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ORIGINAL RESEARCH

Dynamics of Solitary Waves of the Rosenau-RLW Equation

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Abstract In this paper we study the solitary waves of the Rosenau-RLW equation. By using some trigonometric function methods, a family of stable solitary wave solutions are obtained, revealing an intrinsic relationship among the amplitude, frequency and wave speed, for what should be an equation relevant to modeling in a number of fields.

Keywords Rosenau-RLW equation · Solitary waves · Stability

Mathematics Subject Classification (2000) 35Q35 · 35C08 · 35B35

Introduction

This paper is concerned with the existence and stability of exact solitary wave solutions of the Rosenau-RLW equation

$$u_t + \varepsilon u_x + \alpha u_{txx} + \beta u_{txxxx} + uu_x = 0, \tag{1}$$

where $\varepsilon > 0$, $\beta > 0$ and $\alpha \in \mathbb{R}$ are constants. Zuo et al. in [36] considered (1) as a generalization of the Rosenau equation ($\alpha = 0$) which is used to describe the dynamics of dense discrete systems [24–26]. He studied the initial-boundary value problem of the Rosenau-RLW equation by a Crank–Nicolson difference scheme. Equation (1) is a regularized counterpart of the Rosenau-KdV equation

$$u_t + \varepsilon u_x + \alpha u_{xxx} + \beta u_{txxxx} + uu_x = 0, \tag{2}$$

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which has been studied in [17,35]. When $\alpha < 0$ and $\beta = 0$, (1) can be also considered as a generalization of the BBM equation [7,23] which together with the KdV equation,

$$u_t + u_{xxx} + uu_x = 0, \tag{3}$$

arise as models for one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media [1,11,19,23,31], as well as the evolution of weakly nonlinear ion acoustic waves in plasmas [29].

Our interest in the present paper is first to search for exact solutions of (1). Next we investigate the orbital stability of the obtained solitary waves. By a solitary wave solution of the Rosenau-RLW equation, we mean a traveling-wave solution of Eq. (1) of the form

$$u_s(x,t) = \varphi_c(\xi) = \varphi_c(x-ct)$$

decaying at infinity, where $c \in \mathbb{R}$ is the speed of wave propagation. Alternatively, it is a solution φ_c of the equation

$$(c-\varepsilon)\varphi_c + \alpha c \varphi_c{}'' + \beta c \varphi_c{}'''' - \frac{1}{2}\varphi_c{}^2 = 0, \tag{4}$$

where "$' = \mathrm{d}/\mathrm{d}\xi$". To find solutions of (4), because of the well known solitary traveling-wave solution associated with the KdV equation, we use trigonometric methods [8,9,16], based on the sech-method, including two families of traveling wave solutions of (4). Various traveling wave solutions are obtained, revealing an intrinsic relationship among the amplitude, frequency and wave speed.

Following the methods in [6,10,12,18,28,30], we employ the invariants

$$Q(u) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 - \alpha u_x^2 + \beta u_{xx}^2\right) \mathrm{d}x \tag{5}$$

and

$$E(u) = -\int_{\mathbb{R}} \left(\frac{\varepsilon}{2}u^2 + \frac{1}{6}u^3\right) \mathrm{d}x, \tag{6}$$

as well as the fact that a solitary wave φ_c of (1) is a critical point of energy $E(\cdot) + cQ(\cdot)$ subject to constant charge Q, to show that the usual necessary and sufficient condition for the stability of solitary wave holds: defining $d(c) = E(\varphi_c) + cQ(\varphi_c)$, the solitary wave φ_c is stable if $d''(c) > 0$. A direct computation shows that $d'(c) = Q(\varphi_c)$ (see [28]). One can also observe that (1) has the hamiltonian structure. Indeed, one has $u_t = \mathfrak{J}E'(u)$, where

$$\mathfrak{J} = \partial_x M^{-1} \qquad \text{and} \qquad M = I + \alpha\partial_x^2 + \beta\partial_x^4.$$

The main ingredient in stability theory presented in [6,10,12,18,28,30] is to verify the following spectral condition holds ([4]).

Assumption 1 (*Spectral structure*) There exists $(\omega_1, \omega_2) \subset \mathbb{R}$, with $\varepsilon \le \omega \le \omega_2 \le +\infty$, such that $c \mapsto \varphi_c$ is a nontrivial smooth curve, and for each $c \in (\omega_1, \omega_2)$ and a solitary wave φ_c, the linearized operator

$$L = \alpha c \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \beta c \frac{\mathrm{d}^4}{\mathrm{d}\xi^4} - \varphi_c + c - \varepsilon \tag{7}$$

is self-adjoint closed unbounded on a dense subspace of $L^2(\mathbb{R})$ and enjoys the following spectral properties: it has a unique negative simple with eigenfunction χ_c, the zero eigenvalue is simple with eigenfunction $\varphi_c{}'$ and the remainder of the spectrum of L is positive and bounded away zero. Moreover the mapping $c \mapsto \chi_c$ is a continuous curve with values in $H^2(\mathbb{R})$ and $\chi_c(x) > 0$.

It is clear that L is self-adjoint closed unbounded linear operator from $H^2(\mathbb{R})$ into $H^{-2}(\mathbb{R})$, $M^{1/2}LM^{1/2}$ is self-adjoint on $L^2(\mathbb{R})$, and $L(\varphi_c{}') = 0$, where

$$M = I + \alpha \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \beta \frac{\mathrm{d}^4}{\mathrm{d}\xi^4}.$$

To verify the spectral properties of L, we will apply the results of Albert [2], and Albert and Bona [3]. In [2,3], the authors considered the following KdV-type equation

$$u_t + u^p u_x - \mathfrak{M} u_x = 0, \tag{8}$$

where $\mathfrak{M}$ is a Fourier multiplier $\widehat{\mathfrak{M}g}(\xi) = m(\xi)\widehat{g}(\xi)$ and m is a measurable locally bounded even function on $\mathbb{R}$ satisfying

$$C_1|\xi|^{\nu_1} \le m(\xi) \le C_2(1 + |\xi|)^{\nu_2},$$

for $\nu_1 \le \nu_2$, $|\xi| \ge |\xi_0|$ and $C_1, C_2 > 0$, and $m(\xi) > \kappa > 0$, for all $\xi \in \mathbb{R}$. It is shown that, for a solitary wave φ_c of (8), the associated linearized operator $\mathfrak{M} + c - \varphi_c$ satisfies the spectral structure mention above provided that φ_c is a positive solitary wave such that $\widehat{\varphi_c} > 0$ and $\widehat{\varphi_c{}^p}$ belongs to the $PF(2)$-class defined by Karlin in [20] (see Definition 2). It is worth noticing that Souganidis and Strauss in [28] considered Assumption (1) and used the ideas of [12,18,30] to study the instability of solitary waves of the following BBM-type equation

$$u_t + \mathfrak{M} u_t + u_x + u^p u_x = 0, \tag{9}$$

where $\widehat{\mathfrak{M}g}(\xi) = m(\xi)\widehat{g}(\xi)$ is a Fourier multiplier with appropriate conditions, similar to (8), on m. We will apply the ideas of [2,3,12,18,28] to show the stability of our explicit solitary wave solutions of (1). Rest of this paper is divided into three sections. The next section is devoted the local and global well-posedness of (1), by using the properties of the kernel $\mathbb{H}$ (see (12)). In the third section, some explicit solitary waves will be obtained. Finally in the last section, we prove the orbital stability of the obtained solutions. We end this section by introducing some notations that will be used throughout this article.

