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ORIGINAL RESEARCH

Solutions of Super Linear Dirac Equations with General Potentials*

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Abstract This paper is concerned with solutions to the Dirac equation: $-i\sum \alpha_k \partial_k u + a\beta u + M(x)u = g(x, |u|)u$. Here $M(x)$ is a general potential and $g(x, |u|)$ is a self-coupling which grows super-quadratically in u at infinity. We use variational methods to study this problem. By virtue of some auxiliary system related to the "limit equation" of the Dirac equation, we constructed linking levels of the variational functional Φ_M such that the minimax value c_M based on the linking structure of Φ_M satisfies $0 < c_M < \hat{C}$, where $\hat{C}$ is the least energy of the limit equation. Thus we can show the $(C)_c$-condition holds true for all $c < \hat{C}$ and consequently we obtain one solution of the Dirac equation.

Keywords Dirac equations · The Coulomb-type potential · $(C)c$-condition · Super linear · Linking

1. Introduction and the main result

In this paper, we consider the existence of least energy solutions to the following nonlinear Dirac equations

$$\begin{cases} -i\sum \alpha_k \partial_k u + a\beta u + M(x)u = g(x, |u|)u & \text{for } x \in \mathbb{R}^3, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{1.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3, u(x) \in \mathbb{C}^4, \partial_k = \frac{\partial}{\partial x_k}$, a is a positive constant, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 complex matrices (in 2×2 blocks):

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

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with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$M(x)$ denotes a 4×4 real symmetric matrix valued function which in physics represents the external potential (see [1]), and $g \in C(\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$, where $\mathbb{R}^+ := [0, \infty)$.

(1.1) arises in the study of stationary states to the following general Dirac equation

$$-ih\partial_t\psi = ich\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - P(x)\psi + G_\psi(x,\psi), \tag{1.2}$$

where h is the Planck's constant, c is the speed of light, $m > 0$ is the mass of the electron, $P(x)$ is a 4×4 real symmetric matrix standing for the external field, $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ represents the wave function of the state of a relativistic electron, and $G : \mathbb{R} \times \mathbb{C}^4 \to \mathbb{R}$ represents a nonlinear self-coupling.

Stationary states of (1.2) are considered as particle-like solutions. These solutions are solitons in some sense which propagate without changing their shape.

Assume G satisfies $G(x, e^{i\theta}\psi) = G(x,\psi)$ for all $\theta \in [0, 2\pi]$. Stationary solutions are functions of the type

$$\psi(t,x) = e^{\frac{i\theta t}{h}}u(x).$$

Here $u(x)$ is a non-zero localized solution of the following stationary Dirac equation

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + M(x)u = \tilde{G}_u(x,u) \quad \text{for } x \in \mathbb{R}^3 \tag{1.3}$$

with $a = \frac{mc}{h}$, $M(x) = \frac{P(x)}{hc} + \theta I_4$ and $\tilde{G}_u(x,u) = \frac{G_u(x,u)}{hc}$.

In recent years there are many papers dealing with the existence of stationary solutions of (1.3) via variational methods. In [2–5], the authors considered this problem when

$$M = \omega I_4, \ \tilde{G}(u) = \frac{1}{2}H(\tilde{u}u), \ H \in C^2(\mathbb{R},\mathbb{R}), \ H(0) = 0, \tag{1.4}$$

where $\omega \in (-a, 0)$ is a constant and $\tilde{u}u := (\beta u, u)_{\mathbb{C}^4}$. (1.4) corresponds to the so-called Soler Model. In this condition, using a particular ansatz for the solution u, (1.3) can be reduced to a system of ODE's. By a shooting method, infinitely many localized solutions were obtained, see also [6,7]. There are models of self-coupling for which the ansatz is no more valid. For example,

$$\tilde{G}(u) := \frac{1}{2}|\tilde{u}u|^2 + b|\tilde{u}\alpha u|^2, \tag{1.5}$$

where $b > 0, \tilde{u}\alpha u := (\beta u, \alpha u)_{\mathbb{C}^4}$ and $\alpha := \alpha_1\alpha_2\alpha_3$ (see [4,6,7]). Under the additional assumption that $H'(s)s \geq \theta H(s)$ for $\theta > 1$, [6] considered nonlinearities of type (1.5) while with a weaker growth

$$\tilde{G}(u) := \mu|\tilde{u}u|^\tau + b|\tilde{u}\alpha u|^\sigma, \quad 1 < \tau, \sigma < \frac{3}{2}, \ \mu, b > 0.$$

[6] also considered $\tilde{G}$ growing more slowly than $|u|^3$ at infinity and not necessarily satisfying (1.5).

When $M(x)$ and $\tilde{G}(x,u)$ are periodic in x, [8] treated nonlinearity $\tilde{G}(x,u)$ which may be superquadratic or asymptotically quadratic in u as $|u| \to \infty$. If $\tilde{G}(x,u)$ is additionally even in u, the authors obtained infinitely many solutions. They also considered the case where the nonlinearity has a non-vanishing quadratic part in the origin, so that the linearized equation has a potential.

To the non-periodic system, [9] considered function $\tilde{G}(x,u)$ which is asymptotically quadratic in u at infinity and the potential $M(x)$ is of either Coulomb-type or is of the scalar one. Under suitable assumptions, the authors obtained the existence and multiplicity of solutions of (1.3). If $\tilde{G}(x,u)$ is superquadratic in u at infinity, there is much difficulty to obtain solutions of (1.3) via the variational method because the Palais-Smale condition isn't satisfied in general. Just recently, in [10], the authors considered some auxiliary problem related to the "limit equation" of (1.1) which is autonomous and whose least energy solutions with least energy $\hat{C}$ are known. By virtue of this auxiliary system, the authors constructed linking levels of the functional Φ_M such that the minimax value c_M based on the linking structure of Φ_M satisfies $0 < c_M < \hat{C}$. They proved $(C)_c$-condition and thereby obtained one solution of (1.1).

Motivated by [10], in this paper, we also consider (1.1) with $g(x,|u|)|u|$ being super linear in u at infiniy. But here the conditions on g are weaker than [10]. Under our conditions, we also can prove the existence of least energy solutions of (1.1). Our results also apply to the Coulomb-type potential and the Soler model (see [11]).

In the following, for convenience, any real symmetric matrix $U(x)I_4$ will be written simply $U(x)$. For a symmetric real matrix function $L(x)$, let $\underline{\lambda}_L(x)$ (respectively, $\bar{\lambda}_L(x)$) be the minimal (respectively, the maximal) eigenvalue of $L(x)$, $|L(x)| := \max\{|\underline{\lambda}_L(x)|, |\bar{\lambda}_L(x)|\}$, $|L|_\infty :=$ ess $\sup_x |L(x)|$ and $L(\infty) := \lim_{|x|\to\infty} L(x)$ if and only if $|L(x) - L(\infty)| \to 0$ as $|x| \to \infty$. For two given symmetric real matrix functions $L_1(x)$ and $L_2(x)$, we write $L_1(x) \le L_2(x)$ if and only if

$$\max_{\xi\in\mathbb{C}^4, |\xi|=1} (L_1(x) - L_2(x))\xi \cdot \bar{\xi} \le 0.$$

The variational functional of (1.1) is defined by

$$\Phi_M(u) := \int_{\mathbb{R}^3} \left(\frac{1}{2} \left(-i \sum_{k=1}^{3} \alpha_k \partial_k + a\beta + M(x) \right) u \cdot \bar{u} - R(x,u) \right) \mathrm{d}x \tag{1.6}$$

where

$$R(x,u) := \int_0^{|u|} g(x,s)s\mathrm{d}s.$$

Set

$$c_M := \inf\{\Phi_M(u) : u \neq 0 \text{ is a solution of (1.1)}\}.$$

We call a solution $u_0 \neq 0$ of (1.1) a least energy solution if it satisfies $\Phi_M(u_0) = c_M$, and let S_M be the set of all least energy solutions of (1.1).

Set

$$\tilde{R}(x,u) := \frac{1}{2} g(x,|u|)|u|^2 - R(x,u).$$

We make assumptions on the nonlinear term of (1.1) as follows:

(A_1) $g \in C(\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+), g(x,s) > 0$ if $s \neq 0$ and $g(x,s) = o(s)$ as $s \to 0$;

(A_2) $\tilde{R}(x,u) > 0$ for $u \neq 0$ and there exist $c_1 > 0, 0 < \delta < 1$ and $\gamma > 2$ such that $\tilde{R}(x,u) \ge c_1|u|^\gamma$ for $|u| < \delta$;

(A_3) there exist $c_2, c_3 > 0, r > 1$ and $3 < \nu \le 7$ such that $R(x,u) \ge c_2|u|^{\frac{4\nu^2-\nu+3}{2\nu(\nu-1)}}$ and $g(x,|u|)^\nu \le c_3\tilde{R}(x,u)$ for $|u| \ge r$.

