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Existence for the Thermoviscoelastic Thermistor Problem

K. L. Kuttler, M. Shillor* and J. R. Fernández

Abstract

The existence of a weak solution to a dynamic model for a thermistor, which takes into account the thermoelastic properties of the device, is established. The model consists of a coupled system of the equations of dynamic thermoviscoelasticity, the heat equation with the Joule heating term, and the quasistatic charge conservation equation. The system is strongly nonlinear since the electrical conductivity is assumed to be temperature dependent, and the Joule heating term is quadratic in the gradient of the electric potential. The existence of a solution is obtained by considering a sequence of approximate time-retarded problems. After obtaining the necessary a priori estimates, a solution of the problem is found by passing to the approximation limit. The uniqueness of the solution remains an open problem.

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* Corresponding author

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1. Introduction

The "Thermistor Problem," is a model for the combined effects of heat and electric current conduction and Joule's heat generation in a device, the thermistor, made of a material, often a ceramic, which has temperature-dependent electrical conductivity. Such materials are being used in switches and electric surge protecting devices, among other applications. It was brought to the Oxford Study Groups with Industry in the mid 80's and has received, as a result, considerable attention in the mathematical literature, see, e.g., [1, 2, 3, 5, 7, 8, 9, 12, 15, 16, 19, 24, 25, 26, 28] and references therein. Related problems can be found in [21] and [27, 14]. However, in all these references the thermomechanical properties of the device were not taken into account.

A model, which includes the thermomechanical effects of the processes, has been constructed in the companion paper [13]. There, in addition to the model, a numerical algorithm was developed and computer simulations obtained and described. The model consists of a nonlinear coupled system of partial differential equations which includes the heat equation, the electric charge conservation equation and the equations of motion of a thermoviscoelastic medium. From the mathematical point of view the system is somewhat unusual since it consists of an elliptic equation, a parabolic equation and three hyperbolic equations. In this work we establish the existence of a weak solution for the model with thermomechanical effects. Our interest lies in the general problem, and we assume that the electrical conductivity of the material does not vanish, and thus, the mathematical problem is nondegenerate. In the publications ([14, 24, 25, 26, 27]) the degenerate problem has been investigated, which makes it necessary to use the so-called 'capacity solutions' that take into account the vanishing of the electrical conductivity. This, in turn, causes the degeneration of the elliptic equation.

The problem considered in this work is nonlinear, since the electrical conductivity is assumed to depend on the temperature, and the electric heating term, the so-called Joule heating, is quadratic in the electric current.

The paper is organized as follows. The classical model, taken from [13], is presented in Section 2. The weak formulation is derived in Section 3, the assumptions on the problem data specified and the existence result stated in Theorem 3.3. The proof is provided in Section 4, and is based on a sequence of regularized and time-retarded problems. The solutions of the approximate problems follow from the recent results in the theory of set-valued pseudo-monotone operators of [18] and a fixed point argument.

The steady states and the quasistatic problems are discussed shortly in Section 5. Since the steady problem decouples, the existence of a solution follows from known results in the literature. The quasistatic problem is stated, too.

The uniqueness or stability of the solutions for the problem remain unresolved problems. One also may consider the degenerate case when the electrical conductivity vanishes above some prescribed temperature. Finally, the piezoelectric effects may be important, and need to be taken into account.

We note that after this work was submitted, Wu and Xu published the paper [23], which deals with this problem. There, the existence of a very weak solution, the so-called *capacity solution*, was established using a different method of proof. Their main interest was in the case when the electric conductivity vanishes at some high temperature.

2. Classical model

We begin with the description of the model, following [13], where more details can be found. Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$, representing the isothermal reference configuration of the thermistor. We assume that Γ is divided into two relatively open parts Γ_D and Γ_N such that $\Gamma_D \cap \Gamma_N = \emptyset$, Γ_D has positive measure and $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma$. We denote by $\mathbf{n} = (n_1, \ldots, n_d)$ the outward unit normal to Ω on Γ. The setting is depicted in Fig. 1.

The body is held fixed on Γ_D, while on Γ_N it is free. We could have chosen, as in [16], three different ways to divide Γ and then to specify the boundary conditions for the temperature, electrical potential and displacements on each division. However, for the sake of simplicity we assume that the Dirichlet condition holds on Γ_D and the Neumann condition on Γ_N.

We let θ denote the temperature field, ϕ the electric potential and $u = (u_1, \ldots, u_d)$ the displacements field. Let $T > 0$ and set $\Omega_T = \Omega \times (0, T)$.

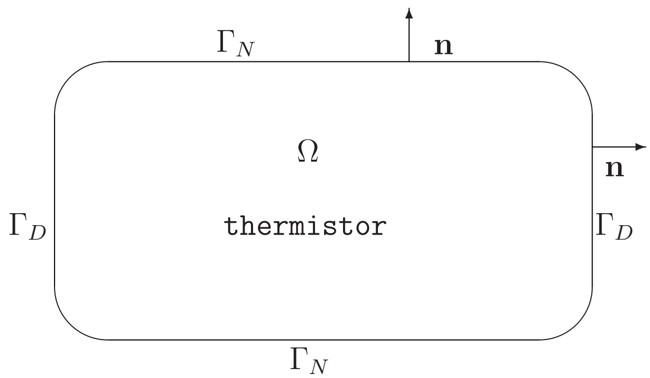

Fig. 1. The setting

The behavior of the system is governed by the energy equation, the equation of charge conservation and the equations of linear thermoviscoelasticity. We may write the system (see, e.g., [4, 6, 11] where the thermoviscoelastic system can be found and any of the references above for a model of the thermistor) as:

$$\rho c_p \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x_j}\left(k_{ij}(\theta) \frac{\partial \theta}{\partial x_i} \right) = \sigma_{el}(\theta)|\nabla \phi|^2 - m_{ij}\Theta_{ref} \frac{\partial^2 u_i}{\partial t \partial x_j}, \tag{2.1}$$

$$\nabla \cdot (\sigma_{el}(\theta)\nabla\phi) = 0, \tag{2.2}$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}(\sigma_{ij}) = f_i, \tag{2.3}$$

$$\sigma_{ij} = a_{ijkl}\frac{\partial u_k}{\partial x_l} + b_{ijkl}\frac{\partial^2 u_k}{\partial t \partial x_l} - m_{ij}\theta. \tag{2.4}$$

Here and below, $i, j, k, l = 1, \ldots, d$ and summation over repeated indices is implied.

