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An Explicit Solution of Joint Estimation-Linear Model Predictive Control

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Abstract Model predictive control (MPC) represents an optimal strategy where constraints on inputs, outputs and system states can be implemented as part of the control law that takes the form of a mathematical program. MPC of linear systems with a quadratic objective function results in a quadratic program (QP) that needs to be solved online at each sampling instant. Multiparametric programming methods that attempt an explicit solution to QPs have been successfully used in context of MPC and is termed as explicit MPC (eMPC). eMPC for linear systems results in a piecewise affine-in-state feedback control law and is determined offline. During online implementation, the control law is selected from among the different pieces based on the real-time value of the states. An ability to verify the controller output over all possible state realizations of the feasible state-space is important in any critical application including health and aerospace and is a unique feature of eMPC. Since state feedback MPC requires full state information, it is always used in conjunction with a state estimator such as a Kalman Filter. Further, fault tolerant control methodologies depend on state and parameter estimation to detect and diagnose faults followed by compensation. Conventionally, the state filtering step is performed prior to MPC and thus the interaction between the estimator and controller remains hidden and can be analyzed only via numerical simulations. In this work, we propose using multiparametric programming to find the explicit solution of the joint estimator-MPC problem. In particular, the eMPC control law now depends linearly on the joint information of states and measurements. This allows explicitly obtaining the sensitivity of the MPC control law to the measurement as well as the estimator parameters. The proposed explicit solution of the joint estimator-MPC problem is demonstrated on a SISO 2-tank system. The effect of the estimator gain on the size of the feasible region is delineated.

Keywords Explicit MPC · Joint Estimator-MPC · Multiparametric programming

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1 Introduction

In most fault-tolerant control applications, state feedback model predictive control (MPC) is used in conjunction with a state estimator. In linear implementations, the estimator uses measurements and a linear state space model to obtain full-state information online which is subsequently used for state feedback by MPC. Popular manifestations of linear estimators include the Luenberger observer and the stationary Kalman filter. The Kalman filter design is dependent on noise covariances. These noise covariances are not known a priori and therefore need to be estimated [1, 2].

In context of nonlinear systems [3], present a one way separation principle, wherein the controller does not affect the stability of the observer, but the controller stability is influenced by the observer dynamics. It is therefore clear that the state estimator also influences the performance of the controller. However, since MPC is typically implemented

implicitly, a quantification of the extent of the interaction between the observer and the controller is only possible through numerical simulations. A recent paper by [4] simultaneously implements explicit approach for MPC and Moving horizon estimation (MHE). However, here the estimation and control are solved as two independent problems. In [5] and [6] the inclusion of state estimation within MPC makes it a stochastic control problem whose solution is obtained by solving an equivalent deterministic MPC problem. The relationship between control and estimator is not explicit in these works and is therefore difficult to characterize their interaction.

Explicit MPC (eMPC), while a sound way for implementation of MPC in its own right, also provides insights into the MPC law, which is not afforded by a numerical implicit implementation. eMPC solves the quadratic program (QP) in a multiparametric (mp) framework resulting in an explicit statement of the control law [7]. The ability to write the MPC law for regulation of a linear system in an analytical form explicitly informs of the piecewise affine-in-state nature of the MPC law. Such an analytical MPC solution provides an opportunity to characterize the interaction between MPC and unconstrained linear state estimation such as a stationary Kalman filter or Luenberger observer. Benefits of quantification of the interaction includes joint design of the estimator and controller with robust performance bound guarantees of the overall system, a requirement for critical applications. This proposal is an attempt to explicitly outline the impact of the state estimator design on the MPC law and its region of validity.

The paper is organized as follows. Section 2 introduces system definition and estimator dynamics. In Section 3, MPC formulation for the estimator dynamics along with equivalent MpQP problem is presented. Section 4 presents (i) explicit relationship between control and the estimator (ii) sensitivity of control to output. Further, Section 5 presents numerical results with the help of an example.

Notations The MPC prediction horizon is $N \in \mathbb{N}_{>0}$. $U \triangleq \{u_k, u_{k+1}, \cdots, u_{k+N-1}\}$ is sequence of control inputs. $\chi \triangleq \{\hat{x}_{k|k}, \hat{x}_{k+1|k}, \cdots, \hat{x}_{k+N|k}\}$ is sequence of predicted state estimates. $\partial_x = \frac{\partial}{\partial x}$ is the partial derivative with respect to x of a function $f(\cdot)$ where x is one of its variable. $\mathbb{E}(\cdot)$ is the conditional expectation. $\bar{X}$ and $\underline{X}$ are vectors of upper and lower bounds on x_k, respectively. $\bar{U}$ and $\underline{U}$ are vectors of upper and lower bounds on u_k, respectively.

2 Preliminaries

2.1 System Description

Consider the linear discrete-time dynamic system

$$y_k = Cx_k + v_k \tag{1}$$

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{2}$$

where the system matrices A, B, and C have appropriate dimensions. $x_k \in \mathbb{X} \subset \mathbb{R}^n$ is the state, $u_k \in \mathbb{U} \subset \mathbb{R}^m$ is the input; $w_k \sim \mathcal{N}(0, R_w)$ is the state disturbance and $v_k \sim \mathcal{N}(0, R_v)$ is measurement noise (both are Gaussian and independent), respectively, and $y_k \in \mathbb{Y} \subset \mathbb{R}^q$ are the measurements. Additionally, the initial state x_0 is assumed to be distributed normally with mean $\hat{x}_0$ and covariance P_0. The constraint sets $\mathbb{X}$ and $\mathbb{U}$ are closed, compact and convex having origin in their interior. The pairs (A, B) and (A, C) are controllable and observable, respectively.