(A_4) there is $g_\infty \in C^1(\mathbb{R}^+, \mathbb{R}^+), g'_\infty(s) > 0$ for $s > 0$ such that $g(x,s) \to g_\infty(s)$ as $|x| \to \infty$ uniformly on bounded sets of s and $g_\infty(s) \le g(x,s)$ for all (x,s).

Our main result reads as follows:

Theorem 1.1 *Let* $(A_1) - (A_4)$ *be satisfied and either*

(R_1) *M is a symmetric continuous real 4×4-matrix function on $\mathbb{R}^3 \setminus \{0\}$ with $0 > M(x) \ge -\frac{k}{|x|}$ where $k < \frac{1}{2}$*
or
(R_2) *M is a symmetric continuous real 4×4-matrix function on $\mathbb{R}^3$ with $|M|_\infty < a, M(x) < M(\infty)$ for all x, and either $(1) M(\infty) \le 0$ or $(2) M(\infty) = m_\infty I_4$ where m_∞ is a constant,*

then (1.1) has at least one least energy solution $u \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \ge 2$ and S_M is compact in $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

Remark 1.1 The function $M(x)$ satisfying (R_1) is called Coulomb-type potential. If $M(x) := -\frac{k}{|x|}$, it is called Coulomb potential. See [1] for discussion on external fields.

Remark 1.2 Under some additional assumptions, for instance (M_3) in [10], we also can check the exponential decay of solutions, we omit it in our paper.

Remark 1.3 There are functions satisfying $(A_1) - (A_4)$. For example,

$$g(x,s) = \left(1 + \frac{1}{1+|x|^2}\right) \alpha s^{\alpha - 2} \times \left[s^2 \ln(1+s) - \frac{1}{2}s^2 + s - \ln(1+s)\right] + 2s^\alpha \ln(1+s)$$

and

$$R(x,u) = \left(1 + \frac{1}{1+|x|^2}\right) |u|^\alpha \left[|u|^2 \ln(1+|u|) - \frac{1}{2}|u|^2 + |u| - \ln(1+|u|)\right],$$

where $0 < \alpha < 1$.

Remark 1.4 $(A_1) - (A_3)$ are weaker than the conditions (g_1) and (g_2) in [10]. Check the following example

$$g(x,s) = \left(1 + \frac{1}{1+|x|^2}\right) \times \left[\mu s^{\mu-2} + (\mu-2)(\mu-\epsilon)s^{\mu-2-\epsilon} \sin^2\left(\frac{s^\epsilon}{\epsilon}\right) + (\mu-2)s^{\mu-2} \sin\left(\frac{2s^\epsilon}{\epsilon}\right)\right]$$

and

$$R(x,u) = \left(1 + \frac{1}{1+|x|^2}\right)\left(|u|^\mu + (\mu-2)|u|^{\mu-\epsilon} \sin^2\left(\frac{|u|^\epsilon}{\epsilon}\right)\right),$$

where $2 < \mu < 3, 0 < \epsilon < \mu - 2$, one can see g satisfies $(A_1) - (A_3)$ but doesn't satisfy (g_2).

2. The variational setting

We will use variational methods to obtain solutions of (1.1). Hence we have to establish a variational setting for the system (1.1). In what follows by $|\cdot|_q$ we denote the usual L^q-norm, and by $(\cdot,\cdot)_2$ the usual L^2-inner product. Let $H_0 := -i\sum_{k=1}^{3}\alpha_k\partial_k + a\beta$ denote the selfadjoint operator on $L^2(\mathbb{R}^3,\mathbb{C}^4)$ with domain $H^1(\mathbb{R}^3,\mathbb{C}^4)$. For any symmetric real matrix value function M, set $H_M := H_0+M$. The spectrum and continuous spectrum of H_M are denoted by $\sigma(H_M)$ and $\sigma_c(H_M)$, respectively.

Lemma 2.1 [10]. *Let M be a symmetric real matrix value function.*

(1) $\sigma(H_0) = \sigma_c(H_0) = \mathbb{R}\backslash(-a,a)$;
(2) *If M satisfies (R_1), then H_M is selfadjoint with $\mathcal{D}(H_M) = H^1(\mathbb{R}^3,\mathbb{C}^4)$ and* $\sigma(H_M)\subset\mathbb{R}\backslash(-(1-2k)a,(1-2k)a)$;
(3) *If M satisfies (R_2), then H_M is selfadjoint with $\mathcal{D}(H_M) = H^1(\mathbb{R}^3,\mathbb{C}^4)$ and* $\sigma(H_M)\subset\mathbb{R}\backslash(-a+|M|_\infty, a-|M|_\infty)$.

By Lemma 2.1, L^2 possesses the orthogonal decomposition

$$L^2 = L^-\oplus L^+, \quad u = u^- + u^+$$

so that H_0 is negative definite on L^-, positive definite on L^+. Let $|H_0|$ and $|H_0|^{\frac{1}{2}}$ respectively be the absolute value and square root of H_0.

Denote $E := \mathcal{D}(|H_0|^{\frac{1}{2}})$ be the domain of the selfadjoint operator $|H_0|^{\frac{1}{2}}$, which is a Hilbert space under the inner product

$$(u,v) = \Re(|H_0|^{\frac{1}{2}}u, |H_0|^{\frac{1}{2}}v)_2$$

with the induced norm $\|u\| = (u,u)^{\frac{1}{2}}$. E possesses the decomposition

$$E = E^-\oplus E^+,$$

where $E^+ = E\cap L^+$ and $E^- = E\cap L^-$ are orthogonal with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products.

By a standard argument, we can obtain the following result. See [8,12]

Lemma 2.2 *E embeds continuously into $H^{\frac{1}{2}}(\mathbb{R}^3,\mathbb{C}^4)$, hence E embeds continuously into L^q for all $q\in[2,3]$ and compactly into L^q_{loc} for all $q\in[1,3)$.*

On E, we define the functional

$$\Phi_M(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} M(x)u\bar{u} - \Psi(u), \tag{2.1}$$

where $\Psi(u) = \int_{\mathbb{R}^3} R(x,u)$.

Note that, by (2) and (3) of Lemma 2.1, $E = \mathcal{D}(|H_M|^{\frac{1}{2}})$ with the equivalent inner product

$$(u,v)_M := \Re(|H_M|^{\frac{1}{2}}u, |H_M|^{\frac{1}{2}}v)_2$$

and norm $\|u\|_M := (u,u)_M^{\frac{1}{2}}$. E has a decomposition

$$E = E_M^-\oplus E_M^+,$$

and Φ_M can be represented as

$$\Phi_M(u) = \frac{1}{2}(\|u^+\|_M^2 - \|u^-\|_M^2) - \Psi(u). \tag{2.2}$$

In order to study the critical points of Φ_M, we now recall some abstract critical point theory developed recently in [13]; see also [14] and [15] for earlier results on that direction.

Let E be a Banach space with direct sum decomposition $E = X \oplus Y$ and P_X, P_Y be projections onto X, Y, respectively. For a functional $\Phi \in C^1(E, \mathbb{R})$ we write $\Phi_a = \{u \in E : \Phi(u) \geq a\}, \Phi^c = \{u \in E : \Phi(u) \leq c\}$ and $\Phi_a^c = \Phi_a \cap \Phi^c$. Recall that Φ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in E one has $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)$, and Φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty} \Phi'(u_n)w = \Phi'(u)w$ for each $w \in E$. A sequence $\{u_n\} \subset E$ is said to be a $(C)_c$-sequence if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$. Φ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence. From now on we assume that X is separable and reflexive, and let $\mathcal{S}$ be a countable dense subset of X^*. For each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R}, \quad p_s = |s(x)| + \|y\|$$

for $u = x + y \in X \oplus Y$, which induces a topology denoted by $\mathcal{T}_{\mathcal{S}}$. Let w^* be the weak*-topology on E^*.

Assume:

(I_1) For any $c \in \mathbb{R}, \Phi_c$ is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous;
(I_2) For any $c > 0$, there exists $\zeta > 0$ such that $\|u\| < \zeta\|P_Y u\|$ for all $u \in \Phi_c$;
(I_3) There exists $\rho > 0$ with $k := \inf \Phi(S_\rho Y) > 0$ where $S_\rho Y := \{u \in Y : \|u\| = \rho\}$.

The following theorem is a special case of the Theorem 3.4 of [12].

