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ORIGINAL RESEARCH

Intermittent Impulsive Synchronization of Chaotic Delayed Neural Networks

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Abstract In this paper, a novel intermittent impulsive synchronization scheme is proposed to realize synchronization of two chaotic delayed neural networks. Intermittent impulsive control breaks through the limitation of the upper bound of the impulsive intervals in general impulsive control. In our synchronization scheme, impulsive control is only activated in the control windows, rather than during the whole time. Several synchronization criteria for chaotic delayed neural networks are established utilizing the method of linear matrix inequalities (LMI) and the Lyapunov–Razumikhin technique. Two numerical examples are given to demonstrate the effectiveness of our main results.

Keywords Intermittent impulsive control · Chaotic delayed neural networks · Chaos synchronization · Time delay · LMI · Lyapunov–Razumikhin technique

Introduction

In the past few decades, neural networks (especially Hopfield neural networks, cellular neural networks and bidirectional associative memory neural networks) have been applied successfully in many areas, such as signal processing, pattern recognition, associative memories, and optimization solvers. In such applications, it is of prime importance to ensure that the designed neural networks are stable. However, it is well known that the finite switching speed of amplifiers and the communication time of neurons may induce time delay in the interaction

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between the neurons when the neural networks are implemented by large-scale integrated electronic circuits, which can affect the stability of neural networks [12]. Marcus and Westervelt firstly introduced a single delay into Hopfield neural networks [27]. It was observed both experimentally and numerically that time delay could destroy a stable network and cause sustaining oscillations, which can be harmful to the network. Consequently, delayed neural networks (DNNs), have also been extensively investigated[24,23,36,35,44].

Chaos synchronization has been an active issue in both mathematics and engineering disciplines over the past two decades due to its potential applications to secure communication. For instance, several chaos-based secure communication schemes have been proposed in recent years [31,6,8,9,11,40]. In these schemes, the plaintext is encrypted by the chaotic signal at the transmitter, and then the ciphertext is transmitted to the receiver across a public channel (unsafe channel). At the receiver, perfect synchronization is usually expected to recover the plaintext, i.e., the decryption of the ciphertext requires the receiver's own copy of the chaotic signal which is synchronized with that of the transmitter [31]. Chaos synchronization plays a critical role in this process. Since the introduction of synchronizing two identical chaotic systems with different initial conditions by Pecora and Carroll in 1990 [31], different types of synchronization techniques have been rapidly developed. Among these include active control between two Lorenz systems [1], a backstepping approach between two Genesio systems [28,34], adaptive control [41,43,32], delay feedback synchronization [4], a variable structure method [20], a sliding model control [42], nonlinear feedback control [29,13], intermittent feedback control [14,37,45], etc. On the other hand, hyperchaos synchronization in which hyperchaotic systems possess more than one positive Lyapunov exponent has been studied in [8,9], utilizing an observer-design method. Recently, chaos synchronization of coupled neural networks has attracted a lot of attention of scholars [21,3,2,30,26] since it was shown that neural networks can handle complicated dynamics and even chaotic behaviors [7,25,5].

It is well known that reducing the redundancy of the synchronization signals can increase the difficulty to break the chaos-based cryptosystem and improve its security. Different from continuous feedback control, impulsive synchronization only requires small synchronizing impulses. These impulses are sampled from the state variables of the drive system at discrete moments and then drive the response system at these moments. When the attractivity of the error system between the drive and the response chaotic systems is achieved, these two coupled systems are said to be synchronized. A generalization of impulsive synchronization with time-varying impulse intervals is investigated in [18,19]. The robustness of impulsive synchronization toward parameter mismatch (with the absence of delay) has been also studied in [17]. Furthermore, impulsive control has been applied to several chaos-based secure communication schemes which exhibit good performance [38,39]. Detailed experiments and performance analysis of impulsive synchronization with the intent of testing its accurate recovery of the plaintext and its applicability to secure communication have appeared in [15,16].

Although impulsive synchronization technology becomes more and more popular in chaos-based secure communication, there always exists an upper bound on the time intervals between the impulses (impulsive intervals) during the synchronization process [38,39]. Usually, the impulsive intervals are small, i.e., the controller in the slave system needs to be activated frequently. Specifically, in some situations such as the orbital transfer of satellite, control of money supply in a financial market, etc., the control windows (the time periods the controller can work) are restricted. If the free windows (the time periods the controller can not be activated) are larger than the upper bound of the impulsive intervals, the general impulsive synchronization approach will fail. On the other hand, to reduce the control windows is a new way to improve the security of chaos-based secure communication by decreasing the redundancy of synchronization signals.

In this paper, we propose a novel intermittent impulsive synchronization scheme for delayed neural networks. Several synchronization criteria for chaotic delayed neural networks are established utilizing the method of linear matrix inequalities (LMI) and the Lyapunov-Razumikhin technique. In our synchronization scheme, the impulsive controller is only activated in the control windows, not during the entire time. To the best of our knowledge, there is no existing work studying this challenging problem. Our synchronization criteria may be used as a guideline for some engineering applications. The remainder of this paper is organized as follows. In "Preliminaries", we present a general impulsive synchronization scheme and an intermittent impulsive synchronization scheme. Some basic lemmas, preliminary definitions and hypotheses are introduced. In "Synchronization Criteria", we establish the general impulsive synchronization criteria and intermittent impulsive synchronization criteria based on Lyapunov-Razumikhin theory and LMI. We demonstrate the effectiveness of our main results by some numerical examples in "Numeral Examples". Finally, conclusions are given.

Preliminaries

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers and R^n the n-dimensional Euclidean linear space equipped with the Euclidean norm $\|\cdot\|$. Throughout this paper, $P > 0$ ($< 0, \leq 0, \geq 0$) denotes a symmetric positive (negative, semi-negative, semi-positive) definite matrix P, P^T the transpose of P and $\lambda_{M(m)}(P)$ the maximum (minimum) eigenvalue of a symmetric matrix P. Let $\varphi(t^+) = \lim_{s\to t^+}\varphi(s)$ and $\varphi(t^-) = \lim_{s\to t^-}\varphi(s)$.

Let $a,\ b \in R$ with $a < b$ and $S \subset R^n$. Define

$$\begin{aligned}&PC([a,b],\ R^n)\\&\quad = \{\varphi : [a,\ b] \to S|\varphi(t^+) = \varphi(t), \forall t \in [a,\ b); \varphi(t^-) \text{ exists in } S, \forall t \in (a,\ b]\\&\qquad \text{and } \varphi(t^-) = \varphi(t) \text{ for all but at most a finite number of points } t \in (a,\ b]\}.\end{aligned}$$

For $\tau > 0$, we equip the linear space $PC([-\tau,\ 0],\ R^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\varphi\|_\tau = \sup_{-\tau\leq s\leq 0}\|\varphi(s)\|$.

Consider a class of delayed neural networks as the master system (drive system), given by

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau(t))) + J_i,\\ &\quad t > 0, \quad i = 1, 2, \ldots, n,\end{aligned} \tag{1}$$

where $x_i(t)$ is the state vector associated with the i-th neuron, $f_i \in C[R,\ R]$ denotes the activation function, by which the neurons respond to each other, $c_i > 0$, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively, which indicate the strengths of the neuron interconnections within the network, J_i is a constant external input to set the desired equilibrium point, $\tau(t)$ is the time-varying delay, satisfying $r = \max\limits_{t\in R^+}\{\tau(t)\}$ and the initial condition of (1) is given by $x_i(t) = \phi_i \in PC([-r,\ 0], R)$.

