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Two-layer models for shallow avalanche flows over arbitrary variable topography

I. Luca · K. Hutter · C. Y. Kuo · Y. C. Tai

Abstract This paper presents a three-dimensional, two-layer model for shallow geophysical mass flows, such as debris flows, hydraulic sediment transport, or sub-aquatic turbidity currents down arbitrary natural topographic terrains. The bottom layer is a dense granular fluid which interacts with the stagnant basal topography through an erosion/deposition mechanism. Above this layer is a lighter fluid layer. There is no mass exchange at the layer interface and at the free upper surface, and the materials in both layers are treated as density preserving. The intrinsic modelling equations are written in non-dimensional form and then formulated relative to a topography-adjusted coordinate system. The mass balance equations and momentum balance equations parallel to the bottom topography are depth-averaged over the layers. The emerging governing system of equations is subsequently simplified on the basis of problem-adapted scales, in which a small parameter ϵ, the *shallowness parameter*, plays a central role. The proposed ordering scheme is motivated by an earlier analysis, [1], and depends on the rheological complexities of the stress parameterizations of the two fluids. The ensuing equations are complemented by constitutive assumptions in each layer, at the bottom topography and at the layer interface.

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1 Introduction

Catastrophic debris flows of fluid–solid mixtures which may occur in typhoon or hurricane induced landslides, in fluvial hydraulic currents during strong precipitation events, or in sub-aquatic turbidity currents under slope instabilities, occasionally develop into a two-layer flow regime of a relatively dense near bottom debris layer carrying the coarse sediment fraction plus a water layer, which carries the clay and silt fractions in suspension and may be regarded as a slurry. The granular fluid system is nourished by soil mass entrained from the stagnant bottom region, over which the solid–fluid system runs. Both layers may in a first approximation be viewed as one-constituent continuous immiscible fluid like bodies, and hence separated from one another by an interface which is material for each layer. Moreover, in a first approximation, the free surface of the upper layer may also be treated as material, thus ignoring the contribution of the precipitation that generally takes place in such events.

In general, the eroding soil has a larger density than the dense debris of the lower layer. However, the increase in density by the eroding soil is approximately counter balanced by the interstitial fluid that may be exchanged between the two layers. In a single constituent treatment of the material in the two regions, this water exchange is not modelled, so that the underlying assumption that the two layer materials are density preserving and that the interface is material with respect to each layer accounts in a gross fashion for the realistic behavior.

The flow of such fluid systems takes place over natural territory, practically "arbitrary" topography, that is best available in digitized files of electronic Geographical Information Systems (GIS) with resolutions of $25 \times 25\,\mathrm{m}^2$, $5 \times 5\,\mathrm{m}^2$, or $1 \times 1\,\mathrm{m}^2$ in exceptional cases. The flow geometries are of large extent in directions parallel to the topographies but shallow perpendicular to them. Moreover, tracks along which these avalanching masses move, may exhibit strong or weak curvatures, so that straight flat trajectories are rather the exception than the rule.

The above description outlines the intentions of the content of this paper: the development of the governing equations describing the dynamics of a shallow two-layer system of a dense granular fluid overlain by a particle-laden slurry down "arbitrary" topography. This system is fed from below by eroding soil, implying that the interface between the moving debris and the stagnant base is non-material. The two fluids are treated as density preserving and are separated from one another by an interface which is material. The balance equations of mass and momentum and the kinematic and dynamic boundary and transition conditions are written in non-dimensional form with the aid of a typical length tangent to the topography, the gravitational acceleration and the mass densities of the layers. Then they are formulated in curvilinear coordinates, adjusted to the geometry of the evolving base – two coordinates are parameters on this base and the third is measured orthogonal to it. The evolving interfaces, one between the moving debris and the stagnant base, and the other between the two layers, are also part of the solution.

Following earlier work in an attempt to reduce the complexity of the theoretical model, the three-dimensional governing equations are integrated over the individual layer thicknesses perpendicular to the basal sliding surface. This reduces the three-dimensional equations to spatially two-dimensional equations of motion, governing the balances of mass and momentum in each layer.

The theory is developed for unspecified phenomenological properties of the two fluids comprising the layer materials, an unspecified erosion parameterization and geometrical peculiarities of the topography and the moving masses. This allows identification of the orders of magnitudes of individual terms on the basis of estimated scales which are thought to be typical for the processes in question. Central in this scaling process is the introduction of a shallowness parameter, $\epsilon \equiv H/L$, which is the ratio of a typical avalanche thickness to a typical topography-parallel length, and which is small, realistically 10^{-2} to 10^{-3}. The dimensionless orders of magnitude or quantities arising in the modelling equations are expressed as certain powers of ϵ, e.g. ϵ^{γ}, $0 < \gamma < 1$. All terms of the non-dimensionalized dynamical equations receive in this way their individual ϵ-weight, which suggests various approximate formulations by dropping those terms which are of higher order small. In this estimating process, earlier experiences with catastrophic avalanche models serve as guide lines, Luca et al. [1–3].

Restricting these equations to those terms which are of order ϵ and larger, the momentum equation normal to the evolving topography reduces to pressure balances that are reminiscent to the hydrostatic pressure balance; the remaining layer balances then comprise equations for the layer thicknesses, layer-depth averaged velocity components parallel to the evolving topography and an evolution equation for the basal surface. In these equations the rheological stress parameterizations (of which only orders of magnitudes in the ϵ-scale are prescribed), and the erosion rate function remain unspecified and are left to the reader for individual parameterization. These equations are formulated for topographies with arbitrary curvature, but the equations considerably simplify when small curvature of the basal topography is assumed.

By specifying the orders of magnitude of the components of the extra stress tensors in planes parallel to the basal topography and perpendicular to it in a fashion as outlined in earlier works, Luca et al. [1,3], we arrive at three different models of avalanching motion of this two-layer dynamical description. We also explain the use of the moving curvilinear coordinate system with two illustrative examples, at least one of them (see Option 1 in Sec. 12) ideally suited to GIS-based formulations.

Even though this paper presents results of independent and original research, results derived in earlier papers are used, perhaps more extensively than is generally common in usual scientific manuscripts. The papers of which knowledge will be helpful to the reader are Luca, Tai and Kuo [2], [3] and, in particular Luca et al. [1].

Related works on two-layer shallow flows using topography-adapted curvilinear coordinates are due to Morales de Luna [4] and Fernández-Nieto et al. [5]. The first cited paper treats the upper layer as a compressible Euler fluid, and the lower layer as an inviscid fluid. In the second cited paper both layers are incompressible, the upper one being an inviscid fluid, and the lower layer being treated as a two-component mixture, in which the grains and the interstitial fluid move with the same velocity. The approaches in both [4] and [5] refer to immiscible layers and to a non-erodible bottom topography with small curvature, the one-dimensional case being considered; numerical results are therein presented.

Now we introduce some notations which we use throughout the paper. Thus, the 2×2 matrices are denoted by capital upright boldface letters, e.g. $\mathbf{A}$, and the 2-column matrices are denoted by small upright boldface letters, e.g. $\mathbf{a}$. A similar notation, but with slanted letters, is used for vectors and tensors, e.g. $\boldsymbol{a}$, $\boldsymbol{A}$. The dyadic product of two column matrices $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{a} \otimes \mathbf{b} \equiv \mathbf{a}\mathbf{b}^T$, where the superscript T stands for the transpose of a matrix; the symbol $\otimes$

also denotes the tensor product of two vectors. The inner product of the 2-column matrices $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{a}\cdot\mathbf{b} \equiv \mathrm{tr}\,(\mathbf{a}\mathbf{b}^T)$, where tr denotes the trace operator, and the inner product of the squared matrices $\mathbf{A}$, $\mathbf{B}$ is defined as $\mathbf{A}\cdot\mathbf{B} \equiv \mathrm{tr}\,(\mathbf{A}\mathbf{B}^T)$. The Greek indices have the values 1, 2, the Latin indices range from 1 to 3, and summation over repeated indices is understood. The 2×2 matrix $\mathbf{I}$ denotes the unit matrix, and $\delta^\alpha{}_\beta$ is the Kronecker symbol.

Then, since we do not change the notation for the function obtained from a given one by a change of variables, to distinguish various differential operators we proceed as follows. For a scalar function f and a 2-column matrix function $\mathbf{v} \equiv (v_1, v_2)^T$ depending on x_1, x_2, we set the gradient as

$$\mathrm{grad}\, f \equiv \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)^T, \quad \mathrm{grad}\,\mathbf{v} \equiv \left(\frac{\partial v_\alpha}{\partial x_\beta}\right).$$

If the independent variables with respect to which the gradient operator is performed are ξ^1, ξ^2, instead of grad f, grad $\mathbf{v}$ we use Grad f, Grad $\mathbf{v}$, respectively. Moreover, we define

$$\mathrm{Div}\,\mathbf{v} \equiv \frac{\partial v_\alpha}{\partial \xi^\alpha}, \quad \mathrm{Div}\,\mathbf{T} \equiv \frac{\partial T_{\alpha\beta}}{\partial \xi^\beta}\mathbf{e}_\alpha,$$

$$\mathbf{e}_1 \equiv (1,0)^T, \quad \mathbf{e}_2 \equiv (0,1)^T,$$

where $\mathbf{T}$ is a 2×2-matrix function depending on ξ^1, ξ^2, with $T_{\alpha\beta}$ as entries.

Finally, concerning differentiation with respect to t (time), we make the following remarks. We use the symbol $\partial/\partial t$ to denote the partial derivative with respect to t of a function depending on x_1, x_2, x_3, t or x, y, t. In Section 2, where the independent variables are ξ^1, ξ^2, t, besides x_1, x_2, x_3, t, there is no need to change this notation (the differentiation is clear). But beginning with Section 3 we deal with 2 time-dependent changes of variables – for fixed t, in (3.17) one relates (x, y) to (ξ^1, ξ^2), and in (3.22) one relates (x_1, x_2, x_3) to (ξ^1, ξ^2, ξ^3). That is why, for the sake of clarity of the derivations, when performing the time differentiation according to these changes of variables we shall distinguish notation for the time derivative as follows: the symbol $\hat{\partial}/\partial t$ is used when the independent variables are ξ^1, ξ^2, t according to (3.17) (see e.g. (3.42)), and $\tilde{\partial}/\partial t$ is used when the independent variables are ξ^1, ξ^2, ξ^3, t according to (3.22) (see e.g. (4.44)).[1] Of course, for a function f depending on ξ^1, ξ^2, t, we have $\hat{\partial} f/\partial t = \tilde{\partial} f/\partial t$.

[1] We hope that, by introducing these notations for the time derivative of a function, no confusion occurs when we refer to the time derivative(s) of the function b which specifies the elevation of the topographic surface. This is so, because we have $b = b(x, y, t)$, and just changing the notation of the independent variables (see the parameterization (3.14)) one may write $b = b(x_1, x_2, t)$. Therefore, apart from $\partial b/\partial t$ (when x, y or x_1, x_2 are envisaged), the derivatives $\hat{\partial} b/\partial t$ and $\tilde{\partial} b/\partial t$ are also defined; what we really need (and use) is $\hat{\partial} b/\partial t$.

2 Basics from the geometry and kinematics of a moving surface

Let $\mathcal{E}$ be a three-dimensional Euclidean point space, of which the translation vector space is denoted by $\mathcal{V}$, Ox_1x_2, x_3 an orthogonal Cartesian coordinate system for $\mathcal{E}$, physically associated to an inertial reference frame $\mathcal{R}$ and such that Ox_3 is the vertical direction, and $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ the orthonormal basis of $\mathcal{V}$ corresponding to $Ox_1x_2x_3$. We consider a moving (in $\mathcal{R}$) surface, that is, a one-parameter family $\{\mathcal{S}_t\}_{t\in I}$, with $I \subset \mathbb{R}$ an open (time) interval, of regular surfaces $\mathcal{S}_t \subset \mathcal{E}$, defined parametrically by

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\xi^1, \xi^2, t) = x_k(\xi^1, \xi^2, t)\boldsymbol{i}_k, \quad (\xi^1, \xi^2) \in \Delta_0, \quad t \in I, \tag{2.1}$$

where $\boldsymbol{\rho} \in \mathcal{V}$ denotes the position vector of a point on $\mathcal{S}_t$ with respect to the origin $O \in \mathcal{E}$ of the Cartesian coordinate system, Δ_0 is an open subset of $\mathbb{R}^2$, and the function $\boldsymbol{\rho}$ is of class C^2 on $\Delta_0 \times I$. We denote by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ the vectors of the natural basis of the tangent space to $\mathcal{S}_t$, that is

$$\boldsymbol{\tau}_\alpha \equiv \frac{\partial \boldsymbol{\rho}}{\partial \xi^\alpha}, \quad \alpha \in \{1, 2\}, \tag{2.2}$$

and define a unit vector field normal to $\mathcal{S}_t$ by

$$\boldsymbol{n} \equiv \frac{\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2}{\|\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2\|}, \tag{2.3}$$

where $\times$ stands for the cross product in $\mathcal{V}$, and $\|\|$ represents the Euclidean norm on $\mathcal{V}$. It is clear that $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$, and hence $\boldsymbol{n}$ too, depend on ξ^1, ξ^2, t, but for simplicity of notation we omit this dependence. The *coefficients of the first fundamental form* of $\mathcal{S}_t$ corresponding to the parameterization (2.1) are

$$\phi_{\alpha\beta} \equiv \boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta, \quad \alpha, \beta \in \{1, 2\},$$

and by the representations

$$\frac{\partial \boldsymbol{n}}{\partial \xi^\beta} = -b_{\alpha\beta}\,\boldsymbol{\tau}^\alpha = -W^\alpha{}_\beta\,\boldsymbol{\tau}_\alpha, \quad \beta \in \{1, 2\}, \tag{2.4}$$

where $\{\boldsymbol{\tau}^1, \boldsymbol{\tau}^2\}$ is the reciprocal basis of $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$, one defines the *coefficients $b_{\alpha\beta}$ of the second fundamental form* of $\mathcal{S}_t$ and the entries $W^\alpha{}_\beta$ of the *Weingarten matrix* (corresponding to the parameterization (2.1) of the *oriented* surface $\mathcal{S}_t$). For the matrices $(\phi^{\alpha\beta})$, $(b_{\alpha\beta})$, $(W^\alpha{}_\beta)$, where $\phi^{\alpha\beta} \equiv \boldsymbol{\tau}^\alpha \cdot \boldsymbol{\tau}^\beta$, we use the symbols

$$\mathbf{M}_0 \equiv (\phi^{\alpha\beta}), \quad \mathbf{H} \equiv (b_{\alpha\beta}), \quad \mathbf{W} \equiv (W^\alpha{}_\beta), \tag{2.5}$$

and mention that

$$(\phi_{\alpha\beta}) = \mathbf{M}_0^{-1}, \quad \mathbf{W} = \mathbf{M}_0\mathbf{H}. \tag{2.6}$$

Then, the *curvature tensor* and the *mean curvature* of the surface $\mathcal{S}_t$ are

$$\mathcal{H} \equiv b_{\alpha\beta}\boldsymbol{\tau}^\alpha \otimes \boldsymbol{\tau}^\beta, \quad \Omega \equiv \frac{1}{2}\mathrm{tr}\,\mathcal{H} = \frac{1}{2}\mathrm{tr}\,\mathbf{W}.$$

Finally, the vector

$$\boldsymbol{u}_\mathcal{S} \equiv \frac{\partial \boldsymbol{\rho}}{\partial t} \tag{2.7}$$

is the *velocity* of the surface parameters (ξ^1, ξ^2) at the moment t. We denote by $\mathcal{U}$ the normal velocity of (ξ^1, ξ^2) at the moment t, that is,

$$\mathcal{U} \equiv \boldsymbol{u}_\mathcal{S} \cdot \boldsymbol{n}, \tag{2.8}$$

and therefore, representing $\boldsymbol{u}_\mathcal{S}$ with respect to the basis $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n}\}$ of $\mathcal{V}$, we can write

$$\boldsymbol{u}_\mathcal{S} = U^\beta \boldsymbol{\tau}_\beta + \mathcal{U}\boldsymbol{n}. \tag{2.9}$$

Moreover, if for $\mathcal{S}_t$ given by (2.1) we use the implicit representation

$$F(x_1, x_2, x_3, t) = 0,$$

with F so chosen, that $\boldsymbol{n}$ defined by (2.3) can be expressed as

$$\boldsymbol{n} = \nabla F/\|\nabla F\|, \quad \nabla F \equiv \frac{\partial F}{\partial x_k}\boldsymbol{i}_k, \tag{2.10}$$

we have

$$\frac{\partial F}{\partial t} + \nabla F \cdot \boldsymbol{u}_\mathcal{S} = 0, \tag{2.11}$$

and hence $\mathcal{U}$ can be deduced as

$$\mathcal{U} = -\frac{\partial F}{\partial t}/\|\nabla F\|; \tag{2.12}$$

it follows that $\mathcal{U}$, unlike $\boldsymbol{u}_\mathcal{S}$, is independent on the parameterization of $\mathcal{S}_t$, and that is why it is called the *speed of displacement* of the (oriented) surface $\mathcal{S}_t$, see e.g. Truesdell and Toupin [6], p. 499.

3 Topography description and change of coordinates near the basal topography

When erosion/deposition processes are present during the avalanche flow, the basal topography (i.e., the surface on which the avalanching mass *flows*) changes in time. Thus, we suppose that the topographic bed (e.g., mountain and deposited material from the avalanching mass, see Figure 1) is at rest in the inertial reference frame $\mathcal{R}$, and model the basal topography by a moving surface $\mathcal{S} \equiv \{\mathcal{S}_t\}_{t\in I}$, given parametrically by

$$\boldsymbol{\rho} = \boldsymbol{\rho}(x, y, t), \quad \boldsymbol{\rho}(x, y, t) \equiv x\boldsymbol{i}_1 + y\boldsymbol{i}_2 + b(x, y, t)\boldsymbol{i}_3, \tag{3.13}$$

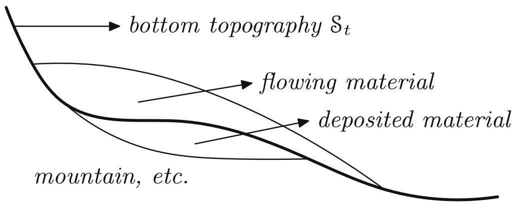

Fig. 1. Basal topography $\mathcal{S}_t$.

where $(x, y) \in \Delta$ and $t \in I$, or, equivalently,

$$x_1 = x, \quad x_2 = y, \quad x_3 = b(x, y, t). \tag{3.14}$$

If b is independent of time t, which happens if there is no erosion of the topographic bed and no deposition on this bed, we say that the *basal topography is fixed*; otherwise, we say that we have a *variable* or *moving basal topography*. The representation of $\mathcal{S}_t$ using Cartesian coordinates is in conformity with the way in which GIS data are usually recorded (the digital data consist of regularly spaced elevation values referenced horizontally).

We let the unit normal vector $\boldsymbol{n}$ to $\mathcal{S}_t$ point into the avalanche body, denote its components with respect to the Cartesian basis $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of $\mathcal{V}$ by $-s_1, -s_2, c$, and collect s_1, s_2 into the 2-column matrix

$$\mathbf{s} \equiv (s_1, s_2)^T.$$

We have

$$c = (1 + \mathrm{grad}\, b \cdot \mathrm{grad}\, b)^{-1/2}, \quad \mathbf{s} = c\, \mathrm{grad}\, b. \tag{3.15}$$

Note that, the implicit representation of the moving surface (3.14), consistent with the above prescription of the unit normal vector $\boldsymbol{n}$ in the sense that (2.10) holds, can be written as

$$F(x_1, x_2, x_3, t) = 0, \quad F(x_1, x_2, x_3, t) \equiv x_3 - b(x_1, x_2, t),$$

and hence the speed of displacement of the bottom topography is given by, see (2.12),

$$\mathcal{U} = c\frac{\partial b}{\partial t}. \tag{3.16}$$

In transport/sediment context, the speed $\mathcal{U}$ is called *erosion/deposition rate*. It is clear that, if $\mathcal{U} > 0$ sediments are deposited, and if $\mathcal{U} < 0$ erosion occurs. The case $\mathcal{U} = 0$ for any $(x, y) \in \Delta$ and for any $t \in I$ corresponds to a fixed bottom topography. At points on $\mathcal{S}_t$ where there is no avalanche mass we consider $\mathcal{U} = 0$.

