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ORIGINAL RESEARCH

Dynamics of Two Interconnected Mass Points in a Resistive Medium

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Abstract The problem of motion of two interconnected mass points in a resistive medium under periodic change of distance between them is considered. It is shown that a necessary condition for the locomotion (displacement of the center of mass) is the nonlinear law of resistance (dry or nonlinear viscous friction). If the mass points are identical, any periodic control law leads to a displacement of the center of mass only in the presence of anisotropic (asymmetrical) friction. If there are different masses, motion is possible also with isotropic friction law under the action of a periodic control law. In the paper such special laws for the control of the velocity and the direction of motion are presented. The friction is assumed to be small and the investigations are based on the method of averaging. By means of this method analytical dependence of the velocity of motion is obtained.

Keywords Method of averaging · Dry friction · Kinematic excitation

Introduction

The motion of technical system and living beings happens in resistive media. This resistance to motion that has to be overcome is of many different kinds—depending on speed, size, and the characteristics of the surrounding medium. Therefore there exist various forms of locomotion in nature and technology. Non-pedal forms of locomotion mostly consist of a conversion of periodic internally driven motions into change of external position. Realization of this type of locomotion is possible in different ways [1]. A large series of papers is devoted to a motion on a surface with dry friction. Nonlinear systems with stick–slip motion under the action of dry friction are especially considered in a series of publications [2–6]. The

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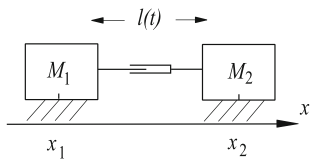
works [7,8] were devoted to snakelike robots, which are modeled as a chain of rigid links connected by actuated revolute joints. Two-link and three-link robots have been investigated in detail. The author in [8] investigated the rectilinear motion of a body with a movable internal mass (moving along a straight line parallel to the line of the body motion) on a rough plane. The maximization of average velocity of the steady-state motion of the system actuated by a periodic motion of the internal masses is an important problem for programming control modes for such system. Optimal periodic velocity-controlled motions were founded in [9,10]. The author in [10] considered motions with both dry-friction resistance, piecewise-linear resistance and quadratic-law viscous resistance. A number of papers dealt with the rectilinear motion of systems of bodies on a rough surface, where the bodies are connected by elastic elements in the case when the force of normal pressure is not changed, and the system is moved by forces that change harmonically acting between the bodies. The non-symmetry of the friction force, required for a motion in a given direction, is provided by the dependence of the friction coefficient on the sign of the velocity of constituent bodies of the system. This effect can be achieved if the contact surfaces of the robot are equipped with a special scaly (needle-shaped) plate with a required orientation of scales (needles). In [11,12] a system of two bodies joined by an elastic element was considered. The motion is excited by harmonic force acting between the bodies. The rectilinear motion of a vibration-driven robot with two internal masses along a rough horizontal plane with symmetric friction coefficients is considered in [13]. One of the internal masses moves relative to the body along the line of its motion, whereas the other mass moves along the normal one, which makes it possible to influence the normal reaction of the supporting surface. Both masses perform harmonic vibrations with the same frequency but shifted in phase. It is shown that by controlling the phase shift and the frequency of the vibrations of the internal masses one can change the direction of motion of the body and the average velocity of the steady-state (velocity-periodic) motion of the robot. The undulatory motion of a finite chain of identical bodies along a straight line in a resistive medium is studied in [14]. The motion is excited and controlled by changing the distances between the bodies of the chain.

In this paper we consider the rectilinear motion of two different mass points in a resistive medium. The force of resistance is described by small non-symmetric dry friction acting on each mass point. The mass points are interconnected with a kinematic constraint. The expression for the stationary "on the average" velocity of the system's motion as a single whole is found.

Mechanical System

We consider the motion of two different mass points M_1 and M_2 along a horizontal straight line OX in a resistive medium. We denote $x_1,\ x_2$ their coordinates. The motion is excited and controlled by changing the distances between the mass points. The medium acts on each point with the force of friction $F(\dot{x}_i)$, depending on the velocity $\dot{x},\ (i = 1,\ 2)$ of this mass point relative to the environment. The mechanical system is shown in Fig. 1.