General Impulsive Synchronization Scheme (GISS):

In GISS, the corresponding slave system (response system) is designed by

$$\begin{cases} \dfrac{dy_i(t)}{dt} = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(y_j(t-\tau(t))) + J_i, \quad t \neq T_k, \\ \Delta y_i(t) = U_{k,i}(x_i(t), y_i(t)), \quad t = T_k, \end{cases} \tag{2}$$

where T_k $(k = 0, 1, \ldots)$ is the k-th impulsive instant, $U_{k,i}(x_i(t), y_i(t)) = B_{k,i}(x_i(t) - y_i(t))$ is the impulsive control of i-th neuron at the k-th impulsive instant and $\{T_k, U_{k,i}\}$ is called an impulsive control law.

The initial conditions for the slave system (2) are given by

$$y_i(t) = \psi_i(t), \quad -r \leq t \leq 0,$$

where $\psi_i \in PC([-r,\ 0],\ R)$.

Let $e_i(t) = x_i(t) - y_i(t)$. The error system is given by

$$\begin{cases} \dfrac{de_i(t)}{dt} = -c_i e_i(t) + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(t) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j(t-\tau(t)), \quad t \neq T_k, \\ \Delta e_i(t) = -B_{k,i} e_i(t), \quad t = T_k, \end{cases} \tag{3}$$

where $\tilde{f}_j(t) = f_j(x_j(t)) - f_j(y_j(t))$.

In our intermittent impulsive synchronization scheme, impulsive control only occurs in control windows, not during the whole time. Define free windows $[m\omega, m\omega + \delta]$ and control windows $[m\omega + \delta, (m+1)\omega]$ where $m = 0, 1, \ldots$ and $0 < \delta < \omega < \infty$.

Intermittent Impulsive Synchronization Scheme (IISS):

In IISS, the corresponding slave system (response system) is designed as follows:

$$\begin{cases} \dfrac{dy_i(t)}{dt} = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(y_j(t-\tau(t))) + J_i, \\ \qquad t \in [m\omega,\ m\omega + \delta], \\ \begin{cases} \dfrac{dy_i(t)}{dt} = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) \\ \qquad + \sum_{j=1}^{n} b_{ij} f_j(y_j(t-\tau(t))) + J_i, \quad t \neq T_{m,l}, \\ \Delta y_i(t) = U_{m,l,i}(x_i(t), y_i(t)), \quad t = T_{m,l}, \end{cases} \\ \qquad t \in [m\omega + \delta,\ (m+1)\omega], \end{cases} \tag{4}$$

where $m = 0, 1, \ldots,\ l = 1, 2, \ldots, M_m,\ M_m$ is a positive integer related to $m,\ T_{m,l}$ denotes the l-th impulsive instant in the m-th control window, $m\omega + \delta = T_{m,1} < T_{m,2} < \cdots < T_{m,M_m} \leq (k+1)\omega$, $U_{m,l,i}(x_i(t), y_i(t)) = B_{m,l,i}(x_i(t) - y_i(t))$ is the impulsive control and $\{T_{m,l}, U_{m,l,i}\}$ is called an intermittent impulsive control law. Let $T_{k,M_k+1} = (k+1)\omega$ and define $\Delta_{k,i} = T_{k,i+1} - T_{k,i}$.

Let $e_i(t) = x_i(t) - y_i(t)$. Also, we obtain the following error system:

$$\begin{cases} \dfrac{de_i(t)}{dt} = -c_i e_i(t) + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(t) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j(t - \tau(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [m\omega,\ m\omega + \delta], \\ \begin{cases} \dfrac{de_i(t)}{dt} = -c_i e_i(t) + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(t) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j(t - \tau(t)), & t \neq T_{m,l}, \\ \Delta e_i(t) = -B_{m,l,i} e_i(t), \quad t = T_{m,l}, \end{cases} \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [m\omega + \delta,\ (m+1)\omega], \end{cases} \tag{5}$$

where $\tilde{f}_j(t) = f_j(x_j(t)) - f_j(y_j(t))$.

Throughout this paper, we assume that f_i $(i = 1, 2, \ldots, n)$ satisfies the Lipschitz condition, i.e.,

H1: There exists a positive constant L_i such that, for all $x,\ y \in R$,

$$\|f_i(x) - f_i(y)\| \le L_i \|x - y\|. \tag{6}$$

Remark 1 Assumption (H1) ensures that the above delayed neural network always has an equilibrium point and all solutions of the delayed neural network exist for all future time.

Chaos Synchronization: Let $x_i(t)$ and $y_i(t)$ be the solutions of the master system and the corresponding slave system, respectively. If

$$\lim_{t \to \infty} \|x_i(t) - y_i(t)\| = 0, \quad i = 1, 2, \ldots, N,$$

then it is said that the slave system is synchronized with the master system.

From the definition of synchronization, if the solution of the error system $e_i(t) = x_i(t) - y_i(t)$ satisfies

$$\lim_{t \to +\infty} \|e_i(t)\| = 0, \quad i = 1, 2, \ldots, N,$$

then synchronization is achieved.

Remark 2 From the definition of chaos synchronization, we know that the goal of chaos synchronization is not to stabilize the slave or master system, but to control the slave system such that the trajectory of the slave system follows that of the master system, while the master and slave systems both are chaotic. Mathematically, synchronization implies attractiveness of the corresponding error system.

To establish the synchronization criteria in the next section, we shall use the following lemmas.

Lemma 1 [33]*: For any vectors $x,\ y \in R^n$ and positive constant ξ, the following matrix inequality holds:*

$$2x^T y \le \xi x^T x + \frac{1}{\xi} y^T y.$$

Lemma 2 [10]*: Function $y(t)$ is non-negative when $t \in (-\tau, \infty)$ and satisfies the following:*

$$\frac{dy(t)}{dt} \le k_1 y(t) + k_2 y(t - \tau), \quad t \ge 0,$$

where k_1 and k_2 are positive constants. We then have the following inequality:

$$y(t) \le \|y(0)\|_\tau e^{(k_1 + k_2)t}, \quad t \ge 0.$$

Lemma 3 [22]*: Let* $\gamma > 0$ *and* $V(t) \in C^1[J, R_+]$, *where* $J = [a-\gamma, b)$, $0 < b-a \leq \Delta$. *Assume that there exist constants* $l > 0$ *and* $\beta \in (0,1)$ *such that*

$$V'(t) \leq lV(t), \quad whenever \quad V(t) \geq \beta V(t+s), \quad s \in [-\gamma, 0]; \tag{7}$$

and there exists a constant $\eta > 0$ *such that* $V(s) \leq \eta,\ s \in [a-\gamma, a),\ V(a) \leq \beta\eta$, *and*

$$l\Delta + \ln\beta < 0. \tag{8}$$

Then $V(t) < \eta,\ t \geq a$.

Remark 3 From condition (8), we know that there always exists a constant $d(\beta < d < 1)$ such that

$$l\Delta + \ln\beta < \ln d.$$

Futhermore, we have $V(t) < d\eta,\ t \geq a$, under the same conditions (7) and (8) of Lemma 3.