As for the case of fixed topography, motivated by computational necessities, we may want to work, at each moment t, with another parameterization than (3.14) of $\mathcal{S}_t$. That is, at the moment t, instead of using the Cartesian parameters

x, y, we want to use other parameters on $\mathcal{S}_t$, say ξ_t^1, ξ_t^2, related to x, y by a one-to-one correspondence

$$x = x(\xi_t^1, \xi_t^2), \quad y = y(\xi_t^1, \xi_t^2).$$

Moreover, we suppose that the choice of ξ_t^1, ξ_t^2 at each moment t is not so "wild", by requiring this transformation to be of the form

$$x = x(\xi^1, \xi^2, t), \quad y = y(\xi^1, \xi^2, t), \quad (\xi^1, \xi^2) \in \Delta_0, \quad t \in I, \tag{3.17}$$

with functions x, y of class C^2 on $\Delta_0 \times I$. According to (3.13) and (3.17), the moving surface $\mathcal{S}$ can be described by $\boldsymbol{\rho} = \boldsymbol{\rho}(\xi^1, \xi^2, t)$, where

$$\boldsymbol{\rho}(\xi^1, \xi^2, t) \equiv x(\xi^1, \xi^2, t)\boldsymbol{i}_1 + y(\xi^1, \xi^2, t)\boldsymbol{i}_2 + b(x(\xi^1, \xi^2, t), y(\xi^1, \xi^2, t), t)\boldsymbol{i}_3, \tag{3.18}$$

or, equivalently,

$$x_1 = x(\xi^1, \xi^2, t), \quad x_2 = y(\xi^1, \xi^2, t),$$
$$x_3 = b(x(\xi^1, \xi^2, t), y(\xi^1, \xi^2, t), t). \tag{3.19}$$

We assume

$$\det \mathbf{F} > 0, \quad \mathbf{F} \equiv \left(\frac{\partial x_i}{\partial \xi^\alpha}\right)_{i,\alpha \in \{1,2\}}, \tag{3.20}$$

which preserves the already defined orientation of $\mathcal{S}_t$. In Sec. 12 we return to transformation (3.17), show possible choices of it, and give a simple example. Finally, we mention that the matrices $\mathbf{M}_0$, $\mathbf{H}$ introduced in (2.5) and corresponding to (3.19) are

$$\mathbf{M}_0 = \mathbf{F}^{-1}(\mathbf{I} - \mathbf{s} \otimes \mathbf{s})\mathbf{F}^{-T}, \quad \mathbf{H} = c\mathbf{F}^T \operatorname{grad}(\operatorname{grad} b)\mathbf{F}, \tag{3.21}$$

see e.g. Luca, Tai and Kuo [2][2].

Now, in order to be able to properly account for the shallowness of the avalanche mass, we define a change of coordinates in the neighborhood of the basal variable topography (3.18). For fixed bottom topography, this change of coordinates was introduced by Bouchut and Westdickenberg [7] and, independently, by De Toni and Scotton [8]; it has been extended by Bouchut et al. [9] and Tai and Kuo [10] for the case of a particular variable basal surface (which we also consider in Sec. 12 as an example). Next we let the variable basal topography to be arbitrary, in a sense that will be made clear in due course.

Thus, if $\boldsymbol{r} = x_i\boldsymbol{i}_i$ is the position vector with respect to $O \in \mathcal{E}$ of a point P lying in that part of $\mathcal{E}$ to which the normal vector $\boldsymbol{n} = n_i\boldsymbol{i}_i$ to $\mathcal{S}_t$ points, and if $\boldsymbol{\rho} = \boldsymbol{\rho}(\xi^1, \xi^2, t) = \rho_i\boldsymbol{i}_i$ is the position vector of the orthogonal projection Q of P onto $\mathcal{S}_t$, the relation

$$\boldsymbol{r} = \boldsymbol{\rho} + \xi\boldsymbol{n} \Longleftrightarrow \boldsymbol{r}(x_1, x_2, x_3) = \boldsymbol{\rho}(\xi^1, \xi^2, t) + \xi\boldsymbol{n}(\xi^1, \xi^2, t)$$
$$\Longleftrightarrow x_i = \rho_i(\xi^1, \xi^2, t) + \xi n_i(\xi^1, \xi^2, t), \quad \xi > 0, \tag{3.22}$$

defines new coordinates ξ^1, ξ^2, ξ of P at the moment t, on the condition that

$$J \neq 0, \quad J \equiv \det \boldsymbol{A}^{-1},$$
$$\boldsymbol{A}^{-1} \equiv \left(\frac{\partial x_i}{\partial \xi^j}\right)_{i,j \in \{1,2,3\}}, \quad \xi^3 \equiv \xi, \tag{3.23}$$

which we next assume to be valid at each moment t, at least in the domain occupied by the avalanche body. It is clear that ξ is the distance between P and Q.

Obviously, since $\{\mathcal{S}_t\}_{t \in I}$ is a moving surface, the projection Q of P onto $\mathcal{S}_t$ changes in time, and hence the coordinates ξ^1, ξ^2, ξ of P depend on t. To be more precise, we should have written

$$\boldsymbol{r}(x_1, x_2, x_3) = \boldsymbol{\rho}(\xi^1(x_1, x_2, x_3, t), \xi^2(x_1, x_2, x_3, t), t) + \xi(x_1, x_2, x_3, t)\, \boldsymbol{n}(\xi^1(x_1, x_2, x_3, t), \xi^2(x_1, x_2, x_3, t), t), \tag{3.24}$$

instead of (3.22). However, both writings, (3.22) and (3.24), are useful:

(i) By keeping t fixed and varying x_1, x_2, x_3, (3.22) and (3.23) clearly show that we have a one-to-one correspondence between (x_1, x_2, x_3) and $(\xi^1, \xi^2, \xi^3 \equiv \xi)$. We point out some properties of this change of coordinates (for fixed t!) by simply taking them over from Bouchut and Westdickenberg [7] and Luca, Tai and Kuo [2].

Thus, the vectors

$$\boldsymbol{g}_k \equiv \frac{\partial \boldsymbol{r}}{\partial \xi^k} = A^{-1}_{jk}\boldsymbol{i}_j, \quad k \in \{1, 2, 3\}, \tag{3.25}$$

form the *natural basis* of $\mathcal{V}$ at P (at the moment t). As a rule, we shall use lower indices to denote the Cartesian components of vectors, and upper indices for their contravariant components with respect to $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$, i.e.

$$\boldsymbol{v} = v_k\boldsymbol{i}_k = v^l\boldsymbol{g}_l, \quad v_k = A^{-1}_{kl}v^l;$$

but e.g. v_1, v_2 should not be confused with $\mathrm{v}_1, \mathrm{v}_2$ in Sec. 7. It is not difficult to see from (3.22) that the vectors of the natural basis are given by

$$\boldsymbol{g}_\beta = (\delta^\alpha{}_\beta - \xi W^\alpha{}_\beta)\boldsymbol{\tau}_\alpha, \quad \beta \in \{1, 2\}, \quad \boldsymbol{g}_3 = \boldsymbol{n}. \tag{3.26}$$

In particular, (3.26) shows that $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ and $\boldsymbol{g}^1$, $\boldsymbol{g}^2$ are tangent vectors to $\mathcal{S}_t$, and that $\boldsymbol{g}_3 = \boldsymbol{g}^3$ are normal to $\mathcal{S}_t$, where $\{\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3\}$ is the reciprocal basis of the natural basis at

[2] Note that $\mathbf{H}$ in (3.21) corresponds to $\tilde{\mathbf{H}}$ in the cited paper.

P. To manage cumbersome calculations, a key point in the approach of Bouchut and Westdickenberg [7], also adopted in Luca, Tai and Kuo [2] and in this paper, was to use these properties of the vectors $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$ to collect the contravariant components of vector and tensor fields in block matrices, as follows. For a vector $\boldsymbol{v} \in \mathcal{V}$ and a symmetric second order tensor $\boldsymbol{\sigma}$ on $\mathcal{V}$, we have

$$\boldsymbol{v} = v^i \boldsymbol{g}_i, \quad \boldsymbol{\sigma} = \sigma^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j.$$

We note that

$$\boldsymbol{v}_n \equiv (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} = v^3 \boldsymbol{g}_3, \quad \boldsymbol{v}_\tau \equiv \boldsymbol{v} - \boldsymbol{v}_n = v^1\boldsymbol{g}_1 + v^2 \boldsymbol{g}_2, \tag{3.27}$$

and introduce the quantities

$$\mathbf{v} \equiv \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad \mathrm{v} \equiv v^3, \quad \mathbf{T} \equiv \begin{pmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{12} & \sigma^{21} \end{pmatrix}, \quad \mathbf{p} \equiv \begin{pmatrix} \sigma^{13} \\ \sigma^{23} \end{pmatrix}. \tag{3.28}$$

The change of basis matrix $\boldsymbol{A}^{-1}$, see (3.25), (3.23), has been written by Bouchut and Westdickenberg [7] in the following block matrix decomposition

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \mathbf{I} & -\mathbf{s} \\ \frac{1}{c}\mathbf{s}^T & c \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{B} & -\mathbf{s} \\ \frac{1}{c}(\mathbf{B}^T\mathbf{s})^T & c \end{pmatrix},$$
$$\mathbf{B} \equiv \mathbf{F}(\mathbf{I} - \xi \mathbf{W}), \tag{3.29}$$

which implies

$$J = J_0 \det(\mathbf{I} - \xi\mathbf{W}), \quad J_0 \equiv \frac{1}{c}\det \mathbf{F}. \tag{3.30}$$

Hence from (3.23) we deduce the restriction

$$\det(\mathbf{I} - \xi\mathbf{W}) \neq 0 \tag{3.31}$$

at the moment t, on the basal topography plus the domain occupied by the avalanche mass; it is (3.31) which clearly states what is meant by "arbitrary" topography. Note that, since condition (3.31) holds if $\xi = 0$, the change of coordinates is in fact valid not only for (some) $\xi > 0$, but also for $\xi \le 0$, at least for sufficiently small negative ξ. If $\xi < 0$, then the modulus of ξ, $|\xi|$, gives the distance between P and its perpendicular projection onto the surface. In short, the change of coordinates (3.22) is valid in a neighborhood of the basal surface, on both sides of it.

Corresponding to (3.22), the covariant coefficients $g_{ij} \equiv \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ and the contravariant coefficients $g^{ij} \equiv \boldsymbol{g}^i \cdot \boldsymbol{g}^j$ of the metric tensor can be expressed with the aid of $\boldsymbol{A}^{-1}$ and of the inverse matrix $\boldsymbol{A}$ of $\boldsymbol{A}^{-1}$, given by

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{B}^{-1}(\mathbf{I} - \mathbf{s}\otimes\mathbf{s}) & c\mathbf{B}^{-1}\mathbf{s} \\ -\mathbf{s}^T & c \end{pmatrix}, \tag{3.32}$$

as

$$(g_{ij}) = (\boldsymbol{A}\boldsymbol{A}^T)^{-1}, \quad (g^{ij}) = \boldsymbol{A}\boldsymbol{A}^T, \quad \boldsymbol{A}\boldsymbol{A}^T = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \tag{3.33}$$

where

$$\mathbf{M} \equiv \mathbf{B}^{-1}(\mathbf{I} - \mathbf{s}\otimes\mathbf{s})\mathbf{B}^{-T}. \tag{3.34}$$

We mention that $\mathbf{M}_0 = \mathbf{M}|_{\xi=0}$, see $(3.21)_1$, $(3.29)_2$. The matrix (g^{ij}) is positive definite, and hence $\mathbf{M}$ is a positive definite matrix.

(ii) Now, we keep the point P fixed in $\mathcal{E}$, implying that the Cartesian coordinates x_1, x_2, x_3 of P are kept fixed, and let t run in I. By differentiating (3.24) with respect to t (we recall the distinct notations for the partial time derivatives, see Introduction) we deduce that the "velocities" $\dot{\xi}^i \equiv \partial\xi^i/\partial t$, $i = 1, 2, 3$, of the coordinates ξ^1, ξ^2, ξ of P have to satisfy the relation

$$\frac{\hat{\partial}\boldsymbol{\rho}}{\partial t} + \xi\frac{\hat{\partial}\boldsymbol{n}}{\partial t} + \dot{\xi}^\alpha\left(\frac{\partial\boldsymbol{\rho}}{\partial\xi^\alpha} + \xi\frac{\partial\boldsymbol{n}}{\partial\xi^\alpha}\right) + \dot{\xi}\boldsymbol{n} = \boldsymbol{0},$$

which can be further written as, see definition (3.25) of $\boldsymbol{g}_\alpha$, $\alpha = 1, 2$,

$$\dot{\xi}^\alpha\boldsymbol{g}_\alpha + \dot{\xi}\boldsymbol{n} = -\boldsymbol{w}, \quad \boldsymbol{w} \equiv \frac{\hat{\partial}\boldsymbol{\rho}}{\partial t} + \xi\frac{\hat{\partial}\boldsymbol{n}}{\partial t}, \tag{3.35}$$

or, equivalently,

$$(\dot{\xi}^1, \dot{\xi}^2)^T = -\mathbf{w}, \quad \dot{\xi} = -\mathrm{w}, \tag{3.36}$$

where definitions (3.28) have been used to introduce the contravariant components $\mathbf{w}$ and w of $\boldsymbol{w}$. It is clear that $\boldsymbol{w}$ is the null vector if the basal topography does not move, as e.g. in Bouchut and Westdickenberg [7], Luca, Tai and Kuo [2]. In passing, note that (3.35) shows that the negative of $\boldsymbol{w}$ can be interpreted as the velocity of the curvilinear coordinates of P. Relations (3.36) will be used to derive rules of differentiation, see (4.44) below; that is why we need to determine $\mathbf{w}$, w. Thus we have

Proposition 3.1 *Let* $\mathbf{w}$, w *be the components with respect to the basis* $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ *of the vector* $\boldsymbol{w}$ *defined in* (3.35)*. Moreover, let* $\mathbf{u}_\mathcal{S}$*,* $\mathcal{U}$ *be the components with respect to* $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n}\}$ *of the velocity* $\boldsymbol{u}_\mathcal{S} \equiv \hat{\partial}\boldsymbol{\rho}/\partial t$ *of the surface parameters* (ξ^1, ξ^2)*, and denote*

$$\mathbf{v}_\mathcal{S} \equiv \left(\frac{\hat{\partial}x}{\partial t}, \frac{\hat{\partial}y}{\partial t}\right)^T, \tag{3.37}$$

where x, y *are the functions which define the transformation* (3.17)*. Then the following results hold:*

$$\mathbf{w} = \mathbf{u}_\mathcal{S} - \xi(\mathbf{I} - \xi\mathbf{W})^{-1}\mathbf{M}_0 \mathrm{Grad}\,\mathcal{U},$$
$$\mathbf{u}_\mathcal{S} = \mathbf{F}^{-1}(\mathbf{v}_\mathcal{S} + \mathcal{U}\mathbf{s}), \quad \mathrm{w} = \mathcal{U}. \tag{3.38}$$

Proof First, from $(3.26)_1$ we deduce

$$\boldsymbol{\tau}_\beta = \underset{\sim}{W}^\alpha{}_\beta \boldsymbol{g}_\alpha, \quad \underset{\sim}{\mathbf{W}} \equiv (\mathbf{I} - \xi \mathbf{W})^{-1},$$

and therefore

$$\frac{\hat{\partial} \boldsymbol{\rho}}{\partial t} \equiv \boldsymbol{u}_{\mathbb{S}} = \underset{\sim}{W}^\alpha{}_\beta U^\beta \boldsymbol{g}_\alpha + \mathcal{U}\boldsymbol{n} \tag{3.39}$$

holds, see (2.7), (2.9). Then, noticing that relation $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ forces $\hat{\partial}\boldsymbol{n}/\partial t$ to be a tangent vector to $\mathbb{S}_t$, we can write

$$\frac{\hat{\partial} \boldsymbol{n}}{\partial t} = a^\alpha \boldsymbol{g}_\alpha,$$

and hence we have to find a^α. Using $(3.26)_1$ we derive

$$a^\alpha = \frac{\hat{\partial} \boldsymbol{n}}{\partial t} \cdot \boldsymbol{g}^\alpha = g^{\beta\alpha} \frac{\hat{\partial} \boldsymbol{n}}{\partial t} \cdot \boldsymbol{g}_\beta = g^{\beta\alpha} (\delta^\gamma{}_\beta - \xi W^\gamma{}_\beta) \frac{\hat{\partial} \boldsymbol{n}}{\partial t} \cdot \boldsymbol{\tau}_\gamma .$$

Moreover, we compute

$$\frac{\hat{\partial} \boldsymbol{n}}{\partial t} \cdot \boldsymbol{\tau}_\gamma = -\boldsymbol{n} \cdot \frac{\hat{\partial} \boldsymbol{\tau}_\gamma}{\partial t} = -\boldsymbol{n} \cdot \frac{\partial \boldsymbol{u}_{\mathbb{S}}}{\partial \xi^\gamma}$$
$$= U^\beta \frac{\partial \boldsymbol{n}}{\partial \xi^\gamma} \cdot \boldsymbol{\tau}_\beta - \frac{\partial \mathcal{U}}{\partial \xi^\gamma} = -b_{\omega\gamma} U^\omega - \frac{\partial \mathcal{U}}{\partial \xi^\gamma},$$

where (2.2), (2.7), (2.9), (2.4) have been used, and therefore

$$\frac{\hat{\partial} \boldsymbol{n}}{\partial t} = -g^{\beta\alpha} (\delta^\gamma{}_\beta - \xi W^\gamma{}_\beta) \left(b_{\omega\gamma} U^\omega + \frac{\partial \mathcal{U}}{\partial \xi^\gamma} \right) \boldsymbol{g}_\alpha. \tag{3.40}$$

There only remains to substitute (3.39), (3.40) into definition (3.35) of $\boldsymbol{w}$, to deduce

$$\boldsymbol{w} = \left\{ \underset{\sim}{W}^\alpha{}_\beta U^\beta - \xi g^{\beta\alpha} (\delta^\gamma{}_\beta - \xi W^\gamma{}_\beta) \left(b_{\omega\gamma} U^\omega + \frac{\partial \mathcal{U}}{\partial \xi^\gamma} \right) \right\} \boldsymbol{g}_\alpha + \mathcal{U}\boldsymbol{n},$$

which, after a routine calculus involving $(3.33)_2$, (3.34), (2.6), implies

$$\mathbf{w} = \mathbf{u}_{\mathbb{S}} - \xi (\mathbf{I} - \xi \mathbf{W})^{-1} \mathbf{M}_0 \mathrm{Grad}\,\mathcal{U},$$
$$\mathrm{w} = \mathcal{U}, \quad \mathbf{u}_{\mathbb{S}} \equiv (U^1, U^2)^T. \tag{3.41}$$

Here we draw attention to the notation in (3.41): $\mathbf{u}_{\mathbb{S}}$ collects 2 of the components of $\boldsymbol{u}_{\mathbb{S}}$ with respect to the basis $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n}\}$, and not with respect to $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ (see (2.9), (3.39)), as we have agreed by (3.28).

