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A Generalization of Opial's Inequality and Applications to Second-Order Dynamic Equations

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Abstract In this paper, we extend to arbitrary time scales some results of [Proc. Amer. Math. Soc., vol. 125, no. 4, pp. 1123–1129, (1997)], where R. C. Brown and D. B. Hinton investigate oscillation of a second-order differential equation. We also provide some examples on nontrivial time scales to illustrate the applicability of the results.

Keywords Disconjugate · Disfocal · Opial inequality · Second-order dynamic equations · Time scale

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1 Introduction

In this paper, in view of a new dynamic generalization of Opial's well-known inequality, we study oscillatory properties of the solutions of the following second-order dynamic equation:

$$y^{\Delta^2} + p(t)y^{\sigma} = 0 \quad \text{for} \quad t \in [a, b]_{\mathbb{T}}, \tag{1.1}$$

where $a, b \in \mathbb{T}$, $p \in \mathrm{C_{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$.

By a *solution* of (1.1), we mean a continuous function $y : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}$, which is twice differentiable on $[a, b]_{\mathbb{T}}$ with y^{Δ^2} rd-continuous. As is well known from [3, Corollary 3.14] that (1.1) admits a unique solution when $y(a)$ and $y^{\Delta}(a)$ are

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prescribed (for further details on second-order dynamic equations, see [3, §3]). We say y has a *generalized zero* at some $c \in [a, \sigma(b)]_{\mathbb{T}}$ provided that $y(c)y^{\sigma}(c) \le 0$ holds, and (1.1) is called *disconjugate* on $[a, b]$ if there is no nontrivial solution of (1.1) with at least two generalized zeros in $[a, b]$. Finally, (1.1) is said to be *disfocal* on $[a, \sigma^2(b)]$ provided there is no nontrivial solution y of (1.1) with a generalized zero in $[a, \sigma^2(b)]$ followed by a generalized zero of y^{Δ} in $[a, \sigma(b)]$.

Opial inequalities and many of their generalizations have various applications in the theories of differential and difference equations. This is very nicely illustrated in the book [1] "Opial Inequalities with Applications in Differential and Difference Equations" by Agarwal and Pang, which is devoted solely to continuous and discrete versions of Opial inequalities. Later in [4] Bohner and Kaymakçalan initiated the time scales unification of a continuous and a discrete analog of a version of Opial's inequality and illustrated some applications of it to dynamic equations on time scales. Here we extend these initial results to Opial's type inequalities corresponding to (3.1) and (3.11) and with the aim of applying them to obtain oscillatory behaviour of solutions of (1.1).

2 Time scale essentials

In this section a brief list of essentials from time scale calculus which are necessary for our results are given. For these and further results we refer the reader to [3].

Definition 2.1 A *time scale* is a nonempty closed subset of real numbers.

Definition 2.2 On an arbitrary time scale $\mathbb{T}$, the following operators are defined: the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t, \infty)_{\mathbb{T}}$ for $t \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$ for $t \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$. For convenience, we set $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$.

Definition 2.3 Let t be a point in $\mathbb{T}$. If $\sigma(t) = t$ holds, then t is called *right-dense*, otherwise it is called *right-scattered*. Similarly, if $\rho(t) = t$ holds, then t is called *left-dense* and a point which is not left-dense is called *left-scattered*.

Definition 2.4 A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided that it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exists (are finite) at left-dense points of $\mathbb{T}$. The set of rd-continuous functions is denoted by $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$, and $\mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R})$ denotes the set of functions whose delta derivatives belong to $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$.

Theorem 2.1 (Existence of antiderivatives) *Let f be a rd-continuous function. Then f has an antiderivative F such that $F^{\Delta} = f$ holds.*

Definition 2.5 If $f \in \mathrm{C_{rd}}(\mathbb{T})$ and $s \in \mathbb{T}$, then the following *integral* is defined

$$F(t) := \int_s^t f(\eta)\Delta\eta \quad \text{for} \quad t \in \mathbb{T}.$$

Theorem 2.2 *Let f, g be a rd-continuous functions, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then the following properties are true:*

1. $\int_a^b [\alpha f(\eta) + \beta g(\eta)]\Delta\eta = \alpha \int_a^b f(\eta)\Delta\eta + \beta \int_a^b g(\eta)\Delta\eta,$
2. $\int_a^b f(\eta)\Delta\eta = -\int_b^a f(\eta)\Delta\eta,$

3. $\int_a^c f(\eta)\Delta\eta = \int_a^b f(\eta)\Delta\eta + \int_b^c f(\eta)\Delta\eta$,
4. $\int_a^b f(\eta)g^{\Delta}(\eta)\Delta\eta = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(\eta)g(\sigma(\eta))\Delta\eta$.

