Cognitive Computation (2023) 15:2216–2231
https://doi.org/10.1007/s12559-023-10183-y

Cognitively Inspired Group Decision-Making with Linguistic q-Rung Orthopair Fuzzy Preference Relations

Tao Li[1] · **Liyuan Zhang**[2]

Received: 7 September 2021 / Accepted: 30 July 2023 / Published online: 14 August 2023

Abstract
In actual decision-making problems, it is very difficult to appropriately depict the cognitive information of the relevant experts because cognition is usually diverse and contains uncertainties and fuzziness. The recently introduced linguistic q-rung orthopair fuzzy set (Lq-ROFS), which determines the linguistic preferred degree and linguistic nonpreferred degree within a wider space, has been shown to be effective in representing cognitive information. However, the corresponding preference relation has yet to be studied. Pairwise comparison is an effective way for decision-makers to express their preferences, especially when cognition is complex and indeterminate. Therefore, this paper employs linguistic q-rung orthopair fuzzy preference relations (Lq-ROFPRs) to express the cognitive information of experts. The additive consistency of Lq-ROFPR is introduced to rank the objects, and a consistency-based model is built to obtain the normalized linguistic q-rung orthopair fuzzy priority weight vector (Lq-ROFPWV). Then, several models are constructed to estimate missing values and improve the additive consistency level. For the group decision-making (GDM) problem, a model is first built with which to gain the weights of decision-makers. When group consensus is not achieved, a consensus-reaching model is designed as a means of increasing the consensus level. This study designs a decision support model to address GDM problem with incomplete Lq-ROFPRs and presents a step-by-step algorithm. The proposed method is utilized to assess four Chinese shopping platforms, and the comprehensive ranking result is reasonable and reliable. This is the first time to investigate GDM with Lq-ROFPRs based on consistency and consensus analysis, the newly studied Lq-ROFPRs not only extend the applications for linguistic preference relations but also endow experts with more flexibility in denoting their cognitive preferences. Compared to the latest published work in this domain, the novel approach conducts a reasonable decision-making process and has some advantages.

Keywords Group decision-making · Linguistic q-rung orthopair fuzzy preference relation · Additive consistency · Consensus

Introduction

A decision-making problem is usually full of social cognition, which is the basis of individual behavior, and thus, experts are needed to provide cognitive information [1]. The representation and processing of cognitive information is an essential step before making a decision. Cognitive information mainly refers to decision-makers' subjective cognition originating from their perceptions and carrying vague and fuzzy information [2]. However, due to the increasing complexities and uncertainties of actual socioeconomic activity, experts may come from different fields and have only limited knowledge, which leads to the provision of complicated, diverse, and indeterminate cognitive information [3].

To deal with cognitive information, Zadeh introduced the fuzzy set, which uses the degree of membership to describe fuzzy information. However, the nonpreferred cognition is missed. To overcome this shortcoming, Atanassov [4] proposed the intuitionistic fuzzy set, which uses membership degree μ and nonmembership degree ν to provide cognitive information. The intuitionistic fuzzy number (μ, ν), which satisfies $\mu + \nu \le 1$, is an important tool in expressing uncertain and complex cognitive information. Yager [5] defined the Pythagorean fuzzy set, where the Pythagorean fuzzy number (μ, ν) satisfies $\mu^2 + \nu^2 \le 1$. Yager [6] further