We still have to derive the formula $(3.38)_2$ for $\mathbf{u}_{\mathbb{S}}$; we do this by taking advantage of knowing the matrix $\boldsymbol{A}$, see (3.32). Thus, first note that (3.26) implies $\boldsymbol{\tau}_\alpha = \boldsymbol{g}_\alpha|_{\xi=0}$, $\alpha = 1, 2$, and hence the change of basis matrix from $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ to $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n}\}$ is $\boldsymbol{A}^{-1}|_{\xi=0}$, that is $\boldsymbol{\tau}_k = A^{-1}_{jk}|_{\xi=0} \boldsymbol{i}_j$, where, to have a compact notation, $\boldsymbol{\tau}_3 \equiv \boldsymbol{n}$ is understood. Then, according to parameterization (3.19), definition (2.7) of the velocity $\boldsymbol{u}_{\mathbb{S}}$ of (ξ^1, ξ^2) gives

$$\boldsymbol{u}_{\mathbb{S}} = \frac{\hat{\partial} x}{\partial t} \boldsymbol{i}_1 + \frac{\hat{\partial} y}{\partial t} \boldsymbol{i}_2 + \frac{\hat{\partial} b}{\partial t} \boldsymbol{i}_3,$$
$$\frac{\hat{\partial} b}{\partial t} = \frac{\partial b}{\partial t} + \mathrm{grad}\, b \cdot \mathbf{v}_{\mathbb{S}}, \tag{3.42}$$

where $\mathbf{v}_{\mathbb{S}}$ is defined in (3.37), and therefore the relation between the components $\hat{\partial} x/\partial t, \hat{\partial} y/\partial t, \hat{\partial} b/\partial t$ of $\boldsymbol{u}_{\mathbb{S}}$ with respect to $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ and the components $U^1, U^2, \mathcal{U}$ of $\boldsymbol{u}_{\mathbb{S}}$ with respect to $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n}\}$ is

$$\begin{pmatrix} \mathbf{u}_{\mathbb{S}} \\ \mathcal{U} \end{pmatrix} = \boldsymbol{A}|_{\xi=0} \begin{pmatrix} \mathbf{v}_{\mathbb{S}} \\ \frac{\partial b}{\partial t} + \mathrm{grad}\, b \cdot \mathbf{v}_{\mathbb{S}} \end{pmatrix}$$
$$\Longleftrightarrow \mathbf{u}_{\mathbb{S}} = \mathbf{F}^{-1} (\mathbf{v}_{\mathbb{S}} + \mathcal{U}\mathbf{s}), \quad \mathcal{U} = c \frac{\partial b}{\partial t}.$$

The formula above for the displacement velocity $\mathcal{U}$ has already been derived, see (3.16).

Relation $(3.38)_3$ shows that $\partial \mathrm{w}/\partial \xi = 0$, which will be on some occasions used, without explicitly mentioning.

4 Rules of differentiation

The rules of differentiation implied by a time-dependent change of coordinates can be found e.g. in Tai and Kuo [10]. The derivation of these formulae is relatively simple, and that is why, for the sake of fluency, we prefer to rededuce them here by referring to the particular time-dependent change of coordinates (3.22). We continue to appeal to geometric reasonings, which make the derivations more transparent. Thus, by means of (3.22), the value $f(x_1, x_2, x_3, t)$ of a scalar function f can be written as

$$f(x_1, x_2, x_3, t) = f(x_1(\xi^1, \xi^2, \xi^3, t), x_2(\xi^1, \xi^2, \xi^3, t),$$
$$x_3(\xi^1, \xi^2, \xi^3, t), t) \equiv f(\xi^1, \xi^2, \xi^3, t),$$

where ξ^1, ξ^2, ξ^3 are the curvilinear coordinates at the moment t of that point P, which at the same moment t has the Cartesian coordinates x_1, x_2, x_3. We want to express the partial derivatives $\partial f/\partial x_i$, $\partial f/\partial t$ in terms of $\partial f/\partial \xi^i$, $\tilde{\partial} f/\partial t$.

First, since by differentiating f with respect to x_i the time t is held constant, we have, see the comments (i) in Sec. 3,

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial \xi^j} \frac{\partial \xi^j}{\partial x_i} \Longleftrightarrow \frac{\partial f}{\partial x_i} = A^T_{ij} \frac{\partial f}{\partial \xi^j}. \tag{4.43}$$

Note, when differentiating f with respect to t the variables x_1, x_2, x_3 are fixed. Therefore we have, see the comments (ii) in Sec. 3,

$$\frac{\partial f}{\partial t} = \frac{\tilde{\partial} f}{\partial t} + \frac{\partial f}{\partial \xi^i}\dot{\xi}^i \Longleftrightarrow \frac{\partial f}{\partial t} = \frac{\tilde{\partial} f}{\partial t} + \text{Grad}\, f \cdot (\dot{\xi}^1, \dot{\xi}^2)^T + \frac{\partial f}{\partial \xi}\dot{\xi},$$

where $\dot{\xi}^1, \dot{\xi}^2, \dot{\xi}$ are subjected to restriction (3.36), which transforms the latter into the rule of differentiation

$$\frac{\partial f}{\partial t} = \frac{\tilde{\partial} f}{\partial t} - \text{Grad}\, f \cdot \mathbf{w} - \frac{\partial f}{\partial \xi}\mathrm{w} = \frac{\tilde{\partial} f}{\partial t} - \frac{\partial f}{\partial \xi^l} w^l. \tag{4.44}$$

The two $\mathbf{w}$ and w in (4.44) are given by (3.38).

Based on (4.44) one can deduce the formulae

$$\frac{\partial A^{-1}_{jk}}{\partial t} = A^{-1}_{jl}\frac{\partial w^l}{\partial \xi^k}, \quad \frac{\tilde{\partial} J}{\partial t} = \frac{\partial J w^k}{\partial \xi^k}, \quad \frac{\partial f}{\partial t} = \frac{1}{J}\left\{\frac{\tilde{\partial} J f}{\partial t} - \frac{\partial J f w^k}{\partial \xi^k}\right\}. \tag{4.45}$$

Indeed, noting that

$$\frac{\tilde{\partial} A^{-1}_{jk}}{\partial t} = \frac{\tilde{\partial}}{\partial t}\left(\frac{\partial x_j}{\partial \xi^k}\right) = \frac{\partial}{\partial \xi^k}\left(\frac{\tilde{\partial} x_j}{\partial t}\right) = \frac{\partial w_j}{\partial \xi^k} = \frac{\partial A^{-1}_{jl} w^l}{\partial \xi^k}, \tag{4.46}$$

in view of (4.44) we have

$$\frac{\partial A^{-1}_{jk}}{\partial t} = \frac{\partial A^{-1}_{jl} w^l}{\partial \xi^k} - \frac{\partial A^{-1}_{jk}}{\partial \xi^l} w^l = \frac{\partial A^{-1}_{jl} w^l}{\partial \xi^k} - \frac{\partial A^{-1}_{jl}}{\partial \xi^k} w^l = A^{-1}_{jl}\frac{\partial w^l}{\partial \xi^k},$$

which verifies relation $(4.45)_1$. Then, by appeal to (4.46), $\partial J/\partial A^{-1}_{jk} = J A_{kj}$ and $\partial A^{-1}_{jl}/\partial \xi^k = \partial A^{-1}_{jk}/\partial \xi^l$, one can immediately derive $(4.45)_2$. Finally, to deduce $(4.45)_3$ one has only to multiply (4.44) by J and to use formula $(4.45)_2$.

5 Local balance equations in curvilinear coordinates

In this section we write the balance equations of mass and linear momentum at regular points of a continuum body in terms of the curvilinear coordinates described in Sec. 3. For a fixed basal topography they have been deduced in Luca, Tai and Kuo [2]. So, in the inertial reference frame $\mathcal{R}$ let

$$\frac{\partial \rho}{\partial t} + \text{div}\, \rho \boldsymbol{v} = 0 \tag{5.47}$$

be the mass balance equation, where ρ is the density and $\boldsymbol{v}$ is the velocity at a point in the avalanche body, while div is the spatial divergence operator. Moreover, let

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \text{div}\,(\rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{\sigma}) = \rho \boldsymbol{b} \tag{5.48}$$

be the momentum balance equation, where $\boldsymbol{\sigma}$ is the stress tensor and $\boldsymbol{b}$ is the specific body force. In this paper we shall be concerned with incompressible avalanche masses, but we write equations (5.47), (5.48) in curvilinear coordinates in full generality, which will be useful for further investigations in the field of avalanche modelling, e.g. when dealing with mixtures over variable topography.

Recalling the notations (3.28), which we use for the velocity $\boldsymbol{v}$, body force $\boldsymbol{b}$ and stress tensor $\boldsymbol{\sigma}$, we state the results in the following

Proposition 5.2 *The balance equations* (5.47), (5.48) *in the curvilinear coordinates defined by* (3.22) *have the form*

$$\frac{\tilde{\partial}}{\partial t}\{J\rho\} + \text{Div}\,\{J\rho(\mathbf{v} - \mathbf{w})\} + \frac{\partial}{\partial \xi}\{J\rho(\mathrm{v} - \mathrm{w})\} = 0, \tag{5.49}$$

$$\frac{\tilde{\partial}}{\partial t}\{J\rho\mathbf{v}\} + \text{Div}\,\{J[\rho\mathbf{v} \otimes (\mathbf{v} - \mathbf{w}) - \mathbf{T}]\} + \frac{\partial}{\partial \xi}\{J[\rho(\mathrm{v} - \mathrm{w})\mathbf{v} - \mathbf{p}]\} + J\boldsymbol{\Gamma}(\mathbf{T}, \mathbf{p}) = J\rho\mathbf{b} + J\rho\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v}) - J\rho\left\{(\text{Grad}\,\mathbf{w})\mathbf{v} + \mathrm{v}\frac{\partial \mathbf{w}}{\partial \xi}\right\}, \tag{5.50}$$

$$\frac{\tilde{\partial}}{\partial t}\{J\rho\mathrm{v}\} + \text{Div}\,\{J[\rho\mathrm{v}(\mathbf{v} - \mathbf{w}) - \mathbf{p}]\} + \frac{\partial}{\partial \xi}\{J[\rho\mathrm{v}(\mathrm{v} - \mathrm{w}) - \sigma^{33}]\} + J\Gamma(\mathbf{T}) = J\rho\mathrm{b} + J\rho\Gamma(\mathbf{v}) - J\rho\left\{\text{Grad}\,\mathrm{w} \cdot \mathbf{v} + \mathrm{v}\frac{\partial \mathrm{w}}{\partial \xi}\right\}, \tag{5.51}$$

where $\boldsymbol{\Gamma}(\mathbf{T}, \mathbf{p})$, $\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v})$, $\Gamma(\mathbf{T})$, $\Gamma(\mathbf{v})$ *are given by*

$$\boldsymbol{\Gamma}(\mathbf{T}, \mathbf{p}) \equiv -\mathbf{B}^{-1}\frac{\partial \mathbf{B}}{\partial \xi^\alpha}\mathbf{T}\mathbf{e}_\alpha + 2\mathbf{B}^{-1}\mathbf{F}\mathbf{W}\mathbf{p} + \Gamma(\mathbf{T})\mathbf{B}^{-1}\mathbf{s},$$

$$\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v}) \equiv -\mathbf{B}^{-1}\frac{\partial \mathbf{B}}{\partial \xi^\alpha}(\mathbf{v} \otimes \mathbf{v})\mathbf{e}_\alpha + 2\mathrm{v}\mathbf{B}^{-1}\mathbf{F}\mathbf{W}\mathbf{v} + \Gamma(\mathbf{v})\mathbf{B}^{-1}\mathbf{s},$$

$$\Gamma(\mathbf{T}) \equiv -\mathbf{B}^T\mathbf{F}^{-T}\mathbf{H}\cdot\mathbf{T}, \quad \Gamma(\mathbf{v}) \equiv -\mathbf{B}^T\mathbf{F}^{-T}\mathbf{H}\cdot(\mathbf{v} \otimes \mathbf{v}), \tag{5.52}$$

respectively, and $\mathbf{w}$, w *are shown in* (3.38).

Proof Since

$$\text{div}\, \rho \boldsymbol{v} = \frac{1}{J}\frac{\partial}{\partial \xi^k}\{J\rho v^k\},$$

see e.g. Luca, Tai and Kuo [2], by using formula $(4.45)_3$ to express the time derivative of ρ in (5.47) we deduce (5.49).

Now we refer to the momentum balance equation (5.48). We have

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} = \frac{\partial \rho v_j}{\partial t}\boldsymbol{i}_j = \frac{\partial}{\partial t}\{\rho A^{-1}_{jk} v^k\} A_{ij}\boldsymbol{g}_i$$

$$= \left\{\frac{\partial \rho v^i}{\partial t} + \rho A_{ij} v^k \frac{A^{-1}_{jk}}{\partial t}\right\}\boldsymbol{g}_i,$$

and it remains to use formulae $(4.45)_1$, $(4.45)_3$ to replace the time derivatives in the relation above, so that

$$J\frac{\partial \rho \boldsymbol{v}}{\partial t} = \left\{\frac{\tilde{\partial}}{\partial t}\{J\rho v^i\} - \frac{\partial}{\partial \xi^k}\{J\rho v^i w^k\} + J\rho \frac{\partial w^i}{\partial \xi^k} v^k\right\}\boldsymbol{g}_i.$$

Then, the divergence term in (5.48) can be taken over from Luca, Tai and Kuo [2] with only minor modifications. It emerges as

$$J\,\mathrm{div}\,(\rho \boldsymbol{v}\otimes \boldsymbol{v} - \boldsymbol{\sigma})$$

$$= \left\{\frac{\partial}{\partial \xi^k}\{J(\rho v^i v^k - \sigma^{ik})\} + J\Gamma^i(\mathbf{T},\mathbf{p}) - J\rho\Gamma^i(\mathbf{v},\mathrm{v})\right\}\boldsymbol{g}_i,$$

where $\Gamma^i(\mathbf{T},\mathbf{p})$, $\Gamma^i(\mathbf{v},\mathrm{v})$ are quantities containing the Christoffel symbols of the transformation, and which are equal to the entries of the block matrices

$$\begin{pmatrix}\boldsymbol{\Gamma}(\mathbf{T},\mathbf{p})\\ \Gamma(\mathbf{T})\end{pmatrix},\quad \begin{pmatrix}\boldsymbol{\Gamma}(\mathbf{v},\mathrm{v})\\ \Gamma(\mathbf{v})\end{pmatrix},$$

see (5.52). There follows that the momentum balance equation (5.48) is equivalent to

$$\frac{\tilde{\partial}}{\partial t}\{J\rho v^i\} + \frac{\partial}{\partial \xi^k}\{J(\rho v^i(v^k - w^k) - \sigma^{ik})\} + J\Gamma^i(\mathbf{T},\mathbf{p})$$

$$= J\rho b^i + J\rho\Gamma^i(\mathbf{v},\mathrm{v}) - J\rho\frac{\partial w^i}{\partial \xi^k}v^k,$$

which can be written as (5.50), (5.51).

We notice that for a fixed basal topography we have $\boldsymbol{w} = \boldsymbol{0}$ at every moment t. The equations of Proposition 5.2 corresponding to this case reduce to those derived in Luca, Tai and Kuo [2].

6 Model equations

We consider an avalanche mass which consists of two immiscible layers, flowing over a fixed or moving bed surface, see Figure 2. We denote by $\mathcal{E}_1 \subset \mathcal{E}$ the domain occupied by the layer near the bottom topography, and label the quantities referring to $\mathcal{E}_1$ by the index 1; the domain occupied by the upper layer is $\mathcal{E}_2$, and the quantities referring to $\mathcal{E}_2$ carry the index 2. It is clear that $\mathcal{E}_1$, $\mathcal{E}_2$ are generally time-dependent, but for simplicity in writing we omit this dependence. The interface between the two layers is denoted by

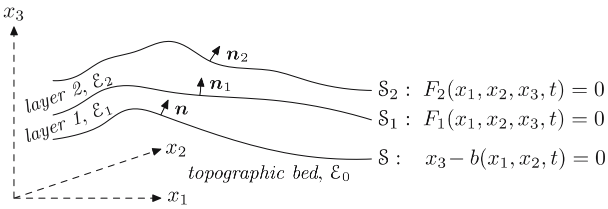

Fig. 2. Two-layer flow over arbitrary topography.

$\mathcal{S}_1$, its unit normal vector pointing into $\mathcal{E}_2$ is $\boldsymbol{n}_1$, and the upper (free) surface is $\mathcal{S}_2$, with the unit normal vector $\boldsymbol{n}_2$. Both $\mathcal{S}_1$ and $\mathcal{S}_2$ are moving surfaces. Each layer is modelled as a one-component body, of a density uniform in space and constant in time, such that $\rho_1 > \rho_2$; in particular, it is therefore assumed that the eventual influx of mass from the topographic bed does not change too much the density in layer 1. We may think of layer 2 as being made by water, and of layer 1 as being a more dense fluid; for an erodible bed surface, as a rough approximation, we may think of submarine landslides. The equations that describe the motion of the two layers over a fixed/moving topography $\mathcal{S}$ are as follows:

- *Local balance equations at regular points*: the mass and momentum balance equations in $\mathcal{E}_\mathfrak{k}$, $\mathfrak{k} = 1, 2$,

$$\mathrm{div}\,\boldsymbol{v}_\mathfrak{k} = 0,\quad \frac{\partial \boldsymbol{v}_\mathfrak{k}}{\partial t} + \mathrm{div}\left\{\boldsymbol{v}_\mathfrak{k}\otimes\boldsymbol{v}_\mathfrak{k} - \frac{1}{\rho_\mathfrak{k}}\boldsymbol{\sigma}_\mathfrak{k}\right\} = \boldsymbol{b}. \tag{6.53}$$

- *Boundary conditions at the basal surface* $\mathcal{S}$:
  a) If the avalanche flows over a fixed basal topography, we have the so-called *non-penetration* condition or *tangency of the velocity field*,

$$\boldsymbol{v}_1\cdot\boldsymbol{n} = 0 \quad \text{at} \quad x_3 - b(x_1,x_2) = 0, \tag{6.54}$$

  which states that the basal topography is a material surface for the lower layer.
  b) If the basal topography is a moving surface, propagating into the domain $\mathcal{E}_1$ (if there is deposition), or into the domain $\mathcal{E}_0$ (if there is erosion) occupied by the bed at rest, assumed to exhibit its own rheology, the mass balance equation

$$[\![\rho(\boldsymbol{v} - \mathcal{U}\boldsymbol{n})]\!]\cdot\boldsymbol{n} = 0 \quad \text{at} \quad x_3 - b(x_1,x_2,t) = 0, \tag{6.55}$$

  and the momentum balance equation

$$[\![\boldsymbol{\sigma} - \rho(\boldsymbol{v} - \mathcal{U}\boldsymbol{n})\otimes(\boldsymbol{v} - \mathcal{U}\boldsymbol{n})]\!]\,\boldsymbol{n} = \boldsymbol{0} \quad \text{at}$$

$$x_3 - b(x_1,x_2,t) = 0 \tag{6.56}$$

  hold. Notation $[\![f]\!]$ stands for the jump of f at the moment t across a given surface. That is, for the case that the surface is $\mathcal{S}_t$, separating $\mathcal{E}_1$ from $\mathcal{E}_0$, at each time t the function f is assumed continuous on $\mathcal{E}_0 \cup \mathcal{S}_t$ and $\mathcal{E}_1 \cup \mathcal{S}_t$, but may be discontinuous on $\mathcal{S}_t$; the difference between the limits of f on $\mathcal{S}_t$ taken from both parts $\mathcal{E}_0$, $\mathcal{E}_1$,

$$[\![f]\!] \equiv f_1 - f_0, \quad f_{\mathfrak{k}} \equiv \lim_{P\in\mathcal{E}_{\mathfrak{k}}\to Q\in\mathcal{S}_t} f, \quad \mathfrak{k} = 0, 1,$$

is the *jump of f across $\mathcal{S}_t$*.

- *Local balance equations at the interface* $\mathcal{S}_1$: the mass balance equation

$$[\![\rho(\boldsymbol{v} - \mathcal{U}_1\boldsymbol{n}_1)]\!] \cdot \boldsymbol{n}_1 = 0 \quad \text{at} \quad F_1(x_1, x_2, x_3, t) = 0, \tag{6.57}$$

and the momentum balance equation

$$[\![\boldsymbol{\sigma} - \rho(\boldsymbol{v} - \mathcal{U}_1\boldsymbol{n}_1) \otimes (\boldsymbol{v} - \mathcal{U}_1\boldsymbol{n}_1)]\!]\boldsymbol{n}_1 = \boldsymbol{0} \quad \text{at} \quad F_1(x_1, x_2, x_3, t) = 0 \tag{6.58}$$

hold across $\mathcal{S}_1$, of which the implicit equation is $F_1(x_1, x_2, x_3, t) = 0$, and speed of displacement is $\mathcal{U}_1$.

- *Evolution equation(s) for the layer interface* $\mathcal{S}_1$: we assume that the materials in the two layers are immiscible, whence the surface separating the two layers is a material surface for each layer, which implies

$$\frac{\partial F_1}{\partial t} + \nabla F_1 \cdot \boldsymbol{v}_1 = 0, \quad \frac{\partial F_1}{\partial t} + \nabla F_1 \cdot \boldsymbol{v}_2 = 0 \quad \text{at} \quad F_1(x_1, x_2, x_3, t) = 0. \tag{6.59}$$

- *Evolution equation for the free surface* $\mathcal{S}_2$: the free surface is a material surface for layer 2, which implies

$$\frac{\partial F_2}{\partial t} + \nabla F_2 \cdot \boldsymbol{v}_2 = 0 \quad \text{at} \quad F_2(x_1, x_2, x_3, t) = 0, \tag{6.60}$$

where $F_2(x_1, x_2, x_3, t) = 0$ represents the equation of $\mathcal{S}_2$.