Notation

We shall denote by $\widehat{\varphi}$ the Fourier transform of φ, defined as

$$\widehat{\varphi}(\zeta) = \int_{\mathbb{R}} \varphi(\omega) \mathrm{e}^{-\mathrm{i}\omega\zeta} \, \mathrm{d}\omega.$$

For $1 \le p < \infty$, $L^p = L^p(\mathbb{R})$ connotes the pth-power Lebesgue-integrable functions with the usual modification for the case $p = \infty$.

For $s \in \mathbb{R}$, we denote by $H^s = H^s(\mathbb{R})$, the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}) = \left\{ \varphi \in \mathscr{S}'(\mathbb{R}) \, : \, \|\varphi\|_{H^s(\mathbb{R})} < \infty \right\},$$

where

$$\|\varphi\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(1 + |\zeta|^2\right)^s |\widehat{\varphi}(\zeta)|^2 \mathrm{d}\zeta \right)^{1/2},$$

and $\mathscr{S}'(\mathbb{R})$ is the space of tempered distributions.

If $\mathfrak{X}$ is any Banach space and $T > 0$, $C(0, T; \mathfrak{X})$ is the class of continuous functions from $[0, T]$ into $\mathfrak{X}$ with its usual norm

$$\|u\|_{C(0,T;\mathfrak{X})} = \max_{t\in[0,T]} \|u(t)\|_{\mathfrak{X}}.$$

If $\mathcal{Y} \subset \mathfrak{X}$ is a subset, then $C(0, T; \mathcal{Y})$ is the collection of elements u in $C(0, T; \mathfrak{X})$ such that $u(t) \in \mathcal{Y}$ for $t \in [0, T]$. When $T = +\infty$, $C(0, +\infty; \mathfrak{X})$ is a Fréchet space with defining set of semi-norms $\max_{t\in[0,N]} \|u(t)\|_{\mathfrak{X}}$, for $N \in \mathbb{N}$. The Banach space $C^1(0, T; \mathfrak{X})$ is the subspace of $C(0, T; \mathfrak{X})$ for which the limit

$$u'(t) = \lim_{h\to 0} \frac{u(t + h) - u(t)}{h}$$

exists in $C(0, T; \mathfrak{X})$. It is equipped with the obvious norm. Inductively, one defines $C^k(0, T; \mathfrak{X})$ and $C^k(0, +\infty; \mathfrak{X})$.

Given a solitary wave φ_c of (1), the orbit of $\mathcal{O}_{\varphi_c}$ is defined by the set $\mathcal{O}_{\varphi_c} = \{\tau_r\varphi_c;\ r \in \mathbb{R}\}$, where $\tau_r\varphi_c(\cdot) = \varphi_c(\cdot + r)$. We also denote by

$$U_\epsilon = U_{\varphi_c,\epsilon} = \left\{ u_0;\ \inf_{\psi\in\mathcal{O}_{\varphi_c}} \|u_0 - \psi\|_{H^2} < \epsilon \right\}$$

the ϵ-neighborhood of the orbit $\mathcal{O}_{\varphi_c}$.

For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant $\Bbbk$ such that $a \leq \Bbbk b$.

Well-Posedness

In this section we are going to study the well-posedness issue for (1). In this section we assume that $\alpha < 2\sqrt{\beta}$.

Definition 1 An evolution equation $u_t = \mathfrak{A}u$, with $u(0) = u_0$, is said to be locally (in time) well-posed in a Banach space $\mathfrak{X}$ if for any $u_0 \in \mathfrak{X}$, there is a positive number T such that the equation possesses a unique solution u which lies in $C(0, T; \mathfrak{X})$. Moreover, the solution u must depend continuously on u_0. The evolution equation is well-posed globally in time if T can be chosen arbitrarily large.

We note that a formal integration in the temporal variable then leads to the Rosenau-RLW integral equation

$$u(t) = u_0(x) + \int_0^t \int_{\mathbb{R}} \mathbb{H}(x - y) \left(\varepsilon u(y, s) + \frac{1}{2} u^2(y, s) \right) \mathrm{d}y\ \mathrm{d}s, \tag{10}$$

where $u_0(x) = u(x, 0)$ is the initial data and

$$\widehat{\mathbb{H}}(\xi) = \frac{\mathrm{i}\xi}{1 - \alpha\xi^2 + \beta\xi^4}. \tag{11}$$

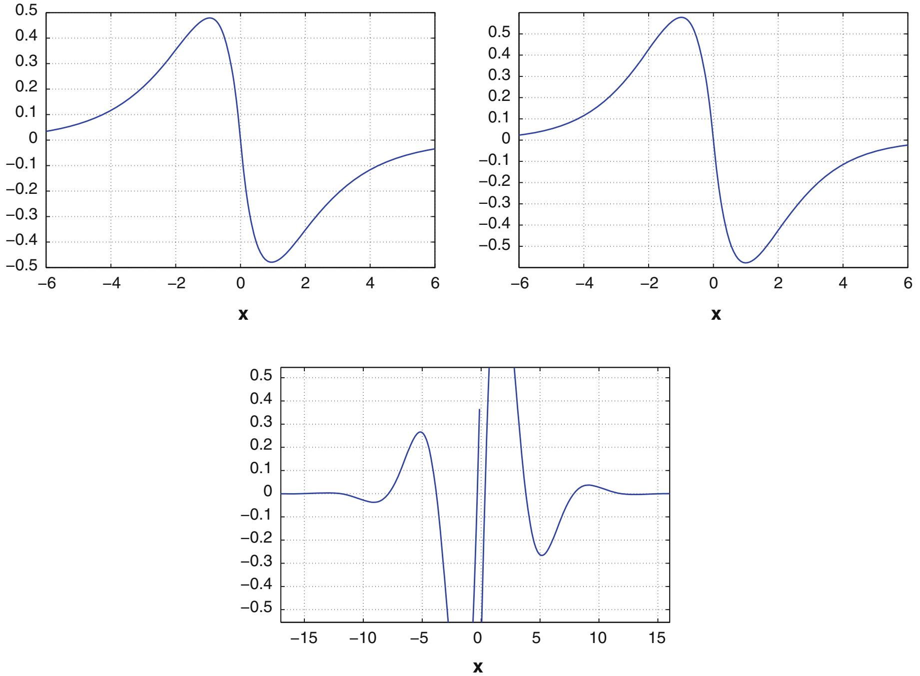


Fig. 1 Kernel $\mathbb{H}$ in (12) for $\beta = 1$. Figures correspond to $\alpha = -3$, $\alpha = -2$ and $\alpha = 1$ respectively from *left to right* and then up to down

