

Depletion and Conservation of Forestry Resources: A Mathematical Model

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Abstract A mathematical model to study the depletion and conservation of forestry biomass in presence of industrialization is proposed and analyzed. In the modeling process, it is assumed that due to forestry biomass industries migrate in the region under consideration and their growth increases due to the availability of forestry biomass. It is also assumed that both the intrinsic growth rate and carrying capacity of forestry biomass depend on industrialization and technological efforts. The model analysis shows that the increase in carrying capacity of forestry biomass due to technological efforts has destabilizing effect. It is found that, by taking growth rate coefficient of carrying capacity of forestry biomass due to technological efforts as a bifurcation parameter, periodic solutions may arise through Hopf-bifurcation. Further, using center manifold theory, the direction and stability of periodic solutions have also been discussed. Numerical simulation is performed to support analytic results.

Keywords Mathematical model · Forestry biomass · Industrialization · Stability · Hopf-bifurcation

Introduction

Forests are important natural resource which maintains the ecological balance in ecosystem. Forests reduce carbon dioxide from environment through photosynthesis and thus global warming. During photosynthesis process trees absorb carbon dioxide and produce oxygen, which is vital for the survival of various species including human on earth. Apart from this important phenomenon, forest also restricts soil erosion and maintains ground water level, biodiversity, etc. [13]. The population of various countries is significantly increasing and thus industrialization [8]. This rapidly growing industrialization affects the forests

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negatively, as the trees are cut down to provide raw material for products and land being cleared to develop new factories. The pollutants emitted from industries also affect the growth rate of forest biomass. Due to continuous increase in industrialization, forests are depleting and now the depletion of forests has become a major problem. A typical example for deforestation due to industrialization is Doon Valley in Uttarakhand, India [20,21]. There are some industries (e.g. pulp-paper, timber, furniture, etc.), those are wholly dependent on forestry biomass. Industrialization affects the growth rate of forestry biomass in two ways, namely; (i) reducing its intrinsic growth rate, and (ii) reducing the forest area and thus the carrying capacity. The emission of pollutants from industries decreases both the intrinsic growth rate and carrying capacity but in the case of development of new industries, the required land is obtained by clearing the forests, which decrease only the carrying capacity of forestry biomass in the region under consideration. Thus for the conservation of forestry biomass it is necessary to increase the carrying capacity of forestry biomass along with its intrinsic growth rate by using appropriate technological efforts. These technological efforts may be considered as genetically engineered plants, which increase the intrinsic growth rate and the carrying capacity of forestry biomass in the region under consideration.

In the last few decades, some mathematical models have been proposed and analyzed to study the depletion of forestry biomass due to population [3,12,19,22,24], pollution [4,18,23] and industrialization ([17,20,21] and the references therein). In particular, Shukla and Dubey [22] presented a mathematical model for the depletion of forest biomass due to population and pollution. In this study, it is shown that the density of forestry biomass decreases as the population or pollution increases and it may extinct if the population and pollution increase without control. They have also proposed a model for the conservation of density of forest biomass and it is shown that it can be maintained at a desired level if appropriate efforts are applied to conserve forest biomass and control the population and pollution. Damage of ecosystem can be minimized by using taxation, license fees, lease of property rights, seasonal harvesting, etc. [16]. Naresh et. al. [18] have studied the effects of toxicant on plant biomass by using a non-linear mathematical model. In this study, it is shown that as the emission rate of pollutants increases, the density of biomass decreases. Toxicants are affecting the growth of plants and the proper dose of fertilizers is the best way to ensure the optimal growth of plants [1]. Dubey et. al [5,8,9] have proposed and analyzed mathematical models to study the depletion of forestry resources due to population, pollution and population pressure augmented industrialization. Further the survival of biological species, which depends on the resource biomass, has been studied by using mathematical models [6,10].

The conservation of forestry biomass is very important for maintaining the ecological balance in ecosystem but a little attention has been paid on this aspect. Keeping this in mind, in this paper, we propose and analyze a nonlinear mathematical model to study the depletion of forestry biomass due to industrialization and its conservation by using some technological efforts.

The organization of the paper is as follows. In the next section we develop our model. In “Equilibrium Analysis” section, we derive criteria for the existence of equilibria. Local stability behavior and the existence of Hopf-Bifurcation are discussed in “Stability Analysis” section and “Existence of Hopf-Bifurcation” sections respectively. In “Direction and Stability of Hopf-Bifurcation” section, we have shown the direction and stability of periodic solutions arising through Hopf-bifurcation. A numerical example to illustrate our results is discussed in “Numerical Simulation” section. A brief discussion takes place in the final section.

Mathematical Model

We consider a forest habitat where forests are depleted due to rapid increase in industrialization. It is assumed that the density of forestry biomass is governed by generalized logistic model. It is further assumed that the intrinsic growth rate as well as carrying capacity of the forestry biomass are depleted due to industrialization and some technological efforts are used to increase them. Again it is considered that some forest based industries migrate in the region and their growth rate increases in accordance with the density of forestry biomass. Thus, the system dynamics is governed by following non-linear ordinary differential equations:

$$\begin{aligned}
 \frac{dB}{dt} &= sB \left(1 - \frac{B}{L}\right) - \beta BI - \beta_1 B^2 I + \phi_1 BT + \phi_2 B^2 T, \\
 \frac{dI}{dt} &= \nu B + \theta \beta BI + \theta_1 \beta_1 B^2 I - \theta_m I^2, \\
 \frac{dT}{dt} &= \phi (L - B) - \phi_0 T,
 \end{aligned}
 \tag{1}$$

where $B(0) \geq 0, I(0) \geq 0, T(0) \geq 0$.

In the model system (1), B is the density of forestry biomass, I is the density of industrialization and T is the intensity of technological efforts applied for the conservation of forestry biomass. The constants s and L are intrinsic growth rate and carrying capacity of forestry biomass respectively in absence of industrialization. The constants β and β_1 represent the depletion rate coefficients in intrinsic growth rate and carrying capacity of forestry biomass respectively, due to industrialization. The constants ϕ_1 and ϕ_2 are the growth rate coefficients of intrinsic growth rate and carrying capacity of forestry biomass respectively due to technological efforts. The constants β_1 and ϕ_2 affect only the carrying capacity of forestry biomass (it has been discussed further in detail). In model system (1), ν is the growth-rate coefficient of industrialization caused by migration of industries to the forest region under consideration, which is assumed to be proportional to the density of forestry biomass. The coefficients θ and θ_1 are the proportionality constants which represent growth of industrialization due to forestry biomass. The constant $\theta_m > 0$ corresponds to the intraspecific interference coefficient between industries. The constants ϕ and ϕ_0 are growth rate and natural depletion rate coefficients of technological efforts applied for the conservation of forestry biomass respectively.