Theorem 2.3 *Let $(I_1) - (I_3)$ be satisfied and suppose there are $R > \rho > 0$ and $e \in Y$ with $\|e\| = 1$ such that $\sup \Phi(\partial Q) \leq k$ where $Q = \{u = x + te : t \geq 0, x \in X, \|u\| < R\}$. Then Φ possesses a $(C)_c$-sequence with $k \leq c \leq \sup \Phi(Q)$. If Φ satisfies the $(C)_c$-condition for all $c \leq \sup \Phi(Q)$ then Φ has a critical point z with $k \leq \Phi(z) \leq \sup \Phi(Q)$.*

3. Autonomous equation-limit problem

In this section we study the following autonomous equation

$$\begin{cases} -i\sum_{k=1}^3 \alpha_k \partial_k u + a\beta u + (b + L)u = g_\infty(|u|)u & \text{for } x \in \mathbb{R}^3, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{3.1}$$

where b is a real number and L is a symmetric real constant matrix with

$$b \in (-a, a) \quad \text{and} \quad b - a < L \leq 0. \tag{3.2}$$

Without loss of generality we may assume $b \geq 0$, otherwise we replace b and L by $\tilde{b} = 0$ and $\tilde{L} = b + L$. (3.2) shows that

$$|L| < a - b. \tag{3.3}$$

Equation (3.1) can be regarded as a "limit-equation" of (1.1), which services to constructing linking levels of the functional Φ_M in the proof of our main results. In our later application, we are concerned with $b = 0$ and $L = 0$ in case (R_1), $b = 0$ and $L = M(\infty)$ in the case (1) of (R_2) and $b = m_\infty$ and $L = 0$ in the case (2) of (R_2).

Let $H_b := H_0 + b$, a selfadjoint operator in L^2 with $\mathcal{D}(H_b) = H^1$ and $\sigma(H_b) \subset \mathbb{R}\backslash(-a+b, a+b)$. On $E = H^{\frac{1}{2}}$ we define an equivalent inner product as follows

$$(u, v)_b = \Re(|H_b|^{\frac{1}{2}} u, |H_b|^{\frac{1}{2}} v)_2 \tag{3.4}$$

with the deduced norm $\|u\|_b := ||H_b|^{\frac{1}{2}} u|_2$.

It is easy to check that the decomposition $E = E^- \oplus E^+$ is also orthogonal with respect to the inner product $(\cdot,\cdot)_b$ and $\|u^\pm\|_b^2 = \|u^\pm\|^2 \pm b|u^\pm|_2^2$ for $u^\pm \in E^\pm$ and

$$\|u\|_b^2 \geq (a-b)|u|_2^2 \tag{3.5}$$

Set $R_\infty(u) = R_\infty(|u|) := \int_0^{|u|} g_\infty(s)s\mathrm{d}s$ and $\gamma' = \frac{4\nu^2-\nu+3}{2\nu(\nu-1)}$.

By (A_3) and (A_4), $R_\infty(u) \geq c_2|u|^{\gamma'}$ for $|u| \geq r$.

Since $g_\infty \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, we obtain for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$R_\infty(u) \geq C_\delta |u|^{\gamma'} \quad \text{for all } |u| \geq \delta. \tag{3.6}$$

Since

$$\tilde{R}(x, u) \leq \frac{1}{2} g(x, |u|)|u|^2,$$

by (A_3), we have

$$g(x, |u|)^\nu \leq c_3 \tilde{R}(x, u) \leq C g(x, |u|)|u|^2, \quad \text{for all } |u| \geq r,$$

where and below C stands for some generic positive constant.

Then

$$g(x, |u|) \leq C|u|^{\frac{2}{\nu-1}} \quad \text{for all } |u| \geq r. \tag{3.7}$$

Together with (A_1) and (A_4), for any $\epsilon > 0$, there is $c_\epsilon > 0$ such that

$$g_\infty(|u|)|u| \leq \epsilon|u| + c_\epsilon |u|^{\frac{\nu+1}{\nu-1}}, \tag{3.8}$$

and

$$R_\infty(u) \leq \epsilon|u|^2 + c_\epsilon |u|^{\frac{2\nu}{\nu-1}} \tag{3.9}$$

Set $\tilde{R}_\infty(u) := \frac{1}{2} g_\infty(|u|)|u|^2 - R_\infty(u)$.

Again by (A_1) and (A_3), for any $\epsilon > 0$, there exist $\rho_\epsilon > 0$ and C_ϵ such that

$$g_\infty(|u|) \leq \epsilon \ \text{ if } |u| < \rho_\epsilon \ \text{ and } g_\infty(|u|) \leq C_\epsilon \big(\tilde{R}_\infty(u)\big)^{\frac{1}{\nu}} \ \text{ if } |u| \geq \rho_\epsilon \tag{3.10}$$

Set $\Psi_\infty(u) := \int_{\mathbb{R}^3} R_\infty(u)$ and define

$$\Phi_b(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} (b+L)u\bar{u} - \Psi_\infty(u)$$

for $u = u^- + u^+ \in E^- \oplus E^+$.

The following lemma shows that Φ_b possesses the linking structure.

Lemma 3.1

(1) *there exist* $r_0 > 0$ *and* $\rho > 0$ *such that* $\Phi_b|_{B^+_{r_0}} \geq 0$ *and* $\Phi_b|_{S^+_{r_0}} \geq \rho$, *where* $B^+_{r_0} := \{u \in E^+ : \|u\|_b \leq r_0\}$ *and* $S^+_{r_0} := \{u \in E^+ : \|u\|_b = r_0\}$;
(2) *For any finite dimensional subspace* $Z \subset E^+, \Phi_b(u) \to -\infty$ *as* $u \in E^- \oplus Z, \|u\|_b \to \infty$.

Proof

(1) By (3.2), (3.3) and (3.9),

$$\begin{aligned}\Phi_b(u) &= \frac{1}{2}\|u\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} Lu\bar{u} - \Psi_\infty(u)\\ &\geq \frac{1}{2}\|u\|_b^2 + \frac{1}{2}\frac{|L|}{b-a}\|u\|_b^2 - \epsilon\|u\|_b^2 - c_\epsilon\|u\|_b^{\frac{2\nu}{\nu-1}},\end{aligned}$$

then one can easily see that (1) holds true.

(2) Arguing indirectly, assume there exist some sequence $(u_j) \subset E^- \oplus Z$ with $\|u_j\|_b \to \infty$, then there is $M > 0$ such that $\Phi_b(u_j) \geq -M$ for all j. Setting $w_j = \frac{u_j}{\|u_j\|_b}$, we have $\|w_j\|_b = 1$, then $w_j \rightharpoonup w, w_j^- \rightharpoonup w^-, w_j^+ \to w^+$ and

$$-\frac{M}{\|u_j\|_b^2} \leq \frac{\Phi_b(u_j)}{\|u_j\|_b^2} = \frac{1}{2}\|w_j^+\|_b^2 - \frac{1}{2}\|w_j^-\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} Lw_j\bar{w}_j - \int_{\mathbb{R}^3}\frac{R_\infty(u_j)}{\|u_j\|_b^2}. \quad (3.11)$$

From (3.11), we can easily see that $w^+ \neq 0$.

Choose $k \in (a+b, \infty) \cap \sigma(H_b)$ satisfying $Z \subset E_k - E_{\lambda_e}$, where $(E_\lambda)_{\lambda\in\mathbb{R}}$ is the spectrum family of H_b and $\lambda_e := \min\{\lambda | \lambda \in \sigma(H_b) \cap [a+b, \infty)\}$.

By (3.6), since $\gamma' > 2$, there exists $M_0 > 1$ such that $R_\infty(u) \geq 3k|u|^2$ for $|u| \geq M_0$. It is clear that

$$\begin{aligned}&\|w^+\|_b^2 - \|w^-\|_b^2 - 3k\int_{\mathbb{R}^3}|w|^2\\ &\leq k|w^+|_2^2 - \|w^-\|_b^2 - 3k|w^+|_2^2 - 3k|w^-|_2^2\\ &\leq -(2k|w^+|_2^2 + \|w^-\|_b^2 + 3k|w^-|_2^2)\\ &< 0.\end{aligned}$$

Therefore, there is $a > 0$ such that

$$\|w^+\|_b^2 - \|w^-\|_b^2 - 3k\int_{|x|\leq a}|w|^2 < 0. \quad (3.12)$$

Note that

$$\begin{aligned}\frac{\Phi_b(u_j)}{\|u_j\|_b^2} &\leq \frac{1}{2}\|w_j^+\|_b^2 - \frac{1}{2}\|w_j^-\|_b^2 - \int_{|x|\leq a}\frac{R_\infty(u_j)}{\|u_j\|_b^2}\\ &= \frac{1}{2}\left(\|w_j^+\|_b^2 - \|w_j^-\|_b^2 - 3k\int_{|x|\leq a}|w_j|^2\right) - \int_{|x|\leq a}\frac{R_\infty(u_j) - \frac{3k}{2}|u_j|^2}{\|u_j\|_b^2}\\ &\leq \frac{1}{2}\left(\|w_j^+\|_b^2 - \|w_j^-\|_b^2 - 3k\int_{|x|\leq a}|w_j|^2\right) + \frac{2kM_0^2a^3}{\|u_j\|_b^2}. \quad (3.13)\end{aligned}$$

