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ORIGINAL RESEARCH

Composition Theorems of Stepanov μ-Pseudo Almost Automorphic Functions and Applications to Nonautonomous Neutral Evolution Equations

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Abstract In this work, we present a new composition theorem of μ-pseudo almost automorphic functions in the sense of Stepanov satisfying some local Lipschitz conditions. Using this results, we establish an existence result of μ-pseudo almost automorphic solutions for some nonautonomous neutral partial evolution equation with Stepanov μ-pseudo almost automorphic nonlinearity. An example is shown to illustrate our results.

Keywords Pseudo almost automorphic solutions · Stepanov almost automorphy · Ergodic perturbations · Neutral evolution equations

Mathematics Subject Classification 35B15 · 43A60 · 34G20

Introduction

In this work, we give a new composition theorem of μ-pseudo almost automorphic functions in the sense of Stepanov, we suppose that the coefficient function satisfies some local Lipschitz conditions.

Then, we study the existence and uniqueness of μ-pseudo almost automorphic mild solutions to the following nonautonomous neutral partial evolution equation:

$$\frac{d}{dt}[u(t) + f(t, u(t))] = A(t)[u(t) + f(t, u(t))] + g(t, u(t)) \quad \text{for } t \in \mathbb{R}, \tag{1}$$

where $A(t)$ generates an hyperbolic evolution family $(U(t,s))_{t\geq s}$, $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a μ-pseudo almost automorphic function and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is Stepanov μ-pseudo almost automorphic.

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Pseudo almost periodic and automorphic functions have many applications in several problems like functional differential equations, integral equations and partial differential equations. The concept of almost automorphy was first introduced in the literature by Bochner [6], as a natural generalization of the almost periodicity. The notion of weighted pseudo almost automorphy has been introduced for the first time by Blot et al. [3]. More recently, using the measure theory, Blot, Cieutat and Ezzinbi [4] introduced the concept of μ-pseudo almost automorphy, as a natural generalization of the classical concept of weighted pseudo almost automorphy. On the other hand, since the work [20] by N'Guéréekata and Pankov, Stepanov-like almost automorphic problems have widely been investigated.

The existence and uniqueness of pseudo almost periodic mild solutions to differential equations in Banach spaces has attracted many researchers [10,14]. This led many authors to develop composition theorems of such functions [5,11,22].

In a recent paper [16], authors gave a result on the existence and uniqueness of pseudo almost periodic solution for the nonautonomous evolution equation (1), where the input function g is S^p-pseudo almost periodic. For contributions on nonautonomous evolution equations in Banach spaces, see [1,16,17].

In this paper, motivated by [4,5,15,16], we use the measure theory to define a Stepanov-ergodic function, we study the composition of μ-pseudo almost automorphic functions in the sense of Stepanov and we give a result of existence of μ-pseudo almost automorphic mild solution of (1).

The organization of this work is as follows. In "Preliminaries" section, we introduce the basic notations and recall the definition of μ-pseudo almost automorphic functions introduced in [5], we also give the new concept of $S^p - \mu$-pseudo almost automorphic functions and we investigate some properties. We present different composition theorems of Stepanov μ-pseudo almost automorphic function in "Composition Theorems" section. In "Evolution Family and Exponential Dichotomy" section, we introduce the basic notations of evolution family and exponential dichotomy. "Pseudo Almost Automorphic Mild Solutions" section is devoted to study the existence and uniqueness of μ-pseudo almost automorphic mild solutions of (1). As an illustration, an example of neutral heat equation with $S^p-\mu$-pseudo almost automorphic coefficients is studied under Dirichlet conditions.

Preliminaries

Pseudo Almost Automorphic Functions

Let $(\mathbb{X}, \|.\|)$ and $(\mathbb{Y}, \|.\|)$ be two Banach spaces, and $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) be the space of bounded continuous functions $f : \mathbb{R} \longrightarrow \mathbb{X}$ (respectively, $f : \mathbb{R}\times\mathbb{Y} \longrightarrow \mathbb{X}$). $BC(\mathbb{R}, \mathbb{X})$ equipped with the supremum norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$$

is a Banach space.

Definition 2.1 [19] A continuous function $f : \mathbb{R} \mapsto \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ and a measurable function g, such that

$$g(t) = \lim_{n\to+\infty} f(t + s_n),$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n \to +\infty} g(t - s_n)$$

for each $t \in \mathbb{R}$.

Let $AA(\mathbb{R}, \mathbb{X})$ be the set of all almost automorphic functions from $\mathbb{R}$ to $\mathbb{X}$. Then $(AA(\mathbb{R}, \mathbb{X}), \|.\|_\infty)$ is a Banach space.

Definition 2.2 [13] A continuous function $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{X}$ is said to be bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$ there exists a subsequence $(s_n)_{n \in \mathbb{N}} \subset (s'_n)_{n \in \mathbb{N}}$ and a measurable function g, such that

$$g(t, s) = \lim_{n \to +\infty} f(t + s_n, s + s_n),$$

is well defined for each $(t, s) \in \mathbb{R}^2$, and

$$f(t, s) = \lim_{n \to +\infty} g(t - s_n, s - s_n)$$

for each $(t, s) \in \mathbb{R}^2$. Denote by $bAA(\mathbb{X})$ the set of all such functions.

In what follows, we give the new concept of the ergodic functions developed in [4], which is a generalization of the ergodicity given in [7,8].

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 2.3 [4] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if

$$\lim_{r \longrightarrow +\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(s)\| d\mu(s) = 0.$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 2.4 [5] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if it is written in the form

$$f = g + h,$$

where $g \in AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$. The collection of such functions will be denoted by $PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Theorem 2.5 [4] *Let* $\mu \in \mathcal{M}$. *Then* $(\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu), \|.\|_\infty)$ *is a Banach space.*

For $\mu \in \mathcal{M}$, we formulate the following hypothesis:

(M1) $\limsup_{r \longrightarrow \infty} \frac{2r}{\mu[-r,r]} < \infty$.

(M2) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta \mu(A) \quad \text{when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

The hypothesis **(M2)** is given in [4].

Definition 2.6 [4] Let $\mu_1, \mu_2 \in \mathcal{M}$. μ_1 is said to be equivalent to μ_2, if there exist constants $\alpha, \beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha \mu_1(A) \leq \mu_2(A) \leq \beta \mu_1(A) \quad \text{for } A \in \mathcal{B} \text{ with } A \cap I = \emptyset.$$

Remark If μ is equivalent to the Lebesgue measure, then μ satisfies **(M1)**.