For the explanation of the terms βBI and $\phi_1 BT$ in the first equation of model system (1), reader is referred to Dubey et al. [9, 11] and Dubey [4]. In the following, we provide an explanation for the terms $\beta_1 B^2 I$ and $\phi_2 B^2 T$. As we know that the logistic equation for the growth rate of forestry biomass with intrinsic growth rate s and carrying capacity L is

$$\frac{dB}{dt} = sB \left(1 - \frac{B}{L}\right).
 \tag{2}$$

By incorporating the terms $\beta_1 B^2 I$ and $\phi_2 B^2 T$, the above equation modifies as

$$\begin{aligned}
 \frac{dB}{dt} &= sB \left(1 - \frac{B}{L}\right) - \beta_1 B^2 I + \phi_2 B^2 T \\
 &= sB - \left(\frac{s}{L} + \beta_1 I - \phi_2 T\right) B^2 \\
 &= sB \left(1 - \frac{B}{L_m}\right),
 \end{aligned}
 \tag{3}$$

where L_m is the modified carrying capacity of forestry biomass and is given by the relation

$$\frac{1}{L_m} = \left(\frac{1}{L} + \frac{\beta_1 I - \phi_2 T}{s} \right).$$

From this relation it is evident that L_m decreases as industrialization increases and it increases as technological efforts increases. Thus from the right hand side of Eq. (3) we can say that the terms $\beta_1 B^2 I$ and $\phi_2 B^2 T$ affect only the carrying capacity of forestry biomass without affecting its intrinsic growth rate.

Equilibrium Analysis

The above model system (1) has following two non-negative equilibria.

$$(i) E_0 \left(0, 0, \frac{\phi L}{\phi_0} \right) \quad \text{and} \quad (ii) E_1(B^*, I^*, T^*).$$

The existence of equilibrium E_0 is obvious, hence omitted. In the following, we prove the existence of equilibrium E_1 .

Existence of $E_1(B^*, I^*, T^*)$

In the equilibrium E_1 , the values of B^* , I^* and T^* are the positive solutions of the following algebraic equations:

$$s \left(1 - \frac{B}{L} \right) - (\beta + \beta_1 B)I + (\phi_1 + \phi_2 B)T = 0, \tag{4}$$

$$\theta_m I^2 - (\theta\beta B + \theta_1\beta_1 B^2)I - \nu B = 0, \tag{5}$$

$$\phi(L - B) - \phi_0 T = 0. \tag{6}$$

From Eq. (5), we get

$$\begin{aligned} I &= \frac{1}{2\theta_m} \left[(\theta\beta B + \theta_1\beta_1 B^2) + \sqrt{(\theta\beta B + \theta_1\beta_1 B^2)^2 + 4\theta_m \nu B} \right] \\ &= f(B) \text{ (say)}. \end{aligned} \tag{7}$$

Again from Eq. (6), we have

$$T = \frac{\phi(L - B)}{\phi_0}. \tag{8}$$

By using above values of I and T in Eq. (4), we get the following equation in B :

$$F(B) = s \left(1 - \frac{B}{L} \right) - (\beta + \beta_1 B)f(B) + (\phi_1 + \phi_2 B) \frac{\phi(L - B)}{\phi_0} = 0. \tag{9}$$

From Eq. (8), we may note that $T < 0$ for $B > L$. Thus for positive value of T^* , we are interested in a root of Eq. (9), which lies in the open interval $(0, L)$.

Now from Eq. (9), we may easily note the following facts:

- (i) $F(0) = s + \phi\phi_1 L / \phi_0$, which is positive,
- (ii) $F(L) = -(\beta + \beta_1 L)f(L)$, which is negative.

The above points (i) and (ii) together imply that there exists a positive root ($B = B^*$) of Eq. (9) in the interval $(0, L)$. Further this root will be unique provided $F'(B)$ is negative in $(0, L)$.

After knowing the value of $B = B^*$, we get the positive values of $I = I^*$ and $T = T^*$ from Eqs. (7) and (8) respectively.

Stability Analysis

The local stability behavior of equilibria E_0 and E_1 may be determined by finding the sign of eigenvalues of the corresponding Jacobian matrix. For this purpose, the Jacobian matrix J for the model system (1) is given as follows:

$$J = \begin{pmatrix} -a_{11} & -(\beta + \beta_1 B)B & (\phi_1 + \phi_2 B)B \\ b_{11} & -b_{12} & 0 \\ -\phi & 0 & -\phi_0 \end{pmatrix},$$

where

$$a_{11} = s \left(1 - \frac{2B}{L} \right) - \beta I - 2\beta_1 B I + \phi_1 T + 2\phi_2 B T,$$

$$b_{11} = (\nu + \theta\beta I + 2\theta_1\beta_1 B I), \quad b_{12} = (-\theta\beta B - \theta_1\beta_1 B^2 + 2\theta_m I).$$

Let J_i be the matrix J , evaluated at the equilibrium E_i ($i = 0, 1$). Now from matrix J_0 , we may note that one eigenvalue of this matrix is always positive, so it is unstable in B -direction.

The stability behavior of interior equilibrium E_1 follows from the following theorem:

Theorem 1 *The interior equilibrium E_1 is locally asymptotically stable iff the following conditions hold:*

$$A_1 > 0, A_3 > 0 \quad \text{and} \quad A_1 A_2 - A_3 > 0. \tag{10}$$

Proof The characteristic equation for J_1 is given as follows:

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0, \tag{11}$$

where

$$A_1 = a_1 + \phi_0 + b_2,$$

$$A_2 = a_1 \phi_0 + a_1 b_2 + b_2 \phi_0 + a_2 b_1 + a_3 \phi,$$

$$A_3 = a_1 b_2 \phi_0 + a_2 b_1 \phi_0 + a_3 b_2 \phi,$$

In the above expressions of A_1, A_2 and A_3 , the values of a_1, a_2, a_3, b_1 and b_2 are given as follows:

$$a_1 = \left(\frac{s}{L} + \beta_1 I^* - \phi_2 T^* \right) B^*, \quad a_2 = (\beta + \beta_1 B^*) B^*, \quad a_3 = (\phi_1 + \phi_2 B^*) B^*,$$

$$b_1 = (\nu + \theta\beta I^* + 2\theta_1\beta_1 B^* I^*), \quad b_2 = \left(\frac{\nu B^*}{I^*} + \theta_m I^* \right).$$

Here it may be noted that if the condition

$$\frac{s}{L} + \beta_1 I^* - \phi_2 T^* > 0 \tag{12}$$

is satisfied then all the conditions stated in (10) are satisfied. Now using Routh–Hurwitz criterion, we can say that the interior equilibria E_1 is locally asymptotically stable.

From the above condition (12), we may easily note that for large values of ϕ_2 this condition may not be satisfied and thus the conditions stated in (10) may also be violated. This implies that system (1) may lose its stability and Hopf-bifurcation may occur for large values of ϕ_2 . Keeping this in mind, we study the behavior of dynamical system (1) in detail with respect to this parameter. \square

Existence of Hopf-Bifurcation

Now we investigate the possibility of Hopf-bifurcation from the interior equilibrium E_1 by choosing ϕ_2 as a bifurcation parameter and leaving the remaining parameters fixed.