2.2 Estimator Dynamics

In linear implementations, a linear state space model along with measurement y_k given by Eq.(2) and Eq.(1), respectively, is used to obtain full-state information online. In Kalman filter the mean of the posterior state density $\hat{x}_{k|k} = \mathbb{E}(x_k | y_k)$ is obtained as an update of the mean of the prior $\hat{x}_{k|k-1} = \mathbb{E}(x_k | y_{k-1})$, thereby yielding a recursive formulation. In case of LTI system, the state covariance P_k becomes time invariant P_∞, resulting in the stationary Kalman filter and can be directly obtained by the discrete algebraic Riccati equation [8] given below.

$$P_\infty = AP_\infty A^T + R_w - AP_\infty C^T (CP_\infty C^T + R_v)^{-1} CP_\infty A^T \tag{3}$$

The time invariant state covariance P_∞ is further used to compute K_∞,

$$K_\infty = AP_\infty C^T (CP_\infty C^T + R_v)^{-1} \tag{4}$$

In conjunction with system dynamic Eq.(2), recursion for the stationary Kalman filter can be written in the following predictor-corrector form [9].

Corrector equation:

$$\hat{x}_{k|k} = (I - K_\infty C)\hat{x}_{k|k-1} + K_\infty y_k \tag{5}$$

Here $K_\infty \in \mathbb{R}^{n \times q}$ is the estimator gain given by.

Predictor equation:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k \tag{6}$$

Equivalently, substituting Eq.(5) in Eq.(6), gives the estimator dynamics as follows.

Estimator dynamics:

$$\hat{x}_{k+1|k} = \bar{A}\hat{x}_{k|k-1} + Bu_k + \Gamma y_k \tag{7}$$

where $\bar{A} = A(I - K_\infty C)$ and $\Gamma = AK_\infty$.

3 Stationary Kalman Filter and Linear MPC: An Explicit Scheme

3.1 Constrained Optimal Control Problem

Consider the following constrained optimal control problem for initial condition $\hat{x}_{k|k-1}$ and prediction horizon N. The purpose of solving this problem is to drive the predicted state $\hat{x}_{k+j|k}, j = 1, \cdots, N$ to origin for the estimator dynamics given by Eq.(7). Hence, it is also a joint estimation and control problem.

$$\mathbb{P}_1 : \min_{\chi, U} J(\chi, U)$$

$$J(\chi, U) = V(\hat{x}_{k+N|k}) + \sum_{j=k}^{k+N-1} L(\hat{x}_{j|k}, u_j) \tag{8}$$

Problem 1
subject to

$$Eq.7, \ \hat{x}_{k|k-1} \in \mathbb{X}, \ \hat{x}_{k+N|k} \in X_T \subseteq \mathbb{X} \text{ and } u_k \in \mathbb{U} \tag{9}$$

$L(\hat{x}_{j|k}, u_j) = \hat{x}_{j|k}^T Q\hat{x}_{j|k}^T + u_j^T Ru_j^T$ and $V(\hat{x}_{k+N|k}) = \hat{x}_{k+N|k}^T P_f \hat{x}_{k+N|k}$ are the stage and terminal cost, respectively, with $L(0,0) = 0$ and $V(0) = 0$ where $P_f > 0, Q > 0$ and $R \succeq 0$. X_T is the terminal set with origin in its interior.

Although random variables w_k, v_k and $x_k = x_0$ induce a stochastic character, $\mathbb{P}_1$ is solved in a deterministic manner based on the realization of y_k.

The main aim of the current work is to explore 1) the possibility of having explicit control laws for problem $\mathbb{P}_1$ 2) if there exists such control law, then to quantify the interaction with the estimator gain K_∞ and other estimator parameters. In this direction, the following subsection tries to analyze $\mathbb{P}_1$ further.

3.2 Multiparametric Quadratic Programming Problem

The problem $\mathbb{P}_1$ can be transformed into equivalent optimization problem which takes the form of a quadratic

program (QP). This transformation is achieved by substituting prediction equations for the dynamics given by Eq.(7) in $\mathbb{P}_1$ (see Appendix for details) and takes following form.

$$\mathbb{P}_2 : \min_U \hat{J}(U, \hat{x}_{k|k-1}, y_k)$$

$$\hat{J}(U, \hat{x}_{k|k-1}, y_k) = \frac{1}{2}U^T HU + (\hat{x}_{k|k-1}^T F_x + y_k^T F_y)U \tag{10}$$

Problem 2
subject to

$$GU \leq W + S^x \hat{x}_{k|k-1} + S^y y_k \tag{11}$$

where $(\hat{x}_{k|k-1}, y_k) \in \mathbb{X} \times \mathbb{Y}$ appear as parameters, U is the decision vector over the horizon N, $H \in \mathbb{R}^{mN \times mN}$ is the Hessian which is positive definite $(\succ 0)$ with $F_x \in \mathbb{R}^{n \times mN}$ and $F_y \in \mathbb{R}^{q \times mN}$ being the linear parts of $\hat{J}(U, \hat{x}_{k|k-1}, y_k)$. The constraint matrices are of the following sizes-$G \in \mathbb{R}^{p \times mN}$, $W \in \mathbb{R}^{p \times 1}$, $S^x \in \mathbb{R}^{p \times n}$ and $S^y \in \mathbb{R}^{p \times q}$-, respectively.

Remark 1 : Note that in Problem $\mathbb{P}_2$ both, the predicted state $\hat{x}_{k|k-1}$ and measurement y_k, parametrize the QP problem. This is in contrast to conventional explicit MPC [7], where only the filtered state $\hat{x}_{k|k}$ constitutes the parameter. The price paid for an increase in the number of parameters is compensated by transparency in the impact of y_k on the control law.