Equation (2.1) is the energy equation expressed in terms of the temperature θ which is measured with respect to a reference absolute temperature Θ_{ref}, given in degrees Kelvin. We assume that Θ_{ref} is also the ambient temperature. The material density ρ and the heat capacity c_p are assumed to be positive constants. $K = K(\theta) = \{k_{ij}(\theta)\}$ and $M = \{m_{ij}\}$ are the heat conduction and thermal expansion tensors, respectively, and for the sake of generality it is assumed that the thermal and mechanical properties of the material are anisotropic. The electrical conductivity $\sigma_{el} = \sigma_{el}(\theta)$ is assumed to depend strongly on the temperature, and will be discussed below. However, we assume that the electrical properties of the material are isotropic. All the results below hold for the case when $\sigma_{el} = \{\sigma_{el,ij}(\theta)\}$ is a tensor. Next, we recall that $I = \sigma_{el}(\theta)\nabla\phi$ is the electric current density, and $J = \sigma_{el}(\theta)|\nabla\phi|^2$ is the Joule heating, the power generated by the electric current.

We note that in [6] the nonlinear dissipation term $\varepsilon' B\varepsilon'$ has been retained in (2.1), and Θ_{ref} was replaced by θ. Such a system may be investigated in the future. The problem above contains the main nonlinearity we are interested in.

Equation (2.2) represents the electric charge conservation, assuming that the only relevant electromagnetic effect is the quasistatic evolution of the electric potential.

Next, (2.3) are the equations of motion of a thermoviscoelastic material, and (2.4) is the constitutive relation. Here, $A = \{a_{ijkl}\}$ is the elasticity tensor; $B = \{b_{ijkl}\}$ is the tensor of viscosity coefficients; $\mathbf{f} = (f_1, \ldots, f_d)$ represents the density of body forces, such as gravity. If we let ε be the linearized strain tensor, then we may write the constitutive equation (2.4) as

$$\sigma = A\varepsilon + B\varepsilon' - M\theta,$$

where a prime above a symbol represents a time derivative. We note that thermistors are usually made of ceramics which exhibit very little viscosity. We use it here for mathematical reasons, and from the practical point of view one may take the viscosity as small as one wishes.

Piezoelectric effects, i.e., effects related to the coupling between the mechanical strain or stress fields and the electric field, are neglected in this model.

Next, we discuss shortly the electrical conductivity σ_{el}. We note that the results in this article apply to any material with temperature dependent electrical conductivity, and thermistors are devices of this type, characterized by a sharp decrease in the electrical conductivity with raising temperature. In a thermistor, a typical dependence of the conductivity on the temperature is depicted in Fig. 2.

In some publications (see, e.g., [25, 26] and references therein) the degenerate case was investigated, where σ_{el} was assumed to vanish above a critical temperature. This is an idealization of the material behavior which introduces considerable mathematical difficulties, and will not be pursued in this work. Indeed, when $\sigma = 0$ the heating term in (2.1) vanishes and the equation for ϕ, (2.2), degenerates. This makes it necessary to consider the so-called *capacity solutions* of the problem, and we refer the reader to [25, 26] and references therein for further details.

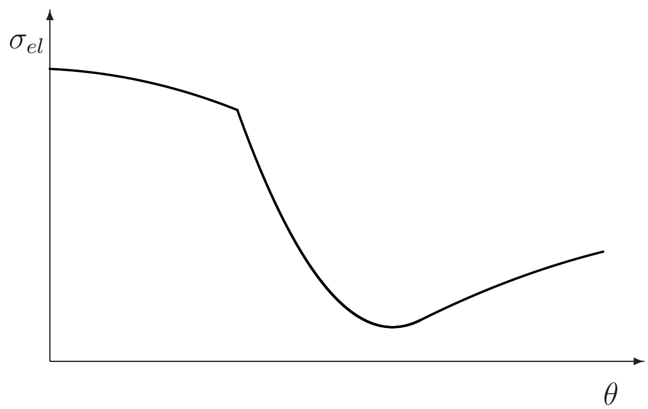

Fig. 2. σ_{el} as a function of the temperature

The boundary conditions for the problem are,

$$u = 0, \quad \theta = \theta_b, \quad \phi = \phi_b \qquad \text{on } \Gamma_D, \quad 0 < t < T, \tag{2.5}$$

since the body is held fixed on Γ_D, and the temperature and potential given. The body exchanged heat with its surroundings, is stress free and insulated on Γ_N, thus,

$$-k_{ij}\frac{\partial\theta}{\partial x_i}n_j = h\theta, \;\; \sigma_{ij}n_j = 0, \quad \frac{\partial\phi}{\partial n} = 0 \;\; \text{on } \Gamma_N, \quad 0 < t < T. \tag{2.6}$$

Here, the applied electric potential is ϕ_b and the temperature of the environment is assumed $\theta_b = 0$. The initial conditions are,

$$u = u_0, \quad u_t = v_0, \quad \theta = 0 \qquad \text{in } \Omega, \qquad t = 0, \tag{2.7}$$

where u_0, v_0 are the initial displacements and velocities, respectively, and $\theta_0 = 0$ is the initial temperature.

The classical formulation of the *thermoviscoelastic thermistor problem* is:

Find a triplet $\{u, \phi, \theta\}$ such that (2.1) – (2.7) hold.

In the next section we investigate the weak formulation of the problem.

3. Weak formulation

We present the assumptions on the problem data, derive a weak formulation for problem (2.1)–(2.7) and state our existence result.

We begin with the assumptions on the data.

Assumptions 3.1. *We assume that:*

(i) There exists $\Phi \in W^{1,4}(\Omega)$ *such that* $\gamma\Phi = \phi_b$ *on* Γ_D.

(ii) The electrical conductivity $\sigma_{el}(\cdot)$ *is Lipschitz continuous and satisfies*

$$0 < \sigma_* \le \sigma_{el}(\cdot) \le M, \tag{3.1}$$

for some constants σ_* *and* M.

(iii) The coefficients of thermal conductivity k_{ij} *are bounded and Lipschitz continuous, satisfying*

$$k_{ij}\xi_i\xi_j \ge \delta |\xi|^2, \tag{3.2}$$

for $0 < \delta$.

(iv) The elasticity and viscosity coefficients satisfy

$$a_{ijkl},\, b_{ijkl} \in L^\infty(\Omega),$$

$$a_{ijkl} = a_{jikl} = a_{klij}, \qquad b_{ijkl} = b_{jikl} = b_{klij},$$

and

$$a_{ijkl}\zeta_{ij}\zeta_{kl} \ge \delta|\zeta|^2, \qquad b_{ijkl}\zeta_{ij}\zeta_{kl} \ge \delta|\zeta|^2,$$

for all symmetric matrices ζ_{ij}.