Proof Suppose that, for the sake of contradiction, there exists a $t^* > a$ such that

$$V(t^*) = d\eta \text{ and } V(t) < d\eta,\ t \in [a,\ t^*).$$

Let $t^1 = \sup\{t \in [a,\ t^*],\ V(t) \leq \beta\eta\}$. Then $t^1 \in [a, t^*),\ V(t^1) = \beta\eta$ and $\beta\eta \leq V(t) \leq d\eta, \quad t \in [t^1, t^*]$. Thus for $t \in [t^1, t^*]$, we have

$$\beta V(t+s) \leq \beta\eta \leq V(t), \quad s \in [-\gamma, 0],$$

which implies

$$V'(t) \leq lV(t), t \in [t^1, t^*].$$

Integrating from t^1 to t^* gives

$$\ln(V(t^*)) - \ln(V(t^1)) \leq l(t^* - t^1) \leq l\Delta. \tag{9}$$

But

$$\ln(V(t^*)) - \ln(V(t^1)) = \ln(d\eta) - \ln(\beta\eta) > l\Delta, \tag{10}$$

which is a contradiction. The proof is complete. □

Synchronization Criteria

In this section, we derive several synchronization criteria via general impulsive control and intermittent impulsive control, respectively. Firstly, utilizing Lyapunov–Razumikhin technique, we obtain the following general impulsive synchronization criterion.

General Impulsive Synchronization Criterion

In GISS, error system (3) can be rewritten in the following compact form:

$$\begin{cases} \frac{de(t)}{dt} = -Ce(t) + A\tilde{f}(t) + B\tilde{f}(t-\tau(t)), & t \neq T_k, \\ \Delta e(t) = -B_k e(t), & t = T_k, \end{cases} \tag{11}$$

where $e(t) = [e_1(t), \ldots, e_n(t)]^T$, $C = diag\{c_1, \ldots, c_n\}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $\tilde{f}(t) = [\tilde{f}_1(t), \ldots, \tilde{f}_n(t)]$ and $B_k = diag\{B_{k,1}, \ldots, B_{k,n}\}$.

Theorem 1 *In GISS, assume that for an impulsive control law* $\{T_k, U_{k,i}\}$

(i) *there exist a positive definite matrix* P *and constants* $\alpha_1 > 0,\ \alpha_2 > 0,\ \xi_1 > 0$ *and* $\xi_2 > 0$ *such that*

$$-2PC + \xi_1 PAA^T P + \xi_2 PBB^T P + \frac{1}{\xi_1} L_f - \alpha_1 P \leq 0, \tag{12}$$

and

$$\frac{1}{\xi_2} L_f - \alpha_2 P \leq 0, \tag{13}$$

where $L_f = diag\{L_1^2, L_2^2, \ldots, L_n^2\}$.

(ii) *there exists a real number* $\beta \in (0,\ 1)$ *such that*

$$(I - B_k^T)P(I - B_k) - \beta P \leq 0. \tag{14}$$

(iii) *there exists a positive number* Δ $(\Delta \geq \Delta_k)$ *such that*

$$\frac{\Delta}{\beta}(\beta\alpha_1 + \alpha_2) + \ln\beta \leq 0, \tag{15}$$

where $\Delta_k = T_{k+1} - T_k$.

Then the slave system (2) is synchronized with the master system (1) by impulsive control law $\{T_k, U_{k,i}\}$.

Proof Define $V(t) = e(t)^T Pe(t)$. When $t \in (T_k,\ T_{k+1})$, we have

$$\begin{aligned} V'(t) &= \dot{e}(t)^T Pe(t) + e(t)^T P\dot{e}(t) \\ &= -2e(t)^T PCe(t) + 2e(t)^T PA\tilde{f}(t) + 2e(t)^T PB\tilde{f}(t - \tau(t)) \end{aligned} \tag{16}$$

By Lemma 1 and condition (6), we obtain

$$\begin{aligned} V'(t) &\leq -2e(t)^T PCe(t) + \xi_1 e(t)^T PAA^T Pe(t) + \frac{1}{\xi_1}\|\tilde{f}(t)\|^2 + \xi_2 e(t)^T PBB^T Pe(t) \\ &\quad + \frac{1}{\xi_2}\|\tilde{f}(t - \tau(t))\|^2 \\ &\leq e(t)^T \left(-2PC + \xi_1 PAA^T P + \xi_2 PBB^T P + \frac{1}{\xi_1} L_f\right) e(t) \\ &\quad + \frac{1}{\xi_2} L_f e(t - \tau(t))^T e(t - \tau(t)) \end{aligned} \tag{17}$$

where $L_f = \text{diag}\{L_1^2, L_2^2, \ldots, L_n^2\}$.

By condition (i), we have

$$V'(t) \leq \alpha_1 V(t) + \alpha_2 V(t - \tau), \tag{18}$$

which implies, if $V(t) \geq \beta V(t + s),\ s \in [-r, 0]$, then

$$V'(t) \leq \frac{1}{\beta}(\beta\alpha_1 + \alpha_2)V(t). \tag{19}$$

When $t = T_k$, we get by condition (ii)

$$\begin{aligned} V(T_k) &= e^T(T_k^-)(I - B_k)^T P(I - B_k)e(T_k^-) \\ &\le \beta e^T(T_k^-)Pe(T_k^-) \\ &= \beta V(T_k^-) \\ &\le \beta \|V(T_k)\|_r. \end{aligned} \tag{20}$$

By (19), (20), condition (iii) and Lemma 3, we have

$$V(t) < \|V(T_k)\|_r \le \rho\|V(T_k)\|_r, \quad t \in (T_k,\ T_{k+1}), \tag{21}$$

where $\beta \le \rho < 1$.

From (20) and (21), we have

$$V(t) \le \rho^k \|V(0)\|_r, \quad t \in [T_k,\ T_{k+1}]. \tag{22}$$

Thus

$$\lim_{t\to\infty} \|e(t)\| \le \lim_{t\to\infty} \frac{V(t)}{\lambda_m(P)} \le \lim_{t\to\infty} \frac{\rho^k\|V(0)\|_r}{\lambda_m(P)} = 0. \tag{23}$$

Therefore, from the definition of synchronization, the slave system (2) is synchronized with the master system (1) by impulsive control law $\{T_k, U_{k,i}\}$.

Remark 4 Conditions (i–ii) of Theorem 1 are related to impulsive controllers $U_{k,i}$ and condition (iii) is about the impulsive intervals Δ_k. If the controllers are strong enough (i.e., $B_k \approx I$) and the impulsive intervals are small enough (i.e., $\Delta_k \approx 0$), then all conditions of Theorem 1 are always satisfied. It implies that we can always realize chaos synchronization by GISS.

If the impulsive control laws $\{T_k, U_{k,i}\}$ $(k = 1, 2, \ldots)$ are the same, i.e., $\Delta_k = \Delta_1$ and $B_k = B_s$ for all k, then we have the following corollary.

Corollary 1 *In GISS, assume that for an impulsive control law* $\{T_k, U_{k,i}\}$

(i) *there exist a positive definite matrix P and constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\xi_1 > 0$ and $\xi_2 > 0$ such that*

$$-2PC + \xi_1 PAA^T P + \xi_2 PBB^T P + \frac{1}{\xi_1}L_f - \alpha_1 P \le 0,$$

and

$$\frac{1}{\xi_2}L_f - \alpha_2 P \le 0,$$

where $L_f = \mathrm{diag}\{L_1^2, L_2^2, \ldots, L_n^2\}$.

(ii) *there exists a real number $\beta \in (0,\ 1)$ such that*

$$(I - B_s^T)P(I - B_s) - \beta P \le 0;$$

(iii) *the impulsive interval Δ_1 satisfies*

$$\frac{\Delta_1}{\beta}(\beta\alpha_1 + \alpha_2) + \ln\beta \le 0.$$

Then the slave system (2) is synchronized with the master system (1) by impulsive control law $\{T_k, U_{k,i}\}$.