Theorem 2 *The necessary and sufficient conditions for the occurrence of Hopf-bifurcation from the interior equilibrium E_1 are that there exists a $\phi_2 = \phi_{2c}$ such that*

- (i) $A_1(\phi_{2c}) > 0$ and $A_3(\phi_{2c}) > 0$,
- (ii) $G(\phi_{2c}) = A_1(\phi_{2c})A_2(\phi_{2c}) - A_3(\phi_{2c}) = 0$,
- (iii) $\text{Re} \left[\frac{d(\lambda_j)}{d\phi_2} \right]_{\phi_2=\phi_{2c}} \neq 0$ for $j = 1, 2$.

Proof For $\phi_2 = \phi_{2c}$, we have $A_3 = A_1A_2$. Using this fact, the characteristic equation (11) reduces to the following form:

$$(\lambda + A_1)(\lambda^2 + A_2) = 0, \quad (13)$$

which has three roots $\lambda_{1,2} = \pm iw$ and $\lambda_3 = \mu$, where $\mu = -A_1 = -(a_1 + b_2 + \phi_0)$ and $w = \sqrt{A_2} = \sqrt{a_1\phi_0 + a_1b_2 + b_2\phi_0 + a_2b_1 + a_3\phi}$. Since $A_2 > 0$ at $\phi_2 = \phi_{2c}$ so for some $\epsilon > 0$ there exists an ϵ -neighborhood of ϕ_{2c} say $(\phi_{2c} - \epsilon, \phi_{2c} + \epsilon)$ for which $\phi_{2c} - \epsilon > 0$ such that $A_2 > 0$ for $\phi_2 \in (\phi_{2c} - \epsilon, \phi_{2c} + \epsilon)$. Thus for $\phi_2 \in (\phi_{2c} - \epsilon, \phi_{2c} + \epsilon)$, the characteristic equation (13) cannot have real positive root.

$$\begin{aligned} \lambda_1(\phi_2) &= u(\phi_2) + iv(\phi_2), \\ \lambda_2(\phi_2) &= u(\phi_2) - iv(\phi_2), \\ \lambda_3 &= -A_1(\phi_2). \end{aligned}$$

To show the transversality condition, we substitute $\lambda = u(\phi_2) + iv(\phi_2)$ in Eq. (11) and separating the real and imaginary parts we have

$$\begin{aligned} u^3(\phi_2) - 3u(\phi_2)v^2(\phi_2) + A_1(\phi_2)(u^2(\phi_2) - v^2(\phi_2)) \\ + A_2(\phi_2)u(\phi_2) + A_3(\phi_2) = 0 \end{aligned} \quad (14)$$

$$3u^2(\phi_2)v(\phi_2) - v^3(\phi_2) + 2A_1(\phi_2)u(\phi_2)v(\phi_2) + A_2(\phi_2)v(\phi_2) = 0. \quad (15)$$

Now differentiating (14) and (15) with respect to ϕ_2 , we get

$$\begin{aligned} F_1(\phi_2)u'(\phi_2) - F_2(\phi_2)v'(\phi_2) + F_3(\phi_2) &= 0, \\ F_2(\phi_2)u'(\phi_2) + F_1(\phi_2)v'(\phi_2) + F_4(\phi_2) &= 0, \end{aligned} \quad (16)$$

where

$$\begin{aligned} F_1(\phi_2) &= 3(u^2(\phi_2) - v^2(\phi_2)) + 2A_1(\phi_2)u(\phi_2) + A_2(\phi_2), \\ F_2(\phi_2) &= 6u(\phi_2)v(\phi_2) + 2A_1(\phi_2)v(\phi_2), \\ F_3(\phi_2) &= A_3'(\phi_2) + (u^2(\phi_2) - v^2(\phi_2))A_1'(\phi_2) + u(\phi_2)A_2'(\phi_2), \\ F_4(\phi_2) &= 2u(\phi_2)v(\phi_2)A_1'(\phi_2) + v(\phi_2)A_2'(\phi_2). \end{aligned}$$

Since $F_1(\phi_2)F_3(\phi_2) + F_2(\phi_2)F_4(\phi_2) \neq 0$, we have

$$\operatorname{Re} \left[\frac{d\lambda_j}{d\phi_2} \right]_{\phi_2=\phi_{2c}} = -\frac{F_1F_3 + F_2F_4}{F_1^2 + F_2^2} \neq 0$$

and

$$\lambda_3(\phi_{2c}) = -A_1(\phi_{2c}) \neq 0.$$

Therefore the transversality condition holds. This implies that Hopf-bifurcation occurs at $\phi_2 = \phi_{2c}$ (see [7, 15]). □

Direction and Stability of Hopf-Bifurcation

To analyze the direction and stability of Hopf-bifurcation, we will follow the method given in [14]. To describe the center manifold and analyze the flow therein, we first translate the origin of the coordinate system to the equilibrium (B^*, I^*, T^*) by writing

$$\bar{B} = B - B^*, \quad \bar{I} = I - I^*, \quad \bar{T} = T - T^*. \tag{17}$$

Then the model system (1) can be written in the form

$$\frac{d}{dt} \begin{pmatrix} \bar{B} \\ \bar{I} \\ \bar{T} \end{pmatrix} = J_1 \begin{pmatrix} \bar{B} \\ \bar{I} \\ \bar{T} \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}. \tag{18}$$

Here the nonlinear terms are

$$\begin{aligned} h_1 &= -\frac{s\bar{B}^2}{L} - \beta\bar{B}\bar{I} - \beta_1\bar{B}^2\bar{I} - \beta_1\bar{B}^2I^* - 2\beta_1B^*\bar{B}\bar{I} \\ &\quad + \phi_1\bar{B}\bar{T} + \phi_2\bar{B}^2\bar{T} + \phi_2\bar{B}^2T^* + 2\phi_2B^*\bar{B}\bar{T}, \\ h_2 &= \theta\beta\bar{B}\bar{I} + \theta_1\beta_1\bar{B}^2\bar{I} + \theta_1\beta_1\bar{B}^2I^* + 2\theta_1\beta_1B^*\bar{B}\bar{I} - \theta_m\bar{I}^2, \\ h_3 &= 0. \end{aligned}$$

If the eigenvectors of J_1 corresponding to $\lambda_{1,2}$ are $W_1 \pm iW_2$ and the eigenvector associated with λ_3 is W_3 (W_1, W_2 and W_3 are real), then it can be shown that the matrix $M = (W_1, W_2, W_3)$ is non-singular and

$$M^{-1}J_1M = \begin{pmatrix} 0 & w & 0 \\ -w & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