By (3.11) and (3.12), we obtain

$$0 \le \liminf_{j\to\infty} \left(\frac{1}{2}\|w_j^+\|_b^2 - \frac{1}{2}\|w_j^-\|_b^2 - \int_{|x|\le a} \frac{R_\infty(u_j)}{\|u_j\|_b^2} \right)$$
$$\le \frac{1}{2}\left(\|w^+\|_b^2 - \|w^-\|_b^2 - 3k\int_{|x|\le a} |w|^2 \right)$$
$$< 0,$$

a contradiction. □

Under our assumptions, it isn't difficult to check that any $(C)_c$-sequence is bounded. Let $\mathcal{K}_b := \{u \in E : \Phi_b'(u) = 0\}$ be the critical set of Φ_b. Since Φ_b is $\mathbb{R}^3$-invariant, i.e., $\Phi_b(a * u) = \Phi_b(u)$ where $(a * u)(x) := u(x + a)$ for all $a \in \mathbb{R}^3$. Using the concentration compactness principle and some abstract critical point theorem (see [16]), one can show $\mathcal{K}_b \backslash \{0\} \neq \emptyset$. In fact, We have

Lemma 3.2 [6,8]. *$\mathcal{K}_b \setminus \{0\} \neq \emptyset$ and $\mathcal{K}_b \subset \cap_{q\ge 2} W^{1,q}$.*

Denote $c_b := \inf\{\Phi_b(u) : u \in \mathcal{K}_b \setminus \{0\}\}$.

Lemma 3.3 *$c_b > 0$. In particular, 0 is an isolated critical point of Φ_b.*

Proof Assume by contradiction that $c_b = 0$, then there exists $u_j \in \mathcal{K}_b \setminus \{0\}$ such that $\Phi_b(u_j) \to 0$.

Observe that

$$\Phi_b(u_j) = \Phi_b(u_j) - \frac{1}{2}\Phi_b'(u_j)u_j = \int_{\mathbb{R}^3} \tilde{R}_\infty(u_j),$$

then

$$\int_{\mathbb{R}^3} \tilde{R}_\infty(u_j) \to 0.$$

Since $0 = \Phi_b'(u_j)(u_j^+ - u_j^-)$, (3.3) and (3.5) imply that

$$\|u_j\|_b^2 = -\int_{\mathbb{R}^3} Lu_j\overline{u_j^+ - u_j^-} + \int_{\mathbb{R}^3} g_\infty(|u_j|)u_j\overline{u_j^+ - u_j^-}$$
$$\le \frac{|L|}{a-b}\|u_j\|_b^2 + \int_{\mathbb{R}^3} g_\infty(|u_j|)u_j\overline{u_j^+ - u_j^-}.$$

By (3.10) and Hölder inequality, one sees

$$\left(1 - \frac{|L|}{a-b}\right)\|u_j\|_b^2 \le \left(\int_{|u_j|<\rho_\epsilon} + \int_{|u_j|\ge\rho_\epsilon} \right) g_\infty(|u_j|)u_j\overline{u_j^+ - u_j^-}$$
$$\le \epsilon|u_j|_2^2 + C_\epsilon \int_{\mathbb{R}^3} \tilde{R}_\infty(u_j)^{\frac{1}{\nu}} |u_j||u_j^+ - u_j^-|$$
$$\le \epsilon|u_j|_2^2 + C_\epsilon \left(\int_{\mathbb{R}^3} \tilde{R}_\infty(u_j) \right)^{\frac{1}{\nu}} |u_j|_{\frac{2\nu}{\nu-1}} |u_j^+ - u_j^-|_{\frac{2\nu}{\nu-1}}$$
$$\le C\epsilon\|u_j\|_b^2 + C\left(\int_{\mathbb{R}^3} \tilde{R}_\infty(u_j) \right)^{\frac{1}{\nu}} \|u_j\|_b^2. \tag{3.14}$$

Hence $1 \le C\epsilon + o(1)$, a contradiction. □

Just as [17], for fixed $u \in E^+$, we introduce the functional $\phi_u : E^- \to \mathbb{R}$ by

$$\phi_u(v) := \Phi_b(u+v).$$

Then we have

$$\begin{aligned}\phi_u''(v)[w,w] &= -\|w\|_b^2 + \int_{\mathbb{R}^3} Lw\bar{w} - \Psi_\infty''(u+v)[w,w] \\ &= -\|w\|_b^2 + \int_{\mathbb{R}^3} Lw\bar{w} - \int_{\mathbb{R}^3} \frac{g_\infty'(|u+v|)}{|u+v|}(\Re[(u+v)\bar{w}])^2 \\ &\quad - \int_{\mathbb{R}^3} g_\infty(|u+v|)|w|^2\end{aligned}$$

for all $v, w \in E^-$, which implies $\phi_u(\cdot)$ is strictly concave. Moreover

$$\phi_u(v) \le \frac{1}{2}(\|u\|_b^2 - \|v\|_b^2) \to -\infty \quad \text{as } \|v\|_b \to \infty.$$

It is easy to check ϕ_u is weakly sequentially upper semicontinuous. Thus there is a unique strict maximum point $h_b(u)$ for $\phi_u(\cdot)$, which is also the only critical point of ϕ_u on E^- and satisfies:

$$v \ne h_b(u) \Leftrightarrow \Phi_b(u+v) < \Phi_b(u + h_b(u)) \tag{3.15}$$

for all $u \in E^+$ and $v \in E^-$.

Moreover,

$$\begin{aligned}\Phi_b'(u+h_b(u))w &= I_b'(h_b(u))w = 0 \\ &\Rightarrow -(h_b(u), w)_b + \Re\int_{\mathbb{R}^3} L(u+h_b(u))\bar{w} \\ &= \Re\int_{\mathbb{R}^3} g_\infty(|u+h_b(u)|)(u+h_b(u))\bar{w}\end{aligned} \tag{3.16}$$

for all $u \in E^+$ and $w \in E^-$.

In the following, we collect some properties of h_b.

Lemma 3.4 [17].

(1) *h_b is $\mathbb{R}^3$-invariant;*
(2) *$h_b \in C^1(E^+, E^-)$ and $h_b(0) = 0$;*
(3) *h_b is a bounded map;*
(4) *If $u_n \rightharpoonup u$ in E^+, then $h_b(u_n) - h_b(u_n - u) \to h_b(u)$ and $h_b(u_n) \rightharpoonup h_b(u)$. The same is true for $|h_b(u)|_2^2$.*

Now we define the reduce functional $I_b : E^+ \to \mathbb{R}$ by

$$\begin{aligned}I_b(u) :&= \Phi_b(u + h_b(u)) \\ &= \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h_b(u)\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} L(u+h_b(u))\overline{u+h_b(u)} - \Psi_\infty(u+h_b(u)).\end{aligned}$$

Then using (3.16), we obtain

$$\begin{aligned}I_b'(u)v &= (u,v)_b - (h_b(u), h_b'(u)v)_b + \Re\int_{\mathbb{R}^3} L(u+h_b(u))\overline{v+h_b'(u)v} \\ &\quad - \Re\int_{\mathbb{R}^3} g_\infty(|u+h_b(u)|)(u+h_b(u))\overline{v+h_b'(u)v} \\ &= \Phi_b'(u+h_b(u))(v+h_b(v)) \\ &= (u,v)_b - (h_b(u), h_b(v))_b + \Re\int_{\mathbb{R}^3} L(u+h_b(u))\overline{v+h_b(v)} \\ &\quad - \Re\int_{\mathbb{R}^3} g_\infty(|u+h_b(u)|)(u+h_b(u))\overline{v+h_b(v)}\end{aligned}$$

for all $u, v \in E^+$. Critical points of I_b and Φ_b are in one to one correspondence via the injective map $u \to u + h_b(u)$ from E^+ into E, which means, if let

$$\mathcal{K}_b^+ := \{u \in E^+ : I_b'(u) = 0\},$$

we have

$$\mathcal{K}_b = \{u + h_b(u) : u \in \mathcal{K}_b^+\}.$$

We now show I_b possesses the mountain pass geometry, that is

Lemma 3.5

(1) *There is* $\rho > 0$ *such that* $\inf I_b|_{S_\rho^+} > 0$;
(2) *For any finite dimensional subspace* $X \subset E^+, I_b(u) \to -\infty$ *as* $u \in X, \|u\|_b \to \infty$.