Theorem 2.7 [5] *Let $\mu \in \mathcal{M}$ satisfy **(M2)**. Then the space $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|.\|_\infty)$ is a Banach space.*

Theorem 2.8 [5] *Let $\mu \in \mathcal{M}$ satisfy **(M2)**. Then $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant that is, if $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ then $f_\tau := f(. + \tau) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ for all $\tau \in \mathbb{R}$.*

Definition 2.9 [5] A continuous function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be almost automorphic if $f(., x) \in AA(\mathbb{R}, \mathbb{X})$, for all $x \in \mathbb{Y}$. The collection of such functions is denoted by $AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Definition 2.10 [5] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be μ-ergodic if $f(., x) \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, for all $x \in \mathbb{Y}$. The collection of such functions is denoted by $\mathcal{E}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Definition 2.11 [5] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if it is written in the form

$$f = g + h,$$

where $g \in AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $h \in \mathcal{E}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$. The collection of such functions is denoted by $PAA(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Pseudo Almost Automorphy in the Sense of Stepanov

Definition 2.12 [18] The Bochner transform $f^b(t, s)$ for $t \in \mathbb{R}$ and $s \in [0, 1]$ of a function $f : \mathbb{R} \longrightarrow \mathbb{X}$ is defined by

$$f^b(t, s) = f(t + s).$$

Definition 2.13 [18] Let $1 \leq p < \infty$. The space $BS^p(\mathbb{R}, \mathbb{X})$ of all Stepanov bounded (or S^p-bounded) functions with the exponent p consists of all measurable functions f on $\mathbb{R}$ with value in $\mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p((0, 1), \mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{BS^p(\mathbb{R},\mathbb{X})} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{1/p}.$$

Remark A function $f \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{X})$ is Stepanov bounded if

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{1/p} < \infty.$$

It is obvious that

$$L^p(\mathbb{R}, \mathbb{X}) \subset BS^p(\mathbb{R}, \mathbb{X}) \subset L^p_{\text{loc}}(\mathbb{R}, \mathbb{X}).$$

Definition 2.14 [18] A function $f \in BS^p(\mathbb{R}, \mathbb{X})$, is said to be almost automorphic in the sense of Stepanov (or S^p-almost automorphic) if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exist a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ such that

$$\left[\int_t^{t+1} \|g(s) - f(s + s_n)\|^p ds \right]^{\frac{1}{p}} \to 0$$

and

$$\left[\int_t^{t+1} \|g(s-s_n)-f(s)\|^p ds\right]^{\frac{1}{p}} \to 0$$

as $n \to +\infty$ pointwise on $\mathbb{R}$. The collection of such functions will be denoted by $AA^p(\mathbb{R}, \mathbb{X})$.

In other words, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \longrightarrow L^p([0,1], \mathbb{X})$ is almost automorphic.

We introduce the following notion of ergodicity:

Definition 2.15 Let $\mu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, \mathbb{X})$, is said to be S^p-μ-ergodic if

$$\lim_{r \longrightarrow +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = 0.$$

The collection of such functions is denoted by $\mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 2.16 Let $\mu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, \mathbb{X})$, is said to be S^p-μ-pseudo almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA^p(\mathbb{R}, \mathbb{X})$ and $\phi \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$. The collection of such functions is denoted by $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$.

Theorem 2.17 *Let $\mu \in \mathcal{M}$ satisfy (**M2**). If $f \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, then $f \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$ for all $p > 1$.*

Proof Let $f \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, since μ is a σ- finite measure, then by Hölder inequality and Fubini's theorem

$$\begin{aligned}
&\int_{[-r,r]} \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&= \int_{[-r,r]} \left(\int_0^1 \|f(s+t)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&\le (\mu[-r,r])^{\frac{1}{q}} \left[\int_{[-r,r]} \left(\int_0^1 \|f(s+t)\|^p ds\right) d\mu(t)\right]^{\frac{1}{p}}\\
&\le \|f\|_\infty^{\frac{1}{q}} (\mu[-r,r])^{\frac{1}{q}} \left[\int_{[-r,r]} \left(\int_0^1 \|f(s+t)\| ds\right) d\mu(t)\right]^{\frac{1}{p}}\\
&= \|f\|_\infty^{\frac{1}{q}} (\mu[-r,r])^{\frac{1}{q}} \left[\int_0^1 \left(\int_{[-r,r]} \|f(s+t)\| d\mu(t)\right) ds\right]^{\frac{1}{p}}\\
&= \|f\|_\infty^{\frac{1}{q}} (\mu[-r,r])^{\frac{1}{q}} (\mu[-r,r])^{\frac{1}{p}} \left[\int_0^1 \frac{1}{\mu[-r,r]} \left(\int_{[-r,r]} \|f(s+t)\| d\mu(t)\right) ds\right]^{\frac{1}{p}}\\
&= \|f\|_\infty^{\frac{1}{q}} (\mu[-r,r]) \left[\int_0^1 \left(\frac{1}{\mu[-r,r]} \int_{[-r,r]} \|f(s+t)\| d\mu(t)\right) ds\right]^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{\mu[-r,r]} \int_{[-r,r]} \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&\quad \le \|f\|_\infty^{\frac{1}{q}} \left[\int_0^1 \left(\frac{1}{\mu[-r,r]} \int_{[-r,r]} \|f(s+t)\| d\mu(t)\right) ds\right]^{\frac{1}{p}}.
\end{aligned}$$

Since $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$ is invariant by translation, then

$$\frac{1}{\mu[-r, r]} \int_{[-r,r]} \|f(s+t)\| d\mu(t) \longrightarrow 0 \quad \text{when } r \longrightarrow \infty$$

for all $s \in [0, 1]$. The Lebesgue Dominated Convergence Theorem implies that

$$\lim_{r \longrightarrow \infty} \frac{1}{\mu[-r, r]} \int_{[-r,r]} \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

□

Corollary 2.18 *Let $\mu \in \mathcal{M}$ satisfy (**M2**). If $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, then $f \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ for all $p > 1$.*

Theorem 2.19 *Let $\mu \in \mathcal{M}$ satisfy (**M2**). Then $PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ is invariant by translation, that is, $f \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$ implies $f_\tau \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, for all $\tau \in \mathbb{R}$.*

Proof It suffices to show that $\mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$ is invariant by translation. Let $f \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$ and $F(t) = \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}}$, then $F \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$, but since $\mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$ is invariant by translation [4], then

$$\begin{aligned} &\frac{1}{\mu([-r, r])} \int_{[-r,r]} \left(\int_t^{t+1} \|f(s+a)\|^p ds \right)^{\frac{1}{p}} d\mu(s) \\ &= \frac{1}{\mu([-r, r])} \int_{[-r,r]} F(t+a) d\mu(t) \longrightarrow 0 \text{ when } r \longrightarrow \infty. \end{aligned}$$

We deduce that $f(.+a) \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$. □

Definition 2.20 Let $AA^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ denote the space of functions $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ such that $f(., y) \in AA^p(\mathbb{R}, \mathbb{X})$, for each $y \in \mathbb{Y}$, $\mathcal{E}^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ denote the space of functions $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ such that $f(., y) \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$, for each $y \in \mathbb{Y}$. Let us set

$$PAA^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu) := AA^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) + \mathcal{E}^p(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu).$$

Now we introduce the space of μ-Stepanov bounded functions:

Definition 2.21 Let $1 \leq p < \infty$. The space $BS^p(\mathbb{R}, \mathbb{X}, \mu)$ of all μ-Stepanov bounded (or $\mu - S^p$-bounded) functions with the exponent p consists of all measurable functions f on $\mathbb{R}$ with value in $\mathbb{X}$ such that

$$f^b \in L^\infty \left(\mathbb{R}, L^p\left((0, 1), \mathbb{X}, d\mu \right) \right).$$

Remark A function $f \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{X}, \mu)$ is $\mu - S^p$ bounded if

$$\|f\|_{BS^p(\mathbb{R},\mathbb{X},\mu)} := \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p d\mu \right)^{1/p} < \infty.$$

It is obvious that

$$L^p(\mathbb{R}, \mathbb{X}, \mu) \subset S^p(\mathbb{R}, \mathbb{X}, \mu) \subset L^p_{\text{loc}}(\mathbb{R}, \mathbb{X}, \mu).$$

Let $\mu \in \mathcal{M}$, we introduce the following hypothesis:

(M3) $\sup_{t \in \mathbb{R}} \mu[t, t+1] < \infty.$

Example If μ is absolutely continuous with respect to Lebesgue measure with a bounded Radon–Nikodym derivative, then **(M3)** naturally holds.