Intermittent Impulsive Synchronization Criterion

In IISS, error system (5) can be rewritten in the following compact form:

$$\begin{cases} \frac{de(t)}{dt} = -Ce(t) + A\tilde{f}(t) + B\tilde{f}(t-\tau(t)), \\ \qquad\qquad\qquad\qquad t \in [m\omega,\ m\omega+\delta], \\ \begin{cases} \frac{de(t)}{dt} = -Ce(t) + A\tilde{f}(t) + B\tilde{f}(t-\tau(t)), \quad t \neq T_{m,l}, \\ \Delta e(t) = -B_{m,l}e(t), \quad t = T_{m,l}, \end{cases} \\ \qquad\qquad\qquad\qquad t \in [m\omega+\delta,\ (m+1)\omega], \end{cases} \tag{24}$$

where $e(t) = [e_1(t), \ldots, e_n(t)]^T$, $C = diag\{c_1, \ldots, c_n\}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $\tilde{f}(t) = [\tilde{f}_1(t), \ldots, \tilde{f}_n(t)]$ and $B_{m,l} = diag\{B_{m,l,1}, \ldots, B_{m,l,n}\}$.

Theorem 2 *In IISS, assume that for an intermittent impulsive control law* $\{T_{m,l}, U_{m,l,i}\}$

(i) *there exist a positive definite matrix* P *and constants* $\alpha_1 > 0$, $\alpha_2 > 0$, $\xi_1 > 0$ *and* $\xi_2 > 0$ *such that*

$$-2PC + \xi_1 PAA^T P + \xi_2 PBB^T P + \frac{1}{\xi_1} L_f - \alpha_1 P \leq 0, \tag{25}$$

and

$$\frac{1}{\xi_2} L_f - \alpha_2 P \leq 0, \tag{26}$$

where $L_f = diag\{L_1^2, L_2^2, \ldots, L_n^2\}$.

(ii) *there exist real numbers* $\beta_{m,l} \in (0,\ 1)$ *such that*

$$(I - B_{m,l}^T)P(I - B_{m,l}) - \beta_{m,l}P \leq 0. \tag{27}$$

(iii) *there exists a real number* $d(\beta_{m,l} < d < 1)$ *such that for each* $m,\ l$,

$$\frac{\Delta_{m,l}}{\beta_{m,l}}(\beta_{m,l}\alpha_1 + \alpha_2) + \ln \beta_{m,l} \leq \ln d, \tag{28}$$

where $\Delta_{m,l} = T_{m,l+1} - T_{m,l}$ *and* $T_{m,M_m+1} = (m+1)\omega$.

(iv) *the upper bound of time delay satisfies* $\Delta \leq r \leq \omega - \delta$ *and*

$$de^{(\alpha_1+\alpha_2)\delta} < 1, \tag{29}$$

where $\Delta = \max_{m,l}\{\Delta_{m,l}\}$.

Then the slave system (4) is synchronized with the master system (1) by intermittent impulsive control law $\{T_{m,l}, U_{m,l,i}\}$.

Proof Define $V(t) = e(t)^T Pe(t)$. When $t \in [m\omega,\ m\omega+\delta]$, the slave system runs in free windows and the impulsive control does not work. Then we have

$$V'(t) \leq \alpha_1 V(t) + \alpha_2 V(t-\tau). \tag{30}$$

In terms of Lemma 2, we have

$$V(t) \leq \|V(m\omega)\|_r e^{(\alpha_1+\alpha_2)(t-m\omega)}, \quad t \in [m\omega,\ m\omega+\delta]. \tag{31}$$

Thus

$$V(m\omega+\delta) \leq \|V(m\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}.$$

When $t \in [m\omega+\delta,\ (m+1)\omega]$, the slave system runs in control windows. Thus the impulsive control works.

When $t \in (T_{m,1},\ T_{m,2})$, we have

$$V'(t) \leq \alpha_1 V(t) + \alpha_2 V(t-\tau),$$

which implies, if $V(t) \geq \beta_{m,l} V(t+s),\ s \in [-r, 0]$,

$$V'(t) \leq \frac{1}{\beta_{m,l}}(\beta_{m,l}\alpha_1 + \alpha_2)V(t). \tag{32}$$

When $t = T_{m,1}$, we get by condition (ii)

$$\begin{aligned} V(T_{m,1}) &= e^T(T_{m,1}^-)(I - B_{m,1})^T P(I - B_{m,1})e(T_{m,1}^-) \\ &\leq \beta_{m,l} e^T(T_{m,1}^-) P e(T_{m,1}^-) \\ &= \beta_{m,l} V(T_{m,1}^-) \\ &\leq \beta_{m,l} \|V(T_{m,1})\|_\tau. \end{aligned} \tag{33}$$

By (32), (33), condition (iii) and Remark 3, we have

$$V(t) < d\|V(T_{m,1})\|_r, \quad t \in [T_{m,1},\ T_{m,2}). \tag{34}$$

Similarly, we have

$$V(t) < d\|V(T_{m,l})\|_r, \quad t \in [T_{m,l},\ T_{m,l+1}),\ \text{for } i = 2, 3, \ldots, M_m, \tag{35}$$

where $T_{m,M_m+1} = (m+1)\omega$.

Since $d < 1$, we have

$$\|V(T_{m,l+1})\|_r \leq \|V(T_{m,l})\|_r \text{ for } i = 1, 2, \ldots, M_m - 1.$$

Thus

$$V(t) < d\|V(T_{m,1})\|_r, \quad t \in [m\omega + \delta, (m+1)\omega]. \tag{36}$$

When $m = 0$, by (31) and (36), we have

$$V(t) \leq \|V(0)\|_r e^{(\alpha_1+\alpha_2)\delta}, \quad t \in [0,\ \delta],$$

and

$$V(t) < d\|V(\delta)\|_r, \quad t \in [\delta,\ \omega].$$

Since $\|V(\delta)\|_r \leq \|V(0)\|_r e^{(\alpha_1+\alpha_2)\delta}$, by condition (iv), we have

$$V(t) < d\|V(0)\|_r e^{(\alpha_1+\alpha_2)\delta} \leq \rho\|V(0)\|_r, \quad t \in [\delta,\ \omega],$$

where $0 < \rho = de^{(\alpha_1+\alpha_2)\delta} < 1$.

Similarly, when $k = 1$, we have

$$V(t) \leq \begin{cases} \|V(\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [\omega,\ \omega+\delta]; \\ \rho\|V(\omega)\|_r, & t \in [\omega+\delta,\ 2\omega]. \end{cases}$$

When $t \in [m\omega,\ (m+1)\omega]$, we have

$$V(t) \leq \begin{cases} \|V(m\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [m\omega,\ m\omega+\delta]; \\ \rho\|V(m\omega)\|_r, & t \in [m\omega+\delta,\ (m+1)\omega]. \end{cases}$$

Since $r \leq \omega - \delta$ in condition (iv), we have

$$\|V((m+1)\omega)\|_r \leq \|V((m+1)\omega)\|_{\omega-\delta} \leq \rho V(m\omega)\|_r.$$

Furthermore,

$$V(t) \leq \begin{cases} \rho^m \|V(0)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [m\omega,\ m\omega+\delta]; \\ \rho^{m+1}\|V(0)\|_r, & t \in [m\omega+\delta,\ (m+1)\omega]. \end{cases}$$

Thus

$$\lim_{t\to\infty} V(t) = 0.$$

Since $V(t) = e^T(t)Pe(t)$, we have

$$\frac{V(t)}{\lambda_M(P)} \leq \|e(t)\|^2 \leq \frac{V(t)}{\lambda_m(P)}.$$

Thus

$$\lim_{t\to\infty} \|e(t)\| \leq \lim_{t\to\infty} \frac{V(t)}{\lambda_m(P)} = 0. \tag{37}$$

Therefore, from the definition of synchronization, the slave system (4) is synchronized with the master system (1) by intermittent impulsive control law $\{T_{m,l}, U_{m,l,i}\}$. □

Remark 5 In the proof of Theorem 2, $V(t)$ converges exponentially to zero along the trajectory of the error system (5). Also, synchronization error $e(t)$ converges exponentially to zero. It implies that chaos synchronization is achieved very fast.