Proof

(1)

$$\begin{aligned}I_b(w) &= \frac{1}{2}\|w\|_b^2 - \frac{1}{2}\|h_b(w)\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} L(w+h_b(w))\overline{w+h_b(w)} \\ &\quad - \Psi_\infty(w+h_b(w)) \\ &= \frac{1}{2}\|w\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} Lw\bar{w} + (\Phi_b(w+h_b(w)) - \Phi_b(w)) - \Psi_\infty(w) \\ &\geq \frac{1}{2}\|w\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} Lw\bar{w} - \Psi_\infty(w) \\ &\geq \frac{1}{2}\|w\|_b^2 - \frac{|L|}{2}|w|_2^2 - \Psi_\infty(w) \\ &\geq \frac{1}{2}\left(1 - \frac{|L|}{a-b}\right)\|w\|_b^2 - \int_{\mathbb{R}^3} R_\infty(w)\end{aligned}$$

for all $w \in E^+$. From (3.9) we can obtain the desired conclusion.

(2) Let $P : L^{\gamma'} \to X$ be the natural projection. Then there is $c_{\gamma'}$ such that $c_{\gamma'}|Pu|_{\gamma'} \leq |u|_{\gamma'}$ for all $u \in L^{\gamma'}$.

By (3.3), (3.5) and (3.6), for any $\epsilon > 0$, one has

$$\begin{aligned}
I_b(u) &= \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h_b(u)\|_b^2 + \frac{1}{2}\int_{\mathbb{R}^3} L(u+h_b(u))\overline{u+h_b(u)} \\
&\quad - \int_{\mathbb{R}^3} R_\infty(u+h_b(u)) \\
&\le \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h_b(u)\|_b^2 + \epsilon|u+h_b(u)|_2^2 - c_\epsilon|u+h_b(u)|_{\gamma'}^{\gamma'} \\
&\le \frac{1}{2}\|u\|_b^2 - \frac{1}{2}\|h_b(u)\|_b^2 + \frac{\epsilon}{a-b}\|u+h_b(u)\|_b^2 - c_{\gamma'}^{\gamma'}c_\epsilon|u|_{\gamma'}^{\gamma'} \\
&\le \left(\frac{1}{2} + \frac{\epsilon}{a-b}\right)\|u\|_b^2 - \left(\frac{1}{2} - \frac{\epsilon}{a-b}\right)\|h_b(u)\|_b^2 - C\|u\|_b^{\gamma'}.
\end{aligned}$$

Since $\gamma' > 2$, the conclusion is true. □

The following lemma shows that any $(PS)_c$-sequence is bounded.

Lemma 3.6 *Assume $I_b(u_j) \to c, I_b'(u_j) \to 0$, then (u_j) is bounded.*

Proof We will show $w_j := u_j + h_b(u_j) = w_j^+ + w_j^-$ is bounded. Argue by contradiction we assume $\|w_j\|_b \to \infty$ as $j \to \infty$.

By (A_2) and (A_3),

$$c_2|w|^{\gamma'} \le R(x,w) \le \frac{1}{2}g(x,|w|)|w|^2 \quad \text{for } |w| \ge r.$$

Then

$$\tilde{R}(x,w) \ge g(x,|w|)^\nu \ge C|w|^{(\gamma'-2)\nu} \quad \text{for } |w| \ge r.$$

Hence

$$\tilde{R}_\infty(w) := \lim_{|x|\to\infty} \tilde{R}(x,w) \ge C|w|^{(\gamma'-2)\nu} = C|w|^{\frac{3(\nu+1)}{2(\nu-1)}} \quad \text{for } |w| \ge r.$$

Therefore, one obtains

$$\begin{aligned}
I_b(u_j) - \frac{1}{2}I_b'(u_j)u_j &= \int_{\mathbb{R}^3}\left[\frac{1}{2}g_\infty(|w_j|)w_j\overline{w}_j - R_\infty(w_j)\right] \\
&= \int_{\mathbb{R}^3}\tilde{R}_\infty(w_j) \ge C\int_{|w_j|\ge r}|w_j|^{\frac{3(\nu+1)}{2(\nu-1)}},
\end{aligned}$$

from which we obtain

$$\int_{|w_j|>r}|w_j|^{\frac{3(\nu+1)}{2(\nu-1)}} \le C\|w_j\|_b. \tag{3.17}$$

By (3.3), (3.5) and (3.16),

$$\begin{aligned}
I_b'(u_j)(u_j) &= \Phi_b'(w_j)(w_j^+ - w_j^-) \\
&= \|w_j\|_b^2 + \int_{\mathbb{R}^3} Lw_j\overline{w_j^+ - w_j^-} - \int_{\mathbb{R}^3} g_\infty(|w_j|)w_j\overline{w_j^+ - w_j^-} \\
&\ge \left(1 - \frac{|L|}{a-b}\right)\|w_j\|_b^2 - \int_{\mathbb{R}^3} g_\infty(|w_j|)w_j\overline{w_j^+ - w_j^-}.
\end{aligned} \tag{3.18}$$

By (A_1), for any small $\epsilon > 0$, there exists $\delta < 1$ satisfying $g_\infty(s) < \epsilon$ for $s \le \delta$. Denote

$$I := \int_{\mathbb{R}^3} g_\infty(|w_j|) w_j \overline{w_j^+ - w_j^-} := I_1 + I_2 + I_3$$
$$= \left(\int_{|w_j| \ge r} + \int_{\delta < |w_j| < r} + \int_{|w_j| \le \delta} \right) g_\infty(|w_j|) w_j \overline{w_j^+ - w_j^-}.$$

Then by (3.7), (3.17),

$$\begin{aligned} |I_1| &\le C \int_{|w_j| \ge r} |w_j|^{\frac{2}{\nu-1}} |w_j| |w_j^+ - w_j^-| \\ &= C \int_{|w_j| \ge r} |w_j|^{\frac{\nu+1}{\nu-1}} |w_j^+ - w_j^-| \\ &\le C \left(\int_{|w_j| \ge r} |w_j|^{\frac{3(\nu+1)}{2(\nu-1)}} \right)^{\frac{2}{3}} \left(\int_{|w_j| \ge r} |w_j^+ - w_j^-|^3 \right)^{\frac{1}{3}} \\ &\le C \|w_j\|_b^{\frac{5}{3}}. \end{aligned} \tag{3.19}$$

For $b > a > 0$, let $\Omega_j(a,b) := \{x \in \mathbb{R}^3, a \le |w_j(x)| \le b\}$ and $C_a^b := \min\left\{ \frac{\tilde{R}_\infty(w(x))}{|w(x)|^2}, a \le |w(x)| \le b \right\}$. By (A_2), $C_a^b > 0$ and $\tilde{R}_\infty(w_j) \ge C_a^b |w_j|^2$ for $x \in \Omega_j(a,b)$. Set $v_j := \frac{w_j}{\|w_j\|_b}$, then $\|v_j\|_b = 1, |v_j|_s \le \gamma_s \|v_j\|_b = \gamma_s$ for $s \in [2,3]$.

Therefore,

$$\begin{aligned} \frac{|I_2|}{\|w_j\|_b^2} &= \left| \int_{\Omega_j(\delta,r)} \frac{g_\infty(|w_j|) w_j \overline{w_j^+ - w_j^-}}{\|w_j\|_b^2} \right| \\ &\le \int_{\Omega_j(\delta,r)} |g_\infty(|w_j|)| |v_j| |v_j^+ - v_j^-| \\ &\le C |v_j|_2 \left(\int_{\Omega_j(\delta,r)} |v_j|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

Since

$$\int_{\Omega_j(\delta,r)} |v_j|^2 = \frac{\int_{\Omega_j(\delta,r)} |w_j|^2}{\|w_j\|_b^2} \le \frac{\int_{\Omega_j(\delta,r)} \tilde{R}_\infty(w_j)}{C_\delta^r \|w_j\|_b^2} \to 0, \tag{3.21}$$

for the above ϵ, by (3.20) and (3.21), there is J such that

$$|I_2| \le \epsilon \|w_j\|_b^2 \quad \text{as } j \ge J. \tag{3.22}$$

Obviously,

$$|I_3| \le \epsilon \|w_j\|_b^2 \tag{3.23}$$

By (3.18), (3.19), (3.22) and (3.23), we obtain

$$\left(1 - \frac{|L|}{a-b} - 2\epsilon\right) \|w_j\|_b^2 - C \|w_j\|_b^{\frac{5}{3}} \le C \|w_j\|_b,$$

which is a contradiction since by assumption (w_j) is unbounded, and hence we obtain (u_j) is bounded. □

Lemma 3.2 shows that 0 is an isolated critical point of I_b. Therefore there is a $M_1 > 0$ such that $\|u\|_b \geq M_1$ for any nontrivial critical point u of I_b. Let

$$\mathcal{N}_b^+ := \{u \in E^+\backslash\{0\} : I_b'(u)u = 0\}.$$

Lemma 3.7 [10]. *For each $u \in E^+\backslash\{0\}$, there is a unique $t = t(u) > 0$ such that $tu \in \mathcal{N}_b^+$.*

Set

$$\begin{aligned} b_1 &:= \inf\{I_b(u) : u \in \mathcal{N}_b^+\}, \\ b_2 &:= \inf\{I_b(u) : u \in \mathcal{K}_b^+\backslash\{0\}\}, \\ b_3 &:= \inf_{f\in\Gamma_b} \max_{t\in[0,1]} I_b(f(t)), \end{aligned}$$

where

$$\Gamma_b := \{f \in C([0,1], E^+) : f(0) = 0, I_b(f(1)) < 0\}.$$

Let $u_0 \in E^+$ be such that $I_b(u_0) < 0$, and set

$$\begin{aligned} \Gamma^b &:= \{f \in C([0,1], E^+) : f(0) = 0, f(1) = u_0\} \\ b_4 &:= \inf_{f\in\Gamma^b} \max_{t\in[0,1]} I_b(f(t)). \end{aligned}$$

Lemma 3.8 $c_b = b_1 = b_2 = b_3 = b_4$.