Proposition 2.22 *Let $\mu \in \mathcal{M}$ satisfy **(M3)**, then constant functions belong to $BS^p(\mathbb{R}, \mathbb{X}, \mu)$.*

Proof Let $f(s) = M$ be a constant function. Then

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p d\mu \right)^{1/p} \leq M \sup_{t\in\mathbb{R}} (\mu[t, t+1])^{1/p} < \infty.$$

□

Composition Theorems

Definition 3.1 Let $UC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ denote the set of all uniformly continuous functions $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$, i.e for each compact set K in $\mathbb{X}$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(t, u) - f(t, v)\| < \varepsilon \tag{2}$$

for all $t \in \mathbb{R}$ and $u, v \in K$ with $\|u - v\| \leq \delta$.

Definition 3.2 Let $UC^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ denote the set of all BS^p-uniformly continuous functions $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$, i.e there is a non-negative function $L \in BS^1(\mathbb{R}, \mathbb{R}, \mu)$ such that for each compact set K in $\mathbb{X}$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left(\int_t^{t+1} \|f(s, u) - f(s, v)\|^p \, ds \right)^{\frac{1}{p}} < L(t)\,\varepsilon \tag{3}$$

for all $t \in \mathbb{R}$ and $u, v \in K$ with $\|u - v\| \leq \delta$.

Lemma 3.3 [13] *Assume that*

$$f \in AA^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}) \cap UC(\mathbb{R} \times \mathbb{X}, \mathbb{X}).$$

If $u \in AA^p(\mathbb{R}, \mathbb{X})$ and $K = \overline{\{u(t) : t \in \mathbb{R}\}}$ is compact, Then

$$f(., u(.)) \in AA^p(\mathbb{R}, \mathbb{X}).$$

Lemma 3.4 *Assume that $\alpha(.) \in AA^p(\mathbb{R}, \mathbb{X})$, $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is a compact subset of $\mathbb{X}$, $h \in \mathcal{E}^p(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu) \cap UC^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and let $\mu \in \mathcal{M}$ satisfy **(M1)**. Then $h(., \alpha(.)) \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$.*

Proof For any fixed $\varepsilon > 0$, let $\delta > 0$ such that (3) holds. Then there exist $\alpha_1 \ldots \alpha_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B(\alpha_i, \delta).$$

For each $t \in \mathbb{R}$, there exists $\alpha_{i(t)}$, $1 \le i(t) \le k$ such that $\left\|\alpha(t) - \alpha_{i(t)}\right\| \le \delta$. Then we get

$$\left(\int_t^{t+1} \|h(s,\alpha(s))\|^p\, ds\right)^{\frac{1}{p}} \le \left(\int_t^{t+1} \left\|h(s,\alpha(s)) - h\left(s,\alpha_{i(t)}\right)\right\|^p ds\right)^{\frac{1}{p}} + \left(\int_t^{t+1} \left\|h\left(s,\alpha_{i(t)}\right)\right\|^p ds\right)^{\frac{1}{p}} \le L(t)\,\varepsilon + \sum_{i=1}^{k}\left(\int_t^{t+1} \|h(s,\alpha_i)\|^p\, ds\right)^{\frac{1}{p}},$$

which gives

$$\begin{aligned}
&\frac{1}{\mu[-r,r]}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha(s))\|^p\,ds\right)^{\frac{1}{p}} d\mu(t)\\
&\le \frac{1}{\mu[-r,r]}\varepsilon\int_{-r}^{r} L(t)\,d\mu(t) + \frac{1}{\mu[-r,r]}\sum_{i=1}^{k}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha_i)\|^p\,ds\right)^{\frac{1}{p}} d\mu(t)\\
&\le \frac{1}{\mu[-r,r]}\varepsilon\int_{-[r]-1}^{[r]+1} L(t)\,d\mu(t) + \frac{1}{\mu[-r,r]}\sum_{i=1}^{k}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha_i)\|^p\,ds\right)^{\frac{1}{p}} d\mu(t)\\
&\le \frac{1}{\mu[-r,r]}\varepsilon\sum_{p=-[r]-1}^{[r]}\int_k^{k+1} L(t)\,d\mu(t)\\
&\quad+\frac{1}{\mu[-r,r]}\sum_{i=1}^{k}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha_i)\|^p\,ds\right)^{\frac{1}{p}} d\mu(t)\\
&\le \varepsilon\|L\|_{BS^1(\mathbb{R},\mathbb{R},\mu)}\frac{(2[r]+2)}{\mu[-r,r]} + \frac{1}{\mu[-r,r]}\sum_{i=1}^{k}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha_i)\|^p\,ds\right)^{\frac{1}{p}} d\mu(t)\\
&\le \varepsilon\|L\|_{BS^1(\mathbb{R},\mathbb{R},\mu)}\frac{(2[r]+2)}{2r}\frac{2r}{\mu[-r,r]}\\
&\quad+\frac{1}{\mu[-r,r]}\sum_{i=1}^{k}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha_i)\|^p\,ds\right)^{\frac{1}{p}} d\mu(t).
\end{aligned}$$

Noting that $h(.,\alpha_i) \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$, $i = 1\ldots k$, and using hypothesis **(M1)** one has

$$\limsup_{r\longrightarrow\infty}\frac{1}{\mu[-r,r]}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha(s))\|^p\,ds\right)^{\frac{1}{p}} d\mu(t) \le M\,\varepsilon\,\|L\|_{BS^1(\mathbb{R},\mathbb{R},\mu)} \quad \text{for all } \varepsilon > 0.$$

Therefore,

$$\lim_{r\longrightarrow\infty}\frac{1}{\mu[-r,r]}\int_{-r}^{r}\left(\int_t^{t+1}\|h(s,\alpha(s))\|^p\,ds\right)^{\frac{1}{p}} d\mu(t) = 0,$$

i.e., $h(.,\alpha(.)) \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$. □

Remark If μ is absolutely continuous with respect to the Lebesgue measure with a Radon Nikodym derivative ρ, then **(M1)** which was used in Lemma 3.4 is equivalent to the condition:

$$\limsup_{r\longrightarrow\infty} \frac{r}{\int_{-r}^{r}\rho(s)\,ds} < \infty. \tag{4}$$

A similar result was given in [22, Lemma3.1] if ρ satisfies

$$\limsup_{r\longrightarrow\infty} \frac{r^{\frac{1}{p}}\left(\int_{-r}^{r}\rho^{q}(s)\,ds\right)^{\frac{1}{q}}}{\int_{-r}^{r}\rho(s)\,ds} < \infty. \tag{5}$$