- *Boundary condition at the free surface* $\mathcal{S}_2$: we assume that the free surface is traction-free, that is,

$$\boldsymbol{\sigma}_2\boldsymbol{n}_2 = \boldsymbol{0} \quad \text{at} \quad F_2(x_1, x_2, x_3, t) = 0. \tag{6.61}$$

Before going further, let us exploit the jump conditions (6.55)–(6.58). We shall denote by ρ_0, $\boldsymbol{v}_0$, $\boldsymbol{\sigma}_0$ the density, velocity and stress tensor, respectively, in the topographic bed. Thus, since $\boldsymbol{v}_0 = \boldsymbol{0}$, (6.55) states

$$\boldsymbol{v}_1 \cdot \boldsymbol{n} = \frac{\rho_1 - \rho_0}{\rho_1}\mathcal{U} \quad \text{at} \quad x_3 - b(x_1, x_2, t) = 0, \tag{6.62}$$

which transforms (6.56) into

$$\boldsymbol{\sigma}_1\boldsymbol{n} + \rho_0\mathcal{U}\boldsymbol{v}_1 = \boldsymbol{\sigma}_0\boldsymbol{n} \quad \text{at} \quad x_3 - b(x_1, x_2, t) = 0. \tag{6.63}$$

Then, since (6.59) implies $\boldsymbol{v}_1 \cdot \boldsymbol{n}_1 = \boldsymbol{v}_2 \cdot \boldsymbol{n}_1 = \mathcal{U}_1$ at each point on the interface $\mathcal{S}_1$, condition (6.57) is satisfied, and (6.58) turns into

$$\boldsymbol{\sigma}_1\boldsymbol{n}_1 = \boldsymbol{\sigma}_2\boldsymbol{n}_1 \quad \text{at} \quad F_1(x_1, x_2, x_3, t) = 0, \tag{6.64}$$

expressing the continuity of the stress vector across the separation surface. Thus, the model equations which we next deal are (6.53), (6.54), (6.62)–(6.64), (6.59)–(6.61).

7 Non-dimensional model equations in curvilinear coordinates

Since the final models proposed in this paper involve ordering approximations, we need to switch to non-dimensional field quantities. Thus, using a typical length L tangent to the topography, the constant gravitational acceleration g, and the densities ρ_1, ρ_2, we introduce non-dimensional field quantities as follows:

$$(x_1, x_2, x_3, t) = L(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{t}/\sqrt{Lg}), \quad b = L\hat{b},$$
$$\boldsymbol{v}_{\mathfrak{k}} = \sqrt{Lg}\,\hat{\boldsymbol{v}}_{\mathfrak{k}}, \quad \boldsymbol{\sigma}_{\mathfrak{k}} = \rho_{\mathfrak{k}}Lg\hat{\boldsymbol{\sigma}}_{\mathfrak{k}}, \quad \mathfrak{k} = 0, 1, 2,$$
$$\boldsymbol{w} = \sqrt{Lg}\,\hat{\boldsymbol{w}}, \quad \boldsymbol{b} = g\hat{\boldsymbol{b}}. \tag{7.65}$$

With the scalings above and notations

$$c_{01} \equiv \frac{\rho_0}{\rho_1}, \quad c_{12} \equiv \frac{\rho_1}{\rho_2},$$

the system of modelling equations (6.53), (6.54), (6.62)–(6.64), (6.59)–(6.61) emerges as follows (we omit the hat):

- *in the domain* $\mathcal{E}_{\mathfrak{k}}$, $\mathfrak{k} = 1, 2$,

$$\operatorname{div} \boldsymbol{v}_{\mathfrak{k}} = 0, \quad \frac{\partial \boldsymbol{v}_{\mathfrak{k}}}{\partial t} + \operatorname{div}\{\boldsymbol{v}_{\mathfrak{k}} \otimes \boldsymbol{v}_{\mathfrak{k}} - \boldsymbol{\sigma}_{\mathfrak{k}}\} = \boldsymbol{b}; \tag{7.66}$$

- *at the basal surface* $\mathcal{S}$,
  a) for fixed bottom topography $x_3 - b(x_1, x_2) = 0$,

$$\boldsymbol{v}_1 \cdot \boldsymbol{n} = 0, \tag{7.67}$$

  b) for moving bottom topography $x_3 - b(x_1, x_2, t) = 0$,

$$\boldsymbol{v}_1 \cdot \boldsymbol{n} = (1 - c_{01})\mathcal{U}, \quad \boldsymbol{\sigma}_1\boldsymbol{n} + c_{01}\mathcal{U}\boldsymbol{v}_1 = c_{01}\boldsymbol{\sigma}_0\boldsymbol{n}; \tag{7.68}$$

- *at the layer interface* $F_1(x_1, x_2, x_3, t) = 0$,

$$\frac{\partial F_1}{\partial t} + \nabla F_1 \cdot \boldsymbol{v}_1 = 0,$$
$$\frac{\partial F_1}{\partial t} + \nabla F_1 \cdot \boldsymbol{v}_2 = 0, \quad c_{12}\boldsymbol{\sigma}_1\boldsymbol{n}_1 = \boldsymbol{\sigma}_2\boldsymbol{n}_1; \tag{7.69}$$

- *at the free surface* $F_2(x_1, x_2, x_3, t) = 0$,

$$\frac{\partial F_2}{\partial t} + \nabla F_2 \cdot \boldsymbol{v}_2 = 0, \quad \boldsymbol{\sigma}_2 \boldsymbol{n}_2 = \boldsymbol{0}. \tag{7.70}$$

Since both layers are assumed incompressible, we introduce the extra-stress tensors $\boldsymbol{\sigma}^E_{\mathfrak{k}}$ by

$$\boldsymbol{\sigma}_{\mathfrak{k}} \equiv -p_{\mathfrak{k}} \boldsymbol{I} + \boldsymbol{\sigma}^E_{\mathfrak{k}},$$

where $p_{\mathfrak{k}}$ is the pressure in the layer $\mathfrak{k}$, and notice the relation[3]

$$\begin{pmatrix} \mathbf{T}_{\mathfrak{k}} & \mathbf{p}_{\mathfrak{k}} \\ \mathbf{p}^T_{\mathfrak{k}} & \sigma^{33}_{\mathfrak{k}} \end{pmatrix} = -p_{\mathfrak{k}} \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} + \begin{pmatrix} \mathbf{P}_{\mathfrak{k}} & \mathbf{p}_{\mathfrak{k}} \\ \mathbf{p}^T_{\mathfrak{k}} & T^{33}_{\mathfrak{k}} \end{pmatrix} \tag{7.71}$$

between the components of the stress tensor (on the left-hand side) and the components of the extra-stress tensor (the second matrix on the right-hand side).

Now we want to write equations (7.66)–(7.70) in terms of the contravariant components of vectors and tensors with respect to the natural basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ corresponding to the (generally time-dependent) change of coordinates in Sec. 3. To this end we assume that, in terms of the curvilinear coordinates ξ^1, ξ^2, ξ, equations $F_1(x_1, x_2, x_3, t) = 0$ and $F_2(x_1, x_2, x_3, t) = 0$ of the layer interface and of the free surface, respectively, can be written as

$$\xi - h_1(\boldsymbol{\xi}, t) = 0, \quad \xi - h_2(\boldsymbol{\xi}, t) = 0, \quad \boldsymbol{\xi} \equiv (\xi^1, \xi^2), \tag{7.72}$$

respectively, with $0 < h_1 < h_2$. Moreover, assuming that ξ^1, ξ^2 are length-dimensional surface parameters, we set

$$(\boldsymbol{\xi}, \xi) = L(\hat{\boldsymbol{\xi}}, \hat{\xi}), \quad (h_1, h_2) = L(\hat{h}_1, \hat{h}_2),$$

and next drop the hat. We have

Proposition 7.1 *In terms of the curvilinear coordinates of Sec. 3, the non-dimensional equations and boundary conditions* (7.66)–(7.70) *can be written as follows:*

- *in the domain* $\mathcal{E}_{\mathfrak{k}}$, $\mathfrak{k} = 1, 2$ *(for simplicity, we omit the index* $\mathfrak{k}$*, which should be attached to* $\mathbf{v}$, v, p, $\mathbf{P}$, $\mathbf{p}$, T^{33}*)*,

$$\mathrm{Div}\,\{J\mathbf{v}\} + \frac{\partial}{\partial \xi}\{J \mathrm{v}\} = 0, \tag{7.73}$$

$$\begin{aligned} &\frac{\tilde{\partial}}{\partial t}\{J\mathbf{v}\} + \mathrm{Div}\,\{J[\mathbf{v} \otimes (\mathbf{v} - \mathbf{w}) + p\mathbf{M} - \mathbf{P})\} \\ &\quad + \frac{\partial}{\partial \xi}\{J[(\mathrm{v} - \mathcal{U})\mathbf{v} - \mathbf{p})]\} + J\boldsymbol{\Gamma}(-p\mathbf{M}, \mathbf{0}) \\ &\quad + J\boldsymbol{\Gamma}(\mathbf{P}, \mathbf{p}) = J\mathbf{b} + J\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v}) \\ &\quad - J\left\{(\mathrm{Grad}\,\mathbf{w})\mathbf{v} + \mathcal{U}\frac{\partial \mathbf{w}}{\partial \xi}\right\}, \end{aligned} \tag{7.74}$$

$$\begin{aligned} &\frac{\tilde{\partial}}{\partial t}\{J\mathrm{v}\} + \mathrm{Div}\,\{J[\mathrm{v}(\mathbf{v} - \mathbf{w}) - \mathbf{p}]\} \\ &\quad + \frac{\partial}{\partial \xi}\{J[\mathrm{v}(\mathrm{v} - \mathcal{U}) - T^{33}]\} + J\frac{\partial p}{\partial \xi} + J\Gamma(\mathbf{P}) \\ &\quad = J\mathrm{b} + J\Gamma(\mathbf{v}) - J\mathrm{Grad}\,\mathcal{U} \cdot \mathbf{v}, \end{aligned} \tag{7.75}$$

where, corresponding to each $\mathfrak{k}$*, the terms* $\boldsymbol{\Gamma}(\mathbf{P}, \mathbf{p})$*,* $\Gamma(\mathbf{P})$*,* $\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v})$*,* $\Gamma(\mathbf{v})$ *can be taken from* (5.52)*, in which* $\mathbf{T}$ *is replaced by* $\mathbf{P}$*, and*

$$\begin{aligned} &\boldsymbol{\Gamma}(-p\mathbf{M}, \mathbf{0}) \\ &\quad \equiv p\left\{\mathbf{B}^{-1}\frac{\partial \mathbf{B}}{\partial \xi^\alpha}\mathbf{M}\mathbf{e}_\alpha + \mathrm{tr}\,(\mathbf{W}(\mathbf{I} - \xi\mathbf{W})^{-1})\mathbf{B}^{-1}\mathbf{s}\right\}; \end{aligned} \tag{7.76}$$

- *at the basal surface* $\xi = 0$,
  a) for fixed topography,

$$\mathrm{v}_1 = 0, \tag{7.77}$$

  b) for moving topography,

$$\mathrm{v}_1 = (1 - c_{01})\mathcal{U}, \tag{7.78}$$

$$\mathbf{p}_1 + c_{01}\mathcal{U}\mathbf{v}_1 = c_{01}\mathbf{p}_0, \quad \sigma^{33}_1 + c_{01}\mathcal{U}\mathrm{v}_1 = c_{01}\sigma^{33}_0; \tag{7.79}$$

- *at the layer interface* $\xi = h_1(\boldsymbol{\xi}, t)$*,*

$$\frac{\tilde{\partial} h_1}{\partial t} + \mathrm{Grad}\,h_1 \cdot (\mathbf{v}_1 - \mathbf{w}) = \mathrm{v}_1 - \mathcal{U}, \tag{7.80}$$

$$\frac{\tilde{\partial} h_1}{\partial t} + \mathrm{Grad}\,h_1 \cdot (\mathbf{v}_2 - \mathbf{w}) = \mathrm{v}_2 - \mathcal{U},$$

$$\begin{aligned} &c_{12}\{(-p_1\mathbf{M} + \mathbf{P}_1)\mathrm{Grad}\,h_1 - \mathbf{p}_1\} \\ &\quad = (-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\,h_1 - \mathbf{p}_2, \\ &c_{12}(\mathbf{p}_1 \cdot \mathrm{Grad}\,h_1 + p_1 - T^{33}_1) \\ &\quad = \mathbf{p}_2 \cdot \mathrm{Grad}\,h_1 + p_2 - T^{33}_2; \end{aligned} \tag{7.81}$$

- *at the free surface* $\xi = h_2(\boldsymbol{\xi}, t)$*,*

$$\frac{\tilde{\partial} h_2}{\partial t} + \mathrm{Grad}\,h_2 \cdot (\mathbf{v}_2 - \mathbf{w}) = \mathrm{v}_2 - \mathcal{U}, \tag{7.82}$$

$$\begin{aligned} &(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\,h_2 - \mathbf{p}_2 = \mathbf{0}, \\ &\mathbf{p}_2 \cdot \mathrm{Grad}\,h_2 + p_2 - T^{33}_2 = 0. \end{aligned} \tag{7.83}$$

Proof In order to deduce (7.73), in the mass balance equation (5.49) we divide by $\rho(=$ constant), switch to non-dimensional quantities to obtain

$$\frac{\tilde{\partial} J}{\partial t} + \mathrm{Div}\,\{J(\mathbf{v} - \mathbf{w})\} + \frac{\partial}{\partial \xi}\{J(\mathrm{v} - \mathrm{w}) = 0, \tag{7.84}$$

[3] To avoid confusion, we note that $\mathbf{P}$, $\mathbf{p}$ and T^{33} are used in Luca et al. [2], [1] to denote the contravariant components of the full stress tensor, while here they stand for the components of the extra-stress tensor.

and take into account formula $(4.45)_2$ for $\tilde{\partial} J/\partial t$. Then, if in (5.50) we divide by ρ, replace $\mathbf{T}$ by $-p\mathbf{M}+\mathbf{P}$, see (7.71), use the non-dimensional quantities, and take account of

$$\boldsymbol{\Gamma}(\mathbf{T},\mathbf{p}) = \boldsymbol{\Gamma}(-p\mathbf{M},\mathbf{0}) + \boldsymbol{\Gamma}(\mathbf{P},\mathbf{p}),$$

see Luca, Tai and Kuo [2], we obtain (7.74). Equation (5.51) can be put into the form (7.75) as in [2].

We go further to condition (7.77), which immediately follows from (7.67) by recalling that $\boldsymbol{g}^3 = \boldsymbol{n}$. In a similar manner, (7.78) can be derived from $(7.68)_1$, and relation

$$\boldsymbol{\sigma n} = (\sigma^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j)\boldsymbol{g}^3 = \sigma^{i3}\boldsymbol{g}_i,$$

which holds for any tensor $\boldsymbol{\sigma}$, when applied to $(7.68)_2$, clearly shows (7.79).

We pass to $(7.69)_1$ and use the rule of differentiation (4.44) to compute

$$\frac{\partial F_1}{\partial t} + \nabla F_1 \cdot \boldsymbol{v}_1 = \frac{\tilde{\partial} F_1}{\partial t} - \frac{\partial F_1}{\partial \xi^i} w^i + \left(\frac{\partial F_1}{\partial \xi^i}\boldsymbol{g}^i\right)\cdot \boldsymbol{v}_1 = \frac{\tilde{\partial} F_1}{\partial t} + \frac{\partial F_1}{\partial \xi^i}(v_1^i - w^i).$$

Now we have only to take into account $(7.72)_1$ to conclude $(7.80)_1$. Condition $(7.80)_2$ is then obvious, and (7.81) is deduced by rewriting $(7.69)_3$ as

$$c_{12}\boldsymbol{\sigma}_1 \nabla F_1 = \boldsymbol{\sigma}_2 \nabla F_1 \Longleftrightarrow c_{12}\sigma_1^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\left(\frac{\partial F_1}{\partial \xi^k}\boldsymbol{g}^k\right) = \sigma_2^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\left(\frac{\partial F_1}{\partial \xi^k}\boldsymbol{g}^k\right),$$

and performing the calculations with the aid of $(7.72)_1$. Finally, (7.82) and (7.83) can be analogously derived. The proof is now complete.

We consider that the body force is the gravitational force, so that (in non-dimensional form) we have, see Luca, Tai and Kuo [2],

$$\mathbf{b} = -c\mathbf{B}^{-1}\mathbf{s}, \quad \mathrm{b} = -c. \tag{7.85}$$

The equations in Proposition 7.1 are meant to stand for the basic unknowns $\mathbf{v}_1$, v_1, $\mathbf{v}_2$, v_2, p_1, p_2, h_1, h_2 and, if the topography is variable (in which case (3.16) must be taken into account), for the vertical height b of the basal surface. To this end they should be complemented by constitutive relations for the stress tensors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and for the frictional stresses at the bottom $\mathcal{S}$ and interface $\mathcal{S}_1$; for the case of moving topography, the erosion/deposition rate must be parameterized in terms of the basic unknown fields, which transforms (3.16) into an evolution equation for b. However, the emerging system of equations is rather complicated, and that is why we use the depth-averaging procedure, which we describe in the next section.

Before going further we want to comment on conditions (7.79). There are several proposals for the erosion/deposition rate $\mathcal{U}$, see e.g. Bouchut et al. [9], Tai and Kuo [10] and the reviews therein on earlier erosion/deposition rate proposals. An interesting approach to give a law for $\mathcal{U}$ is presented in Fraccarollo and Capart [11]. The idea is to assume the topographic bed with its own rheologic properties, and to use the jump of the momentum balance equation at the bottom surface to derive a formula for the erosion/deposition rate. When applied to the model equations in Proposition 7.1, this idea appears as follows: assuming $\mathbf{v}_1 \neq \mathbf{0}$ at $\xi = 0$, and giving the shear stresses on both sides of the bed surface, relation $(7.79)_1$ delivers the erosion/deposition rate $\mathcal{U}$,

$$\mathcal{U}^2 = \frac{(c_{01}\mathbf{p}_0 - \mathbf{p}_1)\cdot(c_{01}\mathbf{p}_0 - \mathbf{p}_1)}{c_{01}^2\mathbf{v}_1\cdot\mathbf{v}_1}, \tag{7.86}$$

where $\mathbf{v}_1$, $\mathbf{p}_0$, $\mathbf{p}_1$ are evaluated at $\xi = 0$; then, condition $(7.79)_2$ gives the value of the normal stress σ_0^{33} at the bed surface, once σ_1^{33} is known. Of course, condition $(7.79)_1$ can be also interpreted as giving the shear stress in the bed once $\mathcal{U}$ and $\mathbf{p}_1$ are known; or, as giving the basal shear stress in the flowing layer, if $\mathcal{U}$ and $\mathbf{p}_0$ are known. It is not our aim in this paper to propose one or another law for the erosion/deposition rate $\mathcal{U}$, and to interpret $(7.79)_1$, since only when corroborating the rheological properties of the avalanche mass and of the topographic bed with experimental and numerical data the model equations can be validated. So, in the remaining analysis we leave aside conditions (7.79), and assume that the erosion/deposition rate is known, be it given by (7.86) or by any other law.