(v) $u_0 \in V$, $v_{0i,j} \in L^2(\Omega)$ *and* $\theta_0 \in V$.

(vi) There exists $\Theta \in H^1(\Omega)$ *such that* $\gamma\Theta = \theta_b$ *on* Γ_D.

Here, γ is the trace operator, and $|\cdot|$ denotes the Frobenius norm of a matrix.

Next, we derive a weak formulation of (2.1)–(2.7). To that end we let

$$V \equiv \left\{\eta \in H^1(\Omega) : \gamma\eta = 0 \text{ on } \Gamma_D\right\}, \tag{3.3}$$

which is the Hilbert space of test functions where we will seek the temperature and the electric potential. By assumption, Γ_D has positive measure and, therefore, we may use on V the norm

$$||\psi||_V \equiv ||\nabla\psi||_{L^2(\Omega)} = \left(\int_\Omega |\nabla\psi|^2\, dx\right)^{1/2}. \tag{3.4}$$

We begin with equation (2.2) for the electric potential. We define $\varphi \in V$ by

$$\varphi + \Phi = \phi.$$

Then, letting $\eta \in V$ and multiplying both sides of (2.2) by η and then integrating by parts yields

$$\int_\Omega \sigma_{el}(\theta)\, \nabla\varphi \cdot \nabla\eta dx = -\int_\Omega \sigma_{el}(\theta)\, \nabla\Phi \cdot \nabla\eta dx. \tag{3.5}$$

Thus, the variational form for (2.2) is to find $\varphi \in V$ such that (3.5) holds for all $\eta \in V$. We shall need the following lemma providing a bound on the solutions to (3.5).

Lemma 3.1. *Suppose φ is a solution to* (3.5)*. Then, there exists a constant C, depending only on σ_* and M in* (3.1)*, such that*

$$\int_\Omega \sigma_{el}(\theta)\, \varphi^2 |\nabla\varphi|^2 \le C. \tag{3.6}$$

Proof: Let Ψ_n be a strictly increasing, bounded, smooth function satisfying $\Psi_n(r) = r^3/3$ whenever $|r| < n$ and $\Psi_n'(r) \uparrow r^2$. We choose $\eta = \Psi_n(\varphi)$ in (3.3). Thus,

$$\begin{aligned} \int_\Omega \sigma_{el}(\theta)\, \nabla\varphi \cdot \nabla\varphi\, \Psi_n'(\varphi) &= -\int_\Omega \sigma_{el}(\theta)\, \nabla\Phi \cdot \nabla\varphi\, \Psi_n'(\varphi) \\ &\le \left(\int_\Omega \sigma_{el}(\theta)\, \Psi_n'(\varphi)\, |\nabla\Phi|^2\right)^{1/2} \\ &\quad \times \left(\int_\Omega \sigma_{el}(\theta)\, \Psi_n'(\varphi)\, |\nabla\varphi|^2\right)^{1/2}, \end{aligned}$$

and dividing both sides by $\left(\int_\Omega \sigma_{el}(\theta)\,\Psi_n'(\varphi)\,|\nabla\varphi|^2\right)^{1/2}$ yields

$$\begin{aligned}\int_\Omega \sigma_{el}(\theta)\,\Psi_n'(\varphi)\,|\nabla\varphi|^2 &\leq \int_\Omega \sigma_{el}(\theta)\,\Psi_n'(\varphi)\,|\nabla\Phi|^2 \\ &\leq M\left(\int_\Omega \Psi_n'(\varphi)^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla\Phi|^4\right)^{1/2} \\ &\leq C\left(\int_\Omega \varphi^4 dx\right)^{1/2} \\ &\leq C\,||\varphi||_{H^1}^2 .\end{aligned}$$

It follows from (3.1) and (3.5) that there exists a bound on $||\varphi||_{H^1}^2$ depending only on σ_* and M. Therefore, by adjusting the constants, we find that

$$\int_\Omega \sigma_{el}(\theta)\,\Psi_n'(\varphi)\,|\nabla\varphi|^2 \leq C,$$

and by letting $n\to\infty$ and using the monotone convergence theorem we obtain (3.6).

Let

$$E \equiv \left\{u \in H^1(\Omega)^d : u = 0 \text{ on } \Gamma_D\right\}. \tag{3.7}$$

As in the case of V, we choose the following norm on E,

$$||u||_E \equiv ||\nabla u||_{L^2(\Omega)} = \left(\int_\Omega u_{i,j}u_{i,j}dx\right)^{1/2}. \tag{3.8}$$

We define the operators $A_d, B_d : E \to E'$ by

$$\langle A_d u,\psi\rangle \equiv \int_\Omega a_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial \psi_i}{\partial x_j}dx, \quad \langle B_d u,\psi\rangle \equiv \int_\Omega b_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial \psi_i}{\partial x_j}dx. \tag{3.9}$$

Next, let $\Theta \in H^1(\Omega)$ be given by Assumption 3.1.(v), and let $\theta = z+\Theta$, thus $z = 0$ on Γ_D. We define $L_d : V \to E'$ by

$$\langle L_d z,\eta\rangle \equiv -\int_\Omega m_{ij}\theta\eta_{i,j}dx. \tag{3.10}$$

Then, the partial differential equation (2.3), along with its boundary conditions, can be set in an abstract form as

$$\rho v' + A_d u + B_d v + L_d \theta = \mathbf{f}, \tag{3.11}$$

$$v(0) = v_0, \tag{3.12}$$

$$u(t) = u_0 + \int_0^t v(s)\, ds. \tag{3.13}$$

Consider now the second term on the left-hand side in (2.1), and let $\eta \in V$. Then, integrating by parts and the use of condition (2.6), yield

$$\begin{aligned} -\int_\Omega (k_{ij}\theta_{,i})_{,j}\, \eta dx &= -\int_{\Gamma_N} k_{ij}\theta_{,i}\eta n_j\, dS + \int_\Omega k_{ij}\theta_{,i}\eta_{,j} dx \\ &= \int_{\Gamma_N} h\theta\eta\, dS + \int_\Omega k_{ij}\theta_{,i}\eta_{,j} dx \\ &= \int_{\Gamma_N} h\theta\eta\, dS + \int_\Omega k_{ij}\Theta_{,i}\eta_{,j} dx + \int_\Omega k_{ij} z_{,i}\eta_{,j} dx. \end{aligned}$$

