fractional powers of operators. J. Math. Anal. Appl. **357**, 232–236 (2009)
5. Tarasov, V.E.: Fractional derivative as fractional power of derivative. Int. J. Math. **18**, 281–299 (2007)
6. Ashyralyev, A., Dal, F., Pınar, Z.: A note on the fractional hyperbolic differential and difference equations. Appl. Math. Comput. **217**(9), 4654–4664 (2011)
7. Agarwal, R.P., O'Regan, D., Stanek, S.: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57–68 (2010)
8. Ahmad, B., Nieto, J.J.: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **58**, 1838–1843 (2009)

9. Ahmad, B., Nieto, J.J.: Class of differential equations of fractional order with multi-point boundary conditions. Georgian Math. J. **21**(3), 243–248 (2014)
10. Ahmad, B., Alsaedi, A.: Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations. Fixed Point Theory Appl. Article ID 364560 (2010). doi:10.1155/2010/364560
11. Ahmad, B., Nieto, J.J.: Riemann–Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. **36**, 1–9 (2011)
12. Ahmad, B.: Nonlinear fractional differential equations with anti-periodic type fractional boundary conditions. Differ. Equ. Dyn. Syst. **21**(4), 387–401 (2013)
13. Feng, M., Zhang, X., Ge, W.: New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions. Bound. Value Probl. **2011**, Art ID 720702 (2011)
14. Chai, Y., Chen, L., Wu, R.: Inverse projective synchronization between two different hyperchaotic systems with fractional order. J. Appl. Math. **2012**, Article ID 762807 (2012)
15. Bai, C., Fang, J.: The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations. Appl. Math. Comput. **150**, 611–621 (2004)
16. Ntouyas, S.K., Obaid, M.: A coupled system of fractional differential equations with nonlocal integral boundary conditions. Adv. Differ. Equ. Article ID 130 (2012). doi:10.1186/1687-1847-2012-130
17. Rehman, M., Khan, R.: A note on boundary value problems for a coupled system of fractional differential equations. Comput. Math. Appl. **61**, 2630–2637 (2011)
18. Salem, H.: On the existence of continuous solutions for a singular system of nonlinear fractional differential equations. Appl. Math. Comput. **198**, 445–452 (2008)
19. Su, X.: Existence of solution of boundary value problem for coupled system of fractional differential equations. Eng. Math. **26**, 134–137 (2009)
20. Su, X.: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. **22**, 64–69 (2009)
21. Ashyralyev, A, Sharifov, Y.A.: Existence and uniqueness of solutions for the system of nonlinear fractional differential equations with nonlocal and integral boundary conditions. Abstr. Appl. Anal. Article ID 594802 (2012)
22. Guezane-Lakoud, A., Khaldi, R.: Solvability of a fractional boundary value problem with fractional integral condition. Nonlinear Anal. **75**, 2692–2700 (2012)
23. Guezane-Lakoud, A., Khaldi, R.: Positive solution to a higher order fractional boundary value problem with fractional integral condition. Rom. J. Math. Comput. Sci. **2**(1), 41–54 (2012)
24. Webb, J.R.L., Infante, G.: Positive solutions of nonlocal boundary value problems involving integral conditions. NoDEA Nonlinear Differ. Equ. Appl. **15**(1–2), 45–67 (2008)
25. Zhao, J., Wang, P., Ge, W.: Existence and nonexistence of positive solutions for a class of third order BVP with integral boundary conditions in Banach spaces. Commun. Nonlinear Sci. Numer. Simulat. **16**, 402–413 (2011)
26. Infante, G., Pietramala, P.: Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations. Nonlinear Anal. **71**, 1301–1310 (2009)
27. Guezane-Lakoud, A., Khaldi, R.: Solvability of a two-point fractional boundary value problem. J. Nonlinear Sci. Appl. **5**, 64–73 (2012)
28. Guezane-Lakoud, A., Khaldi, R.: Solvability of a three-point fractional nonlinear boundary value problem. Differ. Equ. Dyn. Syst. **20**, 395–403 (2012)
29. Graef, J.R., Henderson, J.Y., Yang, B.O.: Positive solutions of a nonlinear higher order boundary value problem. Electron J. Differ. Equ. **45**, 1–10 (2007)
30. Henderson, J., Ntouyas, S.K., Purnaras, I.K.: Positive solutions for systems of generalized three-point nonlinear boundary value problems. Comment. Math. Univ. Carolin. **49**, 79–91 (2008)
31. Wang, J., Xiang, H., Liu, Z.: Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations. Int. J. Differ. Equ. Article ID 186928 (2010). doi:10.1155/2010/186928
32. Guo, D.J., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones Vol 5 of Notes and Reports in Mathematics in Science and Engineering. Academic Press, Boston (1988)
33. Kwong, M.K.: On Krasnoselskii's cone fixed point theorem. Fixed Theory Appl. Article ID164537 2008)