More precisely, by using the residue theorem, one can easily see that

$$\mathbb{H}(x) = \begin{cases} \frac{\pi \operatorname{sgn}(x)}{\beta(\lambda_1^2-\lambda_2^2)} \left(e^{-\lambda_1|x|} - e^{-\lambda_2|x|}\right), & \alpha < -\alpha_*, \\ -\frac{\pi \operatorname{sgn}(x)}{2\sqrt[4]{\beta^3}} |x| e^{-\beta^{-1/4}|x|}, & \alpha = -\alpha_*, \\ \frac{\pi \operatorname{sgn}(x) e^{-\sigma|x|}}{2\beta\sigma\omega(\sigma^2+\omega^2)} \left(\sigma^2 H_1(x) - \omega H_2(x)\right), & \alpha \in (-\alpha_*, \alpha_*), \end{cases} \tag{12}$$

where

$$\left.\begin{aligned} &\alpha_* = 2\beta^{1/2}, \\ &\lambda_1 = \sqrt{-\tfrac{1}{2\beta}(\alpha + \sqrt{\alpha^2 - 4\beta})}, \\ &\lambda_2 = \sqrt{-\tfrac{1}{2\beta}(\alpha - \sqrt{\alpha^2 - 4\beta})}, \\ &\sigma = \tfrac{1}{2}\sqrt{2\beta^{-1/2} - \alpha\beta^{-1}}, \\ &\omega = \tfrac{1}{2}\sqrt{2\beta^{-1/2} + \alpha\beta^{-1}}, \\ &H_1(x) = \cos(\sigma x) - \sin(\sigma|x|), \\ &H_2(x) = \sigma\cos(\omega x) - \omega\sin(\omega|x|). \end{aligned}\right\} \tag{13}$$

Figure 1 illustrates the shape of kernel of (12) for $\beta = 1$, and $\alpha = -3$, $\alpha = -2$ and $\alpha = 1$ respectively.

First we study the local well-posedness, based on Definition 1, in $L^q(\mathbb{R})$-spaces. See [23] for similar results.

Theorem 2 *Let $q \geq 2$. Then* (10) *is well-posed in $L^q(\mathbb{R})$ in the sense of Definition 1. Moreover, the flow map $\mathfrak{G} : u_0 \mapsto u$, that associates to the initial data u_0 the unique solution u, is real analytic.*

Proof First one can observe from (11) that $\mathbb{H} \in L^\ell(\mathbb{R})$, for any $1 \leq \ell \leq \infty$. Thus by the Young inequality,

$$\left\| \mathbb{H} * \left(\varepsilon u + \frac{1}{2}u^2\right) \right\|_{L^q(\mathbb{R})} \leq C \left(\|\mathbb{H}\|_{L^1(\mathbb{R})} \|u\|_{L^q(\mathbb{R})} + \|\mathbb{H}\|_{L^{q/(q-1)}(\mathbb{R})} \|u\|^2_{L^q(\mathbb{R})} \right),$$

which is to say,

$$\mathbb{H} * \left(\varepsilon u + \frac{1}{2}u^2\right) \in L^q(\mathbb{R}).$$

Now we define the constants $r = 2\|u_0\|_{L^q(\mathbb{R})}$ and

$$T = \frac{1}{2C\left(\|\mathbb{H}\|_{L^1(\mathbb{R})} + \|\mathbb{H}\|_{L^{q/(q-1)}(\mathbb{R})} r\right)}$$

and define $\mathscr{X} = \mathscr{X}_{T,r} = C(0, T; B_r(u_0))$, where $B_r(u_0)$ is the closed ball in $L^q(\mathbb{R})$ of the radius r centered at u_0. The set $\mathscr{X}$ is a complete metric space with the distance d induced by the norm on $C(0, T; L^q(\mathbb{R}))$. We show that the operator

$$\Phi(u) = u_0(x) + \int_0^t \int_{\mathbb{R}} \mathbb{H}(x-y) \left(\varepsilon u(y,s) + \frac{1}{2}u^2(y,s)\right) \mathrm{d}y \, \mathrm{d}s$$

is contractive from $\mathscr{X}$ to itself.

For any $u \in \mathscr{X}$, we have

$$d(\Phi(u), 0) = \|\Phi(u)\|_{\mathscr{X}} \leq \|u\|_{L^q(\mathbb{R})} + TC\left(\|\mathbb{H}\|_{L^1(\mathbb{R})} r + \|\mathbb{H}\|_{L^{q/(q-1)}(\mathbb{R})} r^2\right) \leq r, \quad (14)$$

so that Φ maps $\mathscr{X}$ to $\mathscr{X}$. Moreover the Cauchy–Schwarz inequality implies for $u, v \in \mathscr{X}$ that

$$\begin{aligned} &\left\| \mathbb{H} * \left(\varepsilon(u-v) + \frac{1}{2}\left(u^2 - v^2\right)\right) \right\|_{L^q(\mathbb{R})} \\ &\quad \leq C_0 \left(\|\mathbb{H}\|_{L^1(\mathbb{R})} + \|\mathbb{H}\|_{L^{q/(q-1)}(\mathbb{R})} r\right) \|u - v\|_{L^q(\mathbb{R})}, \end{aligned} \quad (15)$$

Consequently, it holds that

$$\|\Phi(u) - \Phi(v)\|_{L^q(\mathbb{R})} \leq C \int_0^t \left[\|\mathbb{H}\|_{L^1(\mathbb{R})} + \|\mathbb{H}\|_{L^{q/(q-1)}(\mathbb{R})} r\right] \|u(\tau) - v(\tau)\|_{L^q(\mathbb{R})} \mathrm{d}\tau.$$

Therefore, we obtain that

$$\begin{aligned} d(\Phi(u), \Phi(v)) &\leq CT\left[\|\mathbb{H}\|_{L^1(\mathbb{R})} + \|\mathbb{H}\|_{L^{q/(q-1)}(\mathbb{R})} r\right] \|u - v\|_{\mathscr{X}} \\ &\leq \frac{1}{2}\|u - v\|_{\mathscr{X}} = \frac{1}{2} d(u, v). \end{aligned}$$

It shows that Φ is contractive which implies the desired local well-posedness result.

To prove the second part of theorem, we will use an argument analogous to [5], [14, Theorem 3], [27, Theorem 3.3] and [32–34] (see also [13,21,22]). Define an operator Ψ as

$$\Psi(u) = \int_0^t \mathbb{H} * (\varepsilon u + g(u))\mathrm{d}s,$$

where $g(u) = \frac{1}{2}u^2$. It is straightforward to see that Ψ is Fréchet differentiable and for $v, z \in C(0, T; L^q(\mathbb{R}))$ we have

$$\Psi'(v)z = \int_0^t \int_{\mathbb{R}} \mathbb{H}(x-y)(1+g'(v))z \,\mathrm{d}y\mathrm{d}\tau.$$

Now we define, for $a, b \in C(0, T; L^q(\mathbb{R}))$, that

$$\Lambda(a, b) = b - a - \Psi(b);$$

so that when $a = u_0$ and $b = u$, where u is the fixed point of the operator Ψ corresponding to initial data u_0, then $\Lambda(u_0, u) = 0$ and

$$D_b\Lambda(u_0, u)z = z - \Psi'(u)z.$$

Furthermore, it is seen from the definition that

$$\begin{aligned}\|\Psi'(u)z\|_{L^q(\mathbb{R})} &\le T \sup_{0\le t\le T} \|\mathbb{H} * ((\varepsilon + g'(u))z)\|_{L^q(\mathbb{R})}\\ &\le CT\left(\|\mathbb{H}\|_{L^1(\mathbb{R})} + \|\mathbb{H}\|_{L^{q/(q-1)}(\mathbb{R})}r\right)\|z\|_{\mathscr{X}}\\ &= \frac{1}{2}\|z\|_{\mathscr{X}}.\end{aligned}$$