To place it in an abstract setting we define the following operators. For $z, \eta \in V$ let

$$\langle P(z), \eta\rangle \equiv \int_{\Gamma_N} h\theta\eta\, dS, \tag{3.14}$$

$$\langle A(z)(z), \eta\rangle \equiv \int_\Omega k_{ij}(z+\Theta)\, z_{,i}\eta_{,j} dx, \tag{3.15}$$

$$\langle f_1, \eta\rangle \equiv -\int_\Omega k_{ij}\Theta_{,i}\eta_{,j} dx, \tag{3.16}$$

$$\langle L(z)(\varphi), \eta\rangle \equiv \int_\Omega \sigma_{el}(z+\Theta)\, \nabla\varphi \cdot \nabla\eta dx, \tag{3.17}$$

$$\langle N(z)\varphi, \eta\rangle \equiv \int_\Omega \sigma_{el}(z+\Theta)\, |\nabla(\varphi+\Phi)|^2\, \eta dx, \tag{3.18}$$

$$\langle G(\mathbf{v}), \eta\rangle \equiv -\int_\Omega m_{ij}\Theta_{ref} v_{i,j}\eta dx. \tag{3.19}$$

In the last definition, we assume that $v_{i,j} \in L^2(\Omega)$ for all $i, j \in \{1, \ldots, d\}$ is a given function. Then, in terms of these operators, the first two equations of (2.1) and (2.2) along with the initial and boundary conditions reduce to finding z and φ such that,

$$\begin{aligned}
&c_p \rho z' + A(z)z + P(z) = f_1 + N(z)(\varphi) + G(v), \\
&z(0) = \theta_0 - \Theta, \\
&L(z)\varphi = -L(z)\Phi,
\end{aligned}$$

together with the balance of momentum which in abstract form is given by (3.11)–(3.13).

Any solution of these abstract equations is a *weak solution* to problem (2.1)–(2.7), upon taking measurable representatives.

We shall use the following notation.

$$\begin{aligned}
&\mathcal{V} \equiv L^2(0,T;V), \qquad \mathcal{U} \equiv L^4\left(0,T; V \cap W^{1,4}(\Omega)\right), \\
&H \equiv L^2(\Omega), \qquad \mathcal{H} \equiv L^2(0,T;H), \qquad \mathcal{E} \equiv L^2(0,T;E).
\end{aligned}$$

All these operators are also considered as acting on the various Lebesgue spaces according to the following convention: $Bu(t) \equiv B(u(t))$.

The dual spaces are,

$$\mathcal{V}' = L^2(0,T;V'), \; \mathcal{U}' = L^{4/3}\left(0,T; \left(V \cap W^{1,4}(\Omega)\right)'\right), \; \mathcal{E}' = L^2(0,T;E').$$

The main result in this work is the existence of weak solutions for the problem.

Theorem 3.2. *Under the Assumptions* 3.1., *there exists a solution* (z, φ, u), *with* $z, \varphi \in \mathcal{V}$ *and* $v \in \mathcal{E}$, $v' \in \mathcal{E}'$, *to the problem*

$$\rho c_p z' + A(z)z + P(z) = f_1 + N(z)(\varphi) + G(v) \quad in \quad \mathcal{U}', \tag{3.20}$$

$$\rho v' + A_d u + B_d v + L_d \theta = \mathbf{f} \quad in \quad \mathcal{E}', \tag{3.21}$$

$$v(0) = v_0, \quad u(t) = u_0 + \int_0^t v(s)\,ds, \tag{3.22}$$

$$z(0) = z_0 \equiv \theta_0 - \Theta \quad in \quad H, \tag{3.23}$$

where

$$L(z)\varphi = -L(z)\Phi \quad in \;\; \mathcal{V}. \tag{3.24}$$

The proof will be given in the next section.

4. Approximate problems and existence

In this section we consider a sequence of approximations based on regularization and time retardation of problem (3.20)–(3.24). After obtaining the necessary a priori estimates, by passing to the limit a solution to the full problem is found.

We begin with some preliminary results. We note that in some treatments of the thermistor problem, see, e.g., [5], use is made of the maximum principle to help deal with the quadratic source term described in $N(z)(\varphi)$. Since the boundary conditions here do not adapt to this approach, we will use the following lemma instead.

Lemma 4.1. *For $\eta \in V \cap L^\infty(\Omega)$ there holds,*

$$\begin{aligned}\langle N(z)(\varphi), \eta\rangle &= -\int_\Omega \sigma_{el}(\theta)\,\varphi\nabla\Phi\cdot\nabla\eta dx - \int_\Omega \sigma_{el}(\theta)\,\varphi\nabla\varphi\cdot\nabla\eta dx \\ &\quad + \int_\Omega \sigma_{el}(\theta)\,\nabla\Phi\cdot\nabla\varphi\eta dx + \int_\Omega \sigma_{el}(\theta)\,|\nabla\Phi|^2\,\eta dx. \qquad (4.1)\end{aligned}$$

Moreover, for such η there exists a constant $C(\Phi)$, independent of φ and η, but dependent on Φ, such that

$$|\langle N(z)(\varphi), \eta\rangle| \le C(\Phi)\,||\eta||_V\,. \qquad (4.2)$$

Proof: For $\eta \in V$ we have,

$$\begin{aligned}\langle N(z)(\varphi), \eta\rangle = &\int_\Omega \sigma_{el}(\theta)\,\nabla\varphi\cdot\nabla\varphi\eta dx + 2\int_\Omega \sigma_{el}(\theta)\,\nabla\varphi\cdot\nabla\Phi\eta dx \\ &+ \int_\Omega \sigma_{el}(\theta)\,|\nabla\Phi|^2\,\eta dx. \end{aligned} \qquad (4.3)$$

We begin with the first term on the right-hand side when $\eta \in V \cap L^\infty(\Omega)$. For such an η, $\varphi\eta \in V$, and (3.24) implies

$$\begin{aligned}&\int_\Omega \sigma_{el}(\theta)\,\nabla\varphi\cdot\nabla\varphi\eta dx \\ &\qquad = \int_\Omega \sigma_{el}(\theta)\,\nabla\varphi\cdot\nabla(\varphi\eta)\,dx - \int_\Omega \sigma_{el}(\theta)\,\nabla\varphi\cdot\nabla(\eta)\varphi dx \\ &\qquad = -\int_\Omega \sigma_{el}(\theta)\,\nabla\Phi\cdot\nabla(\varphi\eta)\,dx - \int_\Omega \sigma_{el}(\theta)\,\nabla\varphi\cdot\nabla\eta\varphi dx\end{aligned}$$

$$= -\int_\Omega \sigma_{el}(\theta)\,\eta\nabla\Phi\cdot\nabla\varphi dx - \int_\Omega \sigma_{el}(\theta)\,\varphi\nabla\Phi\cdot\nabla\eta dx$$
$$-\int_\Omega \sigma_{el}(\theta)\,\nabla\varphi\cdot\nabla\eta\varphi dx.$$

Substituting this into (4.3) yields (4.1).