Excellent algorithms are available to solve problem $\mathbb{P}_2$ online. However, the structure of the problem informs us that $\mathbb{P}_2$ can be further cast into a standard Multiparametric Quadratic Programming (MpQP) form by suitably transforming the decision variable U. The transformation, when estimator dynamics are used for prediction, is simple to derive and is provided here without giving details. Let

$$Z = U + H^{-1}F_x^T \hat{x}_{k|k-1} + H^{-1}F_y^T y_k \tag{12}$$

The problem $\mathbb{P}_2$ can be cast with transformed variable Z into standard MpQP as follows.

$$\mathbb{P}_3 : \min_Z \bar{J}(Z)$$

$$\bar{J}(Z) = \frac{1}{2}Z^T HZ \tag{13}$$

Problem 3
subject to

$$GZ \leq W + E^x \hat{x}_{k|k-1} + E^y y_k \tag{14}$$

Now that $\mathbb{P}_1$ is transformed into problem $\mathbb{P}_3$, a standard MpQP problem, it is clear that explicit solutions to $\mathbb{P}_1$ via $\mathbb{P}_3$ exist in the form of state and measurement-dependent affine control laws. Such explicit solutions can be obtained offline through the sensitivity analysis of KKT conditions

(introduced in the next section) for $\mathbb{P}_3$. Moreover, it also provides insights into the nature of the explicit control law and its interaction with the estimator gain K_∞. The next section derives these explicit solutions and attempts to bring out certain insights regarding the controller-estimator interaction, that are otherwise only known implicitly.

4 Explicit Control Law and Its Sensitivity To Measurement

Certain definitions are introduced for ease of presentation. Let p be the total number of constraints and $\mathcal{M} \triangleq \{1, 2, \cdots, p\}$ the index set for the constraints in Eq.(11). Each candidate active set with its corresponding inactive part can be defined as follows.

Definition 1   For index set $\mathcal{A} \subset \mathcal{M}$ the candidate active set is defined as

$$\mathcal{A}(\hat{x}_{kk-1,y_k}) \triangleq \{(\hat{x}_{k|k-1}, y_k) \in \mathbb{X} \times \mathbb{Y} | G_\mathcal{A} Z - W_\mathcal{A} - E_\mathcal{A}^x \hat{x}_{k|k-1} - E_\mathcal{A}^y y_k = 0\} \tag{15}$$

Similarly the corresponding inactive constraints can be defined as follows

Definition 2   For index set $\mathcal{I} = \mathcal{M} \setminus \mathcal{A}$ the corresponding set of inactive constraints is,

$$\mathcal{I}(\hat{x}_{k|k-1}, y_k) \triangleq \{(\hat{x}_{k|k-1}, y_k) \in \mathbb{X} \times \mathbb{Y} | G_\mathcal{I} Z \leq W_\mathcal{I} + E_\mathcal{I}^x \hat{x}_{k|k-1} - E_\mathcal{I}^y y_k\} \tag{16}$$

Remark 2   : Since $\mathbb{P}_3$ belongs to a class of QP problems the maximum number of candidate active sets that can be enumerated is $\sum_{i=0}^{mN} \binom{p}{i}$ [10].

With the help of Definition 1, the Lagrangian for problem $\mathbb{P}_3$ can be written as follows,

$$\mathcal{L}(Z, \hat{x}_{k|k-1}, y_k) = \bar{J}(Z) + \lambda_\mathcal{A}^T (G_\mathcal{A} Z - W_\mathcal{A} - E_\mathcal{A}^x \hat{x}_{k|k-1} - E_\mathcal{A}^y y_k) \tag{17}$$

Further, if $Z^*(\hat{x}_{k|k-1}, y_k)$ is the optimal solution to problem $\mathbb{P}_3$ with $(\hat{x}_{k|k-1}, y_k) \in \mathbb{X} \times \mathbb{Y}$, then $\mathcal{A}(\hat{x}_{k|k-1}, y_k)$ is called as the optimal active set such that there exist Lagrange multipliers $\lambda_i \geq 0, i \in \mathcal{A}$. Consequently, there is a corresponding solution $U^*(\hat{x}_{k|k-1}, y_k)$ due to Eq.(12) which is optimal for $\mathbb{P}_2$.

This optimal solution $Z^*(\hat{x}_{k|k-1}, y_k)$ satisfies the following equations known as Karush–Kuhn–Tucker (KKT) conditions mentioned below.

KKT conditions:

$$\partial_Z \mathcal{L}(Z^*(\hat{x}_{k|k-1}, y_k), \hat{x}_{k|k-1}, y_k) = 0 \tag{18}$$

$$\lambda_i(G_i Z^*(\hat{x}_{k|k-1}, y_k) - W_i - E_i^x \hat{x}_{k|k-1} - E_i^y y_k) = 0, \ i \in \mathcal{A} \tag{19}$$

$$\lambda_i \geq 0, \ i \in \mathcal{A}$$
$$G_j Z^*(\hat{x}_{k|k-1}, y_k) \leq W_j + E_j^x \hat{x}_{k|k-1} + E_j^y y_k, \ j \in \mathcal{I} \tag{20}$$

Remark 3   : It is assumed that the linear independence constraint qualification (LICQ) and strict complementarity slackness (SCS) property, respectively, holds for Eq.(18)-(20). The satisfaction of LICQ and SCS ensures that λ_i, $i \in \mathcal{A}$ is unique and positive [11]. It indicates that candidate active set $\mathcal{A}$ is indeed an optimal active set.

In order to gain insights about the interaction of U with estimator gain K_∞, *Theorem 2* in [7] is restated below in context of stationary Kalman filter and linear MPC framework.