Hence,

$$D_v\Lambda(u_0, u) = I - \Psi'(u)$$

is invertible and therefore, by Implicit Function Theorem [15], the flow map $\mathfrak{G}(u_0) = u$ is a C^1 map, and

$$D_{u_0}u = -\left(I - \Psi'(u)\right)^{-1} D_a\Lambda(u_0, u);$$

and the second assertion of Theorem 2 follows. □

Theorem 3 *Let $s \ge 0$. Then for any $u_0 \in H^s(\mathbb{R})$, there is a number $T > 0$ and a unique solution $u \in C(0, T; H^s(\mathbb{R}))$ of (10) with $u(0) = u_0$. Moreover, the flow map $\mathfrak{G} : u_0 \mapsto u$, that associates to the initial data u_0 the unique solution u, is real analytic. In addition, $u(t)$ satisfies $E(u(t)) = E(u_0)$ and $Q(u(t)) = Q(u_0)$ for all $t \in [0, T)$.*

Proof Take the Fourier transform in (10) with respect to the spatial variable, we obtain

$$\widehat{u}(\xi, t) = \widehat{u}_0(\xi) + \int_0^t \frac{\mathrm{i}\xi}{1-\alpha\xi^2+\beta\xi^4}\left(\varepsilon\widehat{u} + \frac{1}{2}\widehat{u^2}\right)(\xi, \tau)\mathrm{d}\tau.$$

Now we define, for any $T > 0$, an operator $\mathcal{A} : C(0, T; H^s) \to C(0, T; H^s)$ by

$$\widehat{\mathcal{A}u}(\xi, t) = \widehat{u}_0(\xi) + \int_0^t \frac{\mathrm{i}\xi}{1-\alpha\xi^2+\beta\xi^4}\left(\varepsilon\widehat{u} + \frac{1}{2}\widehat{u^2}\right)(\xi, \tau)\mathrm{d}\tau.$$

When $s \geq 0$, if $u \in H^s$, then for any $\xi \in \mathbb{R}$,

$$(1+|\xi|)^s|\widehat{u^2}(\xi)| \leq \left((1+|\cdot|)^s|\widehat{u}(\cdot)|\right) * \left((1+|\cdot|)^s|\widehat{u}(\cdot)|\right)(\xi) \leq \int_{\mathbb{R}} (1+|\xi|)^{2s}|\widehat{u}(\xi)|^2 \mathrm{d}\xi = \|u\|_{H^s}^2. \tag{16}$$

Consequently,

$$\int_{\mathbb{R}} (1+|\xi|)^{2s} \frac{\xi^2}{(1-\alpha\xi^2+\beta\xi^4)^2}|\widehat{u^2}(\xi)|^2 \mathrm{d}\xi \lesssim \|u\|_{H^s}^4;$$

and it is concluded that $\mathcal{A}u \in C(0,+\infty; H^s(\mathbb{R}))$, if $u \in C(0,+\infty; H^s(\mathbb{R}))$. Following the steps laid out in the proof of Theorem 2, it can be shown that in all cases, when $T > 0$ is chosen sufficiently small, the operator $\mathcal{A}$ is contractive in $C\left([0,T); B_{2\|u_0\|_s}(0)\right)$, where the ball $B_{2\|u_0\|_s}(0)$ is in $H^s(\mathbb{R})$. The contraction mapping principle completes the proof. Invariance of E and Q follows by a standard argument. □

Theorem 4 *Let $s \geq 2$. Then for any $u_0 \in H^s(\mathbb{R})$, there is a unique solution $u \in C(0,+\infty; H^s(\mathbb{R}))$ of (10) with $u(0) = u_0$.*

Proof By Theorem 3, there exists a $T > 0$ and a unique solution u of (1) with $u(0) = u_0$ such that $u \in C(0,T; H^s(\mathbb{R}))$. It remains to show that T can be taken arbitrarily large. First we note for $s = 2$ that the invariant (6) implies that the solution can be extended from $C(0,T; H^2(\mathbb{R}))$ to $C(0,+\infty; H^2(\mathbb{R}))$. Next, when $s > 2$, we multiply both sides of (1) by $2(I+D)^{2s-4}u(x,t)$ and integrate over $\mathbb{R}$ with respect to x to obtain at least for smooth solutions that

$$\begin{aligned}
&2\int_{\mathbb{R}} \left((I+D)^{2s-4}u(x,t)\right) Mu_t(x,t)\mathrm{d}x \\
&\quad = -2\int_{\mathbb{R}} \left((I+D)^{2s-4}u(x,t)\right)\left(\varepsilon u(x,t) + \frac{1}{2}u^2(x,t)\right)_x \mathrm{d}x \\
&\quad = -\int_{\mathbb{R}} \mathrm{i}\xi(1+|\xi|)^{2s-4}\overline{\widehat{u}}(\xi,t)\ \widehat{u^2}(\xi,t)\mathrm{d}\xi,
\end{aligned}$$

where $D = (-\partial_x^2)^{1/2}$ and $M = I + \alpha\partial_x^2 + \beta\partial_x^4$. Using the fact

$$1-\alpha\xi^2+\beta\xi^4 \sim (1+|\xi|)^4, \tag{17}$$

it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (1-\alpha\xi^2+\beta\xi^4)(1+|\xi|)^{2s-4}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi \\
&\quad \leq \int_{\mathbb{R}} (1+|\xi|)^{2s-3}|\widehat{u}(\xi,t)|\left|\widehat{u^2}(\xi,t)\right|\mathrm{d}\xi \\
&\quad \leq \|u(t)\|_{H^s}\|u^2(t)\|_{H^{s-3}} \\
&\quad \leq \|u(t)\|_{H^s}\|u^2(t)\|_{H^s}.
\end{aligned} \tag{18}$$

Now by (17) and the invariance E, we have

$$\|\widehat{u}(t)\|_{L^1}^2 \lesssim \|u\|_{H^2}^2 \lesssim \int_{\mathbb{R}} u^2 - \alpha u_x^2 + \beta u_{xx}^2 \mathrm{d}x$$
$$= \int_{\mathbb{R}} u_0^2 - \alpha(\partial_x u_0)^2 + \beta(\partial_x^2 u_0)^2 \mathrm{d}x \lesssim \|u_0\|_{H^2}^2.$$

This implies that

$$\begin{aligned}\|u^2(t)\|_{H^s}^2 &= \int_{\mathbb{R}} (1+|\xi|)^{2s} |\widehat{u^2}(\xi,t)| \mathrm{d}\xi \\ &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} \left(1+|\xi-\eta|^{2s}+|\eta|^{2s}\right) |\widehat{u}(\xi-\eta,t)\widehat{u}(\eta,t)|^2 \mathrm{d}\eta \mathrm{d}\xi \\ &\lesssim \|\widehat{u}(t) * \widehat{u}(t)\|_{L^2}^2 + \|\widehat{D^s u}(t) * \widehat{u}(t)\|_{L^2}^2 \lesssim \|\widehat{u}(t)\|_{L^1}^2 \|u(t)\|_{H^s}^2 \\ &\leq C\|u(t)\|_{H^s}^2, \end{aligned} \tag{19}$$

where $C = C(\|u_0\|_{H^2})$. Combining (18) and (19) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \left(1 - \alpha\xi^2 + \beta\xi^4\right)(1+|\xi|)^{2s-4} |\widehat{u}(\xi,t)|^2 \mathrm{d}\xi \lesssim \|u(t)\|_{H^s}^2. \tag{20}$$