✉ Liyuan Zhang
zhangliyuancsu@163.com

[1] School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, Shandong, China

[2] Business School, Shandong University of Technology, Zibo 255049, Shandong, China

proposed the q-rung orthopair fuzzy set, which generalized the intuitionistic fuzzy set and the Pythagorean fuzzy set. The q-rung orthopair fuzzy number (μ, ν), satisfying $\mu^q + \nu^q \le 1 (q \ge 1)$, can provide more cognitive information. For example, a decision-maker may give a cognitive value of (0.7,0.8), i.e., the membership degree $\mu = 0.7$ and the nonmembership degree $\nu = 0.8$. We can see that (0.7, 0.8) is neither an intuitionistic fuzzy number nor a Pythagorean fuzzy number because $0.7 + 0.8 > 1$ and $0.7^2 + 0.8^2 > 1$, respectively. However, when $q = 3$, $0.7^3 + 0.8^3 \le 1$ holds, which means that (0.7, 0.8) is a q-rung orthopair fuzzy number for $q \ge 3$. Thus, compared with the intuitionistic fuzzy set and the Pythagorean fuzzy set, the q-rung orthopair fuzzy set has a stronger cognitive information description capability. Recently, these three fuzzy sets have been used to deal with cognitive information in various fields. For instance, Xin et al. [7] proposed an intuitionistic fuzzy three-way decision model based on incomplete cognitive information. Zhou and Chen [8] presented a combined technique for selecting an appropriate blockchain technology provider using Pythagorean fuzzy cognitive information. Krishankumar et al. [9] developed a framework to handle vagueness by reducing human intervention, where q-rung orthopair fuzzy cognitive information was adopted to minimize subjective randomness. For certain other applications, readers can also refer to [10–13].

In real decision-making problems, except for quantitative cognitive information conveyed by using real values in [0, 1], experts may want to use linguistic variables to offer qualitative cognitive judgments such as "good," "fair," and "bad." For a symmetric linguistic set $S = \{s_i \mid i = 0, 1, \cdots, 2\tau\}$, by combining linguistic variables and intuitionistic fuzzy numbers, an intuitionistic fuzzy linguistic number can be given as $\langle s_\alpha, (\mu, \nu)\rangle$, where $s_\alpha \in S$ [14, 15]. Some extended forms have also been considered by many researchers [16–19]. However, if decision-makers want to use linguistic variables to simultaneously express the preferred cognitive and nonpreferred cognitive information, the intuitionistic fuzzy linguistic number is inappropriate. In this case, both the linguistic intuitionistic fuzzy set (LIFS) and linguistic intuitionistic fuzzy number (LIFN) (s_μ, s_ν) were predefined, where $s_\mu \in S, s_\nu \in S$, and $\mu + \nu \le 2\tau$ [20]. LIFSs are very suitable for providing imprecise and uncertain cognitive information because they combine the advantages of both linguistic variables and intuitionistic fuzzy sets, and the membership degree and nonmembership degree only need to be given as linguistic variables rather than as exact values. Liu et al. [21] introduced LIFNs into loss functions and designed an algorithm to determine the thresholds for linguistic intuitionistic fuzzy cognitive information. The linguistic Pythagorean fuzzy set (LPFS) and linguistic Pythagorean fuzzy number (LPFN) (s_μ, s_ν) were proposed by Garg [22], where $\mu^2 + \nu^2 \le (2\tau)^2$. Ping et al. [23] provided an extended alternative queuing method based on LPFS to capture the cognitive opinions of experts. A general Lq-ROFS and linguistic q-rung orthopair fuzzy number (Lq-ROFN) (s_μ, s_ν) were developed by Liu and Liu [24], where $\mu^q + \nu^q \le (2\tau)^q$ for $q \ge 1$. Clearly, as a generalization of LIFN and LPFN, Lq-ROFN contains more qualitative cognitive information.

With the development of social and economic research, human cognition is full of fuzziness and vagueness, and it is convenient to use Lq-ROFNs to describe qualitative cognitive information in real life. Hence, an increasing number of researchers have considered decision-making problems under the linguistic q-rung orthopair fuzzy environment. At present, research on Lq-ROFS has mainly focused on aggregation operators, such as the linguistic q-rung orthopair fuzzy power Bonferroni mean operator [24], linguistic q-rung orthopair fuzzy power Muirhead mean operator [25], linguistic q-rung orthopair fuzzy interactional partitioned Heronian mean operator [26], linguistic q-rung orthopair fuzzy partitioned Maclaurin symmetric mean operator [27], and linguistic q-rung orthopair fuzzy generalized point operator [28]. Moreover, Akram et al. [29] proposed a graph-based GDM method with linguistic q-rung orthopair fuzzy cognitive information based on the Einstein operator. Liu et al. [30] developed the TOPSIS approach, and Peng et al. [31] and Verma [32] both considered similarity measures.