The following table gives the explicit forms of the forward jump operator, delta derivative and the integral on some particular time scales:

$\mathbb{T}$	$\sigma(t)$	$f^{\Delta}(t)$	$\int_s^t f(\eta)\Delta\eta$
$\mathbb{R}$	t	$f'(t)$	$\int_a^b f(\eta)\,\mathrm{d}\eta$
$\mathbb{Z}$	$t+1$	$f(t+1)-f(t)$	$\sum_{\eta=s}^{t-1} f(\eta)$
$q^{\mathbb{N}_0}$, $(q>1)$	qt	$\frac{f(qt)-f(t)}{(q-1)t}$	$(q-1)\sum_{\eta=\log_q(s)}^{\log_q(t)-1} f(q^\eta)q^\eta$
$\mathbb{N}_0^q$, $(q>0)$	$(\sqrt[q]{t}+1)^q$	$\frac{f((\sqrt[q]{t}+1)^q)-f(t)}{(\sqrt[q]{t}+1)^q-t}$	$\sum_{\eta=\sqrt[q]{s}}^{\sqrt[q]{t}-1} f(\eta^q)((\eta+1)^q-\eta^q)$

Lemma 2.1 (Hölder's inequality [4, Theorem 6.13]) *Let $a,b\in\mathbb{T}$. For $f,g\in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$, we have*

$$\int_a^b |f(\eta)g(\eta)|\Delta\eta \le \left(\int_a^b |f(\eta)|^p\Delta\eta\right)^{1/p}\left(\int_a^b |g(\eta)|^q\Delta\eta\right)^{1/q},$$

where $p>1$ and $1/p+1/q=1$.

3 Opial's inequalities

In this section, we give some Opial type inequalities, which will be used to prove our main results in the following section.

Theorem 3.1 *Let $a,b\in\mathbb{T}$ and $w,f\in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ with $f(a)=0$, then we have*

$$\int_a^b w(\eta)|[f(\eta)+f^{\sigma}(\eta)]f^{\Delta}(\eta)|\Delta\eta \le K_w(b,a)\int_a^b [f^{\Delta}(\eta)]^2\Delta\eta, \tag{3.1}$$

where

$$K_w(t,s) := \left(2\int_s^t [w(\eta)]^2[\sigma(\eta)-s]\Delta\eta\right)^{1/2} \quad \text{for} \;\; s,t\in[a,b]_{\mathbb{T}}. \tag{3.2}$$

Proof Clearly, by Hölder's inequality, we have

$$\begin{aligned} f(t) &= \int_a^t f^{\Delta}(\eta)\Delta\eta \\ &\le ((t-a)g(t))^{1/2}, \end{aligned} \tag{3.3}$$

where

$$g(t) := \int_a^t |f^{\Delta}(\eta)|^2\Delta\eta \quad \text{for} \;\; t\in[a,b]_{\mathbb{T}}. \tag{3.4}$$

Hence, we get

$$[g^{\Delta}(t)]^{1/2} = |f^{\Delta}(t)| \quad \text{for all} \;\; t\in[a,b]_{\mathbb{T}}. \tag{3.5}$$

Then, from (3.1), (3.2), (3.3) and (3.5), we have

$$\begin{aligned}
&\int_a^b w(\eta)|[f(\eta)+f^\sigma(\eta)]f^\Delta(\eta)|\Delta\eta \\
&\leq \int_a^b |w(\eta)|(|f(\eta)|+|f^\sigma(\eta)|)|f^\Delta(\eta)|\Delta\eta \\
&\leq \int_a^b |w(\eta)|(\sigma(\eta)-a)^{1/2}([g(\eta)]^{1/2}+[g^\sigma(\eta)]^{1/2})[g^\Delta(\eta)]^{1/2}\Delta\eta \qquad (3.6)\\
&\leq 2\int_a^b |w(\eta)|(\sigma(\eta)-a)^{1/2}([g(\eta)+g^\sigma(\eta)]g^\Delta(\eta))^{1/2}\Delta\eta \qquad (3.7)\\
&= \int_a^b |w(\eta)|(\sigma(\eta)-a)^{1/2}(([g(\eta)]^2)^\Delta)^{1/2}\Delta\eta \qquad (3.8)\\
&\leq K_w(b,a)\left(\int_a^b([g(\eta)]^2)^\Delta\Delta\eta\right)^{1/2} \qquad (3.9)\\
&= K_w(b,a)g(b), \qquad (3.10)
\end{aligned}$$