Proof The proof can be seen in [10]. For reader's convenience, we outline it as follows.

(i) $b_1 \leq b_2$ holds true since $\mathcal{K}_b^+\backslash\{0\} \subset \mathcal{N}_b^+$.

(ii) $b_2 \leq b_3$. Let (u_j) satisfy $I_b(u_j) \to b_3$ and $I_b'(u_j) \to 0$. By lemma 3.6, (u_j) is bounded in E. By the concentration compactness principle, (u_j) is either vanishing or nonvanishing.
By (3.8) and (3.9),

$$\tilde{R}_\infty(u) \leq \epsilon|u|^2 + c_\epsilon|u|^{\frac{2\nu}{\nu-1}} \quad \text{for all } u \in \mathbb{C}^4.$$

If (u_j) is vanishing, then $|u_j|_p \to 0$ for $p \in (2,3)$. Therefore

$$\begin{aligned} b_3 &= \lim_{j\to\infty}\left(I_b(u_j) - \frac{1}{2}I_b'(u_j)u_j\right) = \lim_{j\to\infty}\int_{\mathbb{R}^3}\tilde{R}_\infty(u_j) \\ &\leq \int_{\mathbb{R}^3}(\epsilon|u|^2 + c_\epsilon|u|^{\frac{2\nu}{\nu-1}}) = o(1), \end{aligned}$$

a contradiction.
Thus, (u_j) is non-vanishing, that is, there exist $r, \eta > 0$ and $(a_j) \subset \mathbb{R}^3$ with

$$\limsup_{j\to\infty}\int_{B_r(a_j)}|u_j|^2 \geq \eta.$$

Set $v_j := a_j * u_j$. Since the norm and the functional I_b are invariant under the $*$-action, $I_b(v_j) \to b_3, I_b'(v_j) \to 0$. Therefore, $v_j \rightharpoonup v$ in E with $v \neq 0$ and $I_b'(v) = 0$. A standard argument shows that $I_b(v_j - v) \to b_3 - I_b(v), I_b'(v_j - v) \to 0$

(see [12]). Obviously, $(v_j - v)$ is a $(PS)_{b_3 - I_b(v)}$-sequence. By Lemma 3.6, it is bounded. Therefore,

$$b_3 - I_b(v) = \lim_{j\to\infty}\Big(I_b(v_j - v) - \frac{1}{2}I_b'(v_j - v)(v_j - v)\Big) = \lim_{j\to\infty}\int_{\mathbb{R}^3}\tilde{R}_\infty(v_j - v) \geq 0.$$

Hence, $b_2 \leq I_b(v) \leq b_3$.

(iii) $b_3 \leq b_1$. Let $\hat{S}_b$ be the set of all least energy solutions of I_b. Take $U \in \hat{S}_b$ and define $f(t) := tU(x)$ for $t \geq 0$. Since $I_b'(U) = 0$, one has $t(U) = 1$. Then $f \in \Gamma_b$ and

$$b_3 \leq \max_{t\in[0,1]} I_b(f(t)) = I_b(U) = b_1.$$

(iv) $b_3 \leq b_4$ is clear. Choose $f \in \Gamma_b$. Then $I_b(tf(1))$ and $I_b(tu_0)$ are strictly decreasing for $t \geq 1$, and $I_b(tf(1)) \to -\infty, I_b(tu_0) \to -\infty$ as $t \to \infty$. By Lemma 3.6(2), we can choose a cure $\ell(s)$ jointing $f(1)$ and u_0 in the two-dimensional subspace $\mathrm{span}\{f(1), u_0\}$ such that $I_b(\ell(s)) < 0$ for $1 \leq s \leq 2$. Define $\hat{f}(t)$ by $\hat{f}(t) = f(2t)$ for $t \in \big[0, \frac{1}{2}\big]$ and $\hat{f}(t) = \ell(2t)$ for $t \in \big[\frac{1}{2}, 1\big]$. Then $\hat{f} \in \Gamma^b$ and $\max_{t\in[0,1]} I_b(\hat{f}(t)) = \max_{t\in[0,1]} I_b(f(t))$. Thus $b_4 \leq b_3$.

The proof is completed. □

For estimating the linking level of Φ_M, we need the following result.

Lemma 3.9 *Let $u \in \mathcal{K}_b^+$ be such that $I_b(u) = c_b$, and set $E_u = E^- \oplus \mathbb{R}u$. Then*

$$\sup_{w\in E_u} \Phi_b(w) \leq c_b.$$

Proof For any $w = v + su \in E_u$, by (3.15),

$$\Phi_b(w) \leq \Phi_b(su + h_b(su)) = I_b(su).$$

Since $u \in \mathcal{N}_b^+$, we have

$$\sup_{w\in E_u} \Phi_b(w) \leq \sup_{s\geq 0} I_b(su) = I_b(u) = c_b.$$

□

4. Proof of the main result

We will use Theorem 2.3 to prove our main result.

Observe that

$$R(x,u) \geq c_1|u|^{\gamma'} \quad \text{for all } |u| \geq r. \tag{4.1}$$

$$\tilde{R}(x,u) > 0 \quad \text{if} \quad u \neq 0.$$

By (A_3), for $|u| \geq r$, one has

$$\tilde{R}(x,u) \geq \frac{1}{c_3}g(x,|u|)^\nu \quad \text{and} \quad \frac{1}{2}g(x,|u|)|u|^2 \geq R(x,u) \geq c_2|u|^{\gamma'}.$$

Therefore

$$\tilde{R}(x,u) \geq C|u|^{(\gamma'-2)\nu} = C|u|^{\frac{3(\nu+1)}{2(\nu-1)}} \quad \text{for } |u| \geq r \tag{4.2}$$

Hence $\tilde{R}(x,u) \to \infty$ as $|u| \to \infty$.

For any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$g(x, |u|) \leq \epsilon + c_\epsilon |u|^{\frac{2}{\nu-1}} \tag{4.3}$$

and

$$R(x, u) \leq \epsilon |u|^2 + c_\epsilon |u|^{\frac{2\nu}{\nu-1}} \tag{4.4}$$

for all (x, u).

Now we consider the functional Φ_M defined by (2.1) or equivalently (2.2). Let $\mathcal{K}_M := \{u \in E : \Phi'_M(u) = 0\}$ be the critical set of Φ_M and $c_M := \inf\{\Phi_M(u) : u \in \mathcal{K}_M \setminus \{0\}\}$. Set $\Psi(u) := \int_{\mathbb{R}^3} R(x, u)$.

In virtue of (4.3), (4.4) and Lemma 2.2, we can easily prove the following lemma, which implies (I_1).

Lemma 4.1 *Ψ_M is weakly sequentially lower semicontinuous and Φ'_M is weakly sequentially continuous.*

From the form (2.2), since $R(x, u) \geq 0$, it is clear that Φ_M verifies (I_2). (I_3) is satisfied by the following lemma.