For some GDM problems, the preference relation is a popular technique that can be conducted by comparing each pair of objects. During the last few years, various types of linguistic preference relations have been studied to express decision-makers' cognitive information [33–37]. On the basis of LIFS and LIFN, the linguistic intuitionistic fuzzy preference relation (LIFPR) was introduced and further examined by researchers [38–41]. The LIFPR can be used to provide qualitative positive and negative cognitive judgments. For GDM with preference relations, a common approach is applying the modeling method to obtain the priority weight vector, and consistency and consensus analysis should be conducted. The additive consistency of LIFPR was investigated by Pei et al. [38], Meng et al. [39], and Zhang et al. [40], while the multiplicative consistency was analyzed by Jin et al. [41]. Liu et al. [42] further introduced the linguistic Pythagorean fuzzy preference relation (LPFPR) to describe fuzzy and uncertain cognitive information, and the multiplicative consistency was studied.

In this paper, on the basis of Lq-ROFS and Lq-ROFN, we construct a new preference relation to provide qualitative cognitive information, which is referred to as Lq-ROFPR. The additive consistency of Lq-ROFPR is first defined, and then the relationship between the normalized Lq-ROFPWV and an additively consistent Lq-ROFPR is established. Moreover, several different optimization models are proposed to obtain missing values, derive an acceptably additively consistent Lq-ROFPR, obtain decision-maker weights, and reach a level of consensus. Finally, an algorithm for GDM with Lq-ROFPRs based on consistency and consensus analysis is presented and used to deal with some illustrative examples.

The rest of the paper is organized as follows: The "Basic Concepts" section reviews the literature on LIFPR and LPFPR, which motivated us to introduce the definition of Lq-ROFPR. In the "Additive Consistency Analysis for Incomplete Lq-ROFPR" section, we investigate the additive consistency of Lq-ROFPR and provide a mathematical formula to construct an additively consistent Lq-ROFPR from a normalized Lq-ROFPWV. Then, two programming models are established to address inconsistent and incomplete Lq-ROFPRs. In the "Group Decision-Making with Incomplete Lq-ROFPRs" section, a mathematical model is developed to obtain the decision-maker weights. When the consensus of Lq-ROFPR is unacceptable, another model is presented to improve the consensus level. Moreover, the concrete steps for the GDM method with incomplete Lq-ROFPRs are also described. The "A Case Study and Comparison Analysis" section deals with some numerical examples, and the comparison analysis shows that the proposed approach is effective and feasible. Finally, the "Conclusion" section concludes the paper.

Basic Concepts

Linguistic variables are a useful tool for providing the qualitative cognitive information of decision-makers. Herrera et al. [43] proposed a symmetric linguistic set $S = \{s_i \mid i = 0, 1, \cdots, 2\tau\}$, where s_i represents a possible linguistic variable and τ is a positive integer. Herrera et al. [43] also defined one operational law for linguistic terms, which specifies that a set is ordered, i.e., if $i > j$, then $s_i > s_j$. To save more linguistic cognitive information, the discrete linguistic set S was extended to a continuous linguistic set $\overline{S} = \{s_\alpha \mid \alpha \in [0, 2\tau]\}$ [44]. The linguistic variable s_α is called an original linguistic term when $s_\alpha \in S$ and a virtual linguistic term otherwise [44]. Furthermore, for a linguistic term $s_\alpha \in \overline{S}$, we can introduce a function $I(\cdot) : \overline{S} \to [0, 2\tau]$ to derive its lower index, i.e., $I(s_\alpha) = \alpha$. It is obvious that there is an inverse function $I^{-1}(\cdot) : [0, 2\tau] \to \overline{S}$, such that $I^{-1}(\alpha) = s_\alpha$ for any $\alpha \in [0, 2\tau]$.

