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ORIGINAL RESEARCH

Stabilization of Modified Leslie–Gower Prey–Predator Model

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Abstract The stabilization problem of modified Leslie–Gower type prey–predator model with Holling-type II functional response is investigated. By approximate linearization approach, a feedback control law is obtained which stabilizes the closed loop system. On the other hand, by suitable change of coordinates in the state space, a feedback control law is obtained. This feedback control renders the complex nonlinear system to be linear controllable system such that the positive equilibrium point of the closed-loop system is globally asymptotically stable. Numerical experiments substantiate the analytic findings.

Keywords Prey–predator model · Exact linearization · Asymptotically stable · Feedback control

Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of dominant themes in both system biology and mathematical ecology due to its universal existence and importance. Predator–prey models are a classic and relatively well-studied example of interactions. In this optic, Lotka–Volterra model is one of the earliest predator–prey models based on sound mathematical and ecological principles. It forms the basis of many models in the analysis of population dynamics and received extensive attentions from mathematicians and ecologists [1–4]. In Lotka–Volterra model, the predator is also assumed to be growing logistically with carrying capacity depending on the availability of variable resource (prey). This formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. Based on experiments, Holling [5] suggested three different kinds of functional responses

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(functional response describes the relationship between an individual's rate of consumption and food density) for different species to model the phenomena of predation, which made the standard Lotka–Volterra system more realistic. Many authors investigated the mathematical properties of these models and explained their implication in biology [6–8]. Biologically, it is quite natural to study the existence and asymptotic stability of equilibria, and limit cycles for autonomous predator–prey systems with these functional responses. This prompted us to study the predator–prey system with Holling type-II functional response. For Holling type-II functional response, the predation rate increases as prey density rises, eventually levels off due to the predators handling time. The model also incorporates a modified version of the Leslie–Gower functional response [9–13]. The Leslie–Gower predator–prey model formulation is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. Indeed, Leslie [14] introduced a predator–prey model where the carrying capacity of the predators environment is proportional to the number of prey. This interesting formulation for the predator dynamics has been discussed in Refs. [15] and [16]. One of the most basic and important problems in ecology concerns the survival of species. Owing to the fact that the ecosystems are usually subjected to a lot of perturbation in the real world, it is indispensable to exert suitable control in order to maintain the balance in ecological models. Some investigators applied optimal control techniques to the management of renewable resources in ecosystem [17–19]. Furthermore, some authors studied the optimal control harvesting or optimal control of ecosystem [20,21].

The study on feedback control in complex systems has received considerable interest after the seminal work of Ott et al. [22]. From the point of view of biology, this method seems to be promising due to its simplicity and convenience. However,to the best of our knowledge, the application of feedback linearization control in this field is relatively new [23,24]. In the past few years, feedback linearization approach based on differential geometry provides effective analysis and design of nonlinear systems in engineering. With reference to ecological system, very little has been done so far [25].

This paper investigates the results of modified Leslie–Gower type prey–predator model with Holling type-II functional response, showing that appropriately chosen control approach can achieve global stability of equilibrium states.

The prey–predator dynamics is governed by the following system of differential equations:

$$\begin{aligned} \frac{dX}{dT} &= X\left(a_1 - bX - \frac{c_1 Y}{X + k_1}\right), \\ \frac{dY}{dT} &= Y\left(a_2 - \frac{c_2 Y}{X + k_2}\right). \end{aligned} \tag{1}$$

Here, the prey species $X(t)$ is growing logistically with Holling-type II functional response and modified Leslie–Gower type growth is considered for the predator $Y(t)$ [10]. All model parameters are assumed to be positive and have usual meaning as follows:

a_1 is the logistic growth and b is the strength of interspecies competition amongst the prey. The constant c_1 represents consumption of prey per predator and k_1 is the extent of protection from predator provided to prey species by the environment. For predator species, a_2 describes the logistic growth rate, c_2 is crowding effect and k_2 signifies the extent of another option in surroundings for predation apart from X.