Define $\Delta_0 = \min_{m,l}\{\Delta_{m,l}\}$ and $M_0 = \min_m\{M_m\}$. If the time delay $\tau(t)$ is small and satisfies $\tau(t) \leq r < \Delta_0$, then we have the following synchronization criterion.

Theorem 3 *In IISS, assume that for an intermittent impulsive control law* $\{T_{m,l}, U_{m,l,i}\}$

(i) *there exist a positive definite matrix P and constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\xi_1 > 0$ and $\xi_2 > 0$ such that*

$$-2PC + \xi_1 PAA^TP + \xi_2 PBB^TP + \frac{1}{\xi_1}L_f - \alpha_1 P \leq 0, \tag{38}$$

and

$$\frac{1}{\xi_2}L_f - \alpha_2 P \leq 0, \tag{39}$$

where $L_f = diag\{L_1^2, L_2^2, \ldots, L_n^2\}$.

(ii) *there exist real numbers $\beta_{m,l} \in (0,\ 1)$ such that*

$$(I - B_{m,l}^T)P(I - B_{m,l}) - \beta_{m,l}P \leq 0. \tag{40}$$

(iii) *there exists a real number $d(\beta_{m,l} < d < 1)$ such that for each m, l,*

$$\frac{\Delta_{m,l}}{\beta_{m,l}}(\beta_{m,l}\alpha_1 + \alpha_2) + \ln \beta_{m,l} \leq \ln d, \tag{41}$$

where $\Delta_{m,l} = T_{m,l+1} - T_{m,l}$ and $T_{m,M_m+1} = (m+1)\omega$.

(iv) *the time delay* $\tau(t)$ *satisfies* $\tau(t) \le r \le \Delta_0$ *and*

$$d^{M_0} e^{(\alpha_1+\alpha_2)\delta} < 1, \tag{42}$$

where $\Delta = \max_{m,l}\{\Delta_{m,l}\}$, $\Delta_0 = \min_{m,l}\{\Delta_{m,l}\}$ *and* $M_0 = \min_{m}\{M_m\}$.

Then the slave system (4) is synchronized with the master system (1) by intermittent impulsive control law $\{T_{m,l}, U_{m,l,i}\}$.

Proof Because conditions (i)-(iii) are same with that of Theorem 2, $V(t)$ still satisfies (35). We have

$$V(t) < d\|V(T_{m,l})\|_r, \quad t \in [T_{m,l},\ T_{m,l+1}),\ \text{for } l = 2, 3, \ldots, M_m,$$

where $T_{m,M_m+1} = (m+1)\omega$.

Since $r \le \Delta_0$ in condition (iv), we have

$$\|V(T_{m,l+1})\|_r \le \|V(T_{m,l+1})\|_{\Delta_{m,l}} < d\|V(T_{m,l})\|_r,$$

for $l = 1, 2, \ldots, M_m - 1$. Thus

$$V(t) < d^l\|V(T_{m,1})\|_r, \quad t \in [T_{m,l},\ T_{m,l+1}),\ \text{for } l = 2, 3, \ldots, M_m, \tag{43}$$

where $T_{m,M_m+1} = (m+1)\omega$.

When $m = 0$, by (31) and (43), we have

$$V(t) \le \|V(0)\|_r e^{(\alpha_1+\alpha_2)\delta}, \quad t \in [0,\ \delta),$$

and

$$V(t) < d^l\|V(0)\|_r e^{(\alpha_1+\alpha_2)\delta}, \quad t \in [T_{0,l},\ T_{0,l+1}).$$

Similarly, when $m = 1$, we have

$$V(t) \le \begin{cases} \|V(\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [\omega,\ \omega+\delta); \\ d^l\|V(\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [T_{1,l},\ T_{1,l+1}). \end{cases}$$

When $t \in [m\omega,\ (m+1)\omega]$, we have

$$V(t) \le \begin{cases} \|V(m\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [m\omega,\ m\omega+\delta); \\ d^l\|V(m\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [T_{m,l},\ T_{m,l+1}). \end{cases}$$

Since $\tau(t) \le r \le \Delta_0$ in condition (iv), we have

$$\|V((m+1)\omega)\|_r \le \|V((m+1)\omega)\|_{\Delta_0} \le d^{M_m}\|V(m\omega)\|_r e^{(\alpha_1+\alpha_2)\delta}.$$

In terms of condition (iv), we have

$$\|V((m+1)\omega)\|_r \le d^{M_0}\|V(m\omega)\|_r e^{(\alpha_1+\alpha_2)\delta} \le \theta\|V(m\omega)\|_r,$$

where $\theta = d^{M_0} e^{(\alpha_1+\alpha_2)\delta} < 1$.

Furthermore,

$$V(t) \le \begin{cases} \theta^m\|V(0)\|_r e^{(\alpha_1+\alpha_2)\delta}, & t \in [m\omega,\ m\omega+\delta]; \\ \theta^m d^l\|V(0)\|_r, & t \in [[T_{m,l},\ T_{m,l+1}). \end{cases}$$

Thus

$$\lim_{t\to\infty} V(t) = 0.$$

Also,

$$\lim_{t\to\infty} \|e(t)\| = 0. \tag{44}$$

Therefore, the slave system (4) is synchronized with the master system (1) by intermittent impulsive control law $\{T_{m,l}, U_{m,l,i}\}$.

If the impulsive control laws $\{T_{m,l}, U_{m,l,i}\}$ $(m = 0, 1, \ldots)$ are the same in all control windows, $\Delta_{m,l} = \Delta_1$ and $B_{m,l} = B_s$, then we have the following corollaries.

Corollary 2 *In IISS, assume that for an intermittent impulsive control law* $\{T_{m,l}, U_{m,l,i}\}$

(i) *there exist a positive definite matrix* P *and constants* $\alpha_1 > 0$, $\alpha_2 > 0$, $\xi_1 > 0$ *and* $\xi_2 > 0$ *such that*

$$-2PC + \xi_1 PAA^T P + \xi_2 PBB^T P + \frac{1}{\xi_1} L_f - \alpha_1 P \le 0,$$

and

$$\frac{1}{\xi_2} L_f - \alpha_2 P \le 0,$$

where $L_f = diag\{L_1^2, L_2^2, \ldots, L_n^2\}$.

(ii) *there exists a real number* $\beta \in (0,\ 1)$ *such that*

$$(I - B_s^T)P(I - B_s) - \beta P \le 0.$$

(iii) *there exists a real number* $d(\beta < d < 1)$ *such that*

$$\frac{\Delta_1}{\beta}(\beta\alpha_1 + \alpha_2) + \ln\beta \le \ln d.$$

(iv) *the time delay upper bound* r *satisfies* $\Delta_1 \le r \le \omega - \delta$ *and*

$$de^{(\alpha_1+\alpha_2)\delta} < 1.$$

Then the slave system (4) is synchronized with the master system (1) by intermittent impulsive control law $\{T_{m,l}, U_{m,l,i}\}$.