However, if for example $\rho(t) = e^{t}$, one cannot apply [22, Lemma3.1] since the condition (5) is not verified, in fact:

$$\limsup_{r\longrightarrow\infty} \frac{r^{\frac{1}{p}}\left(\int_{-r}^{r}e^{qs}ds\right)^{\frac{1}{q}}}{\int_{-r}^{r}e^{s}ds} = \limsup_{r\longrightarrow\infty} \frac{r^{\frac{1}{p}}\left(e^{qr}-e^{-qr}\right)^{\frac{1}{q}}}{e^{r}-e^{-r}} = \limsup_{r\longrightarrow\infty} r^{\frac{1}{p}} = \infty.$$

While the condition (4) holds since

$$\limsup_{r\longrightarrow\infty} \frac{r}{\int_{-r}^{r}e^{s}ds} = \limsup_{r\longrightarrow\infty} \frac{r}{e^{r}-e^{-r}} = 0 < \infty.$$

Another example where one cannot apply [22, Lemma 3.1] is when μ has a Radon–Nikodym derivative ρ defined as follows:

$$\rho(t) = \begin{cases} k, & k \le t \le k+\frac{1}{k} \text{ for all } k \in \mathbb{N}^{*},\\ 0, & v \text{ otherwise.} \end{cases} \tag{6}$$

One has

$$[r]-1 \le \int_{-r}^{r}\rho(s)\,ds \le [r] \quad \text{for } r>0. \tag{7}$$

In fact

$$\int_{-r}^{r}\rho(s)\,ds \ge \int_{0}^{[r]}\rho(s)\,ds = \sum_{k=1}^{[r]-1}\int_{k}^{k+\frac{1}{k}}\rho(s)\,ds = [r]-1,$$

and

$$\int_{-r}^{r}\rho(s)\,ds \le \sum_{k=1}^{[r]}\int_{k}^{k+\frac{1}{k}}\rho(s)\,ds = [r].$$

Therefore

$$\lim_{r\longrightarrow\infty}\int_{-r}^{r}\rho(s)\,ds = \infty$$

and then $\mu \in \mathcal{M}$. In addition μ satisfies **(M3)**, in fact for $t \ge 0$ we have

$$\int_{t}^{t+1}\rho(s)\,ds \le \int_{[t]}^{[t]+2}\rho(s)\,ds = \sum_{k=[t]}^{[t]+1}\int_{k}^{k+\frac{1}{k}}\rho(s)\,ds = 2,$$

it follows that

$$\sup_{t\in\mathbb{R}}\mu[t,t+1] = \sup_{t\in\mathbb{R}}\int_{t}^{t+1}\rho(s)\,ds < \infty.$$

On one hand by (7), the condition (4) holds since

$$\limsup_{r\longrightarrow\infty}\frac{2r}{\int_{-r}^{r}\rho(s)\,ds}\leq\limsup_{r\longrightarrow\infty}\frac{2r}{[r]-1}<\infty.$$

On other hand the condition (5) does not hold, in fact

$$\begin{aligned}\int_{-r}^{r}\rho^2(s)\,ds&=\int_{0}^{r}\rho^2(s)\,ds\geq\int_{0}^{[r]}\rho^2(s)\,ds\\&=\sum_{k=1}^{[r]-1}\int_{k}^{k+\frac{1}{k}}\rho^2(s)\,ds=\sum_{k=1}^{[r]-1}k=\frac{([r]-1)[r]}{2}.\end{aligned}$$

Then

$$\frac{r^{\frac{1}{2}}\left(\int_{-r}^{r}\rho^2(s)\,ds\right)^{\frac{1}{2}}}{\int_{-r}^{r}\rho(s)\,ds}\geq\frac{r^{\frac{1}{2}}\left(\frac{([r]-1)[r]}{2}\right)^{\frac{1}{2}}}{[r]}\backsim r^{\frac{1}{2}},\quad\text{when } r \text{ goes to } \infty.$$

Therefore

$$\limsup_{r\longrightarrow\infty}\frac{r^{\frac{1}{2}}\left(\int_{-r}^{r}\rho^2(s)\,ds\right)^{\frac{1}{2}}}{\int_{-r}^{r}\rho(s)\,ds}=\infty.$$

Lemma 3.5 *Let $\mu\in\mathcal{M}$ and $f\in BS^p(\mathbb{R},\mathbb{X})$, then $f\in\mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$ if and only if for any $\varepsilon>0$*

$$\lim_{r\longrightarrow\infty}\frac{\mu\left(\left\{t\in[-r,r]:\ \left(\int_t^{t+1}\|f(s)\|^p\,ds\right)^{\frac{1}{p}}\geq\varepsilon\right\}\right)}{\mu[-r,r]}=0.$$

Proof Since $t\longrightarrow\left(\int_t^{t+1}\|f(s)\|^p\,ds\right)^{\frac{1}{p}}\in\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$, then Lemma 3.5 is a direct result of [4, Theorem2.13]. □

Theorem 3.6 *Let $\mu\in\mathcal{M}$ and $f=g+h\in PAA^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ with $g\in AA^p(\mathbb{R}\times\mathbb{X},\mathbb{X})\cap UC(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $h\in\mathcal{E}^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$. We suppose that there exists a non-negative function $L(.)\in BS^1(\mathbb{R},\mathbb{R},\mu)$ with $p>1$ such that for all $u,v\in L^p_{loc}(\mathbb{R},\mathbb{X})$ and $t\in\mathbb{R}$,*

$$\left(\int_t^{t+1}\|f(s,u(s))-f(s,v(s))\|^p\,ds\right)^{\frac{1}{p}}\leq L(t)\left(\int_t^{t+1}\|u(s)-v(s)\|^p\,ds\right)^{\frac{1}{p}}.\quad(8)$$

*Assume that μ satisfies **(M1)–(M3).***

If $x=\alpha+\beta\in PAA^p(\mathbb{R},\mathbb{X},\mu)$, with $\alpha\in AA^p(\mathbb{R},\mathbb{X})$, $\beta\in\mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$ and $K=\overline{\{\alpha(t):t\in\mathbb{R}\}}$ is compact, then $f(.,x(.))\in PAA^p(\mathbb{R},\mathbb{X},\mu)$.