Integrating the last inequality with respect to t and using (17) yields

$$\|u(t)\|_{H^s}^2 \leq C_1\|u_0\|_{H^s}^2 + C_2 \int_0^t \|u(\tau)\|_{H^s}^2 \mathrm{d}\tau.$$

By the Gronwall lemma, there are two constants C_1 and C_2 in which C_1 is dependent only on $\|u_0\|_{H^s}$ and C_2 only on $\|u_0\|_{H^2}$ such that $\|u(t)\|_{H^s} \leq C_1 \exp(C_2 t)$. This a priori bound allows us to iterate the local theory and achieve a globally defined solution. □

Solitary Waves

In this section, we establish the existence of solitary waves of (1). Here, we propose two types of L^1-solutions of (4). We first consider the sech-ansatz; actually our hypothesis is $\varphi_c(\xi) = A\mathrm{sech}^q(b\xi)$, where A is the amplitude of the solitary wave and b is the inverse width of the solitary wave. One can see after balancing ${\varphi_c}''''$ with ${\varphi_c}^2$ that $q = 4$. Hence plugging $A\mathrm{sech}^4(b\xi)$ into (4), collecting the coefficients $\mathrm{sech}^j(b\xi)$ and equating these coefficients to zero, there obtains

$$\begin{cases} -2\varepsilon + 2c + 512\beta c b^4 + 32c\alpha b^2 = 0, \\ -40\alpha - 208\beta b^2 = 0, \\ -A + 1680\beta c b^4 = 0. \end{cases} \tag{21}$$

After some calculations, we obtain from system (21) that

$$A = A(c) = \frac{35}{12}(c-\varepsilon) \quad \text{and} \quad b = b(c) = \frac{1}{12}\sqrt{\frac{13(\varepsilon - c)}{c\alpha}}, \tag{22}$$

such that $\alpha < 0$, $c > \varepsilon$ and

$$\beta = \frac{36c\alpha^2}{169(c-\varepsilon)}. \tag{23}$$

One can also easily check that $\frac{\mathrm{d}A}{\mathrm{d}c} > 0$ and $\frac{\mathrm{d}b}{\mathrm{d}c} > 0$, for all $c > \varepsilon$; so that the mapping $c \to \varphi_c$ is smooth from $(\varepsilon, +\infty)$ into $H^s(\mathbb{R})$, for all $s \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}\varphi_c(\xi) &= \varphi_c(x-ct)\\ &= \frac{35}{12}(c-\varepsilon)\mathrm{sech}^4\left(\sqrt{\frac{13(\varepsilon-c)}{144c\alpha}}(x-ct)\right).\end{aligned} \tag{24}$$

By using the idea of the sech-ansatz method above, we propose the second type of solution which is

$$\varphi_c(\xi) = \sum_{j=1}^{4} A_j \mathrm{sech}^j(b\xi),$$

where $A_j, b \in \mathbb{R}$. It is straightforward to see after balancing ${\varphi_c}''''$ with ${\varphi_c}^2$ that $A_1 = A_3 = 0$. Hence plugging this form into (4), collecting the coefficients $\mathrm{sech}^j(b\xi)$ and equating these coefficients to zero, there obtains

$$\begin{cases} 32\beta cb^4 + 8c\alpha b^2 + 2c - 2\varepsilon = 0,\\ -A_2^2 - 240\beta cb^4 A_2 - 12c\alpha b^2 A_2 + 2A_4\left(c-\varepsilon+256\beta cb^4+16c\alpha b^2\right) = 0,\\ -40c\alpha b^2 A_4 + 240\beta cb^4 A_2 - 2A_2A_4 - 2080\beta cb^4 A_4 = 0,\\ -A_4 + 1680\beta cb^4 = 0.\end{cases} \tag{25}$$

After some calculations, we obtain from system (25) that

$$A_2 = \frac{910}{293}(c-\varepsilon),\quad A_4 = \frac{1085}{293}(c-\varepsilon)\quad \text{and}\quad b = \frac{1}{1758}\sqrt{\frac{799890\,(\varepsilon-c)}{\alpha c}}, \tag{26}$$

such that $c > \varepsilon$, $\alpha < 0$ and

$$\beta = \frac{27249c\alpha^2}{828100(c-\varepsilon)}. \tag{27}$$

One can also observe that $\frac{\mathrm{d}A_j}{\mathrm{d}c} > 0$, $j = 2, 4$, and $\frac{\mathrm{d}b}{\mathrm{d}c} > 0$, for all $c > \varepsilon$; so that the mapping $c \to \varphi_c$ is smooth from $(\varepsilon, +\infty)$ into $H^s(\mathbb{R})$, for all $s \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}\varphi_c(\xi) &= \varphi_c(x-ct)\\ &= \frac{910}{293}(c-\varepsilon)\mathrm{sech}^2\,(b(x-ct)) + \frac{1085}{293}(c-\varepsilon)\mathrm{sech}^4(b(x-ct)),\end{aligned} \tag{28}$$

where b is as above.

Finally, we propose the ansatz

$$\varphi_c(\xi) = \sum_{j=1}^{2} \frac{A_j}{(B_j + \cosh(b\xi))^j}.$$

One can see after balancing again ${\varphi_c}''''$ with ${\varphi_c}^2$ that $A_1 = 0$. Hence plugging

$$\varphi_c(\xi) = \frac{A}{(B+\cosh(b\xi))^2}$$

into (4), collecting the coefficients $\operatorname{sech}^j(b\xi)$ and equating these coefficients to zero, there obtains

$$\begin{cases} 16\beta cb^4 + 4c\alpha b^2 - \varepsilon + c = 0, \\ -33\beta cb^4 + 3c\alpha b^2 - 2\varepsilon + 2c = 0, \\ 72\beta cb^4 B^2 - 240\beta cb^4 - A + 12cB^2 - 12c\alpha b^2 - 12\varepsilon B^2 = 0, \\ -A + 96\beta cb^4 + 4cB^2 - 4\varepsilon B^2 - 2\beta cb^4 B^2 - 2c\alpha b^2 B^2 - 12c\alpha b^2 = 0, \\ 2cB^4 - 12c\alpha b^2 B^2 + 240\beta cb^4 - 48\beta cb^4 B^2 - 2\varepsilon B^4 - AB^2 = 0. \end{cases} \tag{29}$$

After some computations, we obtain from system (29) that

$$A = \frac{35}{3}(c-\varepsilon), \quad B = \pm 1 \quad \text{and} \quad b = \sqrt{\frac{13(\varepsilon - c)}{36\alpha c}}, \tag{30}$$

such that $c > \varepsilon$, $\alpha < 0$ and

$$\beta = \frac{36c\alpha^2}{169(c-\varepsilon)}. \tag{31}$$

Therefore, we obtain

$$\varphi_c{}^{\pm}(\xi) = \varphi_c{}^{\pm}(x - ct) = \frac{35(c-\varepsilon)}{3\left(1 \pm \cosh\left(\sqrt{\frac{13(\varepsilon-c)}{36\alpha c}}(x-ct)\right)\right)^2}. \tag{32}$$

One can observe that $\varphi_c{}^{+}$ is actually identical with solution (24), while $\varphi_c{}^{-}$ has a singularity in $\xi = 0$ (see Fig. 3).