Corollary 3 *Assume that for an intermittent impulsive control law* $\{T_{m,l}, U_{m,l,i}\}$

(i) *there exist a positive definite matrix* P *and constants* $\alpha_1 > 0$, $\alpha_2 > 0$, $\xi_1 > 0$ *and* $\xi_2 > 0$ *such that*

$$-2PC + \xi_1 PAA^T P + \xi_2 PBB^T P + \frac{1}{\xi_1} L_f - \alpha_1 P \le 0,$$

and

$$\frac{1}{\xi_2} L_f - \alpha_2 P \le 0,$$

where $L_f = diag\{L_1^2, L_2^2, \ldots, L_n^2\}$.

(ii) *there exists a real number* $\beta \in (0,\ 1)$ *such that*

$$(I - B_s^T)P(I - B_s) - \beta P \le 0.$$

(iii) *there exists a real number* $d(\beta < d < 1)$ *such that*

$$\frac{\Delta_1}{\beta}(\beta\alpha_1 + \alpha_2) + \ln\beta \leq \ln d.$$

(iv) *the time delay* $\tau(t)$ *satisfies* $\tau(t) \leq r \leq \Delta_1$ *and*

$$d^{M_1} e^{(\alpha_1+\alpha_2)\delta} < 1,$$

where $M_1 = \lfloor \frac{\omega-\delta}{\Delta_1} \rfloor + 1$ *and* $\lfloor a \rfloor$ *denotes the nearest integers less than or equal to* a.

Then the slave system (4) is synchronized with the master system (1) by intermittent impulsive control law $\{T_{m,l}, U_{m,l,i}\}$.

Numeral Examples

In this section, two numeral examples are given to show the effectiveness of the main results.

(a) Consider a typical Hopfield neural network with constant delay as the master system, described by

$$\frac{dx(t)}{dt} = -Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) + J, \tag{45}$$

where

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2.0 & -0.11 \\ 5.0 & 3.2 \end{bmatrix}, B = \begin{bmatrix} -1.6 & -0.1 \\ 0.18 & -2.4 \end{bmatrix}$$

$$f(x(t)) = \begin{bmatrix} \tanh x_1 \\ \tanh x_2 \end{bmatrix}, \quad \tau(t) = 1, \quad J = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

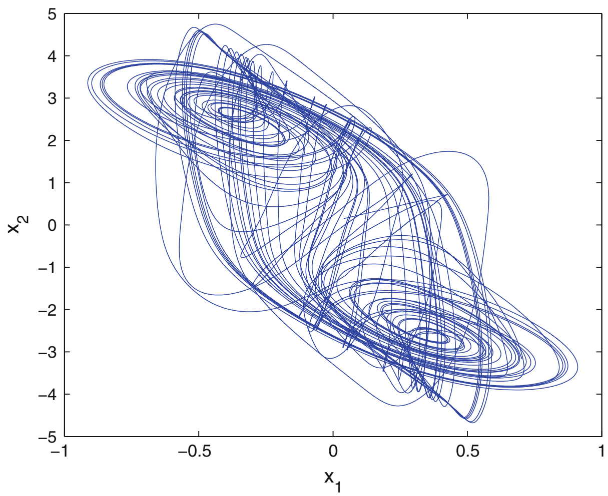

Fig. 1 Phase portrait $x_1(t) - x_2(t)$ of DNNs (45) with $\tau(t) = 1$.

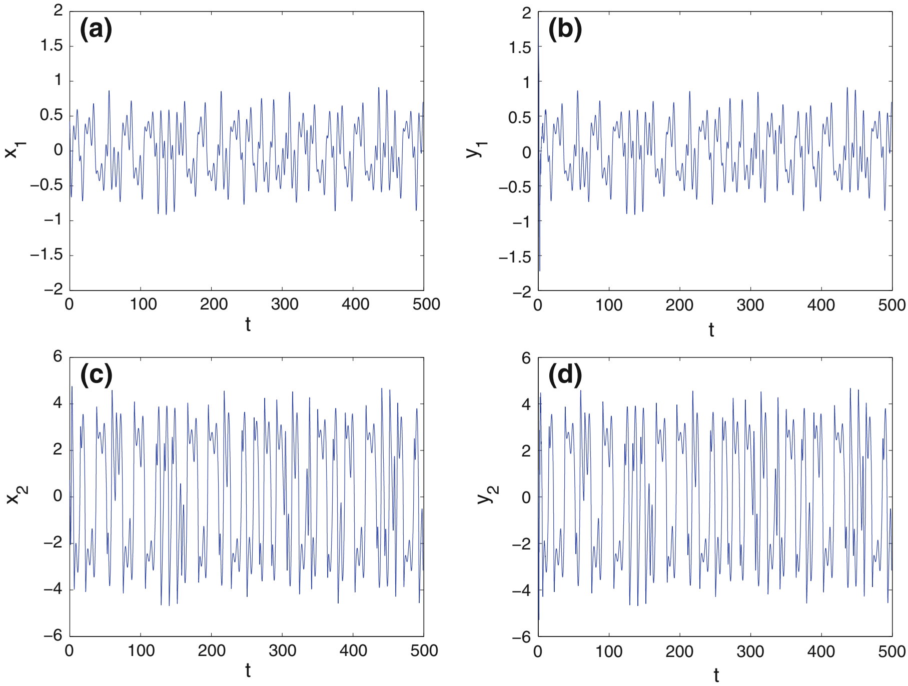

Fig. 2 State trajectories of the master system (*left*) and the slave system (*right*) in GISS.

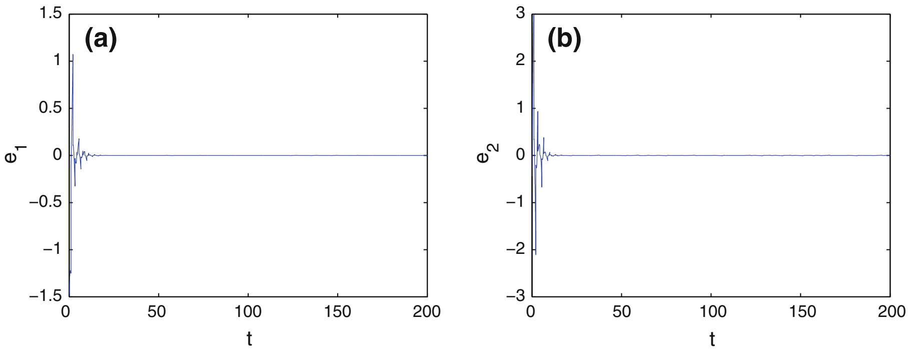

Fig. 3 Synchronization errors in GISS: **a** $e_1(t)$ and **b** $e_2(t)$ with $\Delta_{k,i} = 1.1$ and $B_k = 0.90I$.

The chaotic behavior of system (45) is shown in Fig. 1. Firstly considering GISS, we choose impulsive control parameters: $\Delta_k = 1.1$ and $B_k = 0.90I$. Let $P = I$. Thus, conditions (12–15) are satisfied. By Theorem 1, we know that the corresponding slave system is synchronized with the master system (45) by general impulsive control with $\Delta_k = 1.1$ and $B_k = 0.90I$. The state trajectories of master and slave systems are shown in Fig. 2. The synchronization errors are shown in Fig. 3. Our simulation results show that when the impulsive intervals satisfy $\Delta_k \leq 1.42$ the synchronization can be always achieved.