Fig. 1 Mechanical model



We assume that the distance between the mass points $l(t)$ is a periodic function with the period T, i.e.,

$$x_2 - x_1 = l(t), \quad l(t+T) = l(t) \tag{1}$$

The coordinate X of the center of mass of the system and its velocity V are defined by

$$X = m_1 x_1 + m_2 x_2, \quad V = \dot{X}. \tag{2}$$

Here

$$m_1 = \frac{M_1}{M}, \quad m_2 = \frac{M_2}{M}, \quad M = M_1 + M_2, \quad m_1 + m_2 = 1. \tag{3}$$

The motion of the center of mass is governed by the equation

$$M\dot{V} = F(\dot{x}_1) + F(\dot{x}_2). \tag{4}$$

Using Eqs. (1)–(3) we obtain

$$\dot{x}_1 = V - m_2 u, \quad \dot{x}_2 = V + m_1 u, \quad u(t) = \dot{l}(t). \tag{5}$$

Substitute relations (5) into Eq. (4) becomes the equation of the motion in the form

$$\dot{V} = \frac{1}{M}\big(F(V - m_2 u) + F(V + m_1 u)\big). \tag{6}$$

As $l(t)$ is a periodic function with the period T then

$$\int_0^T u(t)dt = \int_0^T \dot{l}(t)dt = l(T) - l(0) = 0. \tag{7}$$

Thus, the average value of function $u(t)$ for the period T is zero.

Linear Resistance Law

For the linear resistance characterized by the function $F(v) = -kv$, where k is the coefficient of friction, the Eq. (6) becomes

$$\dot{V} = -\frac{k}{M}\big(2V + (m_1 - m_2)u\big). \tag{8}$$

Statement 1 The locomotion in a medium with a linear resistance law at any periodic change of distance between the mass points is impossible.

Proof Let's integrate both sides of Eq. (8) on the first n periods

$$\int_0^{nT} \dot{V}(t)dt = -\frac{2k}{M}\int_0^{nT} V(t)dt - \frac{k}{M}(m_1 - m_2)\int_0^{nT} u(t)dt. \tag{9}$$

The average velocity $\overline{V}$ on an interval between 0 and nT is $\overline{V} = \frac{1}{nT}\int_0^{nT} V(t)dt$. Taking into account the relation (7) we obtain from Eq. (9)

$$\overline{V} = \frac{1}{2kT} \cdot \frac{V(0) - V(nT)}{n}. \tag{10}$$

If $n \to \infty$ the value $V(nT)$ is limited.

Really, we rewrite the Eq. (8) as:

$$\dot{V} = -\alpha V + \beta u(t), \tag{11}$$

where

$$\alpha = \frac{2k}{M} > 0, \quad \beta = -\frac{k(m_1 - m_2)}{M}.$$

The solution of the Eq. (11) is

$$V(t) = V(0)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-\xi)} u(\xi) d\xi,$$

so

$$\begin{aligned} |V(nT)| &= \left| V(0)e^{-\alpha nT} + \beta \int_0^{nT} e^{-\alpha(nT-\xi)} u(\xi) d\xi \right| \\ &< \left| V(0) \right| + |\beta| \int_0^{nT} e^{-\alpha(nT-\xi)} \left| u(\xi) \right| d\xi. \end{aligned}$$

By assumption $u(t)$ is limited, this means that $|u(t)| \le C$.

Then

$$\begin{aligned} |V(nT)| &< V(0) + |\beta| \, C e^{-\alpha nT} \int_0^{nT} e^{\alpha\xi} d\xi \\ &= |V(0)| + \frac{|\beta|}{\alpha} C e^{-\alpha nT} \left(e^{\alpha nT} - 1 \right) \\ &= |V(0)| + \frac{|\beta|}{\alpha} C \left(1 - e^{-\alpha nT} \right) \\ &< |V(0)| + C \frac{|\beta|}{\alpha}. \end{aligned}$$

Then the value $\overline{V} \to 0$.

Discontinuous (Dry) Friction

Dry friction is resistance characterized by the Coulomb's law:

$$f(V) = \begin{cases} k_-, & \text{if } V < 0 \text{ or } V = 0 \text{ and } \Phi < -k_-, \\ -k_+, & \text{if } V > 0 \text{ or } V = 0 \text{ and } \Phi < k_+, \\ -\Phi, & \text{if } V = 0 \text{ and } -k_- < \Phi < k_+. \end{cases} \tag{12}$$

Here k_- and k_+ are positive quantities, Φ is the resultant of the forces, other than frictional ones, applied to the mass points.