The number of parameters can be reduced from 6 to 4 by the following scaling transformations:

$$\begin{aligned} &X = a_1 x/b, \quad Y = a_1^2 y/bc_1, \quad T = t/a_1 \quad \text{and} \\ &a = b_1 K_1/a_1, \quad q = c_2/c_1, \quad p = a_2/a_1, \quad r = bK_2/a_1. \end{aligned}$$

Accordingly, the non-dimensional system takes the form

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{x+a},$$
$$\frac{dy}{dt} = y\left(p - \frac{qy}{x+r}\right). \tag{2}$$

The non-negative initial conditions are associated with system (2)

$$x \geq 0, \quad y \geq 0. \tag{3}$$

Preliminaries

In this section, positive invariance and boundedness for the system (2) are established. Since the state variables x and y represent populations, positivity insures that they never become negative and population always survive. The boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources.

Positive Invariance

Theorem 1 *The positive quadrant int*$(\mathbb{R}_+^2)$ *is invariant for system* (2).

Proof It is observed that boundaries of non-negative quadrant $\mathbb{R}_+^2$ are invariant; this is obvious from system $\mathbb{R}_+^2$. Therefore, the densities $x(t)$, $y(t)$ are positive for $t \geq 0$, if $x(0) > 0$, $y(0) > 0$ then $x(t) > 0$, $y(t) > 0$. The basic existence and uniqueness theorem for differential equation ensures that positive solution and axis cannot intersect [26] i.e. any trajectory starting in $\mathbb{R}_+^2$ cannot cross the coordinate planes. □

Boundedness

Theorem 2 *All the solutions of the system* (2) *with initial conditions* (3) *that initiate in* $\mathbb{R}_+^2$ *are uniformly bounded.*

Proof Let us define

$$W = x + y.$$

The time derivative gives

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = x(1-x) - \frac{xy}{x+a} + y\left(p - \frac{qy}{x+r}\right).$$

For any $L > 0$,

$$\frac{dW}{dt} + LW \leq x(1+L-x) + y\left(p + L - \frac{qy}{1+r}\right) \leq \frac{(1+L)^2}{4} + \frac{(1+r)(p+L)^2}{4}.$$

Thus we can define a constant $M > 0$ such that

$$M = \frac{(1+L)^2}{4} + \frac{(1+r)(p+L)^2}{4} > 0.$$

This shows that

$$\frac{dW}{dt} + LW \leq M.$$

Applying the theory of differential inequality [27], it is obtained

$$0 < W(x, y) \leq \frac{M}{L}(1 - e^{-Lt}) + e^{-Mt} W(x(0), y(0)).$$

Thus for $t \to \infty$, $0 < W(x, y) \leq \frac{M}{L}$. Hence all solutions of (2) that initiate in $\mathbb{R}^2_+$ are confined in the region

$$B = \left\{ (x, y) \in \mathbb{R}^2_+ : W = \frac{M}{L} + \xi \quad \text{for any } \xi > 0 \right\}$$

for all $t \geq T$, where T depends on the initial values $(x(0), y(0))$. This proves the theorem. □

Existence and Local Stability of Equilibrium Points

The system (2) has following three trivial boundary equilibria:

$$E_0 = (0, 0), \quad E_x = (1, 0), \quad E_y = \left(0, \frac{pr}{q}\right).$$

The nonzero unique interior equilibrium point $E^* = (x^*, y^*)$ exists if $pr < qa$ as

$$x^* = \frac{-(aq + p - q) + \sqrt{(aq + p - q)^2 - 4q(pr - qa)}}{2q}, \quad y^* = \frac{p(x^* + r)}{q}.$$

The variational matrix around an arbitrary equilibrium point $(\hat{x}, \hat{y})$ is obtained as

$$\begin{pmatrix} 1 - 2\hat{x} - \frac{a\hat{y}}{(\hat{x}+a)^2} - \lambda & -\frac{\hat{x}}{(\hat{x}+a)} \\ \frac{q\hat{y}^2}{(\hat{x}+r)^2} & p - 2\frac{q\hat{y}}{(\hat{x}+r)} - \lambda \end{pmatrix}.$$

The behavior of boundary equilibrium points and interior equilibrium point of system (2) can be summarized in the following results [28]:

(i) The steady state $E_0(0, 0)$ is unstable node.
(ii) The steady state E_x is saddle which is stable in x-direction and unstable in y-direction.
(iii) The steady state E_y is asymptotically stable provided $pr > qa$. E_y is globally asymptotically stable under above condition.
(iv) The non-trivial equilibrium point $E^*(x^*, y^*)$ if exists, is stable if

$$p > -x^* + \frac{px^*(x^* + r)}{q(x^* + a)^2}.$$

It undergoes Hopf bifurcation around $E^*(x^*, y^*)$ whenever

$$p = -x^* + \frac{px^*(x^* + r)}{q(x^* + a)^2}.$$

(v) The system (2) has unstable unique equilibrium point (x^*, y^*) under

$$p + x^* - \frac{px^*(x^* + r)}{q(x^* + a)^2} < 0. \tag{4}$$

The system (2) admits a limit cycle (closed loop) under condition (4).
The objective of next section is to stabilize the closed loop system (2) under (4) so as to reach a dynamic balance by feedback linearization design.

Feedback Linearization

Feedback linearization transforms original system into equivalent system of a simpler form. The idea is to algebraically transform nonlinear systems dynamics into simpler, fully or partly linear dynamics. This linearization approach differs entirely from conventional linearization (using Taylor series). The feedback linearization is achieved by exact state transformation and feedback [29], rather than by linear approximations of the dynamics. There are two types of approaches of feedback linearization: approximate linearization and exact linearization. These are discussed in the next two subsections.

Approximate Linearization

Here, approximate linearization approach is employed to design feedback control law and to solve local stability problem. Using Taylor expansion, a nonlinear system can always be linearized by a linear system to first degree. But, the idea of approximate linearization here is to approximate a nonlinear system up to highest possible degree.

Theorem 3 *For system* (2)*, there exists a smooth feedback control law which asymptotically stabilizes the closed-loop system:*

$$
\begin{aligned}
u &= k_1(x - x^*) + k_2(y - y^*),\\
k_1 &> \frac{\left(x^* - \frac{px^*(x^*+r)}{q(x^*+a)^2}\right)(-p + k_2)}{\left(\frac{x^*}{x^*+a}\right)} - \frac{p^2}{q},\\
k_2 &< p + x^* - \frac{px^*(x^*+r)}{q(x^*+a)^2}.
\end{aligned}
$$

Proof A linear control u be exerted on the system (2) as

$$
\begin{aligned}
\frac{dx}{dt} &= x(1 - x) - \frac{xy}{x + a},\\
\frac{dy}{dt} &= y\left(p - \frac{qy}{x + r}\right) + u.
\end{aligned} \tag{5}
$$

Using the transformation $v = x - x^*$ and $w = y - y^*$, the system (5) is transformed around the nontrivial equilibrium point $E^*(x^*, y^*)$ as

$$
\begin{aligned}
\dot{v} &= (v + x^*)\left(1 - (v + x^*)\right) - \frac{(v + x^*)(w + y^*)}{(v + x^*) + a},\\
\dot{w} &= (w + y^*)\left(p - \frac{(w + y^*)}{(v + x^*) + r}\right) + u.
\end{aligned} \tag{6}
$$

The linearized form (6) is written as

$$
\dot{\mathbf{U}} = A\mathbf{U} + Bu, \tag{7}
$$

where

$$
\mathbf{U} = \begin{pmatrix} v \\ w \end{pmatrix}, \quad A = \begin{pmatrix} -x^* + \frac{px^*(x^*+r)}{q(x^*+a)^2} & -\frac{x^*}{x^*+a} \\ \frac{p^2}{q} & -p \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

In linear feedback, each control variable is a linear combination of the state variables, so

$$u = K\mathbf{U}. \tag{8}$$

The row vector $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$, K is a constant feedback (or gain) matrix. Substituting (8) into (7), yields

$$\dot{\mathbf{U}} = (A + BK)\mathbf{U} = C\mathbf{U} \tag{9}$$

and

$$C = A + BK = \begin{pmatrix} -x^* + \frac{px^*(x^*+r)}{q(x^*+a)^2} & -\frac{x^*}{x^*+a} \\ \frac{p^2}{q} + k_1 & -p + k_2 \end{pmatrix}.$$