Theorem 1   : *The control law $U_j(\hat{x}_{k|k-1}, y_k)$ obtained as an explicit solution to problem $\mathbb{P}_2$ is a piecewise affine function of the predicted state estimate $\hat{x}_{k|k-1}$ and measurement y_k*

Proof   : To prove that U is a piecewise affine function of the predicted state estimate $\hat{x}_{k|k-1}$ and measurement y_k we collect the KKT conditions given by Eq.(18)-Eq.(20) for problem $\mathbb{P}_3$. Also, let there be N_{CR} active sets each given by Eq.(15). Then for the jth active set the first order necessary condition given by Eq.(18) can be written as follows.

$$HZ_j + G_{\mathcal{A}_j}^T \lambda_{\mathcal{A}_j} = 0 \tag{21}$$

Now, Eq.(21) with Eq.(15) forms a system of KKT equations.

$$\begin{bmatrix} H & G_{\mathcal{A}_j}^T \\ G_{\mathcal{A}_j} & \mathbf{0} \end{bmatrix} \begin{bmatrix} Z_j \\ \lambda_{\mathcal{A}_j} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ W_{\mathcal{A}_j} + E_{\mathcal{A}_j}^x \hat{x}_{k|k-1} + E_{\mathcal{A}_j}^y y_k \end{bmatrix} \tag{22}$$

Solving the above system of equations using schur complement method [11] yields the following solutions.

$$Z_j(\hat{x}_{k|k-1}, y_k) = \tilde{\mathcal{F}}_{z,j}(I - K_\infty C)\hat{x}_{k|k-1} + \tilde{\mathcal{F}}_{z,j} K_\infty y_k + g_{\mathcal{A}_j, z} \tag{23}$$

$$\lambda_{\mathcal{A}_j}(\hat{x}_{k|k-1}, y_k) = \tilde{\mathcal{F}}_{\lambda,j}(I - K_\infty C)\hat{x}_{k|k-1} + \tilde{\mathcal{F}}_{\lambda,j} K_\infty y_k + g_{\mathcal{A}_j, \lambda} \tag{24}$$

See the Appendix for $\tilde{\mathcal{F}}_{z,j}$, $\tilde{\mathcal{F}}_{\lambda,j}$, $g_{\mathcal{A}_j, z}$ and $g_{\mathcal{A}_j, \lambda}$.

The solution $Z_j(\hat{x}_{k|k-1}, y_k)$ is optimal when $\lambda_{\mathcal{A}_j}(\hat{x}_{k|k-1}, y_k) \geq 0$ and satisfies Eq.(20) for some parameters, say $(\hat{x}'_{k|k-1}, y'_k) \in \mathbb{X} \times \mathbb{Y}$, respectively. Thus we get a region of validity (also known as critical region) for Eq.(23).

$$-\tilde{\mathcal{F}}_{\lambda,j}(I - K_\infty C)\hat{x}_{k|k-1} - \tilde{\mathcal{F}}_{\lambda,j}K_\infty y_k \leq g_{\mathcal{A}_j,\lambda} \qquad (25)$$

$$(G_{\mathcal{I}_j}\tilde{\mathcal{F}}_{x,j} - \tilde{E}_{\mathcal{I}_j})(I - K_\infty C)\hat{x}_{k|k-1} + (G_{\mathcal{I}_j}\tilde{\mathcal{F}}_{x,j} - \tilde{E}_{\mathcal{I}_j})K_\infty y_k \leq$$
$$W_{\mathcal{I}_j} - G_{\mathcal{I}_j}g_{\mathcal{A}_j,z} \qquad (26)$$

$$\tilde{H}_j(I - K_\infty C)\hat{x}_{k|k-1} + \tilde{H}_j K_\infty y_k \leq \tilde{L}_j, \; j = 1, \cdots, N_{CR} \qquad (27)$$

$$\tilde{H}_{x,j}(K_\infty)\hat{x}_{k|k-1} + \tilde{H}_{y,j}(K_\infty)y_k \leq \tilde{L}_j, \; j = 1, \cdots, N_{CR} \qquad (28)$$

See the Appendix for $\tilde{H}_j$ and $\tilde{L}_j$. Eq.(28) is obtained after removing redundant inequalities form Eq.(27) and it represents the full dimensional polytope for the jth region of validity for $Z_j(\hat{x}_{k-1}, y_k)$ in the predicted estimate state-measurement space. Similarly, the same result can be shown when none of the constraints are active, i.e., $\mathcal{A} = \{ \ \}$. Further, equating Eq.(23) with Eq.(12) gives piecewise affine control law $U_j(\hat{x}_{k|k-1}, y_k)$ as follows.

$$U_j(\hat{x}_{k|k-1}, y_k) = \tilde{U}_j(I - K_\infty C)\hat{x}_{k|k-1} + \tilde{U}_j K_\infty y_k + g_{\mathcal{A}_j,z} \qquad (29)$$

$$\tilde{U}_j = \tilde{\mathcal{F}}_{z,j} - 2H^{-1}\tilde{B}^T\tilde{Q}\hat{A} \qquad (30)$$

Note that Eq.(28) is also valid for $U_j(\hat{x}_{k|k-1}, y_k)$. This control law is an explicit solution to problem $\mathbb{P}_2$. This completes the proof □

Unlike the standard eMPC where the system states are the only parameters [7], here the parametric space is shared by predicted state estimates $\hat{x}_{k|k-1}$ and measurement y_k, which is evident from Eq.(28). Moreover the critical regions given by Eq.(28) are also parameterized by the estimator gain K_∞ along with $\hat{x}_{k|k-1}$ and y_k. Therefore it will be interesting to see the effect of estimator gain on the parametric partition of $\hat{x}_{k|k-1}$ and y_k which also happens to be the feasible space.

Remark 4 : Excellent methods exist in literature for 1) computing active sets offline and 2) implementation of $\mathbb{P}_3$ in an explicit framework (see [12] for a survey of these methods). The online algorithms for explicit implementation are known as point location algorithms.