8 Depth-averaging procedure

In order to deduce the final modelling equations for the flow of the two shallow layers we use the depth-integration procedure. This section is devoted to the presentation of this method. Note that the equations and boundary conditions corresponding to fixed basal topography can be deduced from those corresponding to a moving basal topography, by taking there $\mathcal{U} = 0$, $\mathbf{w} = \mathbf{0}$ (and disregarding condition (7.79), as already agreed). That is why in the next analysis we shall refer to a moving basal surface, and so we obtain formulae valid for both fixed and moving bottom topography. In the discussion below, without explicitly mentioning, we shall use the Leibniz rules

$$\int_{f(\boldsymbol{\xi},t)}^{g(\boldsymbol{\xi},t)} \mathrm{Div}\,\mathbf{v}\,d\xi = \mathrm{Div}\int_{f(\boldsymbol{\xi},t)}^{g(\boldsymbol{\xi},t)} \mathbf{v}\,d\xi + \mathbf{v}(\boldsymbol{\xi}, f(\boldsymbol{\xi},t))\cdot \mathrm{Grad}\,f - \mathbf{v}(\boldsymbol{\xi}, g(\boldsymbol{\xi},t))\cdot \mathrm{Grad}\,g,$$

$$\int_{f(\boldsymbol{\xi},t)}^{g(\boldsymbol{\xi},t)} \mathrm{Div}\,\mathbf{P}\,d\xi = \mathrm{Div}\int_{f(\boldsymbol{\xi},t)}^{g(\boldsymbol{\xi},t)} \mathbf{P}\,d\xi + \mathbf{P}(\boldsymbol{\xi}, f(\boldsymbol{\xi},t))\mathrm{Grad}\,f - \mathbf{P}(\boldsymbol{\xi}, g(\boldsymbol{\xi},t))\mathrm{Grad}\,g,$$

that hold for a 2-column matrix $\mathbf{v}(\boldsymbol{\xi}, \zeta, t)$ and for a square matrix $\mathbf{P}(\boldsymbol{\xi}, \zeta, t)$ of order 2.

We begin with some remarks. Thus, by integrating the mass balance equation (7.73) for $\mathfrak{k} = 1$ with respect to the normal variable from 0 to $\zeta \in (0, h_1)$, and by taking into account the boundary condition (7.78), we obtain the normal velocity v_1 as

$$\mathrm{v}_1 = \frac{1}{J}\left\{J_0(1 - c_{01})\mathcal{U} - \mathrm{Div}\int_0^{\zeta} J\mathbf{v}_1 d\zeta'\right\}. \tag{8.87}$$

Instead of (7.73) we can integrate its equivalent equation (7.84), and obtain another expression for v_1, that is,

$$\mathrm{v}_1 = \mathcal{U} - \frac{1}{J}\left\{\int_0^{\zeta}\frac{\tilde{\partial} J}{\partial t}d\zeta' + \int_0^{\zeta}\mathrm{Div}\,\{J(\mathbf{v}_1 - \mathbf{w})\}d\zeta' + c_{01}J_0\mathcal{U}\right\},$$

which easily gives

$$\mathrm{v}_1|_{\zeta=h_1} = \mathcal{U} + \frac{\tilde{\partial} h_1}{\partial t} + (\mathbf{v}_1 - \mathbf{w})|_{\zeta=h_1}\cdot \mathrm{Grad}\, h_1 - \frac{1}{J|_{h_1}}\left\{\frac{\tilde{\partial}}{\partial t}\int_0^{h_1} J d\zeta + \mathrm{Div}\int_0^{h_1} J(\mathbf{v}_1 - \mathbf{w})d\zeta + c_{01}J_0\mathcal{U}\right\}.$$

Substitution of the above expression of $\mathrm{v}_1|_{\zeta=h_1}$ into the kinematic boundary condition $(7.80)_1$ transforms the latter into

$$\frac{\tilde{\partial}}{\partial t}\int_0^{h_1} J d\zeta + \mathrm{Div}\int_0^{h_1} J(\mathbf{v}_1 - \mathbf{w})d\zeta = -c_{01}J_0\mathcal{U}. \tag{8.88}$$

Similarly, by integrating the mass balance equation (7.73) valid in the upper layer, also with respect to the normal variable, now from h_1 to $\zeta \in (h_1, h_2)$, and by accounting for the kinematic boundary condition $(7.80)_2$ to express v_2 at $\zeta = h_1$, the normal velocity in layer 2 is obtained as

$$\mathrm{v}_2 = \frac{1}{J}J|_{\zeta=h_1}\left\{\mathcal{U} + \frac{\tilde{\partial} h_1}{\partial t} - \mathrm{Grad}\, h_1 \cdot \mathbf{w}|_{\zeta=h_1}\right\} - \frac{1}{J}\mathrm{Div}\int_{h_1}^{\zeta} J\mathbf{v}_2 d\zeta'. \tag{8.89}$$

On the other hand, if we use (7.84) instead of (7.73), we deduce

$$\mathrm{v}_2 = \mathcal{U} + \frac{1}{J}J|_{\zeta=h_1}\left\{\frac{\tilde{\partial} h_1}{\partial t} + \mathrm{Grad}\, h_1\cdot(\mathbf{v}_2 - \mathbf{w})|_{\zeta=h_1}\right\} - \frac{1}{J}\left\{\int_{h_1}^{\zeta}\frac{\tilde{\partial} J}{\partial t} + \int_{h_1}^{\zeta}\mathrm{Div}\,\{J(\mathbf{v}_2 - \mathbf{w})\}d\zeta'\right\},$$

which implies

$$\mathrm{v}_2|_{\zeta=h_2} = \mathcal{U} + \frac{\tilde{\partial} h_2}{\partial t} + (\mathbf{v}_2 - \mathbf{w})|_{\zeta=h_2}\cdot \mathrm{Grad}\, h_2 - \frac{1}{J|_{h_2}}\left\{\frac{\tilde{\partial}}{\partial t}\int_{h_1}^{h_2} J d\zeta + \mathrm{Div}\int_{h_1}^{h_2} J(\mathbf{v}_2 - \mathbf{w})d\zeta\right\}.$$

Substituting this expression of $\mathrm{v}_2|_{\zeta=h_2}$ into the kinematic boundary condition (7.82) yields

$$\frac{\tilde{\partial}}{\partial t}\int_{h_1}^{h_2} J d\zeta + \mathrm{Div}\int_{h_1}^{h_2} J(\mathbf{v}_2 - \mathbf{w})d\zeta = 0. \tag{8.90}$$

Summarizing, the system of equations and boundary conditions (7.73) (for $\mathfrak{k} = 1, 2$), (7.78), (7.80), (7.82) is equivalent to (8.87)–(8.90). We refer to (8.88), (8.90) as the *depth-averaged mass balance equations*, since they can be deduced by integrating the local mass balance equation (7.84) along each layer depth (while accounting for (7.78), (7.80), (7.82)).

Moreover, one can see that, by integrating the tangential momentum balance equation (7.74) for $\mathfrak{k} = 1$ with respect to the normal variable ζ from 0 to $h_1(\boldsymbol{\xi}, t)$, one obtains a certain expression for the stresses

$$\{(-p_1\mathbf{M} + \mathbf{P}_1)\mathrm{Grad}\, h_1 - \mathbf{p}_1\}_{\zeta=h_1}.$$

Substitution of this expression into condition $(7.81)_1$ transforms the latter into

$$\begin{aligned}&\frac{\tilde{\partial}}{\partial t}\int_0^{h_1} J\mathbf{v}_1 d\zeta \\ &\quad + \mathrm{Div}\int_0^{h_1} J\{\mathbf{v}_1\otimes(\mathbf{v}_1 - \mathbf{w}) + p_1\mathbf{M} - \mathbf{P}_1\}d\zeta \\ &\quad + \frac{1}{c_{12}}\{J\,[(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\, h_1 - \mathbf{p}_2]\}_{\zeta=h_1} \\ &\quad + J_0\,\{\mathbf{p}_1 + c_{01}\mathcal{U}\mathbf{v}_1\}_{\zeta=0} \\ &\quad + \int_0^{h_1} J\{\boldsymbol{\Gamma}(-p_1\mathbf{M}, \mathbf{0}) + \boldsymbol{\Gamma}(\mathbf{P}_1, \mathbf{p}_1)\}d\zeta \\ &= \int_0^{h_1} J\mathbf{b}d\zeta + \int_0^{h_1} J\boldsymbol{\Gamma}(\mathbf{v}_1, \mathrm{v}_1)d\zeta \\ &\quad - \int_0^{h_1} J\left\{(\mathrm{Grad}\,\mathbf{w})\mathbf{v}_1 + \mathcal{U}\frac{\partial \mathbf{w}}{\partial \zeta}\right\}d\zeta.\end{aligned} \tag{8.91}$$

Thus, within the system of modelling equations, condition $(7.81)_1$ can be replaced by (8.91). Similarly, by performing the integration of the tangential momentum balance equation in the upper layer from $h_1(\boldsymbol{\xi}, t)$ to $h_2(\boldsymbol{\xi}, t)$, one can see that condition $(7.83)_1$ can be replaced by

$$\frac{\tilde{\partial}}{\partial t}\int_{h_1}^{h_2} J\mathbf{v}_2 d\xi + \mathrm{Div}\int_{h_1}^{h_2} J\{\mathbf{v}_2\otimes(\mathbf{v}_2-\mathbf{w}) + p_2\mathbf{M} - \mathbf{P}_2\}d\xi$$
$$-\{J[(-p_2\mathbf{M}+\mathbf{P}_2)\mathrm{Grad}\, h_1 - \mathbf{p}_2]\}_{\xi=h_1}$$
$$+\int_{h_1}^{h_2} J\{\boldsymbol{\Gamma}(-p_2\mathbf{M},\mathbf{0}) + \boldsymbol{\Gamma}(\mathbf{P}_2,\mathbf{p}_2)\}d\xi$$
$$=\int_{h_1}^{h_2} J\mathbf{b}d\xi + \int_{h_1}^{h_2} J\boldsymbol{\Gamma}(\mathbf{v}_2,\mathrm{v}_2)d\xi$$
$$-\int_{h_1}^{h_2} J\left\{(\mathrm{Grad}\,\mathbf{w})\mathbf{v}_2 + \mathcal{U}\frac{\partial\mathbf{w}}{\partial\xi}\right\}d\xi. \qquad (8.92)$$

We refer to (8.91), (8.92) as the *depth-averaged tangential momentum balance equations*, since they can be also deduced if one uses the boundary conditions $(7.81)_1$, $(7.83)_1$ in the integration of the tangential momentum balance equation (7.74) along each layer depth.

Now, the idea of the depth-averaging procedure is to use, instead of the original system of equations, i.e.,

$$(7.73)\text{–}(7.75) \quad \text{for} \quad \mathfrak{k}=1,2, (7.78), (7.80)\text{–}(7.83) \quad \text{(I)}$$

or of its equivalent system (as discussed above)

$$(7.74) \text{ and } (7.75) \quad \text{for}$$
$$\mathfrak{k}=1,2, (7.81)_2, (7.83)_2, (8.87)\text{–}(8.92), \quad \text{(II)}$$

the system consisting of

$$(7.75) \text{ for } \mathfrak{k}=1,2, (7.81)_2, (7.83)_2, (8.87)\text{–}(8.92), \quad \text{(III)}$$

in which the depth-integrated mass and tangential momentum balance equations are conceived as the result of an averaging process. This new system (III) is exploited under some ordering approximations which account for the shallowness of the avalanche mass, as follows: both pressures p_1, p_2 are deduced from (7.75), $(7.81)_2$, $(7.83)_2$, and then the remaining equations (8.88), (8.90)–(8.92) are transformed into equations which, complemented by closure relations, stand for the determination of the basic fields $\bar{\mathbf{v}}_1$, $\bar{\mathbf{v}}_2$, h_1, $h \equiv h_2 - h_1$ and, for the case of moving topography, of b, when (3.16) has to be accounted for. Here, if $f_{\mathfrak{k}}$ refers to the layer $\mathfrak{k}$, the *depth-averaged value* $\bar{f}_{\mathfrak{k}}$ is defined as

$$\bar{f}_1 \equiv \frac{1}{h_1}\int_0^{h_1} f_1(\boldsymbol{\xi},\xi,t)d\xi, \quad \bar{f}_2 \equiv \frac{1}{h}\int_{h_1}^{h_2} f_2(\boldsymbol{\xi},\xi,t)d\xi.$$

We note that, by writing the system (I) in the form (II), it became more clear that, if we choose to give the stress tensors by constitutive assumptions in terms of p_1, p_2, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, then the system (I), in which p_1, p_2, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, h_1, h_2 are the basic unknowns, could be overdetermined. To avoid a similar overdeterminateness, in Luca et al. [1–3] we replaced the traction-free boundary condition by the requirement of vanishing pressure on the free surface (the analogous condition being here $(7.83)_2$). However, to replace the requirement $\boldsymbol{\sigma}_2\boldsymbol{n}_2 = \boldsymbol{0}$ on the free surface with $p_2 = 0$ is of no use here, since the mentioned difficulty (if any) is not completely removed, due to condition $(7.81)_1$ at the layer interface. Moreover, (7.81), (7.83) are particular results of a well-established theory of continuum mechanics. In a more trustful approach it would be probably necessary to introduce 4 new unknown scalar fields, e.g., surface stresses at the free surface and at the layer interface. Physically, the surface stresses at the layer interface are justified, since the layers are assumed immiscible.

9 Ordering approximations

Now we make scaling approximations, similar to those performed by Luca et al. [1], in terms of an aspect ratio $\epsilon \equiv H/L$ and a constant $\gamma \in (0,1)$, where H is a typical thickness perpendicular to the topography, and L is a typical length-scale tangent to the topography, already used in the non-dimensionalization procedure. In snow and debris flows, we generally have $\epsilon \approx 10^{-2}$ and $\gamma \approx 0.5$, see Pudasaini and Hutter [12], p. 188.

a) *Geometric approximations*: we assume that both layers are thin, in the sense that $h_2 = O(\epsilon)$, which implies $\xi = O(\epsilon)$ for $\xi \in (0, h_2)$, in particular for $\xi = h_1$[4].

b) *Flow rule approximations*: we suppose that the erosion/deposition rate $\mathcal{U}$ is $O(\epsilon)$, and that the tangential velocities $\mathbf{v}_1$, $\mathbf{v}_2$ are $O(1)$ and satisfy relations of Boussinesq type, that is, in layer 1 we have

$$\int_0^{h_1}\xi\mathbf{v}_1 d\xi = \frac{1}{2}h_1^2 m_1^{(1)}\bar{\mathbf{v}}_1 + O(\epsilon^{2+\gamma}),$$
$$\int_0^{h_1}\mathbf{v}_1\otimes\mathbf{v}_1 d\xi = h_1 m_2^{(1)}\bar{\mathbf{v}}_1\otimes\bar{\mathbf{v}}_1 + O(\epsilon^{2+\gamma}),$$
$$\int_0^{h_1}\xi\mathbf{v}_1\otimes\mathbf{v}_1 d\xi = \frac{1}{2}h_1^2 m_3^{(1)}\bar{\mathbf{v}}_1\otimes\bar{\mathbf{v}}_1 + O(\epsilon^{2+\gamma}), \qquad (9.93)$$

and in layer 2, with $h \equiv h_2 - h_1$, we have

$$\int_{h_1}^{h_2}\xi\mathbf{v}_2 d\xi = \frac{1}{2}h(h+2h_1)m_1^{(2)}\bar{\mathbf{v}}_2 + O(\epsilon^{2+\gamma}),$$
$$\int_{h_1}^{h_2}\mathbf{v}_2\otimes\mathbf{v}_2 d\xi = h m_2^{(2)}\bar{\mathbf{v}}_2\otimes\bar{\mathbf{v}}_2 + O(\epsilon^{2+\gamma}),$$

[4] We use in both layers the same curvilinear coordinate system based on the topography of the bottom surface. In principle, to each layer one could assign its own coordinate system, that corresponds to the surface bounding the layer from below, but this is not done here.

$$\int_{h_1}^{h_2} \xi \mathbf{v}_2 \otimes \mathbf{v}_2 d\xi$$
$$= \frac{1}{2} h(h + 2h_1) m_3^{(2)} \bar{\mathbf{v}}_2 \otimes \bar{\mathbf{v}}_2 + O(\epsilon^{2+\gamma}). \quad (9.94)$$

Notice that we do not assume any law for the tangential velocity profile. Instead, we use the *momentum correction factors* or *Boussinesq coefficients* $m_1^{(1)} \dots m_3^{(2)}$ to account for the deviations of the velocity field from a plug flow (by which we mean that $\mathbf{v}_\mathfrak{k}$ is independent of the avalanche depth ξ), when they are equal to 1. These coefficients are supposed to be scalar functions of $\boldsymbol{\xi}, t$ (possibly by means e.g. of the depth-averaged tangential velocity of the corresponding layer) of order $O(1)$. For a power-law velocity profile they have been deduced in Luca et al. [1]. We mention that (9.93), (9.94) are independent of the choice of the parameterization of the bottom topography.

We shall also need to approximate the integrals

$$\int_0^{h_1} \mathrm{v}_1 \mathbf{v}_1 d\xi, \quad \int_{h_1}^{h_2} \mathrm{v}_2 \mathbf{v}_2 d\xi$$

up to terms $O(\epsilon^{2+\gamma})$. To this end we note that, in view of (3.30), $(3.38)_{1,2}$, $\mathcal{U} = O(\epsilon)$ and $\xi = O(\epsilon)$, we have $J = J_0 + O(\epsilon)$, $\mathbf{w} = \mathbf{F}^{-1}\mathbf{v}_\mathcal{S} + O(\epsilon)$, and hence for v_1, v_2 shown in (8.87), (8.89), we obtain the estimations

$$\mathrm{v}_1 = (1 - c_{01})\mathcal{U} - \frac{1}{J_0} \mathrm{Div} \int_0^{\xi} J_0 \mathbf{v}_1 d\xi' + O(\epsilon^2)$$
$$= O(\epsilon),$$
$$\mathrm{v}_2 = \mathcal{U} + \frac{\tilde{\partial} h_1}{\partial t} - \mathrm{Grad}\, h_1 \cdot \mathbf{F}^{-1}\mathbf{v}_\mathcal{S}$$
$$- \frac{1}{J_0} \mathrm{Div} \int_{h_1}^{\xi} J_0 \mathbf{v}_2 d\xi' + O(\epsilon^2) = O(\epsilon). \quad (9.95)$$

Thus, recalling that $h \equiv h_2 - h_1$, our assumptions are

$$\int_0^{h_1} \mathrm{v}_1 \mathbf{v}_1 d\xi$$
$$= (1 - c_{01}) h_1 \mathcal{U} \bar{\mathbf{v}}_1 + \frac{1}{2} h_1^2 \beta_1 \bar{\mathbf{v}}_1 + O(\epsilon^{2+\gamma}),$$
$$\int_{h_1}^{h_2} \mathrm{v}_2 \mathbf{v}_2 d\xi$$
$$= h \left(\mathcal{U} + \frac{\tilde{\partial} h_1}{\partial t} + \mathrm{Grad}\, h_1 \cdot (\bar{\mathbf{v}}_2 - \mathbf{F}^{-1}\mathbf{v}_\mathcal{S}) \right) \bar{\mathbf{v}}_2$$
$$+ \frac{1}{2} h^2 \beta_2 \bar{\mathbf{v}}_2 + O(\epsilon^{2+\gamma}). \quad (9.96)$$

These assumptions are suggested by the analysis of a plug flow. Thus, since for such a flow $\mathbf{v}_1 = \bar{\mathbf{v}}_1$, $\mathbf{v}_2 = \bar{\mathbf{v}}_2$, from (9.95) we have

$$\mathrm{v}_1 = (1 - c_{01})\mathcal{U} - \frac{1}{J_0} \xi \mathrm{Div}\,(J_0 \bar{\mathbf{v}}_1) + O(\epsilon^2),$$
$$\mathrm{v}_2 = \mathcal{U} + \frac{\tilde{\partial} h_1}{\partial t} + \mathrm{Grad}\, h_1 \cdot (\bar{\mathbf{v}}_2 - \mathbf{F}^{-1}\mathbf{v}_\mathcal{S})$$
$$- \frac{1}{J_0} (\xi - h_1) \mathrm{Div}\,(J_0 \bar{\mathbf{v}}_2) + O(\epsilon^2),$$

and therefore assumptions (9.94) are satisfied with

$$\beta_1 = -\frac{1}{J_0} \mathrm{Div}\,(J_0 \bar{\mathbf{v}}_1), \quad \beta_2 = -\frac{1}{J_0} \mathrm{Div}\,(J_0 \bar{\mathbf{v}}_2).$$

Consequently, Boussinesq coefficients β_1, β_2, different from those above, indicate deviations of the tangential velocity field from a plug flow.

c) *Dynamic stress approximations*: corresponding to the motion of the avalanche mass, the stress tensors $\boldsymbol{\sigma}_\mathfrak{k}$, $\mathfrak{k} = 1, 2$, are postulated to satisfy the conditions (see notations (7.71))

$$p_\mathfrak{k} = O(\epsilon), \quad \mathbf{P}_\mathfrak{k} = O(\epsilon),$$
$$\mathbf{p}_\mathfrak{k} = O(\epsilon^\gamma), \quad T_\mathfrak{k}^{33} = O(\epsilon^{1+\gamma}), \quad (9.97)$$

by which both pressures p_1 and p_2 are assumed of the order of the hydrostatic pressure, and, as we shall see, the normal extra-stresses parallel to the base, i.e. $T_\mathfrak{k}^{11}$, $T_\mathfrak{k}^{22}$, and the shear stresses $T_\mathfrak{k}^{12}$, $\mathbf{p}_\mathfrak{k}$ are small, but still play a role, while the dissipative extra-stresses $T_\mathfrak{k}^{33}$ are negligibly small.