For model system (1), we have calculated the matrix M and it is given as follows:

$$M = \begin{pmatrix} a_3b_2 & a_3w & a_3(b_2 + \mu) \\ a_3b_1 & 0 & a_3b_1 \\ a_1b_2 + a_2b_1 - w^2 & w(a_1 + b_2) & (a_1 + \mu)(b_2 + \mu) + a_2b_1 \end{pmatrix}. \tag{19}$$

Now

$$M^{-1} = \frac{1}{D} \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = Q \quad (\text{say}), \tag{20}$$

where

$$\begin{aligned}
 q_{11} &= -a_3 b_1 w (a_1 + b_2), \\
 q_{12} &= -a_3 w (\mu^2 + a_2 b_1 - b_2^2), \\
 q_{13} &= b_1 w a_3^2, \\
 q_{21} &= -a_3 b_1 (\mu^2 + \mu a_1 + \mu b_2 + w^2), \\
 q_{22} &= a_3 (\mu b_2^2 + \mu^2 b_2 - \mu a_2 b_1 + \mu w^2 + b_2 w^2), \\
 q_{23} &= \mu a_3^2 b_1, \\
 q_{31} &= a_3 b_1 w (a_1 + b_2), \\
 q_{32} &= -a_3 w (b_2^2 + w^2 - a_2 b_1), \\
 q_{33} &= -a_3^2 b_1 w, \\
 \text{and } D &= -(\mu^2 + w^2) a_3^2 b_1 w.
 \end{aligned}$$

Now we use the linear transformation

$$\begin{pmatrix} \bar{B} \\ \bar{I} \\ \bar{T} \end{pmatrix} = M \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad (21)$$

which can be written as

$$\bar{S} = MK, \quad (22)$$

where $\bar{S} = (\bar{B} \bar{I} \bar{T})'$ and M is given by (19).

Now using (22) in Eq. (18), we get

$$\frac{d}{dt}(MK) = J_1 MK + F(MK), \quad (23)$$

where $F(\bar{S}) = (h_1, h_2, h_3)'$. From above equation, we have

$$\frac{dK}{dt} = (M^{-1} J_1 M) K + M^{-1} F(MK), \quad (24)$$

where $K = (k_1, k_2, k_3)'$.

Further we can write (24) in the following form:

$$\begin{aligned}
 \dot{x} &= H_1 x + G_1(x, y), \\
 \dot{y} &= H_2 y + G_2(x, y),
 \end{aligned} \quad (25)$$

where

$$x = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad y = (k_3), \quad H_1 = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}, \quad H_2 = (\mu), \quad G_1 = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} \quad \text{and} \quad G_2 = (g^3).$$

Here both G_1 and G_2 are C^2 functions.

Now system (24) can be written as follows:

$$\frac{d}{dt} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 & w & 0 \\ -w & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} + Q \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}. \quad (26)$$

Now we use the following theorems.

Theorem 3 System (26) has a local center manifold $y = \eta(x)$, $|x| < \delta$ where η is C^2 .

The function $\eta(x)$ can be approximated to any degree of accuracy as proved in the following theorem:

Theorem 4 Let ψ be a C^1 mapping of a neighborhood of the origin in R^2 into R with $\psi(0) = 0$, $\psi'(0) = 0$ and $(N\psi)(x) = O(|x|^p)$ as $x \rightarrow 0$, where $N\psi(x) = \psi'(x)[H_1x + G_1(x, \psi(x))] - H_2\psi(x) - G_2(x, \psi(x))$ and $p > 1$. Then $|\eta(x) - \psi(x)| = O(|x|^p)$ as $x \rightarrow 0$.

For the proofs of Theorems 3 and 4 see Carr [2].

Following the result of Theorem 4, we approximate the center manifold upto a quadratic approximation as

$$k_3 = \eta(k_1, k_2) = \frac{1}{2} (k_{11}k_1^2 + 2k_{12}k_1k_2 + k_{22}k_2^2) + \text{h.o.t.} \tag{27}$$

where h.o.t. stands for higher order terms.

From (27), it follows that

$$\begin{aligned} \frac{dk_3}{dt} &= \left(\frac{\partial \eta}{\partial k_1} \cdot \frac{dk_1}{dt} + \frac{\partial \eta}{\partial k_2} \frac{dk_2}{dt} \right) \\ &= w(-k_{11} + k_{22})k_1k_2 + wk_{12}k_1^2 - wk_{12}k_2^2, \end{aligned} \tag{28}$$

Also from (25), we have

$$\begin{aligned} \frac{dk_3}{dt} &= \mu k_3 + g^3, \\ &= \frac{\mu}{2} (k_{11}k_1^2 + 2k_{12}k_1k_2 + k_{22}k_2^2) + [Q_1k_1^2 + Q_2k_1k_2 + Q_3k_2^2], \end{aligned} \tag{29}$$

where

$$\begin{aligned} Q_1 &= \frac{1}{D} \left[q_{31} \left(-\frac{s}{L} p_{11}^2 - \beta p_{11} p_{21} - \beta_1 I^* p_{11}^2 - 2\beta_1 B^* p_{11} p_{21} + \phi_1 p_{11} p_{31} \right. \right. \\ &\quad \left. \left. + \phi_2 T^* p_{11}^2 + 2\phi_2 B^* p_{11} p_{31} \right) + q_{32} (\theta \beta p_{11} p_{21} + \theta_1 \beta_1 I^{*2} p_{11}^2 \right. \\ &\quad \left. + 2\theta_1 \beta_1 B^* p_{11} p_{21} - \theta_m p_{21}^2) \right], \\ Q_2 &= \frac{1}{D} \left[q_{31} \left(-\frac{2s}{L} p_{11} p_{12} - \beta p_{12} p_{21} - 2\beta_1 I^* p_{11} p_{12} - 2\beta_1 B^* p_{12} p_{21} \right. \right. \\ &\quad \left. \left. + \phi_1 (p_{11} p_{32} + p_{12} p_{31}) + 2\phi_2 T^* p_{11} p_{12} + 2\phi_2 B^* (p_{11} p_{32} + p_{12} p_{31}) \right) \right. \\ &\quad \left. + q_{32} (\theta \beta p_{12} p_{21} + 2\theta_1 \beta_1 I^* p_{11} p_{12} + 2\theta_1 \beta_1 B^* p_{12} p_{21}) \right], \\ Q_3 &= \frac{1}{D} \left[q_{31} \left(-\frac{s}{L} p_{12}^2 - \beta_1 I^* p_{12}^2 + \phi_1 p_{12} p_{32} + \phi_2 T^* p_{12}^2 + 2\phi_2 B^* p_{12} p_{32} \right) \right. \\ &\quad \left. + q_{32} \theta_1 \beta_1 I^* p_{12}^2 \right]. \end{aligned}$$

Equating (28) and (29) and comparing the coefficients k_1^2 , k_1k_2 and k_2^2 , we obtain a system of linear equations for k_{11} , k_{12} and k_{22} . On solving these equations, we get

$$\begin{aligned}
 k_{11} &= - \left[\frac{w^2(Q_1 + Q_3) + \frac{\mu}{2}(wQ_2 + \mu Q_1)}{\frac{\mu^3}{4} + w^2\mu} \right], \\
 k_{12} &= - \left[\frac{\frac{\mu^2 Q_2}{4} + \frac{w\mu}{2}(Q_3 - Q_1)}{\frac{\mu^3}{4} + w^2\mu} \right], \\
 k_{22} &= - \left[\frac{\frac{\mu^2 Q_3}{2} - \frac{w\mu Q_2}{2} + w^2(Q_1 + Q_3)}{\frac{\mu^3}{4} + w^2\mu} \right].
 \end{aligned}
 \tag{30}$$

The flow on the central manifold is governed by the following two-dimensional system:

$$\dot{x} = H_1x + G_1(x, \eta(x)).
 \tag{31}$$

Now, we state a theorem which tells that the behaviour of solutions of (31) contains all information about the behaviour of the solutions of (25).