From the previous theorem we have Eq.(29) that connects u_k with K_∞, $\hat{x}_{k|k-1}$ and y_k. It is obvious that noise in measurement can have a detrimental effect on controller performance. Hence, it is important to assess the sensitivity of

control law $U_j(\hat{x}_{k|k-1}, y_k)$ to such measurements. Sensitivity of the control law to the measurement can be defined using Eq.(29) as follows,

$$\mathcal{S}_{y,j} = \partial_{y_k} U_j(\hat{x}_{k|k-1}, y_k) \qquad (31)$$

where $\mathcal{S}_{y,j} \in \mathbb{R}^{mN \times q}$, this is the sensitivity of $U_j(\hat{x}_{k|k-1}, y_k)$ to a measurement y_k. We begin by recalling Eq.(29) for assessing the sensitivity of $U_j(\hat{x}_{k|k-1}, y_k)$ to measurement y_k. Then differentiating $U_j(\hat{x}_{k|k-1}, y_k)$ with respect y_k yields the following expression for $\mathcal{S}_{y,j}$.

$$\mathcal{S}_{y,j} = \tilde{U}_j K_\infty \qquad (32)$$

where $\tilde{U}_j$ is given by Eq.(30). Thus, Eq.(32) shows that the control sequence is linearly related to the estimator gain K_∞ Now, using Eq.(32) we can rewrite Eq.(29) as shown below.

$$U_j(\hat{x}_{k|k-1}, y_k) = \tilde{U}_j(I - K_\infty C)\hat{x}_{k|k-1} + \mathcal{S}_{y,j}y_k + g_{\mathcal{A}_j,z} \qquad (33)$$

Eq. (33) connects measurement sensitivity $\mathcal{S}_{y,j}$ with the control law $U_j(\hat{x}_{k|k-1}, y_k)$. From this connection it is clear that the impact of y_k on the control law diminishes for lower values of sensitivity $\mathcal{S}_{y,j}$, i.e., for lower values of K_∞ and vice versa. One may similarly quantify the sensitivity of the control law to the estimator parameters such as $K_\infty, P_\infty, R_w, R_v$ by calculating partial derivatives using Eq.(29) and applying chain rule of differentiation.

Remark 5 : Sensitivity $\mathcal{S}_{y,j}^0$ corresponds to the first element of $U_j(\hat{x}_{k|k-1}, y_k)$ that is applied to the system given by Eq.(2)

For knowing the correct critical region index say $j = j^*$, Eq.(28) needs to be satisfied say for point $(\hat{x}'_{k|k-1}, y'_k)$. Once j^* is known Eq.(33) is evaluated at that point to obtain optimal value for u_0 the first element of $U_j(\hat{x}'_{k|k-1}, y'_k)$. This procedure is implemented online with the help of a point location algorithm. For the sake of completeness one such algorithm which is simple in implementation, known as sequential search [7, 12], is presented next.

Algorithm 1 : Sequential search

Input:$\hat{x}'_{k|k-1}$, y'_k
Output:u_0, j^*
stopflag$\leftarrow 1$, j$\leftarrow 1$
while *stopflag* $== 1$ and $j \leq N_{CR}$ **do**
    if $\tilde{H}_{x,j}(K_\infty)\hat{x}_{k|k-1} + \tilde{H}_{y,j}(K_\infty)y'_k \leq \tilde{L}_j$ **then**
        Evaluate Eq.(33) at $(\hat{x}'_{k|k-1}, y'_k)$
        $u_0 \leftarrow$ first element of $U_j(\hat{x}'_{k|k-1}, y'_k)$ and $j^* \leftarrow j$
        stopflag $\leftarrow 0$
    end if
end while

Next section demonstrates the impact of estimator gain K_∞ on critical regions and control laws with the help of an example.

5 Example

Consider the two tank plant dynamics given in [13].The plant is linearized at (6, 6, 8.98) has a noisy measurement with state noise as given below.

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k \tag{34}$$

$$x_{k+1} = \begin{bmatrix} 0.7424 & 0 \\ 0.2211 & 0.7424 \end{bmatrix} x_k + \begin{bmatrix} 0.3441 \\ 0.0487 \end{bmatrix} u_k + w_k \tag{35}$$

$-6 \le x_k^{(1)} \le 25$ and $-6 \le x_k^{(2)}(k) \le 25$
$-8.9823 \le u_k \le 11.0177$ and $-6 \le y_k \le 25$

The state and input weighting is $Q = \begin{bmatrix} 0.0250 & 0 \\ 0 & 0.8000 \end{bmatrix}$ and $R = 0.5$, respectively, with $N = 5$.

To highlight the effects of estimator gain, three different scenarios of state and measurement noise covariance matrices are used to compute the respective estimator gains.

1) For a high estimator gain $K_\infty^H = \begin{bmatrix} 0.9819 \\ 0.9630 \end{bmatrix}$ the state and measurement noise covariances are $Q_w^H = 0.0892I_2$ and $R_v^H = 0.0036$.

2) For an intermediate estimator gain $K_\infty^I = \begin{bmatrix} 0.5623 \\ 0.6080 \end{bmatrix}$ the state and measurement noise covariances are $Q_w^I = 0.0321I_2$ and $R_v^I = 0.0323$.

3) For a low estimator gain $K_\infty^L = \begin{bmatrix} 0.0904 \\ 0.1243 \end{bmatrix}$ the state and measurement noise covariances are $Q_w^L = 0.0036I_2$ and $R_v^L = 0.0896$.