where we have applied the well-known inequality

$$\alpha^{1/2}+\beta^{1/2}\leq 2(\alpha+\beta)^{1/2} \quad \text{for all} \quad \alpha,\beta\in\mathbb{R}^+$$

while passing from (3.6) to (3.7), and Hölder's inequality on obtaining (3.9) from (3.8). Thus, substituting (3.4) into (3.10), we see that (3.1) is true. The proof is therefore completed.

The following result is complementary to Theorem 3.1.

Theorem 3.2 *Let $a,b\in\mathbb{T}$ and $w,f\in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ with $f(b)=0$, then we have*

$$\int_a^b w(\eta)|[f(\eta)+f^\sigma(\eta)]f^\Delta(\eta)|\Delta\eta \leq L_w(b,a)\int_a^b [f^\Delta(\eta)]^2\Delta\eta, \qquad (3.11)$$

where

$$L_w(t,s):=\left(2\int_s^t [w(\eta)]^2[t-\eta]\Delta\eta\right)^{1/2} \quad \text{for} \quad s,t\in[a,b]_{\mathbb{T}}. \qquad (3.12)$$

Proof Setting

$$h(t):=\int_t^b |f^\Delta(\eta)|^2\Delta\eta \quad \text{for} \quad t\in[a,b]_{\mathbb{T}},$$

we then have

$$\begin{aligned}
f(t) &= \int_t^b[-f^\Delta(\eta)]\Delta\eta \\
&\leq ((b-t)h(t))^{1/2}
\end{aligned}$$

for all $t\in[a,b]_{\mathbb{T}}$. Following the steps as in the proof of Theorem 3.1 we obtain the required result, and thus we omit the rest of the proof.

The next result combines Theorem 3.1 and Theorem 3.2.

Theorem 3.3 *Let $a, b \in \mathbb{T}$ and $w, f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $f(a) = f(b) = 0$, then*

$$\int_a^b w(\eta)|[f(\eta) + f^\sigma(\eta)]f^\Delta(\eta)|\Delta\eta \le \max\{K_w(b,c), L_w(c,a)\} \int_a^b [f^\Delta(\eta)]^2 \Delta\eta$$

holds for any $c \in [a,b]_{\mathbb{T}}$, where K, L are as defined in (3.2) *and* (3.12) *respectively.*

Proof The proof follows by applying Theorem 3.1 and Theorem 3.2 on the segments $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$ respectively, and summing the resulting inequalities.

We get the following as a consequence Theorem 3.3.

Corollary 3.1 *Let $a, b \in \mathbb{T}$ and $w, f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $f(a) = f(b) = 0$, then we have*

$$\int_a^b w(\eta)|[f(\eta) + f^\sigma(\eta)]f^\Delta(\eta)|\Delta\eta \le \min_{c\in[a,b]_{\mathbb{T}}} \{\max\{K_w(b,c), L_w(c,a)\}\} \int_a^b [f^\Delta(\eta)]^2 \Delta\eta,$$

where K, L are as defined in (3.2) *and* (3.12) *respectively.*

4 Disfocal problem

In this section, we give our main results for disfocal problems together with an application on a nontrivial time scale to illustrate their applicability.

Theorem 4.1 *Assume that y is a nontrivial solution of the second-order dynamic equation* (1.1) *with $y(a) = y^{\Delta\sigma}(b) = 0$. Then, we have*

$$K_P(\sigma(b), a) \ge 1, \tag{4.1}$$

where

$$P(t) := \int_t^{\sigma(b)} p(\eta)\Delta\eta \quad \text{for} \ \ t \in [a, \sigma(b)]_{\mathbb{T}}. \tag{4.2}$$