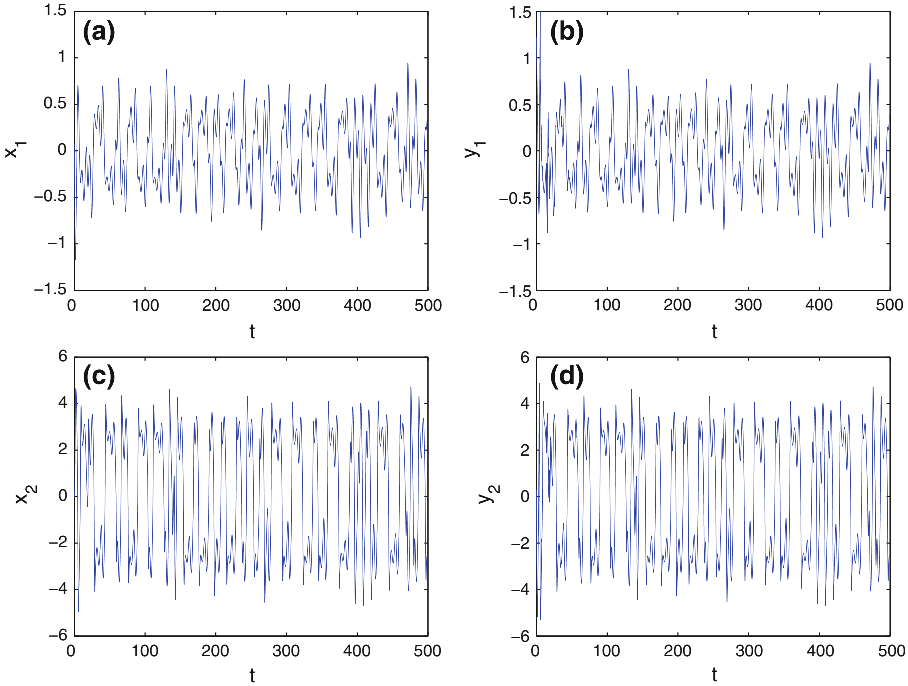

Fig. 4 State trajectories of the master system (*left*) and the slave system (*right*) in IISS.

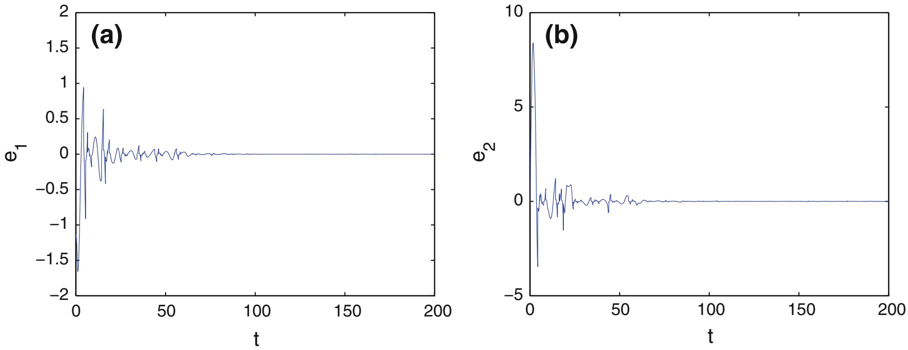

Fig. 5 Synchronization errors in IISS: **a** $e_1(t)$ and **b** $e_2(t)$ with $\Delta_{k,i} = 1.1$ and $B_k = 0.90I$.

Assume that $\omega = 10$ and $\delta = 5$. Thus, the free windows are $[10m,\ 10m+5]$ and the control windows are $[10m+5,\ 10m+10]$. Since the free window width is larger than the impulsive intervals, GISS fails in this scenario. Now, we consider IISS. Choose control parameters $\Delta_{m,l} = 1.1$ and $B_{m,l} = 0.90I$ and let $P = I$. By Corollary 3, we know that the corresponding slave system is synchronized with the master system (45). The state trajectories of master and slave systems, and synchronization errors are shown in Figs. 4, 5, and 6, respectively. Simulation results show that when impulsive intervals satisfy $\Delta_k \leq 1.22$, the synchronization can be always achieved.

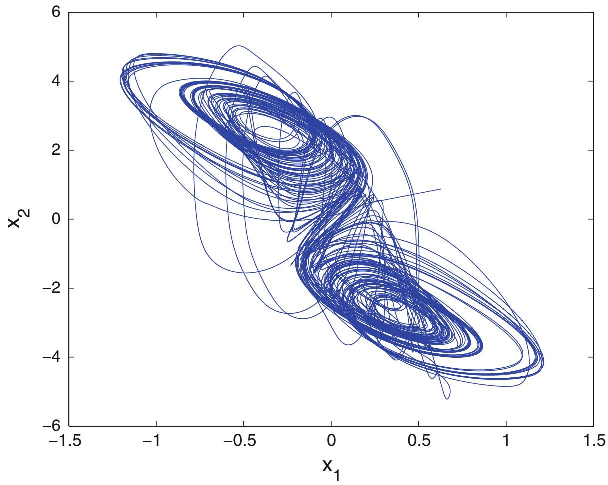

Fig. 6 Phase portrait $x_1(t) - x_2(t)$ of DNNs (45) with $\tau(t) = 1 + 0.4\sin(t)$.

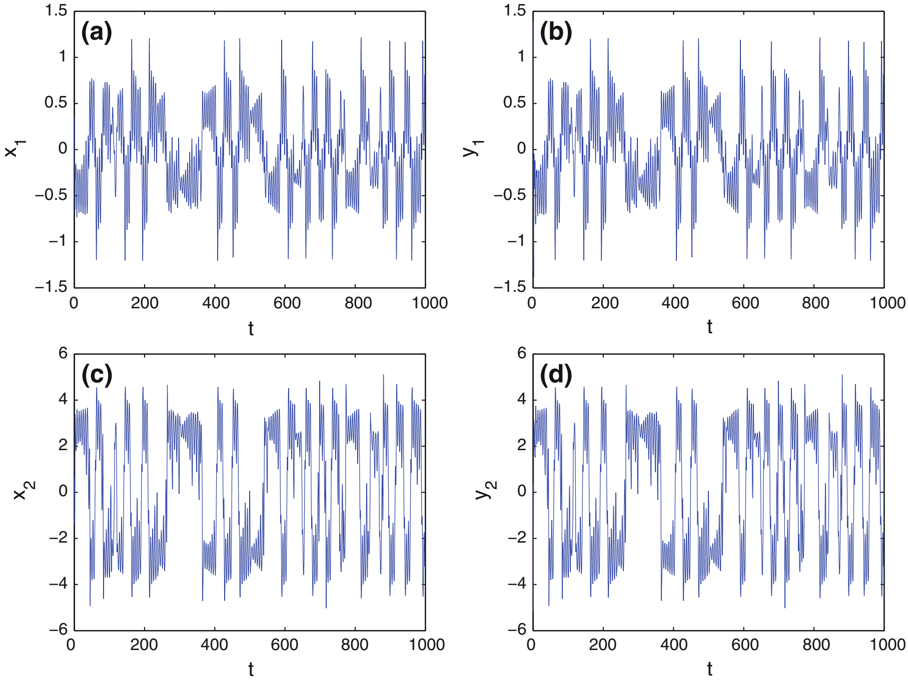

Fig. 7 State trajectories of the master system (*left*) and the slave system (*right*) in GISS.

Remark 6 In the above example, the control window width $\omega - \delta$ is a half of the whole period width ω. General impulsive synchronization approach is not applicable for this Scenario because the free window width δ is larger than the impulsive intervals.