The motion of the center of mass is governed by the equation

$$\dot{V} = m_1 g f(V - m_2 u) + m_2 g f(V + m_1 u). \tag{13}$$

We denote

$$k_0 = \max(k_+, k_-), \quad \mu_+ = k_+/k_0, \quad \mu_- = k_-/k_0.$$

Then

$$f(V) = k_0\mu(V), \quad \mu(V) = \begin{cases} \mu_-, & \text{if } V < 0 \text{ or } V = 0 \text{ and } \Phi < -\mu_-, \\ -\mu_+, & \text{if } V > 0 \text{ or } V = 0 \text{ and } \Phi < \mu_+, \\ -\Phi, & \text{if } V = 0 \text{ and } -\mu_- < \Phi < \mu_+ \end{cases}$$

and Eq. (13) takes the form

$$\dot{V} = m_1 g k_0 \mu(V - m_2 u) + m_2 g k_0 \mu(V + m_1 u). \tag{14}$$

We introduce the dimensionless variables (denoted by an asterisk)

$$t^* = \frac{t}{T}, \quad V^* = V\frac{T}{L}, \quad u^* = u\frac{T}{L}, \tag{15}$$

where L is the maximal distance between the mass points, and T is the period of control $l(t)$. The nondimensionalised Eq. (14) can be represented as (asterisks are omitted)

$$\dot{V} = \varepsilon G(V, t), \quad G(V, t) = m_1\mu(V - m_2 u) + m_2\mu(V + m_1 u), \quad \varepsilon = \frac{k_0 g T^2}{L}. \tag{16}$$

Analytical Investigations: Method of Averaging

We consider a smooth periodic control $l(t)$. The function $l(t)$ is made from parabolas so that on first two intervals ($0 \le t < \tau/2$ and $\tau/2 \le t < \tau$) function changes with constant on absolute value second derivation $2u_+/\tau$. On last two intervals ($\tau \le t < (1+\tau)/2$ and $(1+\tau)/2 \le t \le 1$) the second derivation also is on absolute value constant, but another: $2u_-/(1-\tau)$. The parameter τ characterizes the degree of asymmetry.

Also the functions $l(t)$ and $u(t)$ have the form:

e

$$l(t) = \begin{cases} u_+ t^2/\tau, & 0 \le t < \tau/2, \\ -u_+(t-\tau)^2/\tau + u_+\tau/2, & \tau/2 \le t < \tau, \\ u_+ t^2/\tau, & \tau \le t < (1+\tau)/2, \\ u_+ t^2/\tau, & (1+\tau)/2 \le t \le 1, \end{cases} \tag{17}$$

$$u(t) = \begin{cases} 2u_+ t/\tau, & 0 \le t < \tau/2, \\ -2u_+ t/\tau + 2u_+, & \tau/2 \le t < \tau, \\ -2u_-(t-\tau)/(1-\tau), & \tau \le t < (1+\tau)/2, \\ 2u_-(t-1)/(1-\tau), & (1+\tau)/2 \le t \le 1. \end{cases} \tag{18}$$

We assume that the distance between the mass points $l(t)$ is periodic function, therefor

$$u_+\tau = u_-(1-\tau). \tag{19}$$

The function $l(t)$ and its derivative $u(t)$ are shown in Fig. 2.

We assume the parameter ε is small ($\varepsilon \ll 1$) and apply the method of averaging [15] to Eq. (16) and obtain the time-invariant averaged equation

$$\frac{dv}{dt} = \varepsilon\overline{G}(v), \quad \overline{G}(v) = \int_0^1 G(v, t)dt = \varepsilon(m_2 I_1 + m_1 I_2), \tag{20}$$

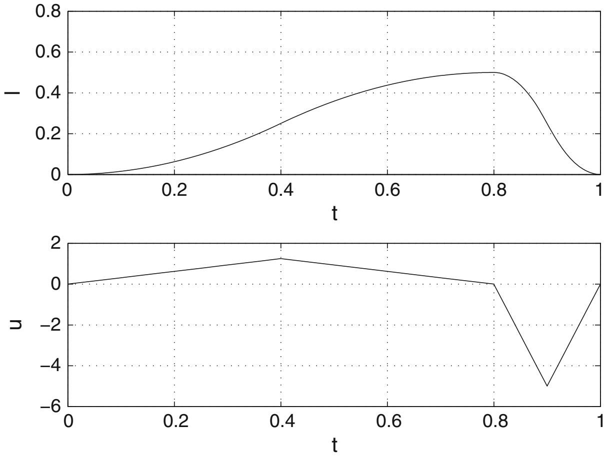


Fig. 2 Functions l (*upper*) and u (*lower*) versus t ($\tau = 0.8,\ u_- = 5,\ u_+ = 1.25$)

where

$$I_1 = \int_0^1 \mu \left(\frac{V}{m_1} + u\right) dt, \quad I_2 = \int_0^1 \mu \left(\frac{V}{m_2} - u\right) dt. \tag{21}$$