Proof We have the following decomposition

$$\begin{aligned}f(t,x(t))&=g(t,\alpha(t))+f(t,x(t))-f(t,\alpha(t))+h(t,\alpha(t))\\&=G(t)+F(t)+H(t),\end{aligned}$$

where $G(t)=g(t,\alpha(t))$, $F(t)=f(t,x(t))-f(t,\alpha(t))$ and $H(t)=h(t,\alpha(t))$. Since $g\in UC(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $K=\overline{\{\alpha(t):t\in\mathbb{R}\}}$ is compact, it follows from Lemma 3.3 that

$g(t,\alpha(t)) \in AA^p(\mathbb{R},\mathbb{X})$. First we prove that $F(.) \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$. By Lemma 3.5 we have for all $\varepsilon > 0$

$$\lim_{r\longrightarrow\infty} \frac{\mu\left(M_{r,\varepsilon}(\beta)\right)}{\mu[-r,r]} = 0,$$

where

$$M_{r,\varepsilon}(\beta) = \left\{ t \in [-r,r] : \left(\int_t^{t+1} \|\beta(s)\|^p ds\right)^{\frac{1}{p}} \geq \varepsilon \right\}$$

Let $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\mu[-r,r]}\int_{-r}^{r}\left(\int_t^{t+1}\|F(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&\leq \frac{1}{\mu[-r,r]}\int_{M_{r,\varepsilon}(\beta)}\left(\int_t^{t+1}\|F(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&\quad+\frac{1}{\mu[-r,r]}\int_{[-r,r]\backslash M_{r,\varepsilon}(\beta)}\left(\int_t^{t+1}\|F(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&\leq \|F\|_{BS^p(\mathbb{R},\mathbb{X})}\frac{\mu\left(M_{r,\varepsilon}(\beta)\right)}{\mu[-r,r]}\\
&\quad+\frac{1}{\mu[-r,r]}\int_{[-r,r]\backslash M_{r,\delta}(\beta)}\left(\int_t^{t+1}\|f(s,x(s))-f(s,\alpha(s))\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&\leq \|F\|_{BS^p(\mathbb{R},\mathbb{X})}\frac{\mu\left(M_{r,\varepsilon}(\beta)\right)}{\mu[-r,r]}\\
&\quad+\frac{1}{\mu[-r,r]}\int_{[-r,r]\backslash M_{r,\delta}(\beta)}L(t)\left(\int_t^{t+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t)\\
&\leq \|F\|_{BS^p(\mathbb{R},\mathbb{X})}\frac{\mu\left(M_{r,\varepsilon}(\beta)\right)}{\mu[-r,r]}+\frac{1}{\mu[-r,r]}\varepsilon\int_{-r}^{r}L(t)\,d\mu(t)\\
&\leq \|F\|_{BS^p(\mathbb{R},\mathbb{X})}\frac{\mu\left(M_{r,\varepsilon}(\beta)\right)}{\mu[-r,r]}+\frac{(2[r]+2)}{\mu[-r,r]}\|L\|_{BS^1(\mathbb{R},\mathbb{R},\mu)}\,\varepsilon.
\end{aligned}$$

Therefore

$$\limsup_{r\longrightarrow\infty}\frac{1}{\mu[-r,r]}\int_{-r}^{r}\left(\int_t^{t+1}\|F(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) \leq M\,\|L\|_{BS^1(\mathbb{R},\mathbb{R},\mu)}\,\varepsilon \quad \text{for all } \varepsilon > 0.$$

Thus $F(.) \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$.

Next we prove that $H(.) \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$. From (8), we can see that $f \in UC^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$. Using Proposition 2.22, it is easy to see that $g \in UC^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and then $h = f - g \in UC^p(\mathbb{R}\times\mathbb{X},\mathbb{X})$. It follows from Lemma 3.4 that $h(.,\alpha(.)) \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$. □

Theorem 3.7 *Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$, $p > 1$ with $g \in AA^p(\mathbb{R}\times\mathbb{X},\mathbb{X}) \cap UC(\mathbb{R}\times\mathbb{X},\mathbb{X})$, $h \in \mathcal{E}^p(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$. We suppose that there exists a non negative function $L(.) \in BS^r(\mathbb{R},\mathbb{R}) \cap BS^1(\mathbb{R},\mathbb{R},\mu)$ with $r \geq \max\left\{p, \frac{p}{p-1}\right\}$ such that for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$,*

$$\|f(t,u)-f(t,v)\| \leq L(t)\|u-v\|.$$

*Assume that μ satisfies **(M1)–(M3).***

If $x = \alpha + \beta \in PAA^p(\mathbb{R},\mathbb{X},\mu)$, with $\alpha \in AA^p(\mathbb{R},\mathbb{X})$, $\beta \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$ and $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, then there exists $q \in [1,p)$ such that $f(.,x(.)) \in PAA^q(\mathbb{R},\mathbb{X},\mu)$.

Proof Since $r \geq \frac{p}{p-1}$, there exists $q \in [1,p)$ such that $r = \frac{pq}{p-q}$. Let

$$p' = \frac{p}{p-q}, \quad q' = \frac{p}{q}.$$

Then $p', q' > 1$ and $\frac{1}{p'} + \frac{1}{q'} = 1$.

We have the following decomposition

$$\begin{aligned} f(t,x(t)) &= g(t,\alpha(t)) + f(t,x(t)) - f(t,\alpha(t)) + h(t,\alpha(t)) \\ &= G(t) + F(t) + H(t), \end{aligned}$$

where $G(t) = g(t,\alpha(t))$, $F(t) = (t,x(t)) - f(t,\alpha(t))$ and $H(t) = h(t,\alpha(t))$. Since $g \in UC(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, it follows from Lemma 3.3 that $g(t,\alpha(t)) \in AA^p(\mathbb{R},\mathbb{X})$. First we prove that $F(.) \in \mathcal{E}^q(\mathbb{R},\mathbb{X},\mu)$.

$$\begin{aligned} \left(\int_t^{t+1} \|F(s)\|^q ds\right)^{\frac{1}{q}} &\leq \left(\int_t^{t+1} \|f(t,x(t)) - f(t,\alpha(t))\|^q ds\right)^{\frac{1}{q}} \\ &\leq \left(\int_t^{t+1} L^q(s)\|\beta(s)\|^q ds\right)^{\frac{1}{q}} \\ &\leq \left(\int_t^{t+1} L^r(s)\, ds\right)^{\frac{1}{p'q}} \left(\int_t^{t+1} \|\beta(s)\|^p ds\right)^{\frac{1}{p}} \\ &\leq \|L\|_{BS^r(\mathbb{R},\mathbb{R})} \left(\int_t^{t+1} \|\beta(s)\|^p ds\right)^{\frac{1}{p}}. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{\mu[-r,r]} \int_{-r}^{r} \left(\int_t^{t+1} \|F(s)\|^q ds\right)^{\frac{1}{q}} d\mu(t) \\ &\quad \leq \frac{\|L\|_{BS^r(\mathbb{R},\mathbb{R})}}{\mu[-r,r]} \int_{-r}^{r} \left(\int_t^{t+1} \|\beta(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t). \end{aligned}$$

Thus $F(.) \in \mathcal{E}^q(\mathbb{R},\mathbb{X},\mu)$.

Next we prove that $H(.) \in \mathcal{E}^q(\mathbb{R},\mathbb{X},\mu)$. We have

$$\begin{aligned} \left(\int_t^{t+1} \|f(s,x) - f(s,y)\|^p ds\right)^{\frac{1}{p}} &\leq \left(\int_t^{t+1} L(s)^p \|x-y\|^p ds\right)^{\frac{1}{p}} \\ &= \left(\int_t^{t+1} L(s)^p ds\right)^{\frac{1}{p}} \|x-y\| \\ &\leq \|L\|_{BS^p(\mathbb{R},\mathbb{R})} \|x-y\| \end{aligned}$$

and $g \in UC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$. Then using Proposition 2.22, it is easy to see that $f, g \in UC^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and then $h = f - g \in UC^p(\mathbb{R} \times \mathbb{X}, \mathbb{X})$. It follows from Lemma 3.4 that $h(., \alpha(.)) \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu) \subset \mathcal{E}^q(\mathbb{R}, \mathbb{X}, \mu)$. □

Evolution Family and Exponential Dichotomy

Definition 4.1 [12,21] A family of bounded linear operators $(U(t,s))_{t \geq s}$, on a Banach space $\mathbb{X}$ is called a strongly continuous evolution family if

1. $U(t,r)U(r,s) = U(t,s)$ and $U(s,s) = I$, for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$,
2. The map $(t,s) \to U(t,s)x$ is continuous for all $x \in \mathbb{X}$, $t \geq s$ and $t, s \in \mathbb{R}$.