Figure 2 shows the wave profiles of (24) and (28) and their Fourier transform.

Stability

In this section, we are going to study the orbital stability of the solitary waves obtained in the previous section. Hereafter we assume that $\alpha < 0$ and $c > \varepsilon$. First we recall the definition of orbital stability.

Definition 2 We say that φ_c is orbitally stable in $H^2(\mathbb{R})$ by the flow generated by the Rosenau-RLW equation (1) if the initial value problem associated to (1) is globally well-posed in $H^2(\mathbb{R})$, and for every ϵ, there is $\delta > 0$ such that for all $u_0 \in U_\delta$ the solution $u(t)$ of (1) with $u(0) = u_0$ satisfies $u(t) \in U_\epsilon$ for all $t > 0$.

Definition 3 A function $w : \mathbb{R} \to \mathbb{R}$ is said to be in the class $PF(2)$ if for all $x \in \mathbb{R}$, $w(x) > 0$,

$$w(x_1 - y_1)w(x_2 - y_2) \geq w(x_1 - y_2)w(x_2 - y_1), \quad \text{for} \quad x_1 < x_2 \quad \text{and} \quad y_1 < y_2 \tag{33}$$

and the strict inequality holds in (33) whenever the intervals (x_1, x_2) and (y_1, y_2) intersect.

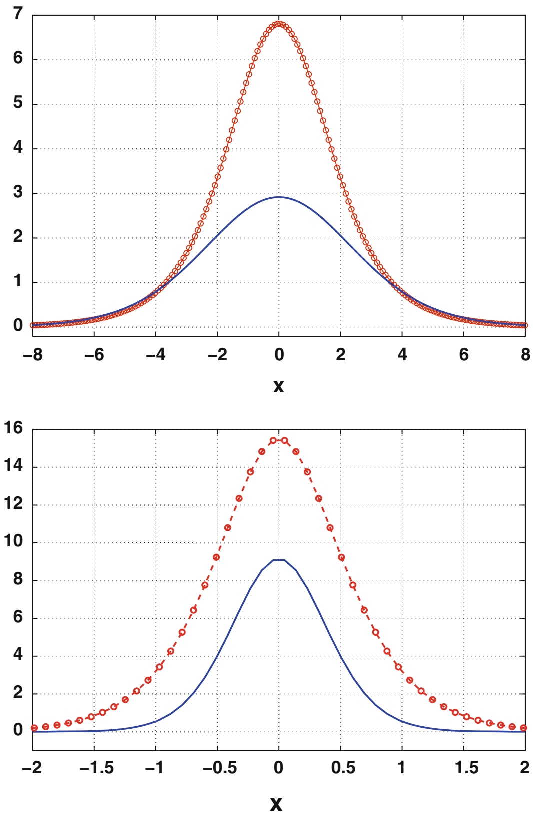
The following result is proved in [3].

Theorem 5 *If f a twice-differentiable positive function on $\mathbb{R}$ satisfying*

$$\frac{d^2}{dx^2}\log f(x) < 0,$$

for $x \neq 0$, then $f \in PF(2)$.

Fig. 2 *Up* is solitary waves of (24) and (28) at $t = 0$, and *down* is their Fourier transform. The *circle-curves* correspond to (28)



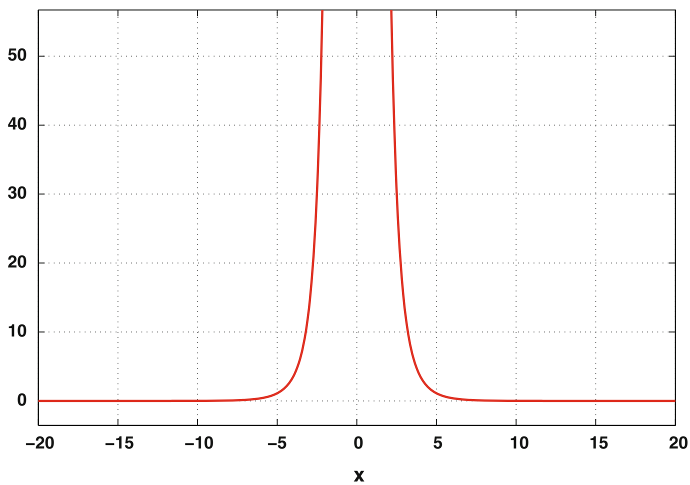


Fig. 3 The graph of solution $\varphi_c{}^{-}$ given by (32)

The following theorem [2] gives some spectral structure of the linearized operator L about a solitary wave φ_c.

Theorem 6 *Let φ_c be an even positive solitary wave of (1). Suppose that $\widehat{\varphi_c} \in PF(2)$, then the operator L satisfies Assumption 1.*

By using Theorem 6, the proof of the following stability theorem can be obtained by using the arguments given in [6,10,18,30].

Theorem 7 *Let φ_c be a positive solitary wave of (1). Suppose that Assumption 1 holds for the linearized operator L. Then φ_c is orbitally stable, if $d''(c) > 0$.*

Theorem 8 *Let $c > \varepsilon$ and $\alpha < 0$. Then the solitary wave φ_c of (1) obtained in (24) is orbitally stable by the flow of the Rosenau-RLW equation.*

The proof of Theorem 7 is a special case of [18, Theorem 3.5], and we will give it for the sake of completeness.

Lemma 1 *Let $d''(c) > 0$. Then $\langle Ly, y\rangle > 0$, if $y \in H^2(\mathbb{R})$ and $\langle y, Q'(\varphi_c)\rangle = \langle y, {\varphi_c}'\rangle = 0$.*

Proof First by using $d'(c) = Q(\varphi_c)$, we have

$$0 < d''(c) = \langle M\varphi_c, \mathrm{d}\varphi_c/\mathrm{d}c\rangle = -\left\langle L\mathrm{d}\varphi_c/\mathrm{d}c, \mathrm{d}\varphi_c/\mathrm{d}c\right\rangle.$$