10 Shallow avalanche equations for a two-layer flow over variable topography

Now, under the scalings introduced in the previous section, we first exploit the normal momentum balance equation (7.75) for both $\mathfrak{k} = 1, 2$, by accounting for the boundary conditions $(7.81)_2$, $(7.82)_2$, to deduce the pressures p_1, p_2. Then, under the same scalings, we transform the depth-integrated balance equations (8.88)–(8.92) to deduce the *shallow avalanche equations* characterizing a two-layer flow over arbitrary moving topography. In doing so, we still deal with arbitrary rheologic properties of the avalanche mass. The procedure closely follows that in Luca et al. [12], which is why some results are simply taken over from the cited paper. We obtain

Proposition 10.1 *Under the scalings a)–c), corresponding to the two layers the mean pressures are given by*

$$\bar{p}_1 = \frac{1}{2} h_1 (c + a_1 m_3^{(1)}) + \frac{1}{c_{12}} h (c + a_2 m_2^{(2)}) + O(\epsilon^{1+\gamma}),$$
$$a_1 \equiv \mathbf{H} \bar{\mathbf{v}}_1 \cdot \bar{\mathbf{v}}_1,$$

$$\bar{p}_2 = \frac{1}{2}h(c + a_2 m_3^{(2)}) + h_1 a_2 (m_3^{(2)} - m_2^{(2)}) + O(\epsilon^{1+\gamma}),$$

$$a_2 \equiv \mathbf{H}\bar{\mathbf{v}}_2 \cdot \bar{\mathbf{v}}_2, \tag{10.98}$$

the depth-integrated mass and momentum balance equations corresponding to the lower layer are

$$\frac{\tilde{\partial}}{\partial t}\{J_0 h_1 (1 - \Omega h_1)\} + \mathrm{Div}\,\{J_0 h_1 [(1 - \Omega h_1 m_1^{(1)})\bar{\mathbf{v}}_1 - \mathbf{u}_S + \Omega h_1 \mathbf{F}^{-1}\mathbf{v}_S]\} = -c_{01} J_0 \mathcal{U} + O(\epsilon^{2+\gamma}), \tag{10.99}$$

$$\begin{aligned}
&\frac{\tilde{\partial}}{\partial t}\{J_0 h_1 (1 - \Omega h_1 m_1^{(1)})\mathbf{F}\bar{\mathbf{v}}_1\} \\
&\quad + \mathrm{Div}\,\{J_0 h_1 \mathbf{F}[(m_2^{(1)} - \Omega h_1 m_3^{(1)})\bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1 \\
&\quad + \bar{p}_1 \mathbf{M}_0 - \bar{\mathbf{P}}_1 - \bar{\mathbf{v}}_1 \otimes \mathbf{u}_S + \Omega h_1 m_1^{(1)} \bar{\mathbf{v}}_1 \otimes \mathbf{F}^{-1}\mathbf{v}_S]\} \\
&\quad + J_0 h_1 \{2\mathbf{F}\mathbf{W}\bar{\mathbf{p}}_1 - (\mathbf{H}\cdot\bar{\mathbf{P}}_1)\mathbf{s}\} \\
&= -\frac{1}{c_{12}}\mathbf{F}\{J[(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\, h_1 - \mathbf{p}_2]\}_{\xi = h_1} \\
&\quad - J_0 \mathbf{F}(\mathbf{p}_1 + c_{01}\mathcal{U}\mathbf{v}_1)|_{\xi=0} \\
&\quad - J_0 h_1 \left\{ c + a_1 m_2^{(1)} - \frac{1}{2} h_1 \tilde{a}_1 m_3^{(1)} \right. \\
&\quad \left. + \frac{1}{c_{12}} 2\Omega h (c + a_2 m_2^{(2)}) + \mathrm{Grad}\,\mathcal{U} \cdot \bar{\mathbf{v}}_1 \right\} \mathbf{s} \\
&\quad - \frac{1}{2} J_0 h_1^2 (c + a_1 m_3^{(1)})\mathbf{F}\mathbf{W}\mathbf{F}^{-1}\mathbf{s} + J_0 h_1^2 \beta_1 \mathbf{F}\mathbf{W}\bar{\mathbf{v}}_1 \\
&\quad + \frac{1}{2} J_0 h_1^2 m_3^{(1)} \frac{\partial}{\partial \xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}(\bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1)\mathbf{e}_\alpha \\
&\quad + (1 - 2c_{01}) J_0 h_1 \mathcal{U}\mathbf{F}\mathbf{W}\bar{\mathbf{v}}_1 \\
&\quad + J_0 h_1 \frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{\mathbf{u}_S \otimes \bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_1 \otimes \mathbf{u}_S\}\mathbf{e}_\alpha \\
&\quad - J_0 \Omega h_1^2 m_1^{(1)} \frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{\mathbf{F}^{-1}\mathbf{v}_S \otimes \bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_1 \otimes \mathbf{F}^{-1}\mathbf{v}_S\}\mathbf{e}_\alpha \\
&\quad + O(\epsilon^{2+\gamma}),
\end{aligned} \tag{10.100}$$

where $\tilde{a}_1 \equiv \mathbf{H}\bar{\mathbf{v}}_1 \cdot \mathbf{W}\bar{\mathbf{v}}_1$*, and the depth-integrated mass and momentum balance equations corresponding to the upper layer emerge as*

$$\begin{aligned}
&\frac{\tilde{\partial}}{\partial t}\{J_0 h [1 - \Omega(h + 2h_1)]\} \\
&\quad + \mathrm{Div}\,\{J_0 h [(1 - \Omega(h + 2h_1) m_1^{(2)})\bar{\mathbf{v}}_2 \\
&\quad - \mathbf{u}_S + \Omega(h + 2h_1)\mathbf{F}^{-1}\mathbf{v}_S]\} \\
&= O(\epsilon^{2+\gamma}),
\end{aligned} \tag{10.101}$$

$$\begin{aligned}
&\frac{\tilde{\partial}}{\partial t}\{J_0 h [1 - \Omega(h + 2h_1) m_1^{(2)}]\mathbf{F}\bar{\mathbf{v}}_2\} \\
&\quad + \mathrm{Div}\,\{J_0 h \mathbf{F}[(m_2^{(2)} - \Omega(h + 2h_1) m_3^{(2)})\bar{\mathbf{v}}_2 \otimes \bar{\mathbf{v}}_2 \\
&\quad + \bar{p}_2 \mathbf{M}_0 - \bar{\mathbf{P}}_2 - \bar{\mathbf{v}}_2 \otimes \mathbf{u}_S \\
&\quad + \Omega(h + 2h_1) m_1^{(2)} \bar{\mathbf{v}}_2 \otimes \mathbf{F}^{-1}\mathbf{v}_S]\} \\
&\quad + J_0 h \{2\mathbf{F}\mathbf{W}\bar{\mathbf{p}}_2 - (\mathbf{H}\cdot\bar{\mathbf{P}}_2)\mathbf{s}\} \\
&= \mathbf{F}\{J[(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\, h_1 - \mathbf{p}_2]\}_{\xi = h_1} \\
&\quad - J_0 h \left\{ (1 - 2\Omega h_1)(c + a_2 m_2^{(2)}) \right. \\
&\quad \left. - \frac{1}{2}(h + 2h_1)\tilde{a}_2 m_3^{(2)} + \mathrm{Grad}\,\mathcal{U} \cdot \bar{\mathbf{v}}_2 \right\} \mathbf{s} \\
&\quad - \frac{1}{2} J_0 h (h + 2h_1)(c + a_2 m_3^{(2)})\mathbf{F}\mathbf{W}\mathbf{F}^{-1}\mathbf{s} \\
&\quad + J_0 h^2 \beta_2 \mathbf{F}\mathbf{W}\bar{\mathbf{v}}_2 + \frac{1}{2} J_0 h (h + 2h_1) m_3^{(2)} \\
&\quad \times \frac{\partial}{\partial \xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}(\bar{\mathbf{v}}_2 \otimes \bar{\mathbf{v}}_2)\mathbf{e}_\alpha + J_0 h \mathcal{U}\mathbf{F}\mathbf{W}\bar{\mathbf{v}}_2 \\
&\quad + 2 J_0 h \left\{ \frac{\tilde{\partial} h_1}{\partial t} + \mathrm{Grad}\, h_1 \cdot (\bar{\mathbf{v}}_2 - \mathbf{F}^{-1}\mathbf{v}_S) \right\} \mathbf{F}\mathbf{W}\bar{\mathbf{v}}_2 \\
&\quad + J_0 h \frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{\mathbf{u}_S \otimes \bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_2 \otimes \mathbf{u}_S\}\mathbf{e}_\alpha \\
&\quad - J_0 \Omega h (h + 2h_1) m_1^{(2)} \frac{\partial \mathbf{F}}{\partial \xi^\alpha} \\
&\quad \times \{\mathbf{F}^{-1}\mathbf{v}_S \otimes \bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_2 \otimes \mathbf{F}^{-1}\mathbf{v}_S\}\mathbf{e}_\alpha + O(\epsilon^{2+\gamma}),
\end{aligned} \tag{10.102}$$

where $\tilde{a}_2 \equiv \mathbf{H}\bar{\mathbf{v}}_2 \cdot \mathbf{W}\bar{\mathbf{v}}_2$.

Proof Using the approximations $\mathbf{v}_\mathfrak{k} = O(1)$, $\mathrm{v}_\mathfrak{k} = O(\epsilon)$, $\mathcal{U} = O(\epsilon)$, (9.97), and recalling that $\mathrm{b} = -c$, equation (7.75) written for $\mathfrak{k} = 1, 2$ emerges as

$$\frac{\partial p_1}{\partial \xi} = -c + \Gamma(\mathbf{v}_1) + O(\epsilon^\gamma), \quad \frac{\partial p_2}{\partial \xi} = -c + \Gamma(\mathbf{v}_2) + O(\epsilon^\gamma), \tag{10.103}$$

where $\Gamma(\mathbf{v}_\mathfrak{k}) = -\mathbf{H} \cdot (\mathbf{v}_\mathfrak{k} \otimes \mathbf{v}_\mathfrak{k}) + O(\epsilon)$, see $(5.52)_4$ with $\mathbf{B} = \mathbf{F} + O(\epsilon)$. We first determine p_2, since in order to obtain p_1 from $(10.103)_1$ we need to know p_2 at $\xi = h_1$. To this end we integrate $(10.103)_2$ from $\xi \in (h_1, h_2)$ to h_2 and use the boundary condition $(7.83)_2$, which, according to the dynamic assumptions (9.97), reads as

$$p_2 = O(\epsilon^{1+\gamma}) \quad \text{at} \quad \xi = h_2(\boldsymbol{\xi}, t).$$

We obtain

$$p_2 = c(h_2 - \xi) + \mathbf{H} \cdot \int_{\xi}^{h_2} \mathbf{v}_2 \otimes \mathbf{v}_2 d\xi' + O(\epsilon^{1+\gamma}),$$

which, by changing the order of integration, gives

$$\bar{p}_2 = \frac{1}{2}hc + \frac{1}{h}\mathbf{H} \cdot \left(\int_{h_1}^{h_2} \xi \mathbf{v}_2 \otimes \mathbf{v}_2 d\xi - h_1 \int_{h_1}^{h_2} \mathbf{v}_2 \otimes \mathbf{v}_2 d\xi \right) + O(\epsilon^{1+\gamma}).$$

Then, by appeal to the Boussinesq approximations (9.94), we derive $(10.98)_2$ and

$$p_2|_{\xi=h_1} = h(c + a_2 m_2^{(2)}) + O(\epsilon^{1+\gamma}). \tag{10.104}$$

Now we integrate $(10.103)_1$ from $\xi \in (0, h_1)$ to h_1 and obtain

$$p_1 = c(h_1 - \xi) + \mathbf{H} \cdot \int_{\xi}^{h_1} \mathbf{v}_1 \otimes \mathbf{v}_1 d\xi' + p_1|_{\xi=h_1} + O(\epsilon^{1+\gamma}). \tag{10.105}$$

With the dynamic assumptions (9.97), condition $(7.81)_2$ appears as

$$c_{12} p_1 = p_2 + O(\epsilon^{1+\gamma}) \quad \text{at} \quad \xi = h_1(\boldsymbol{\xi}, t),$$

which can be used together with (10.104) to replace $p_1|_{\xi=h_1}$ in (10.105). Then, by changing the order of integration, we derive $(10.98)_1$. For further reference we note the value

$$p_1|_{\xi=0} = h_1(c + a_1 m_2^{(1)}) + \frac{1}{c_{12}} h(c + a_2 m_2^{(2)}) + O(\epsilon^{1+\gamma}). \tag{10.106}$$

The depth-integrated balance equations (8.88), (8.91) corresponding to the lower layer are similar to those in Luca et al. [1]. Differences arise due to the terms involving $\mathbf{w}$, $\mathcal{U}$ and to the term evaluated at $\xi = h_1$ in (8.91). Thus, omitting the lower index 1, since $\mathbf{v} = O(1)$, $p = O(\epsilon)$, $\mathbf{P} = O(\epsilon)$, $\mathbf{p} = O(\epsilon^{1+\gamma})$, $\mathrm{v} = O(\epsilon)$, the following approximations of the integrands in (8.88), (8.91) can be taken from the cited paper:

$$J = J_0(1 - 2\Omega\xi) + O(\epsilon^2),$$

$$J\mathbf{v} = J_0(1 - 2\Omega\xi)\mathbf{v} + O(\epsilon^2),$$

$$J(\mathbf{v} \otimes \mathbf{v} + p\mathbf{M} - \mathbf{P}) = J_0\{(1 - 2\Omega\xi)\mathbf{v} \otimes \mathbf{v} + p\mathbf{M}_0 - \mathbf{P}\} + O(\epsilon^{1+\gamma}),$$

$$J\boldsymbol{\Gamma}(-p\mathbf{M}, \mathbf{0}) = J_0 p \left\{ \mathbf{F}^{-1} \frac{\partial \mathbf{F}}{\partial \xi^\alpha} \mathbf{M}_0 \mathbf{e}_\alpha + 2\Omega \mathbf{F}^{-1}\mathbf{s} \right\} + O(\epsilon^{1+\gamma}),$$

$$J\boldsymbol{\Gamma}(\mathbf{P}, \mathbf{p}) = -J_0 \left\{ \mathbf{F}^{-1} \frac{\partial \mathbf{F}}{\partial \xi^\alpha} \mathbf{P}\mathbf{e}_\alpha - 2\mathbf{W}\mathbf{p} + (\mathbf{H}\cdot\mathbf{P})\mathbf{F}^{-1}\mathbf{s} \right\} + O(\epsilon^{1+\gamma})$$

$$J\mathbf{b} = -J_0 c\{(1 - 2\Omega\xi)\mathbf{I} + \xi\mathbf{W}\}\mathbf{F}^{-1}\mathbf{s} + O(\epsilon^2),$$

$$\begin{aligned} J\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v}) = & -J_0\{(1 - 2\Omega\xi)\mathbf{F}^{-1} \frac{\partial \mathbf{F}}{\partial \xi^\alpha} \\ & - \xi \mathbf{F}^{-1} \frac{\partial}{\partial \xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}\}(\mathbf{v} \otimes \mathbf{v})\mathbf{e}_\alpha \\ & + 2J_0 \mathrm{v}\mathbf{W}\mathbf{v} - J_0\{(1 - 2\Omega\xi)(\mathbf{H}\cdot(\mathbf{v} \otimes \mathbf{v}))\mathbf{I} \\ & + \xi(\mathbf{H}\cdot(\mathbf{v} \otimes \mathbf{v}))\mathbf{W} \\ & - \xi(\mathbf{W}^T\mathbf{H}\cdot(\mathbf{v} \otimes \mathbf{v}))\mathbf{I}\}\mathbf{F}^{-1}\mathbf{s} + O(\epsilon^2). \end{aligned} \tag{10.107}$$

For the terms containing $\mathbf{w}$ we use

$$\mathbf{w} = \mathbf{u}_S + O(\epsilon^2) = \mathbf{F}^{-1}\mathbf{v}_S + O(\epsilon), \quad \mathcal{U}\frac{\partial \mathbf{w}}{\partial \xi} = O(\epsilon^2)$$

to obtain

$$J\mathbf{w} = J_0(\mathbf{u}_S - 2\Omega\xi\mathbf{F}^{-1}\mathbf{v}_S) + O(\epsilon^2),$$

$$J\mathbf{v} \otimes \mathbf{w} = J_0\mathbf{v} \otimes (\mathbf{u}_S - 2\Omega\xi\mathbf{F}^{-1}\mathbf{v}_S) + O(\epsilon^2),$$

$$J\left\{(\mathrm{Grad}\,\mathbf{w})\mathbf{v} + \mathcal{U}\frac{\partial \mathbf{w}}{\partial \xi}\right\} = J_0(\mathrm{Grad}\,\mathbf{u}_S)\mathbf{v} - 2J_0\Omega\xi(\mathrm{Grad}\,\mathbf{F}^{-1}\mathbf{v}_S)\mathbf{v} + O(\epsilon^2). \tag{10.108}$$

Now we have only to write approximations (10.107), (10.108) corresponding to layer 1, and to perform the integration in (8.88), (8.91) by taking into account the Boussinesq approximations, to deduce (10.99) and

$$\begin{aligned} & \frac{\tilde{\partial}}{\partial t}\{J_0 h_1(1 - \Omega h_1 m_1^{(1)})\bar{\mathbf{v}}_1\} \\ & + \mathrm{Div}\,\{J_0 h_1[(m_2^{(1)} - \Omega h_1 m_3^{(1)})\bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1 \\ & + \bar{p}_1\mathbf{M}_0 - \bar{\mathbf{P}}_1 - \bar{\mathbf{v}}_1 \otimes \mathbf{u}_S + \Omega h_1 m_1^{(1)}\bar{\mathbf{v}}_1 \otimes \mathbf{F}^{-1}\mathbf{v}_S]\} \\ & + J_0 h_1\{2\mathbf{W}\bar{\mathbf{p}}_1 - (\mathbf{H}\cdot\bar{\mathbf{P}}_1)\mathbf{F}^{-1}\mathbf{s}\} \\ = & -\frac{1}{c_{12}}\{J[(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\,h_1 - \mathbf{p}_2]\}_{\xi=h_1} \\ & - J_0(\mathbf{p}_1 + c_{01}\mathcal{U}\mathbf{v}_1)|_{\xi=0} - J_0 h_1 \\ & \times \left\{ \left[c + a_1 m_2^{(1)} - \frac{1}{2}h_1\tilde{a}_1 m_3^{(1)} \right.\right. \\ & \left.\left. + \frac{1}{c_{12}} 2\Omega h(c + a_2 m_2^{(2)}) \right] \mathbf{I} + \frac{1}{2}h_1(c + a_1 m_3^{(1)})\mathbf{W} \right\} \mathbf{F}^{-1}\mathbf{s} \end{aligned}$$

$$- \mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{J_0 h_1[(m_2^{(1)} - \Omega h_1 m_3^{(1)})\bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1$$

$$+ \bar{p}_1 \mathbf{M}_0 - \bar{\mathbf{P}}_1]\}\mathbf{e}_\alpha + \frac{1}{2}J_0 h_1^2 m_3^{(1)}\mathbf{F}^{-1}$$

$$\times \frac{\partial}{\partial \xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}(\bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1)\mathbf{e}_\alpha + 2(1 - c_{01})J_0 h_1 \mathcal{U}\mathbf{W}\bar{\mathbf{v}}_1$$

$$+ J_0 h_1^2 \beta_1 \mathbf{W}\bar{\mathbf{v}}_1 - J_0 h_1 (\mathrm{Grad}\,\mathbf{u}_{\mathcal{S}})\bar{\mathbf{v}}_1$$

$$+ J_0 \Omega h_1^2 m_1^{(1)}(\mathrm{Grad}\,\mathbf{F}^{-1}\mathbf{v}_{\mathcal{S}})\bar{\mathbf{v}}_1 + O(\epsilon^{2+\gamma}). \qquad (10.109)$$