For the matrix C, the trace and determinant are computed as

$$\text{Trace } C = -p - x^* + \frac{px^*(x^* + r)}{q(x^* + a)^2} + k_2,$$
$$\det C = \left(-x^* + \frac{px^*(x^* + r)}{q(x^* + a)^2}\right)(-p + k_2) + \left(\frac{p^2}{q} + k_1\right)\left(\frac{x^*}{x^* + a}\right).$$

According to Routh–Hurwitz criterion, the necessary and sufficient conditions for stability of controlled system (9) are Trace $C < 0$, $\det C > 0$. For the closed loop system (2), condition (4) holds. However, the controlled system (9) has stable equilibrium state for the following choices of k_1 and k_2:

$$k_2 < p + x^* - \frac{px^*(r + x^*)}{q(a + x^*)^2} < 0 \tag{10}$$

and

$$k_1 > \frac{\left(x^* - \frac{px^*(x^*+r)}{q(x^*+a)^2}\right)(-p + k_2)}{\left(\frac{x^*}{x^*+a}\right)} - \frac{p^2}{q}. \tag{11}$$

This proves the result. □

As a matter of fact, approximate linearization approach employs Taylor series expansion at the equilibrium point to get the linear approximation. The error in linearization process is the main weakness of approximate linearization approach. In particular, the error will be enlarged with the extension of the domain of definition.

Exact Linearization

To overcome the defect of approximate linearization, in this section, a different scenario is considered, namely, the use of exact linearization scheme. It is achieved by exact state transformation. Unlike traditional approximate linearization method, exact linearization design does not ignore any higher order terms. Thus, the design is not only exact, but also globally valid.

Feedback linearization may be applied to nonlinear systems of the form

$$\begin{aligned} \dot{\mathbf{X}} &= f(\mathbf{X}) + g(X)\acute{u}, \quad \mathbf{X}(0) = \mathbf{X}_0, \\ \tilde{X} &= h(\mathbf{X}), \end{aligned} \tag{12}$$

where $\mathbf{X} \in R^n$ is the state vector, $\acute{u} \in R^m$ is the control input vector and the output vector $\tilde{X} \in R^m$ is a smooth function of $\mathbf{X}$. f, g are smooth vector fields on R^n with $f(0) = 0$.

The goal is to develop a state feedback control

$$\acute{u} = \alpha(\mathbf{X}) + \beta(\mathbf{X})v, \tag{13}$$

v being the external reference input and a change of variables $z = \Phi(\mathbf{X})$, with a nonsingular Jacobian, that transforms the nonlinear system into a linear controllable system.

Accordingly, the system (2) admits the form

$$\dot{\mathbf{X}} = \begin{pmatrix} x\,(1-x) - \frac{xy}{x+a} \\ y\left(p - \frac{qy}{x+r}\right) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \acute{u}, \tag{14}$$

where $\acute{u}$ is exerted control and an output $\tilde{X} = x - x^*$ is introduced, which denotes tracking of prey species.

The next theorem provides feedback control law for control the closed loop system (2). For its proof, the following definitions/results will be required:

(i) $L_\Psi \Phi(\mathbf{X})$ denotes the Lie derivative of the vector function $\Phi(\mathbf{X})$ along the vector field Ψ:

$$L_\Psi \Phi(\mathbf{X}) \triangleq \Sigma_{i=1}^{n} \frac{\partial \Phi}{\partial \mathbf{X}} \Psi_i(\mathbf{X}).$$

The constant r designates the relative degree of $X = h(X)$ if and only if $L_g L_f^{r-1} h(\mathbf{X}) \neq 0$ [29].

(ii) The system (12) has relative degree r, then row vectors $dh(\mathbf{X}), dL_f h(\mathbf{X}), \ldots\ d^{r-1} L_f h(\mathbf{X})$ are linearly independent [30].

(iii) A necessary and sufficient condition for the system $\dot{\mathbf{X}} = A\mathbf{X}$ to be globally asymptotically stable is that, for any symmetric positive definite matrix Q, there exists a positive definite matrix P that satisfies the Lyapunov equation $A^T P + PA = -Q$.