Lemma 4.2 *There exist $r_1 > 0$ and $\rho > 0$ such that $\Phi_M|_{B^+_{r_1}} \geq 0$ and $\Phi_M|_{S^+_{r_1}} \geq \rho$.*

Proof We only check the Coulomb potential case and the other case can be treated similarly. Assume (M_1) is satisfied. Let $V_k := \frac{k}{|x|}$. By Kato's inequality,

$$|V_k u|_2^2 \leq 4k^2 |H_0 u|_2^2 = |(2kH_0)u|_2^2,$$

then

$$\int_{\mathbb{R}^3} \frac{k}{|x|} u\bar{u} = |V_k^{\frac{1}{2}} u|_2^2 \leq ||2kH_0|^{\frac{1}{2}} u|_2^2 = 2k||H_0|^{\frac{1}{2}} u|_2^2.$$

By (M_1),

$$-\int_{\mathbb{R}^3} M(x) u\bar{u} \leq 2k||H_0|^{\frac{1}{2}} u|_2^2 = 2k\|u\|^2.$$

For $u \in E^+$,

$$\begin{aligned}
\Phi_M(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} M(x) u\bar{u} - \int_{\mathbb{R}^3} R(x, u) \\
&\geq \left(\frac{1}{2} - k\right)\|u\|^2 - \int_{\mathbb{R}^3} R(x, u) \\
&\geq \left(\frac{1}{2} - k\right)\|u\|^2 - \epsilon|u|_2^2 - c_\epsilon |u|_{\frac{2\nu}{\nu-1}}^{\frac{2\nu}{\nu-1}} \\
&\geq \left(\frac{1}{2} - k\right)\|u\|^2 - C\epsilon\|u\|^2 - Cc_\epsilon \|u\|^{\frac{2\nu}{\nu-1}},
\end{aligned}$$

so the conclusion follows. □

In the sequel, for the case of (M_1), we let $b = 0$ and $L = 0$ in (3.1). Denote the corresponding functional by

$$\Phi_0(u) := \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^3} R_\infty(u),$$

the critical set by $\mathcal{K}_0 := \{u \in E : \Phi_0'(u) = 0\}$, the least energy by $\hat{C}_0 := \min\{\Phi_0(u) : u \in \mathcal{K}_0\backslash\{0\}\}$, the least energy solution set by $\hat{S}_0 := \{u \in \mathcal{K}_0 : \Phi_0(u) = \hat{C}_0\}$, and the induced map from $E^+ \to E^-$ by h_0. For the case $(M_2)(1)$ we consider $b = 0$ and $L = M(\infty)$ in (3.1). Denote the corresponding functional by

$$\Phi_I(u) := \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2}\int_{\mathbb{R}^3} M(\infty)u\bar{u} - \int_{\mathbb{R}^3} R_\infty(u),$$

and the critical set, the least energy, the least energy solution set and the induced map respectively by $\mathcal{K}_I, \hat{C}_I, \hat{S}_I$ and h_I. Similarly in the case of $(M_2)(2)$ we take $b = m_\infty$ and $L = 0$ in (3.1) and denote correspondingly

$$\begin{aligned}\Phi_{II}(u) :&= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{m_\infty}{2}|u|_2^2 - \int_{\mathbb{R}^3} R_\infty(u)\\ &= \frac{1}{2}(\|u^+\|_{m_\infty}^2 - \|u^-\|_{m_\infty}^2) - \int_{\mathbb{R}^3} R_\infty(u)\end{aligned}$$

(where $\|\cdot\|_{m_\infty}$ denotes the norm given by (3.4) with $b = m_\infty$) with notations $\mathcal{K}_{II}, \hat{C}_{II}$, $\hat{S}_{II}$ and h_{II}. If without confusion, sometimes we shall write simply $\Phi, \mathcal{K}, \hat{C}, \hat{S}$ and h standing for one of the cases.

The next lemma shows that Φ_M satisfies the linking condition.

Lemma 4.3 *There is $R > 0$ such that, for any $e \in E^+$ and $E_e := E^- \oplus \mathbb{R}e$,*

$$\Phi_M(u) < 0 \textit{ for all } u \in E_e\backslash B_R. \tag{4.5}$$

Proof It is easy to check that

$$\Phi_M(u) \le \Phi_n(u) \text{ for } n = 0, I, II.$$

By Lemma 3.1,

$$\Phi_n(u) < 0 \text{ for all } u \in E_e\backslash B_R, n = 0, I, II.$$

and henceforth the conclusion holds true. □

By Theorem 2.3, there is a $(C)_c$-sequence (u_j) with $\rho \le c \le \sup \Phi_M(Q)$. We now analyze the $(C)_c$-sequence. First we have

Lemma 4.4 *Any $(C)_c$-sequence for Φ_M is bounded.*

Proof Let $(u_j) \subset E$ satisfy $\Phi_M(u_j) \to c$ and $(1 + \|u_j\|_M)\Phi_M'(u_j) \to 0$.

Then

$$C \ge \Phi_M(u_j) - \frac{1}{2}\Phi_M'(u_j)u_j = \int_{\mathbb{R}^3} \tilde{R}(x, u_j). \tag{4.6}$$

Arguing indirectly, assume up to a subsequence $\|u_j\|_M \to \infty$ as $j \to \infty$. Set $v_j := \frac{u_j}{\|u_j\|_M}$, then $|v_j|_s \le \gamma_s$ for $s \in [2, 3]$.

Obviously, $\frac{3(\nu+1)}{2(\nu-1)} \geq 2$. By (4.2),

$$\tilde{R}(x,u) \geq C|u|^{\frac{3(\nu+1)}{2(\nu-1)}} \geq C|u|^2 \quad \text{for } |u| \geq r.$$

Together with (A_4), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\tilde{R}(x,u) \geq C_\delta |u|^2 \quad \text{for } |u| \geq \delta. \tag{4.7}$$

Set $Q_j(\delta) := \{x \in \mathbb{R}^3 : |u_j(x)| \geq \delta\}$ for $\delta > 0$. It follows from (4.6) and (4.7) that

$$\int_{Q_j(\delta)} |u_j|^2 \leq C. \tag{4.8}$$

Thus we obtain

$$\int_{Q_j(\delta)} |v_j|^2 = \frac{1}{\|u_j\|_M^2} \int_{Q_j(\delta)} |u_j|^2 \leq \frac{C}{\|u_j\|_M^2} \to 0.$$

For $s \in [2,3]$, by Hölder inequality,

$$\begin{aligned} \int_{Q_j(\delta)} |v_j|^s &= \int_{Q_j(\delta)} |v_j|^{2(3-s)} |v_j|^{3(s-2)} \\ &\leq \left(\int_{Q_j(\delta)} |v_j|^2 \right)^{3-s} \left(\int_{Q_j(\delta)} |v_j|^3 \right)^{s-2} \\ &\leq C_3^{3(s-2)} \left(\int_{Q_j(\delta)} |v_j|^2 \right)^{3-s} \to 0. \end{aligned} \tag{4.9}$$

Note that

$$\begin{aligned} \Phi'_M(u_j)(u_j^+ - u_j^-) &= \|u_j\|_M^2 - \int_{\mathbb{R}^3} g(x,|u_j|) u_j \overline{u_j^+ - u_j^-} \\ &= \|u_j\|_M^2 \left(1 - \int_{\mathbb{R}^3} g(x,|u_j|) v_j \overline{v_j^+ - v_j^-} \right), \end{aligned}$$

Hence

$$\int_{\mathbb{R}^3} g(x,|u_j|) v_j \overline{v_j^+ - v_j^-} \to 1. \tag{4.10}$$

By (A_1), for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$g(x,|u|) \leq \epsilon \quad \text{whenever} \quad |u| < \delta.$$

By (A_3) and (A_4), for the above δ, there is $C_\delta > 0$ such that

$$g(x,|u|)^\nu \leq C_\delta \tilde{R}(x,u) \quad \text{for } |u| \geq \delta. \tag{4.11}$$

Therefore, from (4.6) and (4.9), using Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} g(x,|u_j|)v_j\overline{v_j^+ - v_j^-} \\
&= \left(\int_{|u_j|<\delta} + \int_{Q_j(\delta)}\right) g(x,|u_j|)v_j\overline{v_j^+ - v_j^-} \\
&\le \epsilon|v_j|^2 + \left(\int_{Q_j(\delta)} g(x,|u_j|)^\nu\right)^{\frac{1}{\nu}} \left(\int_{Q_j(\delta)} |v_j|^{\nu'}|v_j^+ - v_j^-|^{\nu'}\right)^{\frac{1}{\nu'}} \\
&\le C_2\epsilon + C\left(\int_{Q_j(\delta)} \tilde{R}(x,|u_j|)\right)^{\frac{1}{\nu}} \left(\int_{Q_j(\delta)} |v_j|^{2\nu'}\right)^{\frac{1}{2\nu'}} \\
&\le C_2\epsilon + C\left(\int_{Q_j(\delta)} |v_j|^{2\nu'}\right)^{\frac{1}{2\nu'}} = C_2\epsilon + o(1) \qquad (4.12)
\end{aligned}$$

as $j \to \infty$, where $2\nu' = \frac{2\nu}{\nu-1} < 3$.

(4.10) and (4.12) shows a contradiction, which implies the conclusion holds true. □

By Lemma 4.4, any $(C)_c$-sequenc (u_j) is bounded, hence along a subsequence also denoted by $(u_j), u_j \rightharpoonup u_M$. It is obvious that u_M is a critical point of Φ_M. Moreover there holds the following

Lemma 4.5 *Either*

(1) $u_j \to u_M$, *or*
(2) $c \ge \hat{C}$ *and there exist a positive integer* ℓ, *points* $\bar{u}_1, \ldots, \bar{u}_\ell \in \mathcal{K}\backslash\{0\}$, *a subsequence denoted again by* (u_j), *and sequences* $(a_j^i) \subset \mathbb{Z}^3$, *such that, as* $j \to \infty$,

$$\left\| u_j - u_M - \sum_{i=1}^{\ell}(a_j^i * \bar{u}_i)\right\|_M \to 0,$$

$$|a_j^i| \to \infty, \quad |a_j^i - a_j^k| \to \infty \quad \text{if } i \ne k$$

and

$$\Phi_M(u_M) + \sum_{i=1}^{\ell} \Phi(\bar{u}_i) = c.$$

Proof The proof is well known (see, e.g., [14]), which can be outlined as follows.