If we multiply (10.109) from the left by $\mathbf{F}$ and use

$$\frac{\tilde{\partial}\mathbf{F}}{\partial t} = \mathrm{Grad}\,\mathbf{v}_{\mathcal{S}}, \quad \mathrm{Div}\,\mathbf{F}\mathbf{X} = \mathbf{F}\mathrm{Div}\,\mathbf{X} + \frac{\partial \mathbf{F}}{\partial \xi^\alpha}\mathbf{X}\mathbf{e}_\alpha,$$

$$J_0 h(1 - \Omega h m_1)(\mathrm{Grad}\,\mathbf{v}_{\mathcal{S}})\bar{\mathbf{v}}$$

$$+ J_0 h \frac{\partial \mathbf{F}}{\partial \xi^\alpha}(-\bar{\mathbf{v}} \otimes \mathbf{u}_{\mathcal{S}} + \Omega h m_1 \bar{\mathbf{v}} \otimes \mathbf{F}^{-1}\mathbf{v}_{\mathcal{S}})\mathbf{e}_\alpha$$

$$- J_0 h \mathbf{F}(\mathrm{Grad}\,\mathbf{u}_{\mathcal{S}})\bar{\mathbf{v}} + J_0 \Omega h^2 m_1 \mathbf{F}(\mathrm{Grad}\,\mathbf{F}^{-1}\mathbf{v}_{\mathcal{S}})\bar{\mathbf{v}}$$

$$= -J_0 h(\mathbf{s} \otimes \mathrm{Grad}\,\mathcal{U})\bar{\mathbf{v}} + J_0 h \frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{\mathbf{u}_{\mathcal{S}} \otimes \bar{\mathbf{v}} - \bar{\mathbf{v}} \otimes \mathbf{u}_{\mathcal{S}}\}\mathbf{e}_\alpha$$

$$- J_0 h \mathcal{U}\mathbf{F}\mathbf{W}\bar{\mathbf{v}} - J_0 \Omega h^2 m_1 \frac{\partial \mathbf{F}}{\partial \xi^\alpha}$$

$$\times \{\mathbf{F}^{-1}\mathbf{v}_{\mathcal{S}} \otimes \bar{\mathbf{v}} - \bar{\mathbf{v}} \otimes \mathbf{F}^{-1}\mathbf{v}_{\mathcal{S}}\}\mathbf{e}_\alpha,$$

where $\mathbf{X}$ is a 2×2 matrix, and in which we take h_1, $\bar{\mathbf{v}}_1$, $m_1^{(1)}$ instead of h, $\bar{\mathbf{v}}$ and m_1, respectively, we arrive at (10.100). Note that, in deducing the last relation above, we used the formula $\mathrm{Grad}\,\mathbf{s} = \mathbf{F}\mathbf{W}$, see Luca, Tai and Kuo [2]. Similar calculations can be performed to transform (8.90), (8.92) into (10.101), (10.102).

There are significant simplifications of the *shallow avalanche equations* in Prop. 10.1 if the basal topography is only slightly curved:

Proposition 10.2 *If the bed surface has small curvature, in the sense that $\mathcal{H} = O(\epsilon^\gamma)$, the mean pressures in the two layers are given by*

$$\bar{p}_1 = \frac{1}{2}ch_1 + \frac{1}{c_{12}}ch + O(\epsilon^{1+\gamma}), \quad \bar{p}_2 = \frac{1}{2}ch + O(\epsilon^{1+\gamma}), \qquad (10.110)$$

the depth-integrated mass and momentum balance equations corresponding to the lower layer are

$$\frac{\tilde{\partial}}{\partial t}\{J_0 h_1\} + \mathrm{Div}\,\{J_0 h_1(\bar{\mathbf{v}}_1 - \mathbf{u}_{\mathcal{S}})\} = -c_{01}J_0\mathcal{U} + O(\epsilon^{2+\gamma}), \qquad (10.111)$$

$$\frac{\tilde{\partial}}{\partial t}\{J_0 h_1 \mathbf{F}\bar{\mathbf{v}}_1\} + \mathrm{Div}\,\{J_0 h_1 \mathbf{F}[m_2^{(1)}\bar{\mathbf{v}}_1 \otimes \bar{\mathbf{v}}_1$$

$$+ \bar{p}_1 \mathbf{M}_0 - \bar{\mathbf{P}}_1 - \bar{\mathbf{v}}_1 \otimes \mathbf{u}_{\mathcal{S}}]\} + 2J_0 h_1 \mathbf{F}\mathbf{W}\bar{\mathbf{p}}_1$$

$$= -\frac{1}{c_{12}}\mathbf{F}\{J[(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\,h_1 - \mathbf{p}_2]\}_{\xi=h_1}$$

$$- J_0\mathbf{F}(\mathbf{p}_1 + c_{01}\mathcal{U}\mathbf{v}_1)|_{\xi=0}$$

$$- J_0 h_1(c + a_1 m_2^{(1)} + \mathrm{Grad}\,\mathcal{U} \cdot \bar{\mathbf{v}}_1)\mathbf{s}$$

$$+ J_0 h_1 \frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{\mathbf{u}_{\mathcal{S}} \otimes \bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_1 \otimes \mathbf{u}_{\mathcal{S}}\}\mathbf{e}_\alpha + O(\epsilon^{2+\gamma}), \qquad (10.112)$$

and the depth-integrated mass and momentum balance equations corresponding to the upper layer emerge as

$$\frac{\tilde{\partial}}{\partial t}\{J_0 h\} + \mathrm{Div}\,\{J_0 h(\bar{\mathbf{v}}_2 - \mathbf{u}_{\mathcal{S}})\} = O(\epsilon^{2+\gamma}), \qquad (10.113)$$

$$\frac{\tilde{\partial}}{\partial t}\{J_0 h \mathbf{F}\bar{\mathbf{v}}_2\} + \mathrm{Div}\,\{J_0 h \mathbf{F}(m_2^{(2)}\bar{\mathbf{v}}_2 \otimes \bar{\mathbf{v}}_2$$

$$+ \bar{p}_2 \mathbf{M}_0 - \bar{\mathbf{P}}_2 - \bar{\mathbf{v}}_2 \otimes \mathbf{u}_{\mathcal{S}})\} + 2J_0 h \mathbf{F}\mathbf{W}\bar{\mathbf{p}}_2$$

$$= \mathbf{F}\{J[(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\,h_1 - \mathbf{p}_2]\}_{\xi=h_1}$$

$$- J_0 h(c + a_2 m_2^{(2)} + \mathrm{Grad}\,\mathcal{U} \cdot \bar{\mathbf{v}}_2)\mathbf{s}$$

$$+ J_0 h \frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{\mathbf{u}_{\mathcal{S}} \otimes \bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_2 \otimes \mathbf{u}_{\mathcal{S}}\}\mathbf{e}_\alpha + O(\epsilon^{2+\gamma}). \qquad (10.114)$$

In Sec. 12 we shall show, in particular, how equations (10.99)–(10.102), and consequently (10.111)–(10.114), can be further simplified.

We mention that, if in (10.100) one omits the term evaluated at $\xi = h_1$, as well as the term containing h in the coefficient of $\mathbf{s}$ (line 4 of (10.100)), and if one takes $\bar{p}_1 = \frac{1}{2}h_1(c + a_1 m_3^{(1)})$ in (10.100) (see line 2), one deduces an equation which, together with (10.99), forms the shallow avalanche equations corresponding to a *single* layer made by a one constituent mass flowing over variable topography. (To see this one has only to follow similar derivations for a one-layer avalanche mass for which the free surface is assumed traction-free.) For simplicity we write these shallow avalanche equations for the case of small curvature, that is, we write equations (10.111), (10.112), modified as just mentioned (we omit the lower/upper index 1):

$$\frac{\tilde{\partial}}{\partial t}\{J_0 h\} + \mathrm{Div}\,\{J_0 h(\bar{\mathbf{v}} - \mathbf{u}_{\mathcal{S}})\} = -c_{01}J_0\mathcal{U} + O(\epsilon^{2+\gamma}),$$

$$\frac{\tilde{\partial}}{\partial t}\{J_0 h \mathbf{F}\bar{\mathbf{v}}\} + \mathrm{Div}\,\{J_0 h \mathbf{F}[m_2\bar{\mathbf{v}} \otimes \bar{\mathbf{v}} + \bar{p}\mathbf{M}_0 - \bar{\mathbf{P}} - \bar{\mathbf{v}} \otimes \mathbf{u}_{\mathcal{S}}]\}$$

$$+ 2J_0 h \mathbf{F}\mathbf{W}\bar{\mathbf{p}}$$

$$= -J_0\mathbf{F}(\mathbf{p} + c_{01}\mathcal{U}\mathbf{v})|_{\xi=0}$$

$$- J_0 h(c + am_2 + \mathrm{Grad}\,\mathcal{U}\cdot\bar{\mathbf{v}})\mathbf{s}$$

$$+ J_0 h\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\{\mathbf{u}_\mathbb{S}\otimes\bar{\mathbf{v}} - \bar{\mathbf{v}}\otimes\mathbf{u}_\mathbb{S}\}\mathbf{e}_\alpha + O(\epsilon^{2+\gamma}),$$

where $\bar{p} = \frac{1}{2}ch$.

11 Constitutive assumptions

Equations (10.99)–(10.102) and (3.16) are meant to stand for the determination of $\bar{\mathbf{v}}_1$, $\bar{\mathbf{v}}_2$, h_1, h and b. To this end the mean stresses $\bar{\mathbf{P}}_\mathfrak{k}$, $\bar{\mathbf{p}}_\mathfrak{k}$, the terms evaluated at $\xi = 0$ and $\xi = h_1$ in (10.100), and the erosion/deposition rate $\mathcal{U}$ must be given in terms of these unknown basic fields. Except $\mathcal{U}$, for which we refer the reader e.g. to Bouchut et al. [9], Tai and Kuo [10], Fraccarollo and Capart [11], we propose closure relations for the mentioned quantities, but, if the reader so desires, other laws can be assumed, see e.g. Luca et al. [1].

Thus, we consider the rheological properties of the avalanching mass in the lower layer as being described by one of the three models of granular material described in Luca, Tai and Kuo [3]. We do not repeat here the corresponding constitutive assumptions, we only mention that one of these models is the inviscid fluid, and the other two are topography-adapted versions of Iverson and Denlinger [13,14] and Hutter et al. [15,16] models, respectively. According to these models we have $\bar{\mathbf{p}}_1 = O(\epsilon^{1+\gamma})$, hence negligibly small, and $T_1^{33} = 0$; the stresses $\bar{\mathbf{P}}_1$ can be taken over from [3].

Next, to deduce the basal shear stress $\mathbf{p}_1|_{\xi=0}$, present in the term evaluated at $\xi = 0$ in (10.100), we assume the (dimensional/non-dimensional) Coulomb bottom friction,

$$\boldsymbol{\sigma}_1\boldsymbol{n} - (\boldsymbol{\sigma}_1\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n} = (\tan\delta)(-\boldsymbol{\sigma}_1\boldsymbol{n}\cdot\boldsymbol{n})_+\mathrm{sgn}\,\boldsymbol{v}_{1\tau} \quad \text{at} \quad x_3 = b(x_1, x_2, t), \tag{11.115}$$

where δ is the *basal angle of friction*, $\tan\delta > 0$, $\tan\delta = O(\epsilon^\gamma)$, the index $+$ stands for the positive part of a quantity, i.e. $f_+ \equiv \max\{0, f\}$, and

$$\mathrm{sgn}\,\boldsymbol{v} \equiv \begin{cases} \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}, & \text{if } \boldsymbol{v}\neq\boldsymbol{0}, \quad \boldsymbol{v} \text{ tangent to } \mathbb{S},\\ \text{any tangent vector } \boldsymbol{m} \text{ to } \mathbb{S},\ \|\boldsymbol{m}\|\le 1, & \text{if } \boldsymbol{v} = \boldsymbol{0},\end{cases}$$

see e.g. Bouchut and Westdickenberg [7]. Recalling that $\boldsymbol{v}_\tau = v^1\boldsymbol{g}_1 + v^2\boldsymbol{g}_2$, when written in curvilinear coordinates the law (11.115) emerges as

$$\mathbf{p}_1 = (\tan\delta)(p_1 - T_1^{33})_+\mathrm{sgn}\,\mathbf{v}_1 \quad \text{at} \quad \xi = 0,$$

where $\mathrm{sgn}\,\mathbf{x}$ is defined as

$$\mathrm{sgn}\,\mathbf{x} \equiv \begin{cases} \frac{1}{\sqrt{\mathbf{M}_0^{-1}\mathbf{x}\cdot\mathbf{x}}}\mathbf{x}, & \text{if } \mathbf{x}\neq\mathbf{0},\\ \text{any 2-column } \mathbf{m}, \quad \mathbf{M}_0^{-1}\mathbf{m}\cdot\mathbf{m}\le 1, & \text{if } \mathbf{x} = \mathbf{0}.\end{cases}$$

With the assumption

$$\mathbf{v}_1|_{\xi=0} = \chi_{base}\bar{\mathbf{v}}_1, \quad \chi_{base} > 0, \quad \chi_{base} = O(1),$$

we therefore have

$$(\mathbf{p}_1 + c_{01}\mathcal{U}\mathbf{v}_1)|_{\xi=0} = (\tan\delta)(p_1|_{\xi=0})_+\mathrm{sgn}\,\bar{\mathbf{v}}_1 + c_{01}\chi_{base}\mathcal{U}\bar{\mathbf{v}}_1. \tag{11.116}$$

In the above the pressure p_1 at $\xi = 0$ is given by (10.106), where, with a similar motivation as that from Luca, Tai and Kuo [3] (see also the closure assumptions for the fluid in the upper layer), the neglected terms are of order $O(\epsilon^2)$.

In the upper layer we assume the flowing avalanche mass to be a Newtonian/non-Newtonian fluid with viscosity $\eta = O(\epsilon^{2+\gamma})$, which gives, see e.g. Luca et al. [1],

$$\mathbf{P}_2 = O(\epsilon^{2+\gamma}), \quad \mathbf{p}_2 = O(\epsilon^{1+\gamma}), \quad T_2^{33} = O(\epsilon^{2+\gamma}), \tag{11.117}$$

and hence in equation (10.102) the terms containing $\bar{\mathbf{P}}_2$, $\bar{\mathbf{p}}_2$ are negligibly small. Moreover, in view of $\mathbf{M} = \mathbf{M}_0 + O(\epsilon)$ and (10.104), the term evaluated at $\xi = h_1$ in both (10.100) and (10.102) emerges as

$$\{J[(-p_2\mathbf{M} + \mathbf{P}_2)\mathrm{Grad}\,h_1 - \mathbf{p}_2]\}_{\xi=h_1}$$

$$= J_0\{-p_2\mathbf{M}_0\mathrm{Grad}\,h_1 - \mathbf{p}_2\}_{\xi=h_1} + O(\epsilon^{2+\gamma}).$$

Since p_2 at $\xi = h_1$ is known, see (10.104), we have only to give the shear stress $\mathbf{p}_2$ on the layer interface. To this end at this interface we assume the (dimensional) Chézy-like friction condition, i.e.

$$\boldsymbol{\sigma}_2\boldsymbol{n}_1 - (\boldsymbol{\sigma}_2\boldsymbol{n}_1\cdot\boldsymbol{n}_1)\boldsymbol{n}_1 = \rho_2 c_{int}\|\boldsymbol{v}_{2\tau} - \boldsymbol{v}_{1\tau}\|(\boldsymbol{v}_{2\tau} - \boldsymbol{v}_{1\tau})$$

at $F_1(x_1, x_2, x_3, t) = 0$, where the (non-dimensional, positive) friction coefficient c_{int} is assumed constant, and $\boldsymbol{v}_{\mathfrak{k}\tau}$ is now the tangential velocity with respect to the interface $\mathbb{S}_1$, that is $\boldsymbol{v}_{\mathfrak{k}\tau} \equiv \boldsymbol{v}_\mathfrak{k} - (\boldsymbol{v}_\mathfrak{k}\cdot\boldsymbol{n}_1)\boldsymbol{n}_1$. In non-dimensional form the law above appears as

$$\boldsymbol{\sigma}_2\boldsymbol{n}_1 - (\boldsymbol{\sigma}_2\boldsymbol{n}_1\cdot\boldsymbol{n}_1)\boldsymbol{n}_1 = c_{int}\|\boldsymbol{v}_{2\tau} - \boldsymbol{v}_{1\tau}\|(\boldsymbol{v}_{2\tau} - \boldsymbol{v}_{1\tau}), \tag{11.118}$$

where $\boldsymbol{\sigma}_2 = O(\epsilon^{1+\gamma})$, see (11.117).

In curvilinear coordinates condition (11.118) is

$$\mathbf{p}_2 = c_{int}\sqrt{\mathbf{M}^{-1}(\mathbf{v}_2 - \mathbf{v}_1)\cdot(\mathbf{v}_2 - \mathbf{v}_1)}(\mathbf{v}_2 - \mathbf{v}_1)$$

$$+ O(\epsilon^{2+\gamma}) \quad \text{at} \quad \xi = h_1(\boldsymbol{\xi}, t).$$

Moreover, assumption

$$(\mathbf{v}_2 - \mathbf{v}_1)|_{\xi=h_1} = \chi_{int}(\bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_1), \quad \chi_{int} > 0,$$

$$c_{int}\chi_{int}^2 = O(\epsilon^{1+\gamma}),$$

and approximation $\mathbf{M} = \mathbf{M}_0 + O(\epsilon)$ turn the expression of $\mathbf{p}_2$ at $\xi = h_1$ into

$$\mathbf{p}_2|_{\xi=h_1} = c_{int}\chi_{int}^2\sqrt{\mathbf{M}_0^{-1}(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1)\cdot(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1)} \times(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1)+O(\epsilon^{2+\gamma}).$$

Consequently, at the layer interface we have

$$\begin{aligned}\{J[(-p_2\mathbf{M}+\mathbf{P}_2)\mathrm{Grad}\,h_1-\mathbf{p}_2]\}_{\xi=h_1} &= -J_0h(c+a_2m_2^{(2)})\mathbf{M}_0\mathrm{Grad}\,h_1 - J_0c_{int}\chi_{int}^2\\ &\quad\times\sqrt{\mathbf{M}_0^{-1}(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1)\cdot(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1)}(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1)+O(\epsilon^{2+\gamma}).\end{aligned} \quad (11.119)$$

This completes the task of proposing closure relations for the shallow avalanche equations (10.99)–(10.102).

12 On the parameterization of the basal surface

In the preceding analysis the parameterization (3.19) of the topographic surface was left arbitrary. Here we indicate two possible choices of the transformation (3.17), and hence of this parameterization.

Option 1 The coordinates (ξ^1, ξ^2) of a point Q on $\mathcal{S}_t$ are the arc lengths measured on the curves obtained by intersecting $\mathcal{S}_t$ with the planes $x_1 =$ constant and $x_2 =$ constant passing through Q. For fixed topography these parameters have been used by De Toni and Scotton [8], and for moving topography of the form

$$x_1 = x, \quad x_2 = y, \quad x_3 = b(x,t), \quad (12.120)$$

that is, when b is independent of y, by Bouchut et al. [9]. Both cited papers deal with the description of a one-layer avalanche mass on a surface with small curvature. For simplicity, here we show how equations (10.111)–(10.114) for the two-layer avalanche mass are transformed, if the basal surface with small curvature is given as (12.120) and the parameters ξ^1, ξ^2 are chosen as indicated above. Moreover, we consider the lower layer as being described by Model 2 or Model 3 as presented in Luca, Tai and Kuo [3] (there is no distinction between these models for the case which we deal with), with the bottom law (11.116) and the mean pressure $(10.110)_1$. The upper layer is a Newtonian/non-Newtonian fluid with viscosity of order $O(\epsilon^{2+\gamma})$, and the shear stress at the layer interface given by the Chézy-like friction condition, such that (11.119) holds, see the previous section. Then, denoting $\bar{\mathbf{v}}_\mathfrak{k} \equiv (V_\mathfrak{k}, W_\mathfrak{k})^T$, $\mathfrak{k} = 1, 2$, we assume $W_\mathfrak{k} = O(\epsilon^{1+\gamma})$, and take all the fields independent of ξ^2.