Theorem 4 *If there exists a feedback control law as*

$$\acute{u} = \frac{(x - x^*)(x + a)}{x} + \frac{ay^2}{(x+a)^2} + \frac{qy^2}{(x+r)} - \frac{ay(1-x)}{(x+a)} \\ -py + 2((1-x)^2(x+a) - y), \tag{15}$$

then the closed loop system (14) *is globally asymptotically stable.*

Proof Make a transformation $\bar{x} = x - x^*$, $\bar{y} = y - y^*$, system (14) can be modified in the form

$$\dot{\bar{\mathbf{X}}} = f(\bar{\mathbf{X}}) + g(\bar{\mathbf{X}})\acute{u}, \\ \tilde{X} = h(\bar{\mathbf{X}}) = \bar{x}, \tag{16}$$

where

$$f(\bar{\mathbf{X}}) = \begin{pmatrix} (\bar{x} + x^*)\,(1 - (\bar{x} + x^*)) - \frac{(\bar{x}+x^*)(\bar{y}+y^*)}{(\bar{x}+x^*)+a} \\ (\bar{y} + y^*)\left(p - \frac{(\bar{y}+y^*)}{(\bar{x}+x^*)+r}\right) \end{pmatrix}, \quad g(\bar{\mathbf{X}}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{and} \quad \bar{\mathbf{X}} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.$$

As $h(\bar{\mathbf{X}}) = \bar{x}$, then

$$L_f h(\bar{\mathbf{X}}) = \dot{\bar{x}} = x(1-x) - \frac{xy}{(x+a)}, \tag{17}$$

obviously,

$$L_g L_f^{r-1} h(\bar{\mathbf{X}}) = L_g L_f^{2-1} h(\bar{\mathbf{X}}) = -\frac{x}{x+a} \neq 0. \tag{18}$$

Accordingly, the relative degree, r is equal to 2.
Now, Change of variables

$$z = \Phi(\bar{\mathbf{X}}) = \begin{pmatrix} h(\bar{\mathbf{X}}) \\ L_f h(\bar{\mathbf{X}}) \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix}, \tag{19}$$

$h(\bar{\mathbf{X}})$ and $L_f h(\bar{\mathbf{X}})$ are linearly independent. From result (ii), it is global diffeomorphism. In the new z-coordinate system is described

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= v, \end{aligned} \tag{20}$$

where v is a input that is related to actual input u by

$$v = L_f^2 h(\bar{\mathbf{X}}) + L_g L_f h(\bar{\mathbf{X}}) \acute{u}, \quad L_g L_f h(\bar{\mathbf{X}}) \neq 0,$$

or

$$\acute{u} = \frac{1}{L_g L_f h(\bar{\mathbf{X}})} (-z_1 - z_2 - L_f^2 h(\bar{\mathbf{X}})). \tag{21}$$

This is called Brunovsky canonical form (chain of integrators) [31].

The system (2) becomes equivalent to the linear system. Since the Brunovsky linear system is fully controllable [31], using result (iii), system (2) is globally asymptotic stable.

By using (19), (17) and (18), $\acute{u}$ can be obtained from (21),

$$\acute{u} = \frac{1}{\left(-\frac{x}{x+a}\right)} \left[-(x - x^*) - \left(x(1-x) - \frac{xy}{(x+a)} \right) - L_f \left(x(1-x) - \frac{xy}{(x+a)} \right) \right]. \tag{22}$$

Control law is, thus, obtained in the form

$$\begin{aligned} \acute{u} = \frac{(x-x^*)(x+a)}{x} + \frac{ay^2}{(x+a)^2} + \frac{qy^2}{(x+r)} - \frac{ay(1-x)}{(x+a)} \\ -py + 2\Big((1-x)^2(x+a) - y \Big). \end{aligned} \tag{23}$$