Definition 4.2 [12,21] An evolution family $(U(t,s))_{t \geq s}$ on a Banach space $\mathbb{X}$ is called hyperbolic (or has exponential dichotomy) if there exist projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t, and constants $M > 0$, $\delta > 0$ such that

1. $U(t,s)P(s) = P(t)U(t,s)$, for $t \geq s$ and $t, s \in \mathbb{R}$,
2. The restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible for $t \geq s$ and $t, s \in \mathbb{R}$ (and we set $U_Q(t,s) = U(s,t)^{-1}$).
3. $$\|U(t,s)P(s)\| \leq Me^{-\delta(t-s)} \tag{9}$$

   and

   $$\|U_Q(s,t)Q(t)\| \leq Me^{-\delta(t-s)}, \tag{10}$$

   for $t \geq s$ and $t, s \in \mathbb{R}$.

Here and below we set $Q := I - P$.

Definition 4.3 Given a hyperbolic evolution family, we define its so-called Green's function by

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s) & \text{for } t \geq s,\ t, s \in \mathbb{R}, \\ -U_Q(t,s)Q(s) & \text{for } t < s,\ t, s \in \mathbb{R}. \end{cases}$$

Pseudo Almost Automorphic Mild Solutions

In this section, we investigate the existence and uniqueness of μ-pseudo almost automorphic mild solutions of Eq. (1).

Before starting our main result in this section, we recall the definition of the mild solution to Eq. (1) and we make the following assumptions:

(H0) There exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$, and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \qquad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)\, R\,(\lambda, A(t) - \lambda_0)\,[R(\lambda_0, A(t)) - R(\lambda_0, A(s)]\| \leq L|t-s|^\alpha |\lambda|^{-\beta},$$

for $t, s \in \mathbb{R}$ and $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\}, |\arg \lambda| \leq \theta\}$.

(H1) The evolution family $(U(t,s))_{t\geq s}$ generated by $A(t)$ has an exponential dichotomy with constants $M > 0$, $\delta > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$ and Green's function $\Gamma(t,s)$.

(H2) $t \to R(\lambda_0, A(t)) \in AA(\mathbb{R}, B(\mathbb{X}))$.

We point out that assumption **(H0)** is usually called "Acquistapace-Terreni" condition, which was firstly introduced in [1] and widely used to investigate nonautonomous evolution equations.

(H3) $f = f_1 + f_2 \in PAA(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$, with $f_1 \in AA(\mathbb{R}\times\mathbb{X}, \mathbb{X}) \cap UC(\mathbb{R}\times\mathbb{X}, \mathbb{X})$ and $f_2 \in \mathcal{E}(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$. Assume that f is bounded on $\mathbb{R}\times B$ for each bounded subset B of $\mathbb{X}$ and there exists a constant L_f such that for all $u, v \in \mathbb{X}$ and for all $t \in \mathbb{R}$:

$$\|f(t,u) - f(t,v)\| \leq L_f \|u - v\|.$$

(H4) $g = g_1 + g_2 \in PAA^p(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$, with $g_1 \in AA^p(\mathbb{R}\times\mathbb{X}, \mathbb{X}) \cap UC(\mathbb{R}\times\mathbb{X}, \mathbb{X})$ and $g_2 \in \mathcal{E}^p(\mathbb{R}\times\mathbb{X}, \mathbb{X}, \mu)$. Assume that there exists a non-negative function $L \in BS^r(\mathbb{R},\mathbb{R}) \cap BS^1(\mathbb{R},\mathbb{R},\mu)$ with $r \geq \max\left\{p, \frac{p}{p-1}\right\}$ such that for all $u, v \in \mathbb{X}$ and for all $t \in \mathbb{R}$:

$$\|g(t,u) - g(t,v)\| \leq L(t)\,\|u - v\|.$$

Definition 5.1 A mild solution to Eq. (1) is a continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying

$$u(t)+f(t,u(t)) = U(t,s)[u(s)+f(s,u(s))]+\int_s^t U(t,\sigma)g(\sigma,u(\sigma))d\sigma \quad \text{for } t \geq s. \tag{11}$$

Theorem 5.2 [13] *Let assumptions **(H0)–(H1)** hold and u be a bounded mild solution of (1) on $\mathbb{R}$, then for all $t \in \mathbb{R}$*

$$u(t) = -f(t,u(t)) + \int_{\mathbb{R}} \Gamma(t,s)\, g(s,u(s))ds. \tag{12}$$

Lemma 5.3 [2] *Assume that **(H0)–(H2)** hold. Then $\Gamma \in bAA(\mathbb{X})$.*

Theorem 5.4 *Let $\mu \in \mathcal{M}$ satisfy **(M2)**. Assume that **(H0)–(H2)** hold, if $h \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, for $p > 1$, then*

$$t \longmapsto \int_{\mathbb{R}} \Gamma(t,s)\, h(s)\, ds$$

belongs to $PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Proof Since $h \in PAA^p(\mathbb{R}, \mathbb{X}, \mu)$, we can write $h = h_1 + h_2$, where $h_1 \in AA^p(\mathbb{R}, \mathbb{X})$ and $h_2 \in \mathcal{E}^p(\mathbb{R}, \mathbb{X}, \mu)$. By [13] and using Lemma 5.3, we have $\int_{\mathbb{R}} \Gamma(t,s)\, h_1(s)\, ds \in AA(\mathbb{R}, \mathbb{X})$. To complete the proof, we will prove that $\int_{\mathbb{R}} \Gamma(t,s)\, h_2(s)\, ds \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$. Let us consider for each $t \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\Phi_n(t) := \int_{t-n}^{t-n+1} U(t,\sigma)P(\sigma)h_2(\sigma)d\sigma.$$

We have

$$\begin{aligned}
\|\Phi_n(t)\| &\le M\int_{t-n}^{t-n+1} e^{-\delta(t-\sigma)}\|h_2(\sigma)\|d\sigma\\
&\le M\left[\int_{t-n}^{t-n+1} e^{-q\delta(t-\sigma)}d\sigma\right]^{\frac{1}{q}}\left[\int_{t-n}^{t-n+1}\|h_2(\sigma)\|^p d\sigma\right]^{\frac{1}{p}}\\
&\le M\left[\int_{t-n}^{t-n+1} e^{-q\delta(t-\sigma)}d\sigma\right]^{\frac{1}{q}}\left[\int_{t-n}^{t-n+1}\|h_2(\sigma)\|^p d\sigma\right]^{\frac{1}{p}}\\
&\le M\sqrt[q]{\frac{e^{q\delta}-1}{q\delta}}e^{-n\delta}\left[\int_{t-n}^{t-n+1}\|h_2(\sigma)\|^p d\sigma\right]^{\frac{1}{p}}.
\end{aligned}$$