Now using (4.1) yields

$$\begin{aligned} |\langle N(z)(\varphi),\eta\rangle| \quad \leq \quad & C\left(\int_\Omega |\nabla\Phi|^2 |\varphi|^2 dx\right)^{1/2}\left(\int_\Omega |\nabla\eta|^2 dx\right)^{1/2} \\ & +C\left(\int_\Omega \sigma_{el}(\theta)\,\varphi^2 |\nabla\varphi|^2 dx\right)^{1/2}\left(\int_\Omega |\nabla\eta|^2 dx\right)^{1/2} \\ & +C\left(\int_\Omega |\nabla\Phi|^2 \eta^2 dx\right)^{1/2}\left(\int_\Omega |\nabla\varphi|^2 dx\right)^{1/2} \\ & +C(\Phi)\,||\eta||_{L^2}. \end{aligned}$$

Also, it follows from (3.24) and Assumptions 3.1.(ii) that there exists a constant $C(\Phi)$, which is independent of φ, η and z, such that $||\varphi||_V < C$. Therefore, the above inequality yields (4.2).

We turn now to the regularized and time retarded approximations. To that end, for a function g defined on $[0,T]$ and for $h \in (0,T)$, we denote by g_h the function defined by

$$g_h(t) = \begin{cases} g(t-h) & \text{if} \quad t > h, \\ g_0 \quad \text{if} \quad t \in [0,h], \end{cases}$$

where g_0 will be described shortly. Also, define $F : V \cap W^{1,4}(\Omega) \to (V \cap W^{1,4}(\Omega))'$ by

$$\langle Fu,\eta\rangle \equiv \int_\Omega |\nabla u|^2 \nabla u\cdot\nabla\eta dx.$$

Consider the following regularized and time retarded version of (3.20)-(3.24).

Given a small $h > 0$, find $z^h \in L^2(0,T; V \cap W^{1,4}(\Omega))$ with $(z^h)' \in$

$\mathcal{U}'$, $\varphi^h \in \mathcal{V}$, $v^h \in \mathcal{E}$ with $(v^h)' \in \mathcal{E}'$, such that

$$\rho c_p (z^h)' + A\left(z_h^h\right) z^h + P\left(z^h\right) + hFz^h = f_1 + N\left(z_h^h\right)\left(\varphi_h^h\right) + G\left(v_h^h\right), \quad (4.4)$$

$$z^h(0) = \theta_0 - \Theta, \quad (4.5)$$

$$L\left(z^h\right)\varphi^h = -L\left(z^h\right)\Phi, \quad (4.6)$$

$$\rho (v^h)' + A_d u^h + B_d v^h + L_d z_h^h = \mathbf{f}, \quad (4.7)$$

in which $z_h^h = z^h(t-h)$, and similarly $\varphi_h^h = \varphi^h(t-h)$, and also $z_0 \equiv \theta_0 - \Theta$, and φ_0 is the solution to

$$L(z_0)\varphi_0 = -L(z_0)\Phi. \quad (4.8)$$

For later reference we set $\theta_h^h \equiv z_h^h + \Theta$.

First, we note that there exists a solution to (4.4)-(4.7) on the interval $[0,h]$. This follows from standard results for linear systems together with the observation that $N(z_h^h)(\varphi_h^h)$ and $G(v_h^h)$ are known. It is also an easy consequence of the main existence theorem in [18]. In this way φ^h and z^h are determined on $[0,h]$. In fact φ^h is the solution to (4.6) where z^h solves (4.4) while $\varphi_h^h = \varphi_0$ is defined in (4.8) on this time interval. Now, on the next interval $[h,2h]$ we have that $N\left(z_h^h\right)\left(\varphi_h^h\right)$ is, again, known and so there exists a solution on this interval having the appropriate initial condition coming from the solution on $[0,h]$. Continuing in this way we obtain a solution to (4.4)–(4.7) on $[0,T]$.

To proceed, we need a priori estimates on these solutions.

We begin with an estimate on z^h in the space X, given by

$$X \equiv \{z \in \mathcal{V} : \ z' \in \mathcal{U}'\}, \quad ||z||_X \equiv ||z||_{\mathcal{V}} + ||z'||_{\mathcal{U}'}, \quad (4.9)$$

which is a Banach space since $V \cap W^{1,4}(\Omega)$ is dense in V.

To simplify the notation, we omit the superscript h, and will restore it when needed. It follows from (4.4) and Lemma 4.1., that there exists a constant C, which is independent of h and $t \in [0,T]$, such that

$$\frac{1}{2}\rho c_p |z(t)|_H^2 - \frac{1}{2}\rho c_p |z_0|_H^2 + \delta \int_0^t ||z(s)||_V^2 \, ds \leq C + \frac{1}{2}\delta \int_0^t ||z(s)||_V^2 \, ds + C\int_0^t |z(s)|_H^2 \, ds + C\int_0^t ||v_h(s)||_E^2 \, ds.$$

Therefore, by adjusting the constants, using the assumption that $v_0 \in E$ and Gronwall's inequality, we obtain

$$|z(t)|_H^2 + \int_0^t ||z(s)||_V^2 \, ds \le C + C \int_0^t ||v(s)||_E^2 \, ds. \tag{4.10}$$

Now, it is straightforward to obtain from (4.7), by using the assumptions on the data, the estimate

$$\begin{aligned} &\frac{1}{2}\rho |v(t)|_{H^d}^2 - \frac{1}{2}\rho |v_0|_{H^d}^2 + \delta ||u(t)||_E^2 + \delta \int_0^t ||v(s)||_E^2 \, ds \\ &\qquad \le C + C \int_0^t |z(s)|_H^2 \, ds + \frac{1}{2}\delta \int_0^t ||v(s)||_E^2 \, ds. \end{aligned}$$