Write

$$\frac{\mathrm{d}\varphi_c}{\mathrm{d}c} = a_0\chi_c + b_0{\varphi_c}' + p_0,$$

where p_0 is in the positive subspace of L. Recall that $L\chi_c = -\lambda^2\chi_c$ with $\lambda > 0$ and $L({\varphi_c}') = 0$. It follows that $\langle Lp_0, p_0\rangle < 0$. Now suppose that

$$\langle y, {\varphi_c}'\rangle = \langle y, Q'(\varphi_c)\rangle = 0$$

and decompose y into the sum $a\chi_c + p$ with p in the positive subspace of L. Because

$$0 = \langle L\mathrm{d}\varphi_c/\mathrm{d}c, y\rangle = -a_0 a\lambda^2 + \langle Lp_0, p\rangle,$$

it is inferred that

$$\langle Ly, y\rangle \geq -a^2\lambda^2 + \frac{\langle Lp, p_0\rangle^2}{\langle Lp_0, p_0\rangle} > -a^2\lambda^2 + \frac{(a_0 a\lambda)^2}{a_0^2\lambda^2} = 0,$$

as required. □

It can be proved exactly as in the analogous case of [12] that there exists $\epsilon > 0$ and a unique C^1-map $\varrho : U_\epsilon \to \mathbb{R}$ such that for every $u \in U_\epsilon$ and $r \in \mathbb{R}$, $\langle u(\cdot + \varrho(u)), {\varphi_c}'\rangle = 0$, $\varrho(u(\cdot + r)) = \varrho(u) - r$ and

$$\varrho'(u) = \frac{{\varphi_c}'(\cdot - \varrho(u))}{\int_{\mathbb{R}} u(x){\varphi_c}''(x - \varrho(u))\,\mathrm{d}x}.$$

Lemma 2 *Let $d''(c) > 0$. Then there is $C > 0$ and $\epsilon > 0$ such that*

$$E(u) - E(\varphi_c) \geq C\|u(\cdot + \varrho(u)) - \varphi_c\|_{H^2}^2,$$

for all

$$u \in \widetilde{U}_\epsilon = \{u \in U_\epsilon : \ Q(u) = Q(\varphi_c)\}.$$

Proof Write u in the form

$$u(\cdot+\varrho(u)) = (1+a)\varphi_c + y,$$

where $\langle \varphi_c, y\rangle = 0$ and a is a scalar. Then by the translation invariance Q and Taylor's theorem,

$$Q(\varphi_c) = Q(u) = Q(\varphi_c) + \langle \varphi_c, u(\cdot+\varrho(u)) - \varphi_c\rangle + a,$$

where

$$a = O\left(\|u(\cdot+\varrho(u)) - \varphi_c\|_{H^2}^2\right).$$

Hence

$$\mathcal{S}(u) = \mathcal{S}(u(\cdot+\varrho(u))) = \mathcal{S}(\varphi_c) + \frac{1}{2}\langle Lv, v\rangle + o\left(\|v\|_{H^2}^2\right),$$

where

$$v = u(\cdot+\varrho(u)) - \varphi_c = a\varphi_c + y.$$

Thus

$$E(u) - E(\varphi_c) = \frac{1}{2}\langle Lv, v\rangle + o(\|v\|_{H^2}^2) = \frac{1}{2}\langle Ly, y\rangle + o\left(\|v\|_{H^2}^2\right).$$

Since y is orthogonal to both φ_c and ${\varphi_c}'$, it follows from Lemma 1 that

$$E(u) - E(\varphi_c) \geq 2C\|y\|_{H^2}^2 + o(\|v\|_{H^2}^2),$$

for some constant C. It follows that

$$E(u) - E(\varphi_c) \geq \|v\|_{H^2}^2,$$

by using the fact

$$\|y\|_{H^2} = \|v - a\varphi_c\|_{H^2} \geq \|v\|_{H^2} - O\left(\|v\|_{H^2}^2\right),$$

for $\|v\|_{H^2}$ small. The proof of lemma is now complete. □

Proof of Theorem 7 Assume that $d''(c) > 0$. Let $u_{n,0} \in H^2(\mathbb{R})$ be any sequence such that

$$\inf_r \|u_{n,0} - \varphi_c(\cdot+r)\|_{H^2} \to 0,$$

as $n \to \infty$. If u_n is the unique solution (1) with initial data $u_n(0) = u_{n,0}$, let t_n be an arbitrary sequence of times such that, for each n, $u_n(\cdot, t_n) \in \partial U_{\epsilon/2}$. Since E and Q are continuous on $H^2(\mathbb{R})$ and translation invariant,

$$E(u_n(\cdot, t_n)) = E(u_{n,0}) \to E(\varphi_c)$$

and

$$Q(u_n(\cdot, t_n)) = Q(u_{n,0}) \to Q(\varphi_c).$$

Next choose $w_n \in U_\epsilon$ so that $Q(w_n) = Q(\varphi_c)$ and

$$\|w_n - u_n(\cdot, t_n)\|_{H^2} \to 0.$$

By Lemma 2,

$$0 \leftarrow E(w_n) - E(\varphi_c) \geq C\|w_n(\cdot+\varrho(w_n)) - \varphi_c\|_{H^2}^2 = C\|w_n - \varphi_c(\cdot - \varrho(w_n))\|_{H^2},$$

and therefore

$$\|u_n(\cdot, t_n) - \varphi_c(\cdot - \varrho(w_n))\|_{H^2} \to 0.$$

This means that $u_n(\cdot, t_n)$ tends to $\mathcal{O}_{\varphi_c}$. This contradiction completes the proof of Theorem 7. □

Now we are in position to prove Theorem 8.

Proof of Theorem 8 By Theorem 7, first of all we should study the behavior of the first two eigenvalues associated with the operator

$$L = \alpha c \frac{d^2}{d\xi^2} + \beta c \frac{d^4}{d\xi^4} - \varphi_c + c - \varepsilon.$$

By applying Theorem 6, it suffices to show that $\widehat{\varphi_c} \in PF(2)$. But a straightforward calculation reveals from (24) that

$$\widehat{\varphi}(\xi) = A \frac{\pi \xi}{3b^2} \left(1 + \frac{\xi^2}{4b^2}\right) \operatorname{csch}\left(\frac{\pi \xi}{2b}\right)$$

and

$$\frac{d^2}{d\xi^2} \log \widehat{\varphi_c}(\xi)$$
$$= -C_{(24)} \frac{(64b^6 + 12\xi^4 b^2)\cosh^2\left(\frac{\pi\xi}{2b}\right) - 12\xi^4 b^2 - \xi^6\pi^2 - 16\xi^2\pi^2 b^4 - 8\xi^4\pi^2 b^2 - 64b^6}{\left(\cosh^2\left(\frac{\pi\xi}{2b}\right) - 1\right)\xi^2\left(4b^2 + \xi^2\right)^2},$$

where $C_{(24)} = \frac{A\pi}{12b^4}$, where A and b is are as in (22). By using the Taylor expansion

$$\cosh^2\left(\frac{\pi\xi}{2b}\right) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{1}{2(2n)!}\left(\frac{\pi\xi}{b}\right)^{2n}, \tag{34}$$

it is readily seen that

$$\frac{d^2}{d\xi^2} \log \widehat{\varphi_c}(\xi) < 0,$$

for $\xi \neq 0$. Then by Theorem 7, we need to calculate $\frac{d}{dc} Q(\varphi_c)$. Actually, we have from (24) that

$$Q(\varphi_c) = \left(\frac{1}{b} - \frac{16}{9}\alpha b + \frac{1024}{99}\beta b^3\right)\frac{16A^2}{35},$$

and

$$\frac{d}{dc} Q(\varphi_c) = K_{(24)} \frac{\left(1400c^3 - 1856\varepsilon c^2 + 403c\varepsilon^2 + 53\varepsilon^3\right)}{c\sqrt{\frac{(\varepsilon - c)c}{\alpha}}} > 0.$$

where

$$K_{(24)} = \frac{70\sqrt{13}}{11583}.$$

This completes the proof of Theorem 8. □

Theorem 9 *Let $c > \varepsilon$ and $\alpha < 0$. Then the solitary wave φ_c of (1) obtained in (28) is orbitally stable by the flow of the Rosenau-RLW equation.*