Theorem 5 *Suppose that the zero solution of (31) is stable (asymptotically stable) (unstable), then the zero solution of (25) is stable (asymptotically stable) (unstable) [2].*

Now, (31) can be written as

$$\frac{d}{dt} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} g^1 \\ g^2 \end{pmatrix},
 \tag{32}$$

where

$$\begin{aligned}
 g^1 &= \frac{1}{D} (q_{11}h_1' + q_{12}h_2' + q_{13}h_3') + \text{h.o.t.}, \\
 g^2 &= \frac{1}{D} (q_{21}h_1' + q_{22}h_2' + q_{23}h_3') + \text{h.o.t.}
 \end{aligned}$$

and also the values of h_1', h_2' and h_3' are given below:

$$\begin{aligned}
 h_1' &= -\frac{s}{L}(p_{11}k_1 + p_{12}k_2)^2 - \frac{sp_{13}Z}{L}(p_{11}k_1 + p_{12}k_2) - \beta p_{21}k_1(p_{11}k_1 + p_{12}k_2) \\
 &\quad - \frac{1}{2}\beta p_{13}p_{21}Zk_1 - \frac{1}{2}\beta p_{23}Z(p_{11}k_1 + p_{12}k_2) - \beta_1 p_{21}k_1(p_{11}k_1 + p_{12}k_2)^2 \\
 &\quad - \beta_1 I^*(p_{11}k_1 + p_{12}k_2)^2 - \beta_1 I^* p_{13}Z(p_{11}k_1 + p_{12}k_2) - 2\beta_1 B^* p_{21}k_1(p_{11}k_1 + p_{12}k_2) \\
 &\quad - \beta_1 B^* p_{23}Z(p_{11}k_1 + p_{12}k_2) - \beta_1 B^* p_{13}p_{21}k_1Z + \phi_1(p_{11}k_1 + p_{12}k_2)(p_{31}k_1 + p_{32}k_2) \\
 &\quad + \frac{1}{2}\phi_1 p_{33}Z(p_{11}k_1 + p_{12}k_2) + \frac{1}{2}\phi_1 p_{13}Z(p_{31}k_1 + p_{32}k_2) + \phi_2 B^* p_{13}Z(p_{31}k_1 + p_{32}k_2) \\
 &\quad + \phi_2 T^*(p_{11}k_1 + p_{12}k_2)^2 + \phi_2 T^* p_{13}Z(p_{11}k_1 + p_{12}k_2) + \phi_2 B^* p_{33}Z(p_{11}k_1 + p_{12}k_2) \\
 &\quad + 2\phi_2 B^*(p_{11}k_1 + p_{12}k_2)(p_{31}k_1 + p_{32}k_2) + \phi_2(p_{11}k_1 + p_{12}k_2)^2(p_{31}k_1 + p_{32}k_2), \\
 h_2' &= \theta\beta p_{21}k_1(p_{11}k_1 + p_{12}k_2) + \frac{1}{2}\theta\beta p_{13}p_{21}Zk_1 + \frac{1}{2}\theta\beta p_{23}Z(p_{11}k_1 + p_{12}k_2) \\
 &\quad + \theta_1\beta_1 p_{21}k_1(p_{11}k_1 + p_{12}k_2)^2 + \theta_1\beta_1 I^*(p_{11}k_1 + p_{12}k_2)^2 + \theta_1\beta_1 I^* p_{13}Z(p_{11}k_1 + p_{12}k_2) \\
 &\quad + 2\theta_1\beta_1 B^* p_{21}k_1(p_{11}k_1 + p_{12}k_2) + \theta_1\beta_1 B^* p_{23}Z(p_{11}k_1 + p_{12}k_2) + \theta_1\beta_1 B^* p_{13}p_{21}Zk_1 \\
 &\quad - \theta_m p_{21}^2 k_1^2 - \theta_m p_{21} p_{23} k_1 Z, \\
 h_3' &= 0,
 \end{aligned}$$

here

$$Z = (k_{11}k_1^2 + 2k_{12}k_1k_2 + k_{22}k_2^2).$$

Now, the sign of the following quantity ‘ W ’ will determine the stability of limit cycle arising from Hopf-bifurcation:

$$W = \Gamma \operatorname{Re} \left[\frac{d\lambda_j}{d\phi_2} \right]_{\phi_2=\phi_{2c}} = \frac{1}{w} [g_{12}^1(g_{11}^1 + g_{22}^1) - g_{12}^2(g_{11}^2 + g_{22}^2) - g_{11}^1g_{11}^2 + g_{22}^1g_{22}^2] - [g_{111}^1 + g_{112}^2 + g_{122}^1 + g_{222}^2], \tag{33}$$

here g_{ij}^1 denotes the partial derivative $\partial^2 g^1 / \partial k_i \partial k_j$ at the origin and the quantities with three subscripts represent third-order partial derivatives evaluated at origin. If $\Gamma > 0$ ($\Gamma < 0$), then the Hopf-bifurcation is supercritical (subcritical). Further, if $W > 0$ ($W < 0$), the bifurcating limit cycle is stable (unstable). Now, we find that

$$g_{11}^1 = \frac{1}{D} \left[q_{11} \left(-\frac{2}{L} s p_{11}^2 - 2\beta p_{11} p_{21} - 2\beta_1 I^* p_{11}^2 - 4\beta_1 B^* p_{11} p_{21} + 2\phi_1 p_{11} p_{31} + 2\phi_2 T^* p_{11}^2 + 4\phi_2 B^* p_{11} p_{31} \right) + q_{12} (2\theta \beta p_{11} p_{21} + 2\theta_1 \beta_1 I^* p_{11}^2 + 4\theta_1 \beta_1 B^* p_{11} p_{21} - 2\theta_m p_{21}^2) \right], \tag{34}$$