Effect of Estimator gain on feasible parametric space $(\hat{x}_{k|k-1}, y_k)$: The constraints in Eq.(14) were obtained for three different gains K_∞^H, K_∞^I and K_∞^L. The total number of constraints thus obtained is $p = 2(n(N + 1) + mN)$ having mN decision variables with $\hat{x}_{k|k-1}$ and y_k being parameters. To show the effect of gain on the partitioned space its total volume (V_{Tot}) is computed. The total volume also signifies the amount of available feasible space and is computed in a Monte-Carlo fashion using the expression
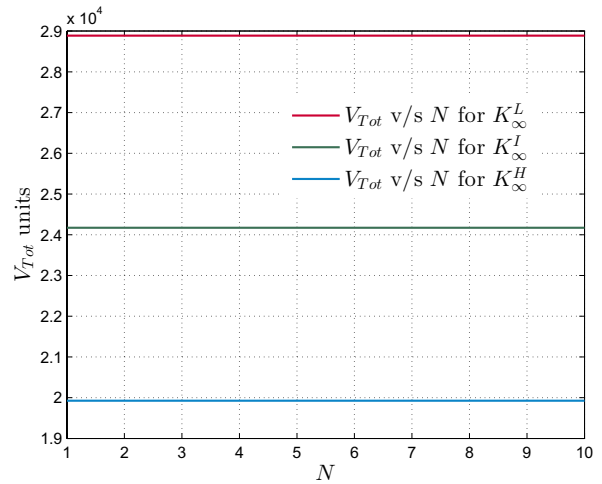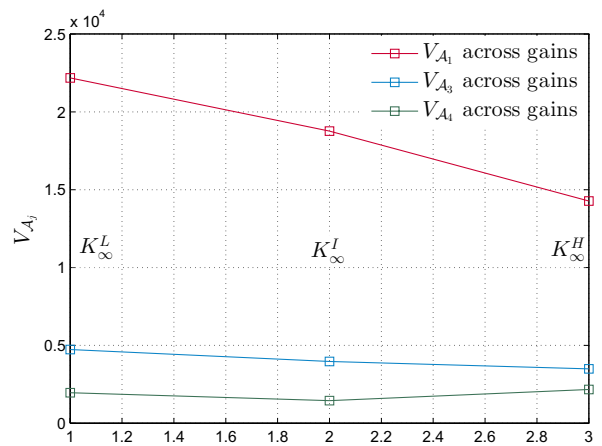

Fig. 1 V_{Tot} v/s N

Fig. 2 V_{A_j} across gains for $N = 7$

$V_{Tot} = (NP - NF)V_{cube}/NP$, where NP is total number of parametric points $(\hat{x}_{k|k-1}, y_k)$ randomly realized from the interval bounds for the state and measurement and NF is the number of those points $(\hat{x}_{k|k-1}, y_k)$ that do not belong to the critical regions Eq.(28). The total volume therefore signifies the amount of feasible parametric space available. These were computed using $NP = 50000$ uniformly distributed values of $\hat{x}_{k|k-1}$ and y_k. For the sake of comparison, total volume (V_{Tot}) of the partitioned space corresponding to different values of N (from 1 to 10) involving these three gains, were computed. Across prediction horizon V_{Tot} remains constant

Table 1 Volume of each critical region for different estimator gains

Gains	Critical Region Volumes				
	V_{A_1}(units)	V_{A_2}(units)	V_{A_3}(units)	V_{A_4}(units)	V_{A_5}(units)
K_∞^H	1.4269×10^4	0.0004×10^4	0.348×10^4	0.2163×10^4	0.0004×10^4
K_∞^I	1.8762×10^4	0.0004×10^4	0.3964×10^4	0.1443×10^4	0.0001×10^4
K_∞^L	2.2178×10^4	0.0024×10^4	0.4731×10^4	0.1954×10^4	0.0001×10^4
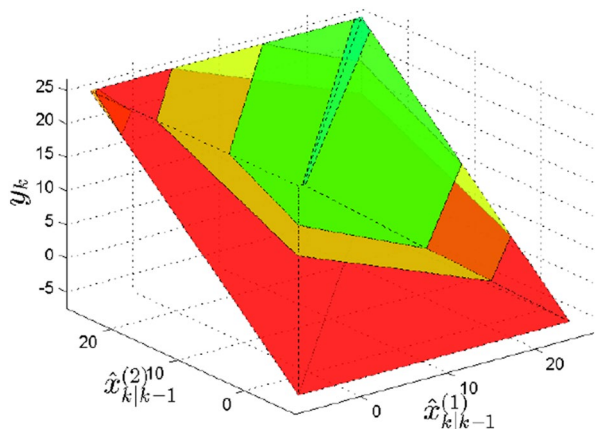
Fig. 3 Critical regions ($N_{CR} = 5$) for K_∞^H
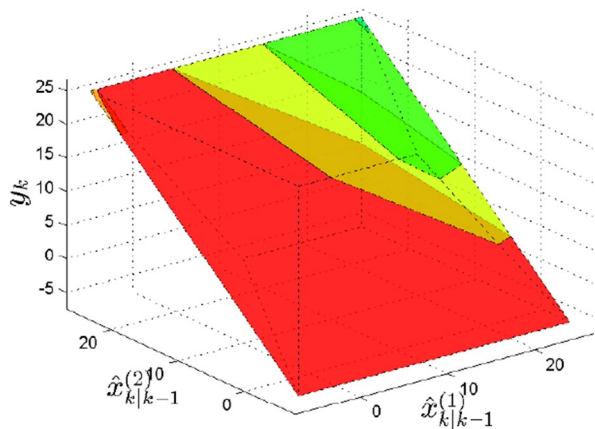
Fig. 4 Critical regions ($N_{CR} = 5$) for K_∞^I
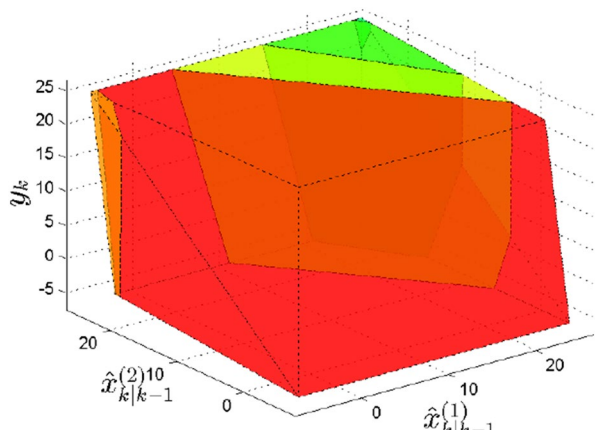
Fig. 5 Critical regions ($N_{CR} = 5$) for K_∞^L
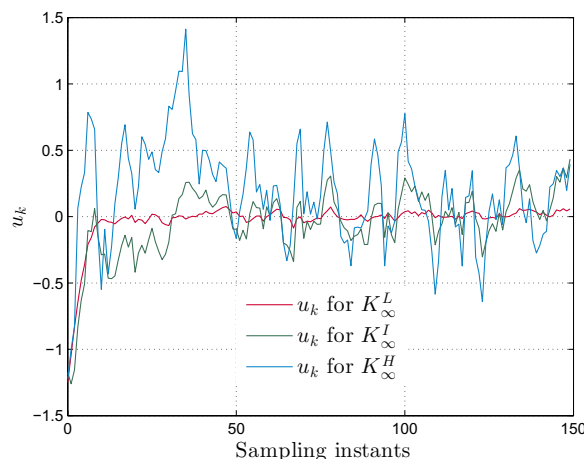
Fig. 6 Control inputs for K_∞^L, K_∞^I and K_∞^H