□

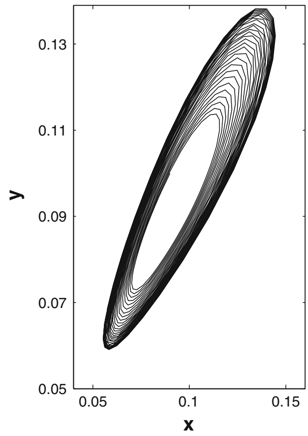
Numerical Simulation

In this section, numerical experiments are performed to substantiate the stabilization of the system (2). The parameters are chosen as follows [32]:

$$a = 0.01, \quad p = 2.0, \quad q = 0.721, \quad r = 0.001. \tag{24}$$

For this choice of parameters, the equilibrium point $E^*(0.004, 0.014)$ of the system (2) is unstable as condition (4) is satisfied. Figures 1 and 2 show a closed loop (unstable focus) and corresponding oscillating time series, respectively for system (2).

Fig. 1 Closed loop (unstable focus) of original system (2)



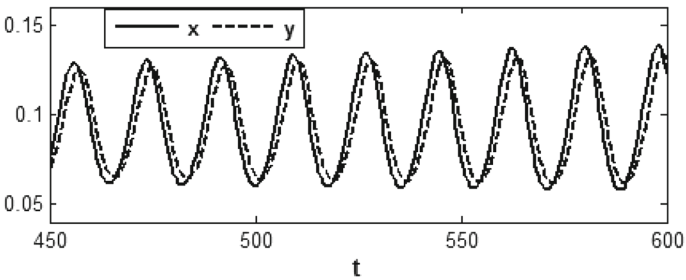


Fig. 2 Oscillating time series of original system (2)

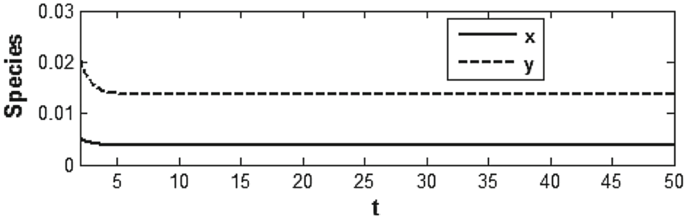


Fig. 3 State curve of system (2) after approximate linearization

When feedback control is applied, then the feedback control law $u = K\mathbf{U},\ K = \left(2\ -\frac{1}{2}\right)$ is obtained by approximate linearization and closed loop system gets asymptotically stabilized. The time series, plotted in Fig. 3, shows the asymptotic stability of the closed loop system (2).

From Theorem 3.2, the system (2) is globally asymptotically stable under feedback control law (23),

$$\acute{u} = \frac{x(x+0.01)}{x} + \frac{0.01y^2}{(x+0.01)^2} + \frac{0.721y^2}{(x+0.001)} - \frac{0.01y(1-x)}{(x+0.01)} \\ -0.721y + 2\bigg((1-x)^2(x+0.01) - y\bigg). \tag{25}$$

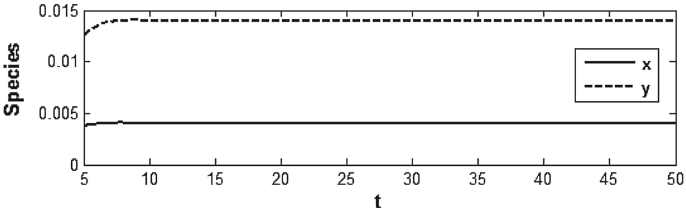


Fig. 4 State curve of system (2) after exact linearization

Stable state curves are drawn in Fig. 4 which show the asymptotic stability to $E^*(0.004, 0.014)$ of the system (2) through exact linearization.

Conclusions

In this paper, it is analyzed the stability of closed loop system using feedback linearization. It is shown that appropriate control can change the undesirable behavior and a dynamic balance can be achieved. The approximate linearization approach is used for local stabilization while exact linearization approach is used for global stability. The main feature of the exact linearization approach is to algebraically transform a nonlinear dynamics into a linear one, so that the well-developed linear control techniques can be applied. Numerical simulations are also given to substantiate the analytical findings. It is expected that the results will contribute to construct an effective control policy to make the system permanent.

Linearization approaches are important methods of dealing with nonlinear dynamical models. Now, biological systems may also be studied by this approach.

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