$$g_{11}^2 = \frac{1}{D} \left[q_{21} \left(-\frac{2}{L} s p_{11}^2 - 2\beta p_{11} p_{21} - 2\beta_1 I^* p_{11}^2 - 4\beta_1 B^* p_{11} p_{21} + 2\phi_1 p_{11} p_{31} + 2\phi_2 T^* p_{11}^2 + 4\phi_2 B^* p_{11} p_{31} \right) + q_{22} (2\theta \beta p_{11} p_{21} + 2\theta_1 \beta_1 I^* p_{11}^2 + 4\theta_1 \beta_1 B^* p_{11} p_{21} - 2\theta_m p_{21}^2) \right], \tag{35}$$

$$g_{12}^1 = \frac{1}{D} \left[q_{11} \left(-\frac{2}{L} s p_{11} p_{12} - \beta p_{12} p_{21} - 2\beta_1 I^* p_{11} p_{12} - 2\beta_1 B^* p_{12} p_{21} + \phi_1 p_{12} p_{31} + \phi_1 p_{11} p_{32} + 2\phi_2 T^* p_{11} p_{12} + 2\phi_2 B^* p_{12} p_{31} + 2\phi_2 B^* p_{11} p_{32} \right) + q_{12} (\theta \beta p_{12} p_{21} + 2\theta_1 \beta_1 I^* p_{11} p_{12} + 2\theta_1 \beta_1 B^* p_{12} p_{21}) \right], \tag{36}$$

$$g_{12}^2 = \frac{1}{D} \left[q_{21} \left(-\frac{2}{L} s p_{11} p_{12} - \beta p_{12} p_{21} - 2\beta_1 I^* p_{11} p_{12} - 2\beta_1 B^* p_{12} p_{21} + \phi_1 p_{12} p_{31} + \phi_1 p_{11} p_{32} + 2\phi_2 T^* p_{11} p_{12} + 2\phi_2 B^* p_{12} p_{31} + 2\phi_2 B^* p_{11} p_{32} \right) + q_{22} (\theta \beta p_{12} p_{21} + 2\theta_1 \beta_1 I^* p_{11} p_{12} + 2\theta_1 \beta_1 B^* p_{12} p_{21}) \right], \tag{37}$$

$$g_{22}^1 = \frac{1}{D} \left[q_{11} \left(-\frac{2}{L} s p_{12}^2 - 2\beta_1 I^* p_{12}^2 + 2\phi_1 p_{12} p_{32} + 2\phi_2 T^* p_{12}^2 + 4\phi_2 B^* p_{12} p_{32} \right) + 2q_{12} \theta_1 \beta_1 I^* p_{12}^2 \right], \quad (38)$$

$$g_{22}^2 = \frac{1}{D} \left[q_{21} \left(-\frac{2}{L} s p_{12}^2 - 2\beta_1 I^* p_{12}^2 + 2\phi_1 p_{12} p_{32} + 2\phi_2 T^* p_{12}^2 + 4\phi_2 B^* p_{12} p_{32} \right) + 2q_{22} \theta_1 \beta_1 I^* p_{12}^2 \right], \quad (39)$$

$$g_{111}^1 = \frac{1}{D} \left[q_{11} \left(-\frac{6}{L} s p_{11} p_{13} k_{11} - 3\beta p_{11} p_{23} k_{11} - 3\beta p_{13} p_{21} k_{11} - 6\beta_1 p_{21} p_{11}^2 - 6\beta_1 I^* p_{11} p_{13} k_{11} - 6\beta_1 B^* p_{11} p_{23} k_{11} - 6\beta_1 B^* p_{13} p_{21} k_{11} + 3\phi_1 p_{13} p_{31} k_{11} + 3\phi_1 p_{11} p_{33} k_{11} + 6\phi_2 p_{11}^2 p_{31} + 6\phi_2 T^* p_{11} p_{13} k_{11} + 6\phi_2 B^* p_{11} p_{33} k_{11} + 6\phi_2 B^* p_{13} p_{31} k_{11} \right) + q_{12} (3\theta\beta p_{13} p_{21} k_{11} + 3\theta\beta p_{11} p_{23} k_{11} + 6\theta_1 \beta_1 p_{21} p_{11}^2 + 6\theta_1 \beta_1 I^* p_{11} p_{13} k_{11} + 6\theta_1 \beta_1 B^* p_{11} p_{23} k_{11} + 6\theta_1 \beta_1 B^* p_{13} p_{21} k_{11} - 6\theta_m p_{21} p_{23} k_{11}) \right], \quad (40)$$

$$g_{22}^2 = \frac{1}{D} \left[q_{21} \left(-\frac{6}{L} s p_{12} p_{13} k_{22} - 3\beta p_{12} p_{23} k_{22} - 6\beta_1 I^* p_{12} p_{13} k_{22} - 6\beta_1 B^* p_{12} p_{23} k_{22} + 3\phi_1 p_{13} p_{32} k_{22} + 3\phi_1 p_{12} p_{33} k_{22} + 6\phi_2 p_{12}^2 p_{32} + 6\phi_2 T^* p_{12} p_{13} k_{22} + 6\phi_2 B^* p_{12} p_{33} k_{22} + 6\phi_2 B^* p_{13} p_{32} k_{22} \right) + q_{22} (3\theta\beta p_{12} p_{23} k_{22} + 6\theta_1 \beta_1 I^* p_{12} p_{13} k_{22} + 6\theta_1 \beta_1 B^* p_{12} p_{23} k_{22}) \right], \quad (41)$$

$$g_{122}^1 = \frac{1}{D} \left[q_{11} \left(-\frac{4}{L} s p_{12} p_{13} k_{12} - \frac{2}{L} s p_{11} p_{13} k_{22} - \beta p_{13} p_{21} k_{22} - 2\beta p_{12} p_{23} k_{12} - \beta p_{11} p_{23} k_{22} - 2\beta_1 p_{21} p_{12}^2 - 4\beta_1 I^* p_{12} p_{13} k_{12} - 2\beta_1 I^* p_{11} p_{13} k_{22} - 4\beta_1 B^* p_{12} p_{23} k_{12} - 2\beta_1 B^* p_{11} p_{23} k_{22} - 2\beta_1 B^* p_{13} p_{21} k_{22} + 2\phi_1 p_{12} p_{33} k_{12} + \phi_1 p_{11} p_{33} k_{22} + 2\phi_1 p_{13} p_{32} k_{12} + \phi_1 p_{31} p_{13} k_{22} + 2\phi_2 p_{12}^2 p_{31} + 4\phi_2 p_{11} p_{12} p_{32} + 4\phi_2 T^* p_{12} p_{13} k_{12} + 2\phi_2 T^* p_{11} p_{13} k_{22} + 4\phi_2 B^* p_{12} p_{33} k_{12} + 2\phi_2 B^* p_{11} p_{33} k_{22} + 4\phi_2 B^* p_{13} p_{32} k_{12} + 2\phi_2 B^* p_{13} p_{31} k_{22} \right) + q_{12} (\theta\beta p_{11} p_{23} k_{22} + 2\theta\beta p_{12} p_{23} k_{12} + \theta\beta p_{13} p_{21} k_{22} + 2\theta_1 \beta_1 p_{21} p_{12}^2 + 4\theta_1 \beta_1 I^* p_{12} p_{13} k_{12} + 2\theta_1 \beta_1 I^* p_{11} p_{13} k_{22} + 4\theta_1 \beta_1 B^* p_{12} p_{23} k_{12} + 2\theta_1 \beta_1 B^* p_{11} p_{23} k_{22} + 2\theta_1 \beta_1 B^* p_{13} p_{21} k_{22} - 2\theta_m p_{21} p_{23} k_{22}) \right], \quad (42)$$