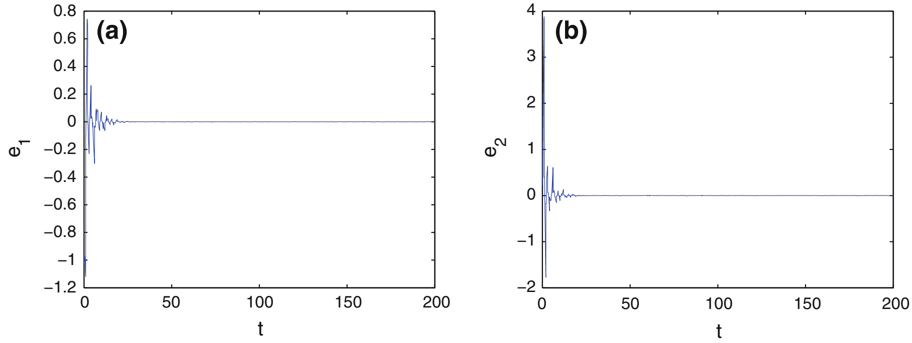

Fig. 8 Synchronization errors in GISS: **a** $e_1(t)$ and **b** $e_2(t)$ with $\Delta_{k,i} = 1.0$ and $B_k = 0.90I$.

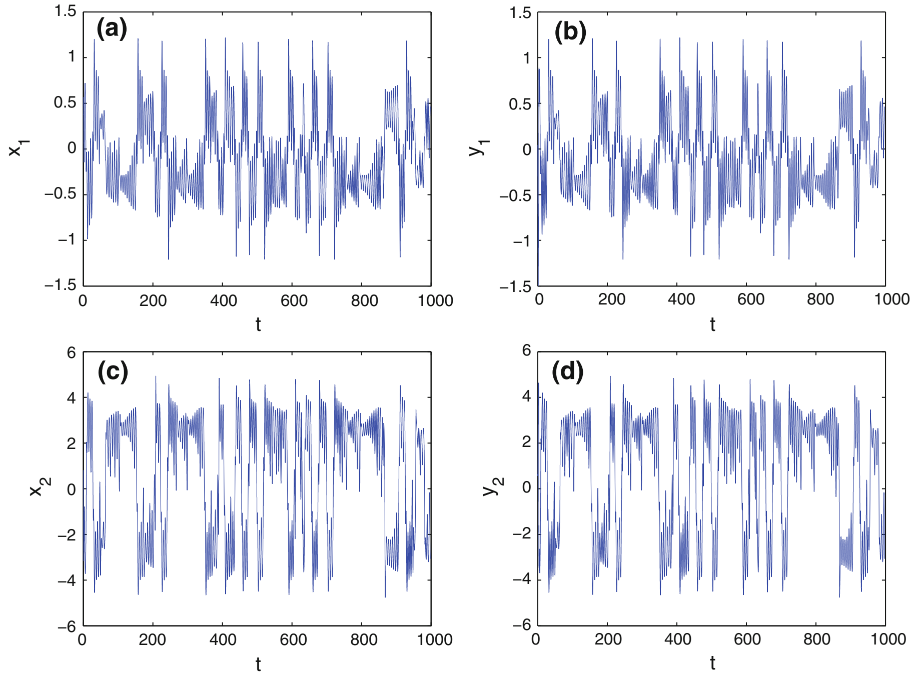

Fig. 9 State trajectories of the master system (*left*) and the slave system (*right*) in IISS.

(b) Consider the above Hopfield neural network with time-varying delay, i.e.,

$$\tau(t) = 1 + 0.4\sin(t).$$

The chaotic behavior of system (45) is shown in Fig. 7. Firstly considering GISS, let $\Delta_k = 1.0$, $B_k = 0.90I$ and $P = I$. The state trajectories of master and slave systems are shown in Fig. 8. The synchronization errors are shown in Fig. 9. Our simulation results show that when the impulsive intervals satisfy $\Delta_k \le 1.50$ the synchronization can be always achieved.

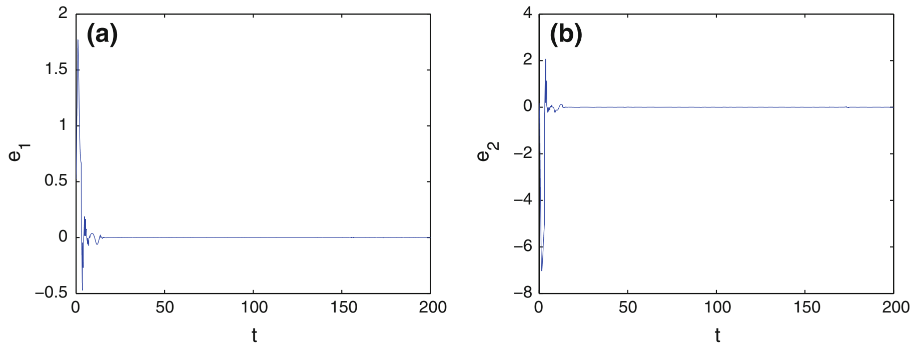

Fig. 10 Synchronization errors in IISS: **a** $e_1(t)$ and **b** $e_2(t)$ with $\Delta_{k,i} = 0.6$ and $B_k = 0.90I$.

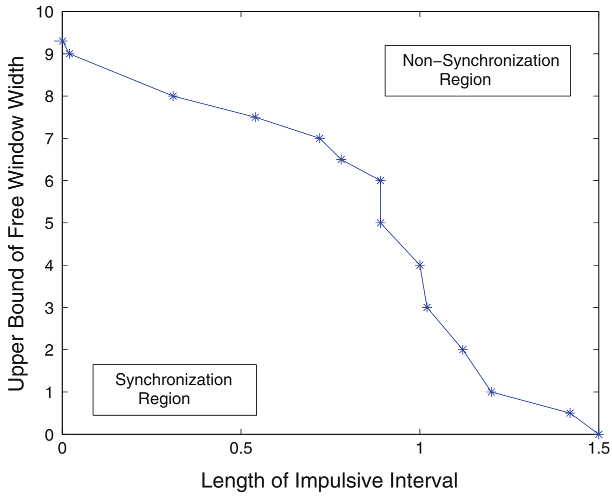

Fig. 11 The relationship between the length of impulsive interval and the free window width to guarantee synchronization.

Assume that $\omega = 10$ and $\delta = 7$. In this scenario, since the free window width $\delta = 7$ is larger than the impulsive intervals $\Delta_k = 1.50$, GISS fails. Let's consider IISS. Select $\Delta_{m,l} = 0.6$ and $B_{m,l} = 0.90I$ and let $P = I$. By Corollary 3, we know that the corresponding slave system is synchronized with the master system (45). The state trajectories of master and slave systems, and synchronization errors are shown in Figs. 10 and 11, respectively. Simulation results show that when impulsive intervals satisfy $\Delta_k \leq 0.72$, the synchronization can be always achieved.

Fixing the control parameter $B_{m,l} = 0.90I$, we try to find out the relationship between the length of impulsive interval and the free window width to guarantee synchronization. Simulation results are shown in Fig. 11.

Remark 7 Figure 11 shows that the upper bound of the free window width decreases as the length of impulsive interval increases. In other words, to guarantee synchronization, if one wants to reduce the control window width, more frequent impulsive controls are needed.

Conclusion

In this paper, we have investigated intermittent impulsive synchronization of two chaotic delayed neural networks. We have presented a novel intermittent impulsive synchronization scheme to break through the limitation of the general impulsive synchronization scheme. In our synchronization scheme, the impulsive controller is only activated in the control windows, not in the free windows. Several criteria to guarantee chaos synchronization of two coupled neural networks, based on Lyapunov–Razumikhin theory and LMI. Two numeral examples are given to show how IISS breaks through the limit of the upper bound of the impulsive intervals, different from GISS. IISS can be flexibly applied to the scenario where the control windows are restricted. On the other hand, via reducing the control window width and decreasing the redundancy of the synchronization signals, one can further improve the security of chaos-based secure communication. Thus, IISS should have a great application and perspective.

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