Thus, we define the arc length

$$s(x,t) \equiv \int_{x_0}^{x}\sqrt{1+(\partial b/\partial x)^2}dx'$$

along the curve $x_1 = x$, $x_2 =$ constant, $x_3 = b(x,t)$, where $x_0 =$ constant is such that the plane $x = x_0$ does not intersect the avalanche body at any moment t, and consider the time-dependent transformation

$$\xi^1 = s(x,t), \quad \xi^2 = y \Longleftrightarrow x = x(\xi^1,t), \quad y = \xi^2. \quad (12.121)$$

This induces the parameterization

$$x_1 = x(\xi^1,t), \quad x_2 = \xi^2, \quad x_3 = b(x(\xi^1,t),t)$$

of $\mathcal{S}_t$ as given by (12.120). Corresponding to the transformation (12.121) we have to determine the quantities $\mathbf{F}$, J_0, $\mathbf{M}_0$, $\mathbf{u}_\mathcal{S}$ which appear in the modelling equations (10.111)–(10.114); $\mathbf{s}$, c, a_1, a_2 have also to be determined and expressed in terms of $\xi^1 \equiv s$. Thus, noting that

$$c = \frac{1}{\sqrt{1+(\partial b/\partial x)^2}}, \quad \mathbf{s} = \left(c\frac{\partial b}{\partial x}, 0\right)^T,$$

$$\frac{\partial s}{\partial x} = \frac{1}{c}, \quad \frac{\partial x}{\partial s} = c,$$

we obtain

$$\frac{\partial b}{\partial s} = c\frac{\partial b}{\partial x} \Longleftrightarrow \frac{\partial b}{\partial x} = \frac{\partial b/\partial s}{\sqrt{1-(\partial b/\partial s)^2}},$$

implying

$$c = \sqrt{1-(\partial b/\partial s)^2}, \quad \mathbf{s} = \left(\frac{\partial b}{\partial s}, 0\right)^T, \quad \frac{\partial^2 b}{\partial x^2} = \frac{1}{c^4}\frac{\partial^2 b}{\partial s^2}.$$

Now we show that

$$\mathbf{F} = \begin{pmatrix} c & 0\\ 0 & 1\end{pmatrix}, \quad J_0 = 1, \quad \mathbf{M}_0 = \mathbf{I},$$

$$\mathbf{H} = \mathbf{W} = \begin{pmatrix} 2\Omega & 0\\ 0 & 0\end{pmatrix},$$

$$\Omega = \frac{1}{2}c^3\frac{\partial^2 b}{\partial x^2} = \frac{1}{2c}\frac{\partial^2 b}{\partial s^2}, \quad a_\mathfrak{k} = 2\Omega V_\mathfrak{k}^2, \quad \mathbf{u}_\mathcal{S} = O(\epsilon^{1+\gamma}) \quad (12.122)$$

hold. The first two relations above are immediate consequences of (12.121) and of definitions (3.20), (3.30) of $\mathbf{F}$ and J_0, for $\mathbf{M}_0$ and $\mathbf{H}$ one uses formulae (3.21), then $\mathbf{W} = \mathbf{M}_0\mathbf{H}$ and $a_\mathfrak{k} \equiv \mathbf{H}\bar{\mathbf{v}}_\mathfrak{k}\cdot\bar{\mathbf{v}}_\mathfrak{k}$ are accounted for to obtain $\mathbf{W}$ and $a_\mathfrak{k}$. In order to derive the last relation in (12.122), valid for small curvature $\mathcal{H} = O(\epsilon^\gamma)$, we need $\mathbf{v}_\mathcal{S}$, which, in view of (12.121), reads as $\mathbf{v}_\mathcal{S} = (\hat{\partial}x/\partial t, 0)^T$. By differentiating the identity $x = x(s(x,t),t)$ with respect to t we obtain

$$\frac{\hat{\partial}x}{\partial t} = -\frac{\partial x}{\partial s}\frac{\partial s}{\partial t} = -c\frac{\partial s}{\partial t} = c\int_{x_0}^{x}c^{-2}\frac{\partial c}{\partial t}dx', \quad (12.123)$$

and recalling that $\mathcal{U} = c\partial b/\partial t$, see (3.16), we have

$$\frac{\partial c}{\partial t} = -c^3 \frac{\partial b}{\partial x}\frac{\partial^2 b}{\partial t \partial x} = -c^3 \frac{\partial b}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{c}\mathcal{U}\right)$$
$$= c^4 \mathcal{U}\frac{\partial^2 b}{\partial^2 x} - c^2 \frac{\partial}{\partial x}\left(\mathcal{U}\frac{\partial b}{\partial x}\right).$$

Since $\Omega = O(\epsilon^\gamma)$ and $\mathcal{U} = O(\epsilon)$, we deduce

$$\frac{\partial c}{\partial t} = -c^2 \frac{\partial}{\partial x}\left(\mathcal{U}\frac{\partial b}{\partial x}\right) + O(\epsilon^{1+\gamma}), \tag{12.124}$$

which, when replaced in (12.123), gives

$$\frac{\hat{\partial} x}{\partial t} = -c\mathcal{U}\frac{\partial b}{\partial x} + c\left\{\mathcal{U}\frac{\partial b}{\partial x}\right\}_{x=x_0} + O(\epsilon^{1+\gamma})$$
$$= -c\mathcal{U}\frac{\partial b}{\partial x} + O(\epsilon^{1+\gamma}) = O(\epsilon),$$

due to the choice of x_0 and of the assumption that $\mathcal{U} = 0$ at those places on $\mathcal{S}_t$ where there is no avalanche mass. We finally derive

$$\mathbf{v}_{\mathcal{S}} = \begin{pmatrix} -c\mathcal{U}\partial b/\partial x \\ 0 \end{pmatrix} + O(\epsilon^{1+\gamma}), \tag{12.125}$$

and hence $\mathbf{u}_{\mathcal{S}} = O(\epsilon^{1+\gamma})$, see $(3.38)_2$. We mention the formula

$$\frac{\hat{\partial} c}{\partial t} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x}\frac{\hat{\partial} x}{\partial t} = -c^2 \frac{\partial b}{\partial x}\frac{\partial \mathcal{U}}{\partial x} + O(\epsilon^{1+\gamma})$$
$$= -\frac{\partial b}{\partial s}\frac{\partial \mathcal{U}}{\partial s} + O(\epsilon^{1+\gamma}), \tag{12.126}$$

which holds in view of (12.124), $\Omega = O(\epsilon^\gamma)$, $\hat{\partial} x/\partial t = O(\epsilon)$ and $\partial c/\partial x = O(\epsilon^\gamma)$.

Now, noting that for the one-dimensional case discussed here there is no distinction between the Models 2 and 3 in Luca, Tai and Kuo [3], and that the earth pressure coefficient $\tilde{k}$ defined therein takes the form[5]

$$\tilde{k} \equiv \tilde{k}^1_{\text{act}} \quad \text{if} \quad \partial V_1/\partial s > 0,$$
$$\tilde{k} \equiv \tilde{k}^1_{\text{pass}} \quad \text{if} \quad \partial V_1/\partial s < 0,$$
$$\tilde{k}^1_{\text{act/pass}} \equiv 2\sec^2\varphi - 1 \mp 2\sec\varphi\tan\varphi,$$

where φ = constant is the *internal angle of friction*, straightforward calculations in (10.111)–(10.114) using in particular condition (11.116) at the bottom topography, (11.119) with $a_2 = O(\epsilon^\gamma)$ at the layer interface, (12.126), $\partial c/\partial s = -2\Omega\partial b/\partial s$ and $\partial c/\partial s = O(\epsilon^\gamma)$[6], show that the avalanche depths h_1, h, the velocities V_1, V_2 and the height b of the basal surface satisfy the following system of equations (the terms $O(\epsilon^{2+\gamma})$ are omitted):

- in the lower layer,

$$\frac{\tilde{\partial} h_1}{\partial t} + \frac{\partial}{\partial s}\{h_1 V_1\} = -c_{01}\mathcal{U},$$
$$\frac{\tilde{\partial}}{\partial t}\{h_1 V_1\} + \frac{\partial}{\partial s}\left\{h_1\left[m_2^{(1)}(V_1)^2 + b + \tilde{k}c\left(\frac{1}{2}h_1 + \frac{1}{c_{12}}h\right)\right]\right\}$$
$$= \left(b + \frac{1}{c_{12}}ch\right)\frac{\partial h_1}{\partial s} + \frac{1}{c_{12}}c_{int}\chi^2_{int}|V_2 - V_1|(V_2 - V_1)$$
$$- \tan\delta\left\{h_1(c + a_1 m_2^{(1)}) + \frac{1}{c_{12}}h(c + a_2 m_2^{(2)})\right\}_+ \times \operatorname{sgn} V_1 - c_{01}\chi_{base}\mathcal{U}V_1,$$

- in the upper layer,

$$\frac{\tilde{\partial} h}{\partial t} + \frac{\partial}{\partial s}\{hV_2\} = 0,$$
$$\frac{\tilde{\partial}}{\partial t}\{hV_2\} + \frac{\partial}{\partial s}\left\{h[m_2^{(2)}(V_2)^2 + b + ch_1 + \frac{1}{2}ch]\right\}$$
$$= (b + ch_1)\frac{\partial h}{\partial s} - c_{int}\chi^2_{int}|V_2 - V_1|(V_2 - V_1).$$

We recall that the erosion/deposition rate $\mathcal{U} = O(\epsilon)$ has to be given, and that relation

$$\mathcal{U} = c\frac{\partial b}{\partial t} \Longleftrightarrow \frac{\hat{\partial} b}{\partial t} = c\mathcal{U} + O(\epsilon^{1+\gamma})$$

constitutes an equation which must be added to the system above. The precedent equivalence holds in view of

$$\frac{\hat{\partial} b}{\partial t} = \frac{\partial b}{\partial t} + \operatorname{grad} b \cdot \mathbf{v}_{\mathcal{S}} = \frac{\partial b}{\partial t} + \frac{1}{c}\mathbf{s} \cdot \mathbf{v}_{\mathcal{S}},$$

of (12.125) and of the expressions for c, $\mathbf{s}$.

[5] In view of (3.26), definition $(3.27)_2$ yields $\bar{\boldsymbol{v}}_{1\tau} = (v_1^\alpha - \xi W^\alpha_{\ \beta} v_1^\beta)\boldsymbol{\tau}_\alpha$, so that, when $\mathcal{H} = O(\epsilon^\gamma)$, we have $\bar{\boldsymbol{v}}_{1\tau} = \bar{v}_1^\alpha \boldsymbol{\tau}_\alpha + O(\epsilon^{1+\gamma})$. Therefore, to deduce the eigenvalues $\partial V_1/\partial s$ and 0 of the mean surface stretching tensor, formula (B4) in Luca et al. [1] can be used with $\mathbf{u} = (V_1, 0)^T$. Then, the two eigenvectors $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ of this tensor, ordered as required by Model 3, are $\boldsymbol{f}_1 = \boldsymbol{\tau}_1$, $\boldsymbol{f}_2 = \boldsymbol{\tau}_2$, implying $\bar{\mathbf{P}}_1 = \bar{p}_1\mathbf{M}_0 - \bar{p}_1\begin{pmatrix} \tilde{k}_1 & 0 \\ 0 & \tilde{k}_2 \end{pmatrix}$, see (8.22) in Luca, Tai and Kuo [3].

[6] This condition is repeatedly used, e.g. $ch\frac{\partial h_1}{\partial s} = \frac{\partial}{\partial s}(chh_1) - ch_1\frac{\partial h}{\partial s} + O(\epsilon^{2+\gamma})$.

The system of equations in the upper layer coincides with the corresponding system of equations in the paper by Fernández-Nieto et al. [5], if the Boussinesq coefficient $m_2^{(2)}$ in the above is taken equal to 1, and if the term $g \sin\theta(d_X\theta)$ $h_1^2/2$ in the cited paper is neglected, as being of order $O(\epsilon^{2+\gamma})$[7]. In the bottom layer, even with $\mathcal{U} = 0$, the system of equations above is different from the system in [5], due to different constitutive assumptions.

Option 2 Another possibility of choosing the transformation (3.17) was proposed by Tai and Kuo [10], as suggested by the approach of "unified coordinates", see Hui, Li and Li [17], Hui [18]. Thus, following Tai and Kuo [10] with only minor modifications, we require the transformation (3.17) to be so chosen, that

$$\mathbf{u}_{\mathbb{S}} = \mathbf{0} \Longleftrightarrow \mathbf{v}_{\mathbb{S}} = -\mathcal{U}\mathbf{s} \Longleftrightarrow \frac{\hat{\partial} x}{\partial t} = -c\,\mathcal{U}\frac{\partial b}{\partial x},$$
$$\frac{\hat{\partial} y}{\partial t} = -c\,\mathcal{U}\frac{\partial b}{\partial y} \tag{12.127}$$

holds at each moment $t > 0$[8]. At $t = 0$, when the erosion/deposition starts, transformation (12.127) must be known, say

$$x = x(\xi^1, \xi^2, 0) \equiv \underset{\sim}{x}(\xi^1, \xi^2),$$
$$y = y(\xi^1, \xi^2, 0) \equiv \underset{\sim}{y}(\xi^1, \xi^2). \tag{12.128}$$

We take the variables ξ^1, ξ^2 of $\underset{\sim}{x}$, $\underset{\sim}{y}$ above to be the arc lengths on $\mathbb{S}_{t=0}$ as indicated in Option 1. Clearly, since $\mathbf{u}_{\mathbb{S}} = \mathbf{0}$ and $\mathbf{v}_{\mathbb{S}} = O(\epsilon)$, equations (10.99)–(10.102) simplify considerably.

The two partial differential equations in (12.127) defining the transformation (3.17), accompanied by the corresponding initial conditions (12.128), have to be added to the system ensuing from (10.99)–(10.102) by accounting for $\mathbf{u}_{\mathbb{S}} = \mathbf{0}$ and $\mathbf{v}_{\mathbb{S}} = O(\epsilon)$, and solved at each step of integration in the numerical procedure. Even if the number of equations is now larger than the number of equations corresponding to Option 1, the system to be solved simplifies due to the restriction $\mathbf{u}_{\mathbb{S}} = \mathbf{0}$. Here we mention that, following Option 1, even for the one-dimensional case we have already encountered a difficulty: we could explicitly calculate the integral in (12.123), needed to determine $\mathbf{u}_{\mathbb{S}}$, only if the curvature of the basal surface was small. Following Option 2, this inconvenience is eliminated. On the other hand, condition (12.127) can cause problems in the numerical procedure if the moving topographic surface has large curvature and "shrinks", instead of "swelling". Moreover, in this paper and related works by Luca et al. [2]–[3] it is tacitely assumed that the vectors $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, tangent to the basal surface, are $O(1)$, so that $\boldsymbol{g}_\alpha = \boldsymbol{\tau}_\alpha + O(\epsilon) = O(1)$, $\alpha = 1, 2$. This property of $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ clearly holds in Option 1, but it can be violated in Option 2, even if on $\mathbb{S}_{t=0}$ the parameters ξ^1, ξ^2 are the arc lengths indicated in Option 1. Nevertheless, this difficulty can be avoided by defining the unit vectors $\boldsymbol{e}_\alpha \equiv \boldsymbol{g}_\alpha/\|\boldsymbol{g}_\alpha\|$, $\alpha = 1, 2$, and deriving the final modelling equations by considering the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{n}\}$ instead of $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{n}\}$.

It is interesting to note that for the moving surface (12.120) with small curvature, transformation (3.17) is practically the same in both Option 1 and Option 2, see (12.125) and (12.127).

13 Conclusions

In this paper a three-dimensional two-layer model of debris flow dynamics down arbitrary, natural topography is presented. The dynamical equations are formulated with respect to curvilinear coordinates following the basal sliding surface as it evolves in time due to the erosion–deposition processes which may take place at this interface between the moving and stagnant portions of the soil-debris regions.

Mechanically, the debris flow is supposed to be in a relatively mature state of its motion. Such stages of a fluid–granular mixture often manifest themselves as materially separated disjoint regimes of (i) a dense granular flow (with some interstitial fluid, which dynamically can be lumped with the solid phase, but together formally comprise a one-constituent fluid with complex rheological properties) and, (ii) a particle laden fluid, say water with clay suspended, of much simpler rheological properties. These two material layers have been treated here as density preserving single component materials. This assumption is physically better satisfied for the upper than the lower layer fluid, because it is in applications of e.g. fluvial hydraulics or sub-aquatic turbidity currents natural water with a small mass fraction of a fine grained particle suspension, in hydraulics referred to as wash-load. The assumption of constant density of the dense granular material is less justified, but we believe that the energetically active pulsating debris flow in the lower layer is thought to instantly adjust the entrained soil density to that in its immediate vicinity.

We have assumed that mass exchanges take place between the stagnant base and the dense debris in the lower layer, whilst the interfaces between the two layers and the free surface are material. This assumption is not so convincingly justified for the middle interface, especially not under

[7] Most likely this term is kept in Fernández-Nieto et al. [5] for numerical reasons, as e.g. in Bouchut et al. [9].

[8] Strictly following Tai and Kuo [10] we should have required $\mathbf{w} = \mathbf{0}$. However, the transformation between (x_1, x_2, x_3, t) and (ξ^1, ξ^2, ξ^3, t) is not arbitrary, so that $\mathbf{w}$ is given by $(3.38)_1$, and the requirement $\mathbf{w} = \mathbf{0}$ would imply the dependence of $\mathbf{u}_{\mathbb{S}}$ on ξ. What is however arbitrary is $\mathbf{u}_{\mathbb{S}}$. Taking $\mathbf{u}_{\mathbb{S}} = \mathbf{0}$, since $\xi = O(\epsilon)$, $\mathcal{U} = O(\epsilon)$, we deduce $\mathbf{w} = O(\epsilon^2)$, which is negligibly small.

conditions of high turbulent intensity. In a further version of a two-layer model of catastrophic debris flow, the bottom layer should therefore be a binary fluid–solid mixture overlain by a single fluid with fluid mass exchange between the two layers across the interface.

Geometrically, the dynamical equations have been derived for an avalanching debris mass that is thin when measured on length scales of its extent, but moving on topographies with arbitrary curvature. "Arbitrary" here means that any local thickness of the two layers together is smaller than the inverse of each positive (if any) principal curvature, but can otherwise vary as we please. The restriction of the "arbitrariness" is dictated by the chosen curvilinear coordinate system, which generally leads to the intersection of neighboring coordinate lines which are perpendicular to the instantaneous basal surface. The rather complicated modelling equations are considerably more simple when the curvatures are $O(\epsilon^\gamma)$, $0 < \gamma < 1$. Most applications will fit into this geometric regime.

Rheologically, from the very beginning we have not explicitly specified the stress deformation relations of the layer materials. We have simply stated order of magnitude relations for the stress components on planes perpendicular to the instantaneous evolving topography and being parallel and perpendicular to it, respectively. Based on earlier analysis, Luca et al. [1], the various stress components do not have the same orders of magnitude, but three classes of models have been defined in [1], which may be applied to each of the stress states of the two layers. The reader is free to choose any one of the models outlined in [1]. For instance, the upper layer may be treated as an ideal fluid (which then only involves the fluid pressure as a stress component), or it may be based on a Newtonian fluid with constant viscosity or a non-Newtonian fluid with a viscosity parameterization of a visco-plastic fluid (for details see [1]). The lower layer fluid is a relatively dense granular array with an interstitial fluid which affects the constitutive closure of the stress tensor. It needs a more sophisticated stress parameterization, in which visco-plastic behavior and normal stress effects may be significant, as demonstrated by the extension of the Savage–Hutter model, outlined in Luca et al. [1]. Just for reference, in the present paper for the upper layer we have chosen a Newtonian/non-Newtonian fluid with small viscosity, so small that only the shear stress at the layer interface survives, parameterized by a Chézy-like friction condition, and the lower layer is one of the three models in Luca, Tai and Kuo [3].

Application of the model equations to concrete situations are deferred to a subsequent study. Such a study requires first an explicit erosion–deposition parameterization as well as selection of the rheological model. A particularly interesting test will be the construction of a steady flow of river water with sediment and wash loads down a prescribed river bed.

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