$$g_{112}^2 = \frac{1}{D} \left[q_{21} \left(-\frac{2}{L} s p_{12} p_{13} k_{11} - \frac{4}{L} s p_{11} p_{13} k_{12} - 2\beta p_{13} p_{21} k_{12} - \beta p_{12} p_{23} k_{11} - 2\beta p_{11} p_{23} k_{12} - 4\beta_1 p_{11} p_{12} p_{21} - 2\beta_1 I^* p_{12} p_{13} k_{11} - 4\beta_1 I^* p_{11} p_{13} k_{12} - 2\beta_1 B^* p_{12} p_{23} k_{11} - 4\beta_1 B^* p_{11} p_{23} k_{12} - 4\beta_1 B^* p_{13} p_{21} k_{12} + \phi_1 p_{12} p_{33} k_{11} \right) \right]$$

$$\begin{aligned}
 &+2\phi_1 p_{11} p_{33} k_{12} + \phi_1 p_{13} p_{32} k_{11} + 2\phi_1 p_{13} p_{31} k_{12} + 4\phi_2 p_{11} p_{12} p_{31} + 2\phi_2 p_{32} p_{11}^2 \\
 &+4\phi_2 T^* p_{11} p_{13} k_{12} + 2\phi_2 T^* p_{12} p_{13} k_{11} + 2\phi_2 B^* p_{12} p_{33} k_{11} + 4\phi_2 B^* p_{11} p_{33} k_{12} \\
 &+2\phi_2 B^* p_{13} p_{32} k_{11} + 4\phi_2 B^* p_{13} p_{31} k_{12} \Big) + q_{22}(\theta\beta p_{12} p_{23} k_{11} + 2\theta\beta p_{11} p_{23} k_{12} \\
 &+2\theta\beta p_{13} p_{21} k_{12} + 4\theta_1\beta_1 p_{11} p_{12} p_{21} + 2\theta_1\beta_1 I^* p_{12} p_{13} k_{11} + 4\theta_1\beta_1 I^* p_{11} p_{13} k_{12} \\
 &+2\theta_1\beta_1 B^* p_{12} p_{23} k_{11} + 4\theta_1\beta_1 B^* p_{11} p_{23} k_{12} + 4\theta_1\beta_1 B^* p_{13} p_{21} k_{12} - 4\theta_m p_{21} p_{23} k_{12} \Big].
 \end{aligned}
 \tag{43}$$

For finding the sign of W , we can evaluate the various quantities present in expression of W in terms of the system parameters and substituting the resulting expression into Eq. (33).

Numerical Simulation

To check the feasibility of our analysis regarding existence of interior equilibrium and its stability conditions, we have conducted some numerical computation using MATLAB by choosing the following set of parameter values in model system (1):

$$\begin{aligned}
 s = 0.5, \quad L = 100, \quad \beta = 0.09, \quad \beta_1 = 0.0001, \quad \nu = 0.43, \quad \theta = 0.02, \quad \theta_1 = 0.01, \\
 \theta_m = 0.1 \quad \phi_1 = 0.02, \quad \phi_2 = 0.001, \quad \phi = 0.01, \quad \phi_0 = 0.03.
 \end{aligned}
 \tag{44}$$

The following equilibrium values are obtained for the above set of parameters:

$$B^* = 49.2475, \quad I^* = 15.0145, \quad T^* = 16.9174.$$

For the above set of parameter values, it may be checked that the stability conditions stated in theorem 1 are satisfied.

The eigenvalues of the Jacobian matrix J_1 corresponding to the equilibrium E_1 for the model system (1) are:

$$-2.0952, \quad -0.1668 + 0.1699i, \quad \text{and} \quad -0.1668 - 0.1699i.$$

We note that one eigenvalue of Jacobian matrix J_1 is negative and the other two eigenvalues are with negative real part. Hence, the interior equilibrium E_1 is locally asymptotically stable.

For the above set of parameter values, the critical value of ϕ_2 (i.e. ϕ_{2c}) is 0.002. The variation of different variables with respect to time for $\phi_2 = 0.001$ and $\phi_2 = 0.0022$ has been shown in Figs. 1 and 2, respectively. From these figures, it is clear that for the value of ϕ_2 less than ϕ_{2c} , the system get stabilized but for ϕ_2 greater than ϕ_{2c} , sustained oscillations occur. Bifurcation diagrams of the model system (1) are drawn with respect to the parameter ϕ_2 in Fig. 3. This figure clearly shows that as ϕ_2 passes through a critical value, the system loses its stability and periodic solutions arise through Hopf-bifurcation. This means that the carrying capacity of forestry biomass can be increased only up to a certain level by applying suitable technological efforts. If we try to increase the carrying capacity of forestry biomass beyond a certain limit by using technological efforts, the system loses its stability. For the above given set of parameter values, the sign of Γ and W are calculated and found to be positive. Hence, in this case, supercritical Hopf-bifurcation occurs and bifurcating periodic solutions are stable. Figure 4 shows the periodic solutions are orbitally stable.

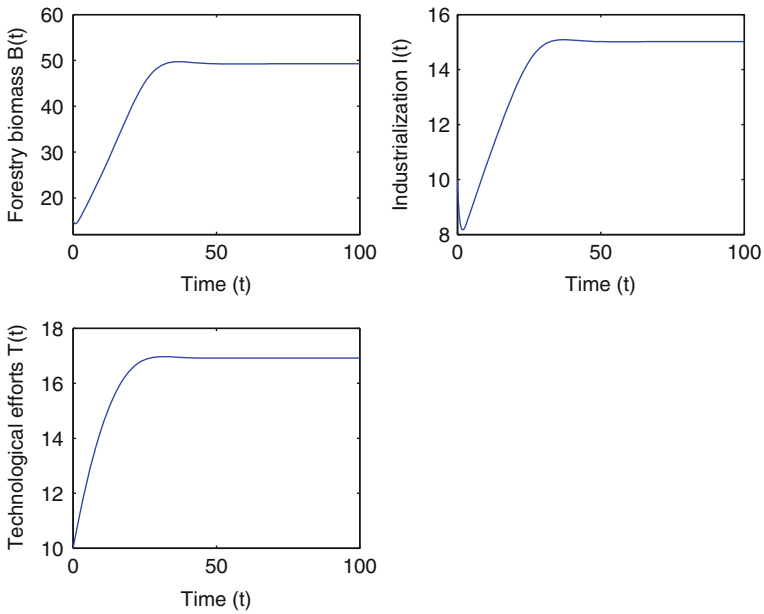


Fig. 1 Variation of forestry biomass, industrialization and technological efforts with respect to time for $\phi_2 = 0.001$