The results of the averaging are:

$$I_1 = \begin{cases} \mu_-(\tau_2^1 - \tau_1^1) - \mu_+(1 + \tau_1^1 - \tau_2^1), & \text{if} \quad 0 < V/m_1 < u_-, \\ \mu_-(1 + \tau_3^1 - \tau_4^1) - \mu_+(\tau_4^1 - \tau_3^1), & \text{if} \quad - u_+ V/m_1 < 0, \\ \mu_-, & \text{if} \quad V/m_1 < -u_+, \\ -\mu_+, & \text{if} \quad V/m_1 > u_-, \end{cases} \tag{22}$$

where

$$\tau_1^1 = \tau + \frac{(1-\tau)V}{2m_1u_-}, \quad \tau_2^1 = 1 - \frac{(1-\tau)V}{2m_1u_-}, \quad \tau_3^1 = -\frac{\tau V}{2m_1u_+}, \quad \tau_4^1 = \tau + \frac{\tau V}{2m_1u_+}.$$

$$I_2 = \begin{cases} \mu_-(\tau_2^2 - \tau_1^2) - \mu_+(1 + \tau_1^2 - \tau_2^2), & \text{if} \quad 0 < V/m_2 < u_+, \\ \mu_-(1 + \tau_3^2 - \tau_4^2) - \mu_+(\tau_4^2 - \tau_3^2), & \text{if} \quad - u_- < V/m_2 < 0, \\ -\mu_+, & \text{if} \quad V/m_2 > u_+, \end{cases} \tag{23}$$

where

$$\tau_1^2 = \frac{(1-\tau)V}{2m_2u_+}, \quad \tau_2^2 = \tau\left(1 - \frac{(1-\tau)V}{2m_2u_+}\right), \quad \tau_3^2 = \tau - \frac{(1-\tau)V}{2m_2u_-}, \quad \tau_4^2 = 1 + \frac{(1-\tau)V}{2m_2u_-}.$$

If the friction is isotropic ($\mu_+ = \mu_- = \mu_*$), Eq. (20) takes the form

$$\begin{aligned} \frac{dv}{dt} &= \varepsilon\mu_*(a - bv), \\ a &= (m_1 - m_2)(2\tau - 1), \\ b &= 2\left(\frac{m_1}{m_2} \cdot \frac{\tau}{u_+} + \frac{m_2}{m_1} \cdot \frac{1-\tau}{u_-}\right), \end{aligned} \tag{24}$$

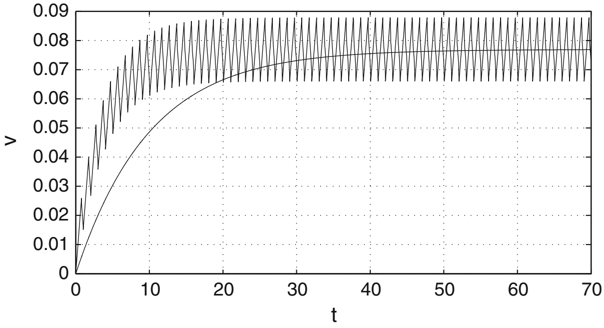


Fig. 3 The solution of the exact Eq. (16) and the corresponding averaged Eq. (24)

and has the solution ($v(0) = 0$)

$$v(t) = \frac{a}{b}\left(1 - e^{-\varepsilon\mu_* t}\right). \tag{25}$$

The exact motion $V(t)$ obtained by numerical integration of the Eq. (20) and the analytical solution $v(t)$ are shown in Fig. 3. The parameters are: $\varepsilon = 0.1,\ \mu_+ = \mu_- = 1$.

We are interested in steady-state motions, i.e. in motions with a constant velocity. The steady-state velocity of the system is

$$v_* = a/b = \frac{1}{2}(m_1 - m_2)(2\tau - 1)/\left(\frac{m_1}{m_2}\cdot\frac{\tau}{u_+} + \frac{m_2}{m_1}\cdot\frac{1-\tau}{u_-}\right). \tag{26}$$

Statement 2 In the case of isotropic friction necessary and sufficient conditions for locomotion are different masses of mass points ($m_1 \neq m_2$) and an asymmetry in the control law ($\tau \neq 1/2$).

The locomotion is possible when and only when the right hand side of the Eq. (24) is not equal 0.

Conclusions

The motion of two interconnected different mass points along a horizontal straight line in a resistive medium is considered. It is shown that the motion of different mass points can be controlled by changing the distance between the mass points in an environment with a nonlinear resistance law. If the mass points are identical, any periodic control law leads to a displacement of the center of mass only in the presence of anisotropic (asymmetrical) friction. If there are different masses, motion is possible also with isotropic friction law under the action of a periodic control law. One of such a special law for the control of the velocity and the direction of motion is presented. The dry friction is assumed to be small and the investigations are based on the method of averaging. By means of this method analytical dependence of the velocity of motion is obtained.

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