but decreases as the gain varies from K_∞^L to K_∞^H as shown in Fig. 1. However, for $N = 5$ and 7 volume pertaining to critical regions $\mathcal{A}_j$, $j = 1, 3, 4$ varies with N and the estimator gains as shown in Table 1 and Fig. 2.

For the two tank example the number of states is $n = 2$, number of inputs $m = 1$ and prediction horizon $N = 5$ having total number of constraints $p = 34$ with 5 decision variables for each of the three gains. The two tank example is chosen because the partitioned space involves only three parameters, i.e., 2 predicted state estimates $\hat{x}_{k|k-1}$ and 1 measurement y_k which makes the partitioned space easy to visualize. Three fully partitioned spaces were obtained for respective gains each having $N_{CR} = 5$ critical regions as shown in Figs. 3, 4 and 5. Each of these partitioned space has an empty set { } with the following set of constraints ({15}, {30}, {30, 31}, {30, 31, 32}) that are active. The critical regions given by Eq.(28) corresponding to each of these sets are represented by different colours in Figs. 3, 4 and 5. The partitions have their total volumes as 19926*units* ($NF = 16557$), 24174*units* ($NF = 9428$) and 28888*units* ($NF = 1515$), respectively. The un-partitioned space for the above example is a cube of side 31*units* having volume $V_{cube} = 29791units$. The individual volume $V_{\mathcal{A}_j}$, $j = 1, 2, 3, 4, 5$ for each critical region for the three gains is given in Table.1. Note that the total volume of these three partitions is less than the volume of the un-partitioned space and that reflects the effect of the estimator gains (K_∞^H, K_∞^I, K_∞^L) on the feasible space as shown in Figs. 3, 4 and 5. Moreover, Fig. 1 stresses that the volume V_{Tot} decreases with increase in estimator gain K_∞, which also means that volume of the partitioned space sets limit over the full state estimate information $\hat{x}_{k|k-1}$ and measurement y_k, that will be available

Table 2 ∞-norm of each $\mathcal{S}_{y,j}^0$ for different estimator gains

Gains	$\|\mathcal{S}_{y,1}^0\|_\infty$	$\|\mathcal{S}_{y,2}^0\|_\infty$	$\|\mathcal{S}_{y,3}^0\|_\infty$	$\|\mathcal{S}_{y,4}^0\|_\infty$	$\|\mathcal{S}_{y,5}^0\|_\infty$
K_∞^H	0.6227	2.1184	0	0	1.0796×10^{-16}
K_∞^I	0.3745	1.2131	0	0	6.4966×10^{-17}
K_∞^L	0.0686	0.1949	0	0	1.1915×10^{-17}

Table 3 output y_k mean squared error (MSE) values for different estimator gains

System variable	$MSE(K_\infty^H)$	$MSE(K_\infty^I)$	$MSE(K_\infty^L)$
y_k	0.4342	0.0669	0.0353

for the control input u_k and the statistics of Table 1 emphasizes it. The shrunken parametric space shown in Figs. 3, 4 and 5 basically points out that there is no admissible control input available for some parameters.

Effect of Estimator gain on u_k: It is The MPC problem $\mathbb{P}_1$ is solved in MpQP fashion for 150 sampling instants with $N = 5$ and initial condition $x^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

The control inputs are obtained for K_∞^H, K_∞^I and K_∞^L, respectively. To assess the sensitivity of control u_k to measurement y_k the following norm $\|\mathcal{S}_{y,j}^0 y_k\|_\infty$ is computed across the gains and for each control law given by Eq. (33) as given in Table 2. Table 2 and Fig. 6 emphasize that for lower gain values the impact of y_k on $U_j(\hat{x}_{k|k-1}, y_k)$ given by Eq.(33) diminishes. For assessing the impact of u_k on the system given by Eq.(34)-(35) mean squared error between system output y_k and the origin is computed as given in Table 3. Figure 6 and Table 3 indicate that as the MSE values across gains decreases the ability to control system output y_k improves.