An application of Gronwall's inequality yields the following estimate,

$$|v(t)|_{H^d}^2 + ||u(t)||_E^2 + \int_0^t ||v(s)||_E^2 \, ds \le C + C \int_0^t |z(s)|_H^2 \, ds. \tag{4.11}$$

It follows from (4.10) that

$$\begin{aligned} &|v(t)|_{H^d}^2 + ||u(t)||_E^2 + \int_0^t ||v(s)||_E^2 \, ds \\ &\qquad \le C + C \int_0^t \left(C + C \int_0^s ||v(r)||_E^2 \, dr \right) ds \\ &\qquad \le C + C \int_0^t \int_0^s ||v(r)||_E^2 \, dr ds. \end{aligned}$$

Now, another application of Gronwall's inequality and (4.10) yield the existence of a constant C, which is independent of h and $t \in [0, T]$, such that

$$\begin{aligned} &|v(t)|_{H^d}^2 + ||u(t)||_E^2 + \int_0^t ||v(s)||_E^2 \, ds + |z(t)|_H^2 \\ &\qquad + \int_0^t ||z(s)||_V^2 \, ds \le C. \end{aligned} \tag{4.12}$$

Thus, (4.2), (4.10) and (4.12) imply that z' and v' are also bounded in $\mathcal{U}'$ and $\mathcal{E}'$. It follows from the result in Lions [20, p. 57] that there exists a subsequence $\{h_n\}$, which we still denote by h, such that as $h \to 0$,

$$z^h \to z \text{ strongly in } \mathcal{H}. \tag{4.13}$$

In fact, a stronger result is possible, but (4.13) is sufficient for our purposes.

Lemma 4.2. *In addition to* (4.13), *there exists a further subsequence and there exist* $\varphi \in \mathcal{V}$ *and* $v \in \mathcal{E}$ *such that*

$$\varphi^h \rightarrow \varphi \text{ strongly in } \mathcal{V}, \tag{4.14}$$
$$v^h \rightarrow v \text{ strongly in } \mathcal{E}. \tag{4.15}$$

Proof: Consider (3.24) and let $\varphi \in \mathcal{V}$ be the solution to

$$L(z)\varphi = -L(z)\Phi,$$

where z is given in (4.13). Then,

$$\begin{aligned} \langle L(z)(\varphi), \eta\rangle &\equiv \int_\Omega \sigma_{el}(z+\Theta)\nabla\varphi\cdot\nabla\eta\,dx \\ &= -\int_\Omega \sigma_{el}(z+\Theta)\nabla\Phi\cdot\nabla\eta\,dx, \\ \langle L(z^h)(\varphi^h), \eta\rangle &\equiv \int_\Omega \sigma_{el}(z^h+\Theta)\nabla\varphi^h\cdot\nabla\eta\,dx \\ &= -\int_\Omega \sigma_{el}(z^h+\Theta)\nabla\Phi\cdot\nabla\eta\,dx, \end{aligned}$$

for all $\eta \in \mathcal{V}$. Subtracting the two equalities and rearranging yields,

$$\begin{aligned} &\int_\Omega \left(\sigma_{el}(z+\Theta) - \sigma_{el}(z^h+\Theta)\right)\nabla\varphi\cdot\nabla\eta\,dx \\ &\quad + \int_\Omega \left(\sigma_{el}(z^h+\Theta)\nabla\varphi - \sigma_{el}(z^h+\Theta)\nabla\varphi^h\right)\cdot\nabla\eta\,dx \\ &= \int_\Omega \left(\sigma_{el}(z^h+\Theta) - \sigma_{el}(z+\Theta)\right)\nabla\Phi\cdot\nabla\eta\,dx. \end{aligned}$$

Therefore, by letting $\eta = \varphi - \varphi^h$ and $\theta^h = z^h + \Theta$, while $\theta = z + \Theta$, we find

$$\begin{aligned} &\sigma_* \int_\Omega \left|\nabla\varphi - \nabla\varphi^h\right|^2 dx \\ &\quad \leq \int_\Omega \left|\sigma_{el}(\theta) - \sigma_{el}(\theta^h)\right| (|\nabla\varphi| + |\nabla\Phi|) \left|\nabla\varphi - \nabla\varphi^h\right| \, dx. \end{aligned} \tag{4.16}$$

Now, (4.13) means that $\theta^h \rightarrow \theta$ strongly in $L^2(0,T;H)$. Therefore, if we first take measurable representatives and then a further subsequence, we obtain

$$\theta^h \rightarrow \theta \text{ in } L^2(\Omega_T), \quad \theta^h \rightarrow \theta \text{ pointwise a.e. in } \Omega_T. \tag{4.17}$$

It follows from (4.16) that

$$\begin{aligned}
&\sigma_* \int_0^T \int_\Omega \left|\nabla\varphi - \nabla\varphi^h\right|^2 \, dxdt \\
&\quad \leq 2 \left(\int_0^T \int_\Omega \left|\sigma_{el}(\theta) - \sigma_{el}\left(\theta^h\right)\right|^2 \left(|\nabla\varphi|^2 + |\nabla\Phi|^2\right) \, dxdt \right)^{1/2} \\
&\qquad \times \left(\int_0^T \int_\Omega \left|\nabla\varphi - \nabla\varphi^h\right|^2 dxdt \right)^{1/2} \\
&\quad < \varepsilon \left(\int_0^T \int_\Omega \left|\nabla\varphi - \nabla\varphi^h\right|^2 dxdt \right)^{1/2},
\end{aligned}$$

whenever h is sufficiently small, due to the dominated convergence theorem, the boundedness of σ_{el} and (4.17). Since $\varepsilon > 0$ is arbitrary, this proves the first part of the lemma, (4.14).