Proof Considering the boundary conditions, we have

$$\begin{aligned}\int_a^{\sigma(b)} y^\sigma(\eta) y^{\Delta^2}(\eta)\Delta\eta &= y^\sigma(b)y^{\Delta\sigma}(b) - y(a)y^\Delta(a) - \int_a^{\sigma(b)} [y^\Delta(\eta)]^2\Delta\eta \\ &= -\int_a^{\sigma(b)} [y^\Delta(\eta)]^2\Delta\eta \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_a^{\sigma(b)} p(\eta)[y^\sigma(\eta)]^2\Delta\eta &= -\int_a^{\sigma(b)} P^\Delta(\eta)[y^\sigma(\eta)]^2\Delta\eta \\ &= P(a)[y(a)]^2 + \int_a^{\sigma(b)} P(\eta)[[y(\eta)]^2]^\Delta\Delta\eta \\ &= \int_a^{\sigma(b)} P(\eta)[[y(\eta)]^2]^\Delta\Delta\eta \\ &= \int_a^{\sigma(b)} P(\eta)([y(\eta) + y^\sigma(\eta)]y^\Delta(\eta))\Delta\eta \\ &\le \int_a^{\sigma(b)} |P(\eta)|([y(\eta) + y^\sigma(\eta)]y^\Delta(\eta))\Delta\eta \end{aligned} \tag{4.4}$$

by (4.2). Multiplying (1.1) by y^σ and integrating from a to $\sigma(b)$, we get

$$\begin{aligned}\int_a^{\sigma(b)}[y^\Delta(\eta)]^2\Delta\eta &\leq \int_a^{\sigma(b)}|P(\eta)|([y(\eta)+y^\sigma(\eta)]y^\Delta(\eta))\Delta\eta\\ &\leq K_P(\sigma(b),a)\int_a^{\sigma(b)}[y^\Delta(\eta)]^2 \qquad (4.5)\end{aligned}$$

from Theorem 3.1, (4.3) and (4.4). Clearly, (4.1) follows from (4.5) by dividing through

$$\int_a^{\sigma(b)}[y^\Delta(\eta)]^2\Delta\eta$$

on both sides, and the proof is therefore completed.

Remark 4.1 The conclusion of Theorem 4.1 also holds for the following second order dynamic inequality:

$$y^{\Delta^2}+p(t)y^\sigma \geq 0 \quad \text{for} \quad t\in[a,b]_{\mathbb{T}} \qquad (4.6)$$

with $y(a)=0$ and $y(b)y^{\Delta\sigma}(b)\leq 0$.

Now, we give the following example.

Example 4.1 Let $\mathbb{T}=2^{\mathbb{N}_0}$, and consider the following second-order dynamic equation:

$$y^{\Delta^2}+\frac{3}{(2t-1)(1535-2t)}y(2t)=0 \quad \text{for} \quad t\in[1,256]_{2^{\mathbb{N}_0}}. \qquad (4.7)$$

Here, we have $\sigma(t)=2t$ for $t\in 2^{\mathbb{N}_0}$, $a=1, b=256$ and

$$p(t)=\frac{3}{(2t-1)(1535-2t)} \quad \text{for} \quad t\in[1,256]_{2^{\mathbb{N}_0}}.$$

One can easily verify that a solution of (4.7) is $y(t)=(t-1)(1535-t)$ for $t\in[1,1024]_{2^{\mathbb{N}_0}}$. For this solution, by simple calculations, we obtain $y^\Delta(t)=3(512-t)$ for $t\in[1,512]_{2^{\mathbb{N}_0}}$. We hence have $y(1)=0$ and $y^{\Delta\sigma}(256)=y^\Delta(512)=0$, which shows that all the assumptions of Theorem 4.1 are held. Therefore, we must have $K_P(512,1)\geq 1$, where

$$P(t)=\int_t^{512}\frac{3}{(2\eta-1)(1535-2\eta)}\Delta\eta \quad \text{for} \quad t\in[1,512]_{2^{\mathbb{N}_0}}.$$

We get by calculating with *Mathematica 6.0* that

$$\begin{aligned}K_P(512,1)&=\left(2\int_1^{512}\left(\int_\eta^{512}\frac{3}{(2\zeta-1)(1535-2\zeta)}\Delta\zeta\right)^2[2\eta-1]\Delta\eta\right)^{1/2}\\ &=1,17794\geq 1,\end{aligned}$$

which coincides with the theorem's claim.

Analogous to the above theorem, by considering Theorem 3.2 and Theorem 4.1 one can easily prove the following result.