The parametric space and the control law computations are performed using MPT3 Toolbox [14]

6 Conclusion

Any practical use of MPC requires simultaneous implementation of state estimation, including estimation of disturbance or fault parameters. Since online implementation of MPC conventionally uses numerical methods for solving the QP, the effect of the estimator parameters on the controller performance is unclear. In this work, an attempt has been made to analytically quantify the effect by making use of multiparametric methods to obtain an explicit solution

of the joint estimator-MPC problem. The resulting solution is affine in the predicted state and measurement. It is shown that the estimator gain directly impacts the size of the critical regions in MPC. Our future work will consider the explicit solution of simultaneous state and parameter estimation along with MPC. This will enable an explicit verifiable solution for various fault scenarios.

Declarations

Appendix

Prediction Equations: While Eq.(7) propagates in future, all future measurements within prediction horizon N are considered to be constant and equal to y_k. The prediction equations for the estimator dynamics are as follows.

$$\chi = \tilde{A}\hat{x}_{k|k-1} + \tilde{B}U + \tilde{\Gamma}y_k \tag{1}$$

where

$$\tilde{B} = \begin{bmatrix} \mathbf{0}_{n\times m} & \cdots & \cdots & \mathbf{0}_{n\times m} \\ B & \ddots & & \vdots \\ AB & B & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0}_{n\times m} \\ A^{N-1}B & \cdots & AB & B \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} I_{n\times n} \\ A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{bmatrix},$$

$\tilde{A} = \hat{A}(I - K_\infty C)$ and $\tilde{\Gamma} = \hat{A}K_\infty$

The prediction equations in Eq.(1) are linear in K_∞.

Constraint Equations: The constraint matrices for Eq.(11) are as follows.

$$G = \begin{bmatrix} \tilde{B} \\ -\tilde{B} \\ I_{mN\times mN} \\ -I_{mN\times mN} \end{bmatrix}, \quad S^x = S(I - K_\infty C),$$

$$S^y = SK_\infty, \quad S = \begin{bmatrix} -\hat{A} \\ \hat{A} \\ \mathbf{0}_{mN\times n} \\ \mathbf{0}_{mN\times n} \end{bmatrix} \text{ and } W = \begin{bmatrix} \overline{X} \\ -\underline{X} \\ \overline{U} \\ -\underline{U} \end{bmatrix}$$

Objective Function: The matrices corresponding to $\hat{J}(U, \hat{x}_{k|k-1}, y_k)$ are as follows.

$H = 2(\tilde{B}^T\tilde{Q}\tilde{B} + \tilde{R})$, $F_x = 2(I - K_\infty C)^T\hat{A}^T\tilde{Q}\tilde{B}$ and $F_y = 2K_\infty^T\hat{A}^T\tilde{Q}\tilde{B}$

where $\tilde{Q} = diag(Q, Q, \cdots, Q, P_f)$ and $\tilde{R} = diag(R, R, \cdots, R, R)$.

Constraint Matrices for Standard MpQP: The constraint matrices for the RHS of Eq.(14) are given below

$E^x = \tilde{E}(I - K_\infty C)$, $E^y = \tilde{E}K_\infty$ and

$\tilde{E} = (S + 2GH^{-1}\tilde{B}^T\tilde{Q}\hat{A})$

Details related to Theorem 1: The matrices corresponding to Eq.(23) are as follows.

$\mathcal{F}_{\mathcal{A}_j,z_x} = \tilde{\mathcal{F}}_{z,j}(I - K_\infty C)$, $\mathcal{F}_{\mathcal{A}_j,z_y} = \tilde{\mathcal{F}}_{z,j}K_\infty$ and $g_{\mathcal{A}_j,z} = H^{-1}G_{\mathcal{A}_j}^T(G_{\mathcal{A}_j}H^{-1}G_{\mathcal{A}_j}^T)^{-1}W_{\mathcal{A}_j}$.

where $\tilde{\mathcal{F}}_{x,j} = H^{-1}G_{\mathcal{A}_j}^T(G_{\mathcal{A}_j}H^{-1}G_{\mathcal{A}_j}^T)^{-1}\tilde{E}_{\mathcal{A}_j}$.

The matrices corresponding to Eq.(24) are as follows.

$\mathcal{F}_{\mathcal{A}_j,\lambda_x} = \tilde{\mathcal{F}}_{\lambda,j}(I - K_\infty C)$, $\mathcal{F}_{\mathcal{A}_j,\lambda_y} = \tilde{\mathcal{F}}_{\lambda,j}K_\infty$ and $g_{\mathcal{A}_j,\lambda} = -(G_{\mathcal{A}_j}H^{-1}G_{\mathcal{A}_j}^T)^{-1}W_{\mathcal{A}_j}$

where $\tilde{\mathcal{F}}_{\lambda,j} = (G_{\mathcal{A}_j}H^{-1}G_{\mathcal{A}_j}^T)^{-1}\tilde{E}_{\mathcal{A}_j}$.

The matrices corresponding to Eq.(28) are as follows.

$\tilde{H}_{\mathcal{A}_j,x} = \tilde{H}_j(I - K_\infty C)$, $\tilde{H}_{\mathcal{A}_j,y} = \tilde{H}_jK_\infty$ and $\tilde{L}_j = \begin{bmatrix} g_{\mathcal{A}_j,\lambda} \\ W_{\mathcal{I}_j} - G_{\mathcal{I}_j}g_{\mathcal{A}_j,z} \end{bmatrix}$

where $\tilde{H}_j = \begin{bmatrix} -(G_{\mathcal{A}_j}H^{-1}G_{\mathcal{A}_j}^T)^{-1}\tilde{E}_{\mathcal{A}_j} \\ G_{\mathcal{I}_j}H^{-1}G_{\mathcal{A}_j}^T(G_{\mathcal{A}_j}H^{-1}G_{\mathcal{A}_j}^T)^{-1}\tilde{E}_{\mathcal{A}_j} - \tilde{E}_{\mathcal{I}_j} \end{bmatrix}$

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