Now, we turn to the second part and define v to be the solution to

$$\rho v' + A_d u + B_d v + L_d \theta = \mathbf{f}, \text{ in } \mathcal{E}', \tag{4.18}$$

$$v(0) = v_0, u(t) = u_0 + \int_0^t v(s) \, ds, \tag{4.19}$$

where $\theta = z + \Theta$ with z given in (4.13). Then, (4.12) yields

$$\begin{aligned}
&\frac{1}{2}\rho \left|v^h(t) - v(t)\right|_{H^d}^2 + \delta \left|\left|u^h(t) - u(t)\right|\right|_E^2 \\
&\qquad\qquad + \delta \int_0^t \left|\left|v^h(s) - v(s)\right|\right|_E^2 ds \\
&\quad \leq C_\delta \int_0^t \left|\theta_h^h - \theta\right|_H^2 ds + \frac{1}{2}\delta \int_0^t \left|\left|v^h(s) - v(s)\right|\right|_E^2 ds,
\end{aligned} \tag{4.20}$$

where C_δ depends on δ but not on h. Now, we have

$$\left|\left|z_h^h - z\right|\right|_{\mathcal{H}} \leq \left|\left|z_h^h - z_h\right|\right|_{\mathcal{H}} + \left|\left|z_h - z\right|\right|_{\mathcal{H}} \leq \left|\left|z^h - z\right|\right|_{\mathcal{H}} + \left|\left|z_h - z\right|\right|_{\mathcal{H}}.$$

The last term converges to 0 by the continuity of translations in L^2, and so $z_h^h \to z$ in $\mathcal{H}$. Similar considerations apply to φ_h^h and v_h^h, and then $\theta_h^h \to \theta$ in $\mathcal{H}$. Now (4.20) implies the second conclusion of the lemma.

Lemma 4.2 and the estimates (4.12) providing bounds for v, u and z, imply the following convergences for a suitably chosen subsequence:

$$z^h \to z \text{ strongly in } \mathcal{H} \text{ and pointwise}; \tag{4.21}$$

$$\varphi^h \to \varphi \text{ strongly in } \mathcal{V} \text{ and pointwise;} \tag{4.22}$$

$$\nabla\varphi^h \to \nabla\varphi \text{ strongly in } \mathcal{H} \text{ and pointwise;} \tag{4.23}$$

$$z_h^h \to z \text{ strongly in } \mathcal{H} \text{ and pointwise;} \tag{4.24}$$

$$z^h \to z \text{ weakly in } \mathcal{V}; \tag{4.25}$$

$$z^{h\prime} \to z' \text{ weakly in } \mathcal{U}'; \tag{4.26}$$

$$v^{h\prime} \to v' \text{ weakly in } \mathcal{E}'; \tag{4.27}$$

$$v^h \to v \text{ in } \mathcal{E}. \tag{4.28}$$

Next, we pass to the limit $h \to 0$ in (4.4)–(4.7) using the above subsequence.

Lemma 4.3. *For the subsequence described in* (4.21)–(4.28)*, the following hold true:*

$$\begin{aligned}
A\left(z_h^h\right) z^h &\to A(z)\, z \textit{ weakly in } L^2(0,T;V'), && (4.29)\\
hFz^h &\to 0 \textit{ strongly in } \mathcal{U}', && (4.30)\\
N\left(z_h^h\right)\left(\varphi_h^h\right) &\to N(z)(\varphi) \textit{ weakly in } \mathcal{U}', && (4.31)\\
L\left(z^h\right)\varphi^h &\to L(z)\,\varphi \textit{ weakly in } \mathcal{V}', && (4.32)\\
L_d(z_h^h) &\to L_d(z) \textit{ strongly in } \mathcal{E}' && (4.33)
\end{aligned}$$

and

$$-L\left(z^h\right)\Phi \to -L(z)\,\Phi \textit{ weakly in } \mathcal{V}'. \tag{4.34}$$

Proof: First, we note that where appropriate, we use measurable representatives. We begin with (4.29). Letting $\theta_h^h = z_h^h + \Theta$ and $\theta^h = z^h + \Theta$, as above and let $\eta \in \mathcal{U}$ be a given function, then

$$\begin{aligned}
\langle A\left(z_h^h\right) z^h - A(z)\, z, \eta\rangle_{\mathcal{V}',\mathcal{V}} &= \int_0^T \int_\Omega k_{ij}\left(\theta_h^h\right)\left(z_{,i}^h - z_{,i}\right)\eta_{,j}\, dxdt \\
&+ \int_0^T \int_\Omega \left(k_{ij}\left(\theta_h^h\right) - k_{ij}(\theta)\right) z_{,i}\eta_{,j}\, dxdt. && (4.35)
\end{aligned}$$

From the assumption that the k_{ij} are Lipschitz continuous, it follows

$$k_{ij}\left(\theta_h^h\right) \to k_{ij}(\theta) \text{ pointwise,}$$

and (4.25) implies that $z^h_{,i} \to z_{,i}$ weakly in $L^2(\Omega_T)$. Then, the boundedness of the k_{ij} yields

$$k_{ij}\left(\theta^h_h\right)\eta_{,j} \to k_{ij}(\theta)\,\eta_{,j} \text{ strongly in } L^2(\Omega_T).$$

Therefore,

$$\int_0^T\int_\Omega k_{ij}\left(\theta^h_h\right)\left(z^h_{,i}-z_{,i}\right)\eta_{,j}dxdt \to 0.$$

Now, the integrand in the last term in (4.35) converges to zero pointwise, and the boundedness of the k_{ij} implies that we can use the dominated convergence theorem and conclude that this term converges to zero, as well. This proves (4.29).

We note that (4.30) follows from the observation that $h\int_0^T\langle Fz^h, z^h\rangle dt$ is bounded and from the inequality,

$$\begin{aligned}\int_0^T h\langle Fz^h,\eta\rangle dt &\le h^{1/4}\int_0^T\left(h\langle Fz^h,z^h\rangle\right)^{3/4}||\nabla\eta||_{L^4(\Omega)}\,dt\\ &\le h^{1/4}\left(\int_0^T h\langle Fz^h,z^h\rangle dt\right)^{3/4}\left(\int_0^T||\nabla\eta||^4_{L^4(\Omega)}\,dt\right)^{1/4}\\ &\le Ch^{1/4}\,||\eta||_{L^4(0,T;V\cap W^{1,4}(\Omega))}\,.\end{aligned}$$

We turn to consider (4.31),

$$\begin{aligned}&\langle N\left(z^h_h\right)\left(\varphi^h\right),\eta\rangle\\ &\quad=-\int_0^T\int_\Omega\sigma_{el}\left(\theta^h_h\right)\varphi^h\nabla\Phi\cdot\nabla\eta dx-\int_0^T\int_\Omega\sigma_{el}\left(\theta^h_h\right)\varphi^h\nabla\varphi^h\cdot\nabla\eta dx\\ &\qquad+\int_0^T\int_\Omega\sigma_{el}\left(\theta^h_h\right)\nabla\Phi\cdot\nabla\varphi^h\eta dx+\int_0^T\int_\Omega\sigma_{el}\left(\theta^h_h\right)|\nabla\Phi|^2\,\eta dx. \quad (4.36)\end{aligned}$$