Multiply both sides of the inequality by $\frac{1}{\mu([-r,r])}$ and integrating, we obtain

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|\Phi_n(t)\|d\mu(t)\\
&\le M\sqrt[q]{\frac{e^{q\delta}-1}{q\delta}}e^{-n\delta}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left[\int_t^{t+1}\|h_2(\sigma-n)\|^p d\sigma\right]^{\frac{1}{p}}d\mu(t).
\end{aligned}$$

Since $h_2 \in \mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$ and μ satisfies (**M2**), then by Theorem 2.19, $\mathcal{E}^p(\mathbb{R},\mathbb{X},\mu)$ is invariant by translation and the left side of the inequality goes to 0 when r goes to infinity. Therefore

$$\Phi_n(t) \in \mathcal{E}(\mathbb{R},\mathbb{X},\mu).$$

From

$$M\sqrt[q]{\frac{e^{q\delta}-1}{q\delta}}\sum_{n\ge 0}e^{-n\delta} < \infty,$$

we deduce that $\sum_{n\ge 0}\Phi_n$ converges uniformly to

$$\Phi(t) = \int_{-\infty}^{t} U(t,\sigma)P(\sigma)h_2(\sigma)d\sigma,$$

it follows that

$$\Phi \in \mathcal{E}(\mathbb{R},\mathbb{X},\mu).$$

Using the same argument, we show that

$$\Psi(t) = \int_t^{+\infty} U_Q(t,\sigma)Q(\sigma)h_2(\sigma)d\sigma \in \mathcal{E}(\mathbb{R},\mathbb{X},\mu).$$

We conclude that $\int_{\mathbb{R}}\Gamma(t,s)\,h_2(s)\,ds \in \mathcal{E}(\mathbb{R},\mathbb{X},\mu)$. □

Lemma 5.5 *Let $\mu \in \mathcal{M}$ satisfy **(M1)–(M3)**. Assume that **(H0)–(H4)** hold. The operator Λ defined by*

$$(\Lambda u)(t) = \int_{\mathbb{R}}\Gamma(t,s)\,g(s,u(s))ds,$$

maps $PAA(\mathbb{R},\mathbb{X},\mu)$ to $PAA(\mathbb{R},\mathbb{X},\mu)$.

Proof We can easily obtain this result from Theorems 3.7 and 5.4. □

Theorem 5.6 *Let $\mu \in \mathcal{M}$ satisfy* ***(M1)–(M3)****. Assume that* ***(H0)–(H4)*** *hold. Then Eq.* (1) *admits a unique μ-pseudo almost automorphic mild solution if*

$$\left(L_f + \|L\|_{BS^r(\mathbb{R},\mathbb{R})} \frac{2M}{1-e^{-\delta}} \left(\frac{1-e^{-\delta r'}}{\delta r'}\right)^{\frac{1}{r'}}\right) < 1,$$

where $\frac{1}{r} + \frac{1}{r'} = 1.$

Proof Define the nonlinear operator Λ on $BC(\mathbb{R}, \mathbb{X})$ by :

$$\begin{aligned}(\Lambda u)(t) = -f(t, u(t)) + \int_{-\infty}^{t} U(t,s)P(s)g(s,u(s))ds \\ - \int_{t}^{+\infty} U_Q(t,s)Q(s)g(s,u(s))ds \quad \text{for } t \in \mathbb{R}.\end{aligned}$$

Let $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$, using Lemma 5.5 and [5, Theorem5.7], we deduce that Λ is well defined and maps $PAA(\mathbb{R}, \mathbb{X}, \mu)$ into itself. Let $u, v \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. It follows that for each $t \in \mathbb{R}$:

$$\begin{aligned}&\| (\Lambda u)(t) - (\Lambda v)(t) \| \\ &\quad \le \|f(t,u(t)) - f(t,v(t))\| + \int_{-\infty}^{t} Me^{-\delta(t-s)} \|g(s,u(s)) - g(s,v(s))\| ds \\ &\qquad + \int_{t}^{+\infty} Me^{-\delta(s-t)} \|g(s,u(s)) - g(s,v(s))\| ds \\ &\quad \le L_f \|u-v\|_\infty + \|u-v\|_\infty \int_{-\infty}^{t} Me^{-\delta(t-s)} L(s)\, ds \\ &\qquad + \|u-v\|_\infty \int_{t}^{+\infty} Me^{-\delta(s-t)} L(s)\, ds \\ &\quad \le \left(L_f + L_1 + L_2\right) \|u-v\|_\infty ,\end{aligned}$$

where

$$L_1 = \int_{-\infty}^{t} Me^{-\delta(t-s)} L(s)\, ds$$

and

$$L_2 = \int_{t}^{+\infty} Me^{-\delta(s-t)} L(s)\, ds.$$

Since $L \in BS^r(\mathbb{R},\mathbb{R})$, we get

$$\begin{aligned} L_1 &= M\sum_{k=1}^{\infty}\int_{t-k}^{t-k+1} e^{-\delta(t-s)}L(s)\,ds \\ &\le M\sum_{k=1}^{\infty}\left(\int_{t-k}^{t-k+1} e^{-\delta r'(t-s)}ds\right)^{\frac{1}{r'}}\left(\int_{t-k}^{t-k+1} L(s)^r\,ds\right)^{\frac{1}{r}} \\ &\le M\,\|L\|_{BS^r(\mathbb{R},\mathbb{R})}\sum_{k=1}^{\infty}\left(\int_{t-k}^{t-k+1} e^{-\delta r'(t-s)}ds\right)^{\frac{1}{r'}} \\ &\le M\,\|L\|_{BS^r(\mathbb{R},\mathbb{R})}\frac{1}{1-e^{-\delta}}\left(\frac{1-e^{-\delta r'}}{\delta r'}\right)^{\frac{1}{r'}}. \end{aligned}$$

Similarly we have

$$L_2 \le M\,\|L\|_{BS^r(\mathbb{R},\mathbb{R})}\frac{1}{1-e^{-\delta}}\left(\frac{1-e^{-\delta r'}}{\delta r'}\right)^{\frac{1}{r'}}.$$

Thus

$$\|(\Lambda u)-(\Lambda v)\|_\infty \le \left(L_f + M\,\|L\|_{BS^r(\mathbb{R},\mathbb{R})}\frac{2}{1-e^{-\delta}}\left(\frac{1-e^{-\delta r'}}{\delta r'}\right)^{\frac{1}{r'}}\right)\|u-v\|_\infty .$$

By the well known contraction principle, we can show that Λ has a unique fixed point

$$u \in PAA(\mathbb{R},\mathbb{X},\mu)$$

which satisfies

$$u(t) = -f(t,u(t)) + \int_{\mathbb{R}} \Gamma(t,s)\,g(s,u(s))ds.$$

□

Application

Let μ be a measure with a Radon–Nikodym derivative ρ defined by

$$\rho(t) = \begin{cases} e^t & \text{if } t \le 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Since

$$\lim_{t\to+\infty}\frac{\rho(t+c)}{\rho(t)} = 1 \quad \text{and} \quad \lim_{t\to-\infty}\frac{\rho(t+c)}{\rho(t)} = e^c,$$

then by [4], μ satisfies **(M2)**. In addition, μ satisfies **(M1)** since

$$\limsup_{r\to\infty}\frac{2r}{\mu[-r,r]} = \limsup_{r\to\infty}\frac{2r}{\int_{-r}^{r}\rho(s)\,ds} = \limsup_{r\to\infty}\frac{2r}{1-e^{-r}+r} < \infty.$$

The fact that the derivative ρ is bounded implies that μ satisfies **(M3)**.