It is obvious (u_j) is a $(PS)_c$-sequence and

$$c \leftarrow \Phi_M(u_j) - \frac{1}{2}\Phi_M'(u_j)u_j = \int_{\mathbb{R}^3} \tilde{R}(x,u_j) \ge 0,$$

Assume (1) is false. It is easy to see that $u_j^1 := u_j - u_M$ is a $(PS)_{c_0}$-sequence for Φ with $c_0 = c - \Phi_M(u_M)$ and $u_j^1 \rightharpoonup 0$. Since Φ is invariant under the $*$-action of $\mathbb{R}^3$ by the concentration compactness principle, there exist a sequence $(a_j^1) \subset \mathbb{R}^3$ with $|a_j^1| \to \infty$ and a critical point $\bar{u}_1 \neq 0$ of Φ satisfying $a_j^1 * u_j^1 \rightharpoonup \bar{u}_1$ and

$$\Phi(a_j^1 * u_j^1) \to c - \Phi_M(u_M) - \Phi(\bar{u}_1) \geq 0.$$

Since $\Phi_M(u_M) \geq 0$ and $\Phi(\bar{u}_1) \geq \hat{C}$, one sees that $c \geq \hat{C}$.

If $a_j^1 * u_j^1 \to \bar{u}_1$, the proof is completed. Otherwise, repeating the above argument, after at most finitely many steps we finish the proof. □

As a straight consequence of Lemma 4.5 we have

Lemma 4.6 *Φ_M satisfies the $(C)_c$-condition for all $c < \hat{C}$.*

By Theorem 2.3 and Lemma 4.6, in order to obtain nontrivial least energy solutions of (1.1), we only have to prove the linking level $\sup \Phi_M(Q) < \hat{C}$.

Let $U_n \in \hat{S}_n$ for $n = 0, I, II$. Set $e := U_n^+$ and $E_e := E^- \oplus \mathbb{R}e$.

Lemma 4.7 $d := \sup\{\Phi_M(u) : u \in E_e\} < \hat{C}$.

Proof See [10]. We outline it as follows.

By lemma 4.2 and the linking property we have $d \geq \rho$.

Assume (M_1) is satisfied. Since $M(x) < 0$, $\Phi_M(u) \leq \Phi_0(u)$ for all $u = v + sU_0^+$, and

$$\begin{aligned}\Phi_0(u) &= \Phi_0(v + sU_0^+) \leq \Phi_0(sU_0^+ + h_0(sU_0^+)) \\ &= I_0(sU_0^+) \leq I_0(U_0^+) \\ &= \Phi_0(U_0) = \hat{C}_0.\end{aligned}$$

Hence $d \leq \hat{C}_0$. Assume by contradiction that $d = \hat{C}_0$. Let $w_j = v_j + s_j U_0^+ \in E_e$ be such that $d - \frac{1}{j} \leq \Phi_M(w_j) \to d$. It follows from Lemma 4.3 that (w_j) is bounded and we can assume that $w_j \rightharpoonup w$ in E with $v_j \rightharpoonup v \in E^-$ and $s_j \to s$. It is clear that $s > 0$ (otherwise $d = \hat{C}_0 = 0$, a contradiction). Then

$$\begin{aligned}d - \frac{1}{j} \leq \Phi_M(w_j) &\leq \Phi_0(w_j) + \frac{1}{2}\int_{\mathbb{R}^3} M(x) w_j \bar{w}_j \\ &\leq \hat{C}_0 + \frac{1}{2}\int_{\mathbb{R}^3} M(x) w_j \bar{w}_j.\end{aligned}$$

Taking the limit one has $\hat{C}_0 \leq \hat{C}_0 + \frac{1}{2}\int_{\mathbb{R}^3} M(x) w\bar{w}$. Hence $w = 0$, a contradiction.

If $(M_2)(1)$ holds, for $u = v + sU_I^+ \in E_e, \Phi_M(u) \leq \Phi_I(u) \leq \hat{C}_I$, hence $d \leq \hat{C}_I$. Just as above, from

$$\Phi_M \leq \Phi_I(u) + \frac{1}{2}\int_{\mathbb{R}^3} (M(x) - M(\infty)) u\bar{u},$$

we can see $d < \hat{C}_I$.

Similarly, if $(M_2)(2)$ appears, we can check $d < \hat{C}_{II}$. □

Set

$$Q_n := \{u = u^- + sU_n^+ : u^- \in E^-, s \geq 0, \|u\| < R\}, n = 0, I, II.$$

As a consequence of Lemma 4.7 one has

Lemma 4.8 $\sup \Phi_M(Q_n) < \hat{C}$ *for* $n = 0, I, II$.

We now in a position to complete the proof of Theorem 1.1.

(Existence of Least Energy Solutions) By Lemma 4.1–Lemma 4.8, there exists a $(C)_c$-sequence (u_j) with $\rho \le c \le \sup \Phi_M(Q_n) < \hat{C}$ and $u_j \to u$ as $j \to \infty$. Then $\Phi'_M(u) = 0$ and $\Phi_M(u) \ge \rho$. Therefore $\mathcal{K}\backslash\{0\} \ne \emptyset$.

Recall $c_M := \inf\{\Phi_M(u) : u \in \mathcal{K}_M\backslash\{0\}\}$. Along the same lines of proof of lemma 3.2, one can check that $c_M > 0$. Let (u_j) satisfy that $\Phi_M(u_j) \to c_M, \Phi'(u_j) = 0$. Since $c_M < \hat{C}$, we have $u_j \to u$ in E with $\Phi_M(u) = c_M$ and $\Phi'_M(u) = 0$, hence $S_M \ne \emptyset$. □

In order to show the compactness of S_M, we need the following result which can be proved using the same iterative argument of [6, proposition 3.2].

Lemma 4.9 *If* $u \in \mathcal{K}_M$ *with* $|\Psi_M(u)| \le C_1$ *and* $|u|_2 \le C_2$, *then for any* $q \in [2, \infty)$, $u \in W^{1,q}(\mathbb{R}^3)$ *with* $\|u\|_{W^{1,q}} \le \Lambda_q$, *where* Λ_q *depends only on* C_1, C_2 *and* q.

For any $(u_j) \subset S_M$, one has $\Phi_M(u_j) = c_M$ and $\Phi'_M(u_j) = 0$, which implies (u_j) is a $(C)_{c_M}$-sequence. By Lemma 4.4 (u_j) is bounded and henceforth S_M is bounded in E. By Lemma 2.2, $|u|_s \le C_s$ for all $u \in S_M, s \in [2, 3]$, and then from (4.4) one can see $|\Psi_M| \le C_1$ for some $C_1 > 0$. By Lemma 4.9, for each $q \in [2, \infty)$, there is $\Lambda_q > 0$ such that

$$\|u\|_{W^{1,q}} \le \Lambda_q \quad \text{for all } u \in S_M.$$

By the Sobolev embedding theorem, there is $C_\infty > 0$ such that

$$|u|_\infty \le C_\infty \quad \text{for all } u \in S_M. \tag{4.13}$$

(Compactness of S_M) Let $(u_j) \subset S_M$, then (u_j) is a $(C)_{c_M}$-sequence. Since $c_M < \hat{C}$, it follows from Lemma 4.6 that $u_j \to u$ in E along a subsequence. Obviously, $u \in S_M$. By

$$H_0 u_j = -M(x)u_j + g(x, |u_j|)u_j$$

and

$$H_0 u = -M(x)u + g(x, |u|)u$$

one has

$$\begin{aligned} |H_0(u_j - u)|_2 &\le |M(u_j - u)|_2 + |g(\cdot, |u_j|)u_j - g(\cdot, |u|)u|_2 \\ &\le o(1) + |g(\cdot, |u_j|)(u_j - u)|_2 + |(g(\cdot, |u_j|) - g(\cdot, |u|))u|_2. \end{aligned}$$

By 4.13, $|u_j|_\infty \le C_\infty$ and $u_j \to u$ in E,

$$\int_{\mathbb{R}^3} |(g(x, |u_j|) - g(x, |u|))u|^2 = \left(\int_{|x|<R} + \int_{|x|\ge R} \right) |(g(x, |u_j|) - g(x, |u|))u|^2 \to 0.$$

Therefore, one obtains $|H_0(u_j - u)|_2 \to 0$, i.e., $u_j \to u$ in H^1. □

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