Theorem 4.2 *Assume that* y *is a nontrivial solution of* (1.1) *with* $y^{\Delta}(a) = y^{\sigma^2}(b) = 0$*. Then, we have*

$$L_P(\sigma^2(b), a) \geq 1,$$

where

$$P(t) := \int_a^t p(\eta)\Delta\eta \quad \text{for } t \in [a, \sigma(b)]_{\mathbb{T}}.$$

Remark 4.2 The conclusion of Theorem 4.2 also holds for (4.6) with $y(a)y^{\Delta}(a) \geq 0$ and $y^{\sigma^2}(b) = 0$.

5 Disconjugacy condition

In this section, we study disconjugacy property of solutions to (1.1). We start the section with the following main result, and then give an application.

Theorem 5.1 *Assume that* y *is a nontrivial solution of* (1.1) *with* $y(a) = y^{\sigma^2}(b) = 0$*, and let* $P \in \mathrm{C}^1_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$ *be a function satisfying* $P^{\Delta} \equiv p$ *on* $[a, b]_{\mathbb{T}}$*. Then, we have*

$$\min_{c \in [a, \sigma^2(b)]_{\mathbb{T}}} \{\max\{K_P(\sigma^2(b), c), L_P(c, a)\}\} \geq 1. \tag{5.1}$$

Proof Very similar to the proof of Theorem 4.1, one can obtain the desired inequality (5.1) by applying Corollary 3.1 instead of Theorem 3.1.

The following example illustrates Theorem 5.1.

Example 5.1 Let $\mathbb{T} = \mathbb{N}_0^2$, and for $t \in [0, 64]_{\mathbb{N}_0^2}$, consider the following second-order dynamic equation:

$$y^{\Delta^2} + \frac{4}{(99 - 2t^{1/2} - t)(1 + 3t^{1/2} + 2t)} y((t^{1/2} + 1)^2) = 0. \tag{5.2}$$

Here, we have $\sigma(t) = (t^{1/2} + 1)^2$ for $t \in 2^{\mathbb{N}_0}$, $a = 0, b = 64$ and

$$p(t) = \frac{4}{(99 - 2t^{1/2} - t)(1 + 3t^{1/2} + 2t)} \quad \text{for } t \in [0, 64]_{\mathbb{N}_0^2}.$$

A solution of (5.2) satisfying $y(0) = 0$ and $y^{\sigma^2}(64) = y(100) = 0$ is $y(t) = t(100 - t)$ for $t \in [0, 100]_{\mathbb{N}_0^2}$. Therefore, due to Theorem 5.1, we must have

$$\min_{c \in [0, 100]_{\mathbb{N}_0^2}} \{\max\{K_P(100, c), L_P(c, 0)\}\} \geq 1,$$

where

$$P(t) = \int_0^t \frac{4}{(99 - 2\eta^{1/2} - \eta)(1 + 3\eta^{1/2} + 2\eta)} \Delta\eta \quad \text{for } t \in [0, 81]_{\mathbb{N}_0^2}$$

since $P^{\Delta} = p$ holds on $[0, 64]_{\mathbb{N}_0^2}$. The following table constructed by *Mathematica 6.0* denotes the values of $K_P(100, c)$ and $L_P(c, 0)$ for $c \in [0, 100]_{\mathbb{N}_0^2}$:

c	0	1	4	9	16	25	36	49	64	81	100
$K_P(100,c)$	3,50	3,47	3,38	3,24	3,03	2,76	2,43	2,03	1,55	0,97	0,00
$L_P(c,0)$	0,00	0,00	0,57	1,02	1,45	1,84	2,20	2,52	2,80	3,02	3,17
$\max\{K_P,L_P\}$	3,50	3,47	3,38	3,24	3,03	2,76	2,43	2,52	2,80	3,02	3,17

Indeed, we see that $\max\{K_P(100,c), L_P(c,0)\}$ attains its minimum value at $c = 36$ with $2,43$, and as is ensured by Theorem 5.1, this value is not less than 1.

The following corollary is an immediate consequence of Theorem 5.1.

Corollary 5.1 *Assume that y is a nontrivial solution of* (1.1) *with $y(a) = 0$, and let $P \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ be a function as in Theorem 5.1. If*

$$\min_{c\in[a,\sigma^2(b)]_{\mathbb{T}}} \{\max\{K_P(\sigma^2(b),c), L_P(c,a)\}\} < 1,$$

then $y^{\sigma^2}(b) \neq 0$.

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