To illustrate the above results we examine the existence of μ-pseudo almost automorphic solutions to the following model:

$$\begin{cases} \frac{\partial}{\partial t}[u(t,\xi) - f(t,u(t,\xi))] = \frac{\partial^2}{\partial \xi^2}[u(t,\xi) - f(t,u(t,\xi))] \\ \quad + \alpha(t)[u(t,\xi) - f(t,u(t,\xi))] + g(t,u(t,\xi)), & t \in \mathbb{R},\ \ \xi \in [0,\pi] \\ u(t,0) = u(t,\pi) = 0, & t \in \mathbb{R} \\ u(t,0) - f(t,u(t,0)) = u(t,\pi) - f(t,u(t,\pi)) = 0, & t \in \mathbb{R}, \end{cases} \tag{13}$$

with

$$f(t,x) = a(t)\psi(x) + b(t)\varphi(x)$$

and

$$g(t,x) = c(t)\psi(x) + d(t)\varphi(x),$$

where $\alpha,\ a,\ c : \mathbb{R} \to \mathbb{R}$ are almost automorphic functions such that $\alpha(t) \le -M < 0$, for all $t \in \mathbb{R}$, $b \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$ and $d \in \mathcal{E}^2(\mathbb{R}, \mathbb{R}, \mu)$. The functions $\psi, \varphi : \mathbb{R} \to \mathbb{R}$ are bounded Lipschitz continuous. It is clear that f belongs to $PAA(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu)$ and satisfies:

$$|f(t,x) - f(t,y)| \le L_f\, |x-y| \quad \text{for all } \ t, x, y \in \mathbb{R}, \tag{14}$$

where $L_f = L_\psi\, |a|_\infty + L_\varphi\, |b|_\infty$. We can see also that g belongs to $PAA^2(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu)$ and satisfies:

$$|g(t,x) - g(t,y)| \le L(t)\, |x-y| \quad \text{for all } t, x, y \in \mathbb{R}, \tag{15}$$

where $L(t) = L_\psi\, |c(t)| + L_\varphi\, |d(t)|$. The boundedness of ρ implies that $L \in BS^2(\mathbb{R}, \mathbb{R}) \cap BS^1(\mathbb{R}, \mathbb{R}, \mu)$.

To represent the system (13) in the abstract form (1), we choose the space $\mathbb{X} = L^2([0,\pi], \mathbb{R})$, endowed with its natural topology. We also consider the operator $A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$, given by

$$A\phi = \phi'' \quad \text{for } \phi \in D(A),$$

where

$$D(A) = \left\{\phi \in \mathbb{X} :\ \phi'' \in \mathbb{X},\ \ \phi(0) = \phi(\pi) = 0\right\}.$$

Let us set for $t \in \mathbb{R}$ and $\xi \in [0,\pi]$:

$$\begin{aligned} U(t)(\xi) &:= u(t,\xi), \\ F(t,v)(\xi) &:= f(t, v(\xi)) \quad \text{for } v \in \mathbb{X}, \\ G(t,v)(\xi) &:= g(t, v(\xi)) \quad \text{for } v \in \mathbb{X}. \end{aligned}$$

Using (14) and (15), it is clear that F and G satisfies **(H3)** and **(H4)** with $p = r = 2$. Moreover, it is well known ([9]) that A is the generator of an analytic C_0-semigroup $\{T(t)\}_{t\ge 0}$ on $\mathbb{X}$ with $\|T(t)\| \le e^{-t}$, for $t \ge 0$.

Define a family of linear operators $A(t)$ by:

$$\begin{cases} D(A(t)) = D(A), \\ A(t)\phi = A\phi + \alpha(t)\phi \quad \text{for } \phi \in D(A). \end{cases}$$

Equation (13) takes the following abstract form

$$\frac{d}{dt}[U(t) - F(t, U(t))] = A(t)[U(t) - F(t, U(t))] + G(t, U(t)).$$

The operators $A(t)$ generate an evolution family $(U(t,s))_{t\geq s}$ given by

$$U(t,s)\phi = e^{\int_s^t \alpha(\tau)d\tau} T(t-s)\phi \quad \text{for all } \phi \in \mathbb{X} \text{ and } t \geq s,$$

with

$$\|U(t,s)\| \leq e^{-(M+1)(t-s)} \quad \text{for } t \geq s.$$

It follows that $(U(t,s))_{t\geq s}$ has an exponential dichotomy. Let $(s_n')_{n\geq 0}$ be a real sequence, then there is a subsequence $(s_n)_{n\geq 0} \subseteq (s_n')_{n\geq 0}$ and a real measurable function $t \to \tilde{\alpha}(t)$ such that for all $t \in \mathbb{R}$

$$\begin{cases} |\alpha(t+s_n) - \tilde{\alpha}(t)| \to 0 & \text{as } n \to +\infty, \\ |\tilde{\alpha}(t-s_n) - \alpha(t)| \to 0 & \text{as } n \to +\infty. \end{cases}$$

Consider $\tilde{A}(t) := A + \tilde{\alpha}(t)$, then we have

$$R(\lambda, A(t+s_n)) - R\left(\lambda, \tilde{A}(t)\right) = R(\lambda, A(t+s_n))\left[\alpha(t+s_n) - \tilde{\alpha}(t)\right] R\left(\lambda, \tilde{A}(t)\right).$$

It follows that

$$\begin{aligned} &\left\| R(\lambda, A(t+s_n)) - R\left(\lambda, \tilde{A}(t)\right) \right\| \\ &\leq \|R(\lambda, A(t+s_n))\| \, |\alpha(t+s_n) - \tilde{\alpha}(t)| \left\| R\left(\lambda, \tilde{A}(t)\right) \right\| \\ &\leq N\, |\alpha(t+s_n) - \tilde{\alpha}(t)| \to 0, \quad \text{as } n \text{ goes to } \infty. \end{aligned}$$

Similarly, we show that $\left\| R\big(\lambda, \tilde{A}(t-s_n)\big) - R\big(\lambda, A(t)\big) \right\| \to 0$. Therefore, the family $A(t)$ satisfies **(H2)**. Consequently all assumptions **(H0)**$-$**(H4)** are satisfied, by Theorem 5.6 we deduce that (13) has a unique μ-pseudo almost automorphic mild solution on $\mathbb{R}$, under the condition

$$\left(L_f + \|L\|_{BS^2(\mathbb{R},\mathbb{R})} \frac{2}{1-e^{-(M+1)}} \left(\frac{1-e^{-2(M+1)}}{2(M+1)} \right)^{\frac{1}{2}} \right) < 1.$$

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