Proof First we note that

$$\widehat{\varphi_c}(\xi) = \frac{\pi\xi}{b^2}\left(\frac{A_2}{2} + \frac{A_4}{3}\left(1 + \frac{\xi^2}{4b^2}\right)\right)\operatorname{csch}\left(\frac{\pi\xi}{2b}\right)$$

and

$$\frac{d^2}{d\xi^2}\log(\widehat{\varphi_c}) = -C_{(28)}\frac{\eta(\xi)}{\left(\cosh\left(\frac{\pi\xi}{2b}\right)^2 - 1\right)\xi^2(6A_2b^2 + 4A_4b^2 + A_4\xi^2)^2},$$

where $C_{(28)} = \frac{\pi}{4b^4}$ and

$$\begin{aligned}\eta(\xi) &= (12A_4^2\xi^4b^2 + 192A_2b^6A_4 + 144A_2^2b^6 + 64A_4^2b^6)\cosh\left(\frac{\pi\xi}{2b}\right)^2\\ &\quad -36\xi^2\pi^2A_2^2b^4 - 16\xi^2\pi^2A_4^2b^4 - 64A_4^2b^6\\ &\quad -144A_2^2b^6 - 48\xi^2\pi^2A_2b^4A_4\\ &\quad -12\xi^4\pi^2A_2b^2A_4 - 12A_4^2\xi^4b^2 - \xi^6\pi^2A_4^2 - 192A_2b^6A_4 - 8\xi^4\pi^2A_4^2b^2\end{aligned}$$

By a straightforward calculation, it is readily seen from (34) that

$$\frac{d^2}{d\xi^2}\log(\widehat{\varphi_c}) < 0$$

for all $\xi \neq 0$. By the proof of Theorem 8, it is enough to calculate $\frac{d}{dc}Q(\varphi_c)$. Indeed, we have from (28) that

$$\begin{aligned}Q(\varphi_c) &= -\frac{8\alpha b}{15}\left(A_2^2 + \frac{16}{7}A_4A_2 + \frac{32}{21}A_4^2\right) + \frac{32\beta b^3}{21}\left(A_2^2 + \frac{512}{165}A_4^2 + \frac{16}{5}A_4A_2\right)\\ &\quad + \frac{2}{b}\left(\frac{A_2^2}{3} + \frac{8}{15}A_4A_2 + \frac{8}{35}A_4^2\right).\end{aligned}$$

and

$$\frac{d}{dc}Q(\varphi_c) = K_{(28)}\frac{\left(235578848c^3 - 323443550c^2\varepsilon + 73379707c\varepsilon^2 + 14484995\varepsilon^3\right)}{c\sqrt{\frac{(\varepsilon-c)c}{\alpha}}} > 0,$$

where

$$K_{(28)} = \frac{20\sqrt{799890}}{32372885259},$$

and A_2, A_4 and b is are as in (26); and the proof of Theorem 9 is now complete. □

Figure 1 illustrates the shape of kernel of (12) for $\beta = 1$, and $\alpha = -3$, $\alpha = -2$ and $\alpha = 1$ respectively. Finally, we observed in Theorem 3 that the solutions of (1) satisfies the conservation laws Q and E in (5) and (6). We calculate these conserved quantities by using the solitary waves given by (24), (28) and (32). Actually, we obtain

$$E_{(24)}(\varphi_c) = -\frac{16\varepsilon A^2}{35b} - \frac{256A^3}{2079b},$$

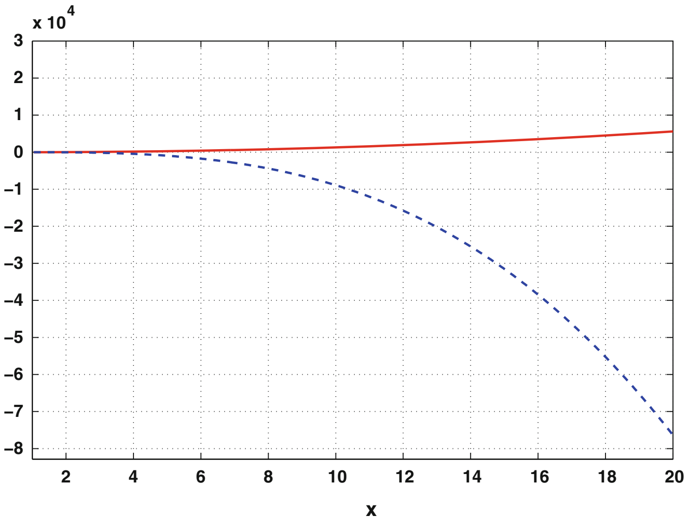
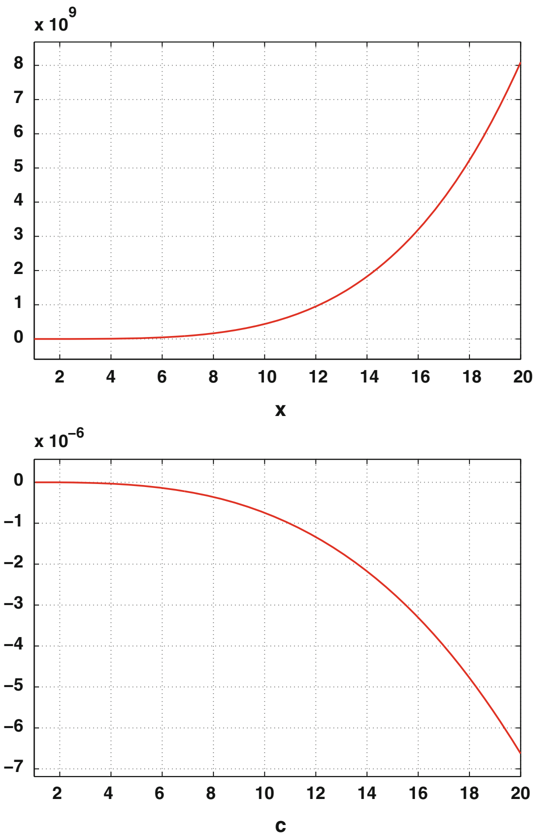


Fig. 4 Invariants of Q and E of solution (24) as the functions in terms of c. The *dash-curve* corresponds to E

Fig. 5 Invariants of E in down and Q in up of solution (28) as the functions in terms of c



where A and b is are as in (22),

$$E_{(28)}(\varphi_c) = -\frac{8A_2^3}{45b} - \left(\frac{\varepsilon}{3} + \frac{8A_4}{35}\right)\frac{2A_2^2}{b} - \left(\varepsilon + \frac{8A_4}{21}\right)\frac{16A_2A_4}{15b} - \left(\frac{\varepsilon}{5} + \frac{16A_4}{297}\right)\frac{16A_4^2}{7b}$$

where A_2, A_4 and b is are as in (26).

Figures 4 and 5 illustrate the graphs of the invariants E and Q in terms of c, when $\alpha = -1$ and $\varepsilon = 1$, for solutions (24) and (28), respectively.

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