It is straightforward to pass to the limit in each of the terms, except for the second one on the right-hand side. However, thanks to (4.21)–(4.28) the integrand converges to $\sigma_{el}(\theta)\,\varphi\nabla\varphi\cdot\nabla\eta$ pointwise. Therefore, the result will follow from the Vitali convergence theorem once we show that the integrands are uniformly integrable. To that end let $r = 12/11$, then

$$\left|\sigma_{el}\left(\theta^h\right)\varphi^h\nabla\varphi^h\cdot\nabla\eta\right|^r \le C\left|\varphi^h\right|^r\left|\nabla\varphi^h\right|^r|\nabla\eta|^r,$$

and choosing $p = 11/6$ leads to

$$\begin{aligned}
&\int_0^T \int_\Omega \left|\sigma_{el}\left(\theta^h\right)\varphi^h \nabla\varphi^h \cdot \nabla\eta\right|^r dxdt \\
&\quad \le C\left(\int_0^T \int_\Omega \left|\varphi^h\right|^{pr} \left|\nabla\varphi^h\right|^{pr} dxdt\right)^{1/p} \left(\int_0^T \int_\Omega \left|\nabla\eta\right|^{rp'} dxdt\right)^{1/p'} \\
&\quad = C\left(\int_0^T \int_\Omega \left|\varphi^h\right|^{2} \left|\nabla\varphi^h\right|^{2} dxdt\right)^{6/11} \left(\int_0^T \int_\Omega \left|\nabla\eta\right|^{12/5} dxdt\right)^{5/11} \\
&\quad \le C\left(\int_0^T \int_\Omega \left|\varphi^h\right|^{2} \left|\nabla\varphi^h\right|^{2} dxdt + \int_0^T \int_\Omega \left|\nabla\eta\right|^{12/5} dxdt\right) \\
&\quad \le C\left(\int_0^T \int_\Omega \left|\varphi^h\right|^{2} \left|\nabla\varphi^h\right|^{2} dxdt + \int_0^T \int_\Omega \left|\nabla\eta\right|^{4} dxdt\right) \\
&\quad \le D,
\end{aligned}$$

where $D < \infty$ is a constant independent of h thanks to (3.6) and the assumptions made on σ_{el}, (3.1). Thus, the integrands in the second term on the right-hand side of (4.36) form a uniformly integrable set and so we can pass to the limit in this term as well as the others.

Formulas (4.32) and (4.34) follow easily from (4.21)–(4.28), especially the strong convergence of φ^h. Finally consider (4.33). Letting $v \in \mathcal{E}$ and using the definition of L_d given in (3.10) we obtain

$$\left|\langle L_d z_h^h - L_d z, \mathbf{v}\rangle_{\mathcal{E}}\right| \le C\left|\left|\theta_h^h - \theta\right|\right|_{\mathcal{H}} \left|\left|\mathbf{v}\right|\right|_E,$$

which establishes the desired conclusion because of (4.21). This proves the lemma.

We now use the results of the preceding lemma to pass to the limit $h \to 0$ in (4.4)-(4.7), thus proving Theorem 3.2..

5. Quasistatic and steady problems

In this short section we present the quasistatic and the steady or static versions of the problem.

The quasistatic problem is obtained by neglecting the inertial terms in the equations of motion (2.3). Thus, the problem is to find a triplet

$\{u, \phi, \theta\}$ such that

$$\rho c_p \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x_j}\left(k_{ij}(\theta)\frac{\partial \theta}{\partial x_i}\right) = \sigma_{el}(\theta)|\nabla\phi|^2 - m_{ij}\Theta_{ref}\frac{\partial^2 u_i}{\partial t \partial x_j}, \tag{5.1}$$

$$\nabla \cdot (\sigma_{el}(\theta)\nabla\phi) = 0, \tag{5.2}$$

$$-\frac{\partial}{\partial x_j}(a_{ijkl}\frac{\partial u_k}{\partial x_l} + b_{ijkl}\frac{\partial^2 u_k}{\partial t \partial x_l} - m_{ij}\theta) = f_i, \tag{5.3}$$

along with the boundary conditions,

$$u = 0, \quad \theta = \theta_b, \quad \phi = \phi_b \qquad \text{on } \Gamma_D, \tag{5.4}$$

$$-k_{ij}\frac{\partial \theta}{\partial x_i}n_j = h\theta, \;\; \sigma_{ij}n_j = 0, \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{5.5}$$

and the initial conditions at $t = 0$,

$$u = u_0, \quad \theta = 0 \qquad \text{in } \Omega. \tag{5.6}$$

An initial condition for u is needed because of the viscosity term in (5.3). We note that the system is fully coupled.

This problem is new, is elliptic-parabolic, and remains an open problem, to be investigated in the future.

The steady or static problem is obtained by neglecting any time dependence. It consists of the following system of elliptic equations, Find a triplet $\{u, \phi, \theta\}$ such that

$$-\frac{\partial}{\partial x_j}\left(k_{ij}(\theta)\frac{\partial \theta}{\partial x_i}\right) = \sigma_{el}(\theta)|\nabla\phi|^2, \tag{5.7}$$

$$\nabla \cdot (\sigma_{el}(\theta)\nabla\phi) = 0, \tag{5.8}$$

$$-\frac{\partial}{\partial x_j}(a_{ijkl}\frac{\partial u_k}{\partial x_l} - m_{ij}\theta) = f_i, \tag{5.9}$$

along with the boundary conditions,

$$u = 0, \quad \theta = \theta_b, \quad \phi = \phi_b \qquad \text{on } \Gamma_D, \tag{5.10}$$

$$-k_{ij}\frac{\partial \theta}{\partial x_i}n_j = h\theta, \;\; \sigma_{ij}n_j = 0, \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_N. \tag{5.11}$$

We note that the steady problem decouples. Equations (5.7) and (5.8), together with the relevant boundary conditions form the classical steady thermistor problem (see, e. g., [16]). Once the temperature and the electric potential have been found, the displacements and the thermal stresses can be found from (5.9) and the boundary conditions (5.10) and (5.11).

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Kenneth L. Kuttler: Department of Mathematics, Brigham Young University, Provo, UT 84602, USA, **E-mail:** klkuttle@math.byu.edu

Meir Shillor: Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA, **E-mail:** shillor@oakland.edu

José R. Fernández: Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, Escola Politécnica Superior, Campus Univ. s/n, 27004 Lugo, Spain, **E-mail:** jose.fernandez@usc.es