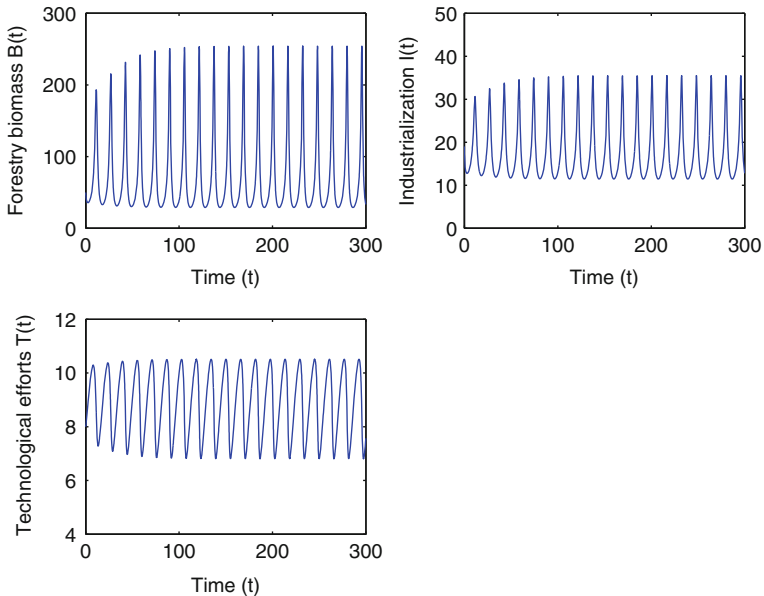


Fig. 2 Variation of forestry biomass, industrialization and technological efforts with respect to time for $\phi_2 = 0.0022$

Conclusion

In this paper, a non-linear mathematical model is proposed and analyzed to study the depletion of forestry biomass due to industrialization and its conservation through suitable technologi-

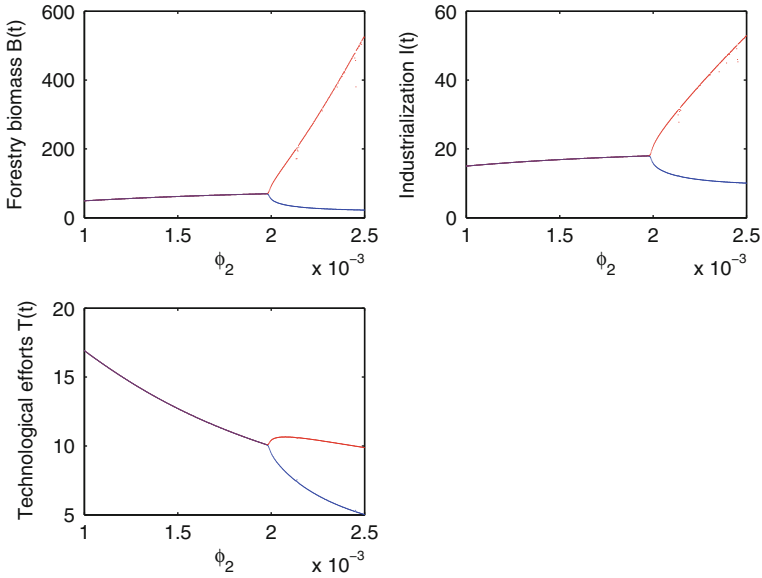


Fig. 3 Bifurcation diagram of forestry biomass, industrialization and technological efforts with respect to ϕ_2 . The other parameters are same as given in (44)

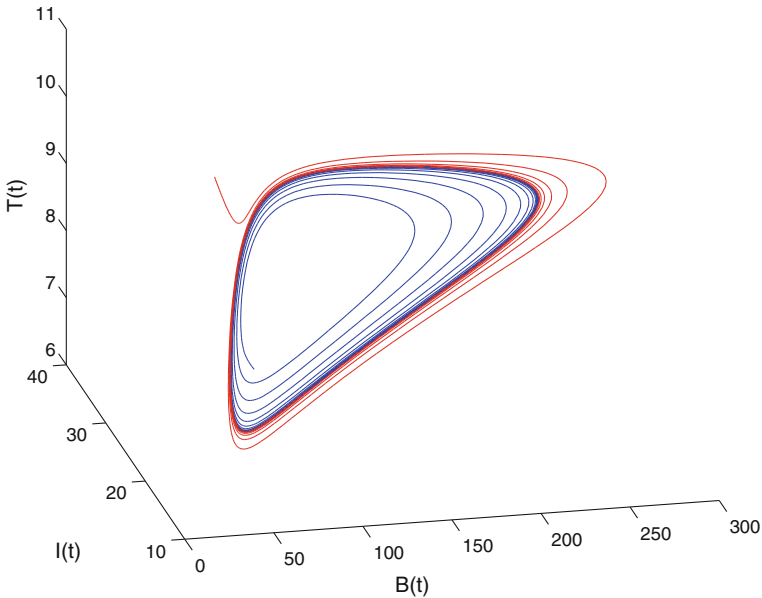


Fig. 4 Stable limit cycle for $\phi_2 = 0.0022$

cal efforts, like genetically engineered plants. In the modeling process, we have assumed that the intrinsic growth rate as well as carrying capacity of the forestry biomass are depleted due to industrialization. It is also assumed that technological efforts are applied to increase the intrinsic growth rate and carrying capacity of forestry biomass. The model is analyzed using

stability theory of differential equations. It is found that the proposed model exhibits one axial equilibrium and an interior equilibrium. Conditions for stability of interior equilibrium have been derived which are stated in Theorem 1. It is found that these conditions may be violated for higher values of ϕ_2 implying that the growth rate coefficient of carrying capacity of forestry biomass due to technological efforts have destabilizing effect on the dynamics of the system. The Hopf-bifurcation analysis is performed by taking ϕ_2 as a bifurcation parameter. It is shown that when ϕ_2 passes through a critical value, the interior equilibrium E_1 loses its stability and periodic solutions arise through Hopf-bifurcation. The stability and direction of Hopf-bifurcation have also been discussed in detail by using the center manifold theorem. The obtained results reveal that with the help of technological efforts, we can increase the carrying capacity of forestry biomass only up to a certain level. Further attempts to increase the carrying capacity of forestry biomass results into the destabilization of the system i.e. the density of forestry biomass oscillates and does not settle down to its equilibrium level. Thus through applying technological efforts, it is not possible to increase the carrying capacity of forestry biomass up to any desired level.

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