The definitions of LIFPR and LPFPR are given as follows:

Definition 1 [38–41] An LIFPR on a finite object set $X = \{x_1, x_2, \cdots, x_n\}$ for the continuous linguistic set $\overline{S}$ is defined as $R = (r_{ij})_{n\times n}$, where $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$, $s_{\mu_{ij}} \in \overline{S}, s_{\nu_{ij}} \in \overline{S}$, $s_{\mu_{ij}} = s_{\nu_{ji}}$, $s_{\nu_{ij}} = s_{\mu_{ji}}$, $s_{\mu_{ii}} = s_{\nu_{ii}} = s_\tau$, and $I(s_{\mu_{ij}}) + I(s_{\nu_{ij}}) \le 2\tau$, for all $i, j = 1, 2, \cdots, n$.

Definition 2 [42] An LPFPR on a finite object set $X = \{x_1, x_2, \cdots, x_n\}$ for the continuous linguistic set $\overline{S}$ is defined as $R = (r_{ij})_{n\times n}$, where $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$, $s_{\mu_{ij}} \in \overline{S}, s_{\nu_{ij}} \in \overline{S}$, $s_{\mu_{ij}} = s_{\nu_{ji}}$, $s_{\nu_{ij}} = s_{\mu_{ji}}$, $s_{\mu_{ii}} = s_{\nu_{ii}} = s_{\sqrt{2}\tau}$, and $I^2(s_{\mu_{ij}}) + I^2(s_{\nu_{ij}}) \le (2\tau)^2$, for all $i, j = 1, 2, \cdots, n$.

Inspired by LIFPR and LPFPR, we can introduce the following concept of Lq-ROFPR.

Definition 3 An Lq-ROFPR on a finite object set $X = \{x_1, x_2, \cdots, x_n\}$ for the continuous linguistic set $\overline{S}$ is defined as $R = (r_{ij})_{n\times n}$, where $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$, $s_{\mu_{ij}} \in \overline{S}, s_{\nu_{ij}} \in \overline{S}$, $s_{\mu_{ij}} = s_{\nu_{ji}}$, $s_{\nu_{ij}} = s_{\mu_{ji}}$, $s_{\mu_{ii}} = s_{\nu_{ii}} = s_{2\tau\sqrt[q]{0.5}}$, and $I^q(s_{\mu_{ij}}) + I^q(s_{\nu_{ij}}) \le (2\tau)^q$, for all $i, j = 1, 2, \cdots, n$, $q \ge 1$.

In Definition 3, $s_{\mu_{ij}}$ can be explained as the qualitative preferred degree of x_i over x_j, while $s_{\nu_{ij}}$ means the qualitative nonpreferred degree of x_i over x_j. Moreover, the qualitative hesitation degree can be given by $s_{\pi_{ij}} = I^{-1}\left(\sqrt[q]{(2\tau)^q - I^q(s_{\mu_{ij}}) - I^q(s_{\nu_{ij}})}\right)$. In this paper, for a real number $a \ge 0$, $\sqrt[q]{a}$ is defined as a when $q = 1$.

For convenience, $\widetilde{s} = (s_\mu, s_\nu)$ is called an Lq-ROFN if $s_\mu \in \overline{S}$, $s_\nu \in \overline{S}$, and $I^q(s_\mu) + I^q(s_\nu) \le (2\tau)^q$ [24]. Clearly, when $q = 1$, the Lq-ROFN $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$ becomes an LIFN, and the Lq-ROFPR $R = (r_{ij})_{n\times n}$ reduces to an LIFPR [38–41]. When $q = 2$, the Lq-ROFN $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$ becomes an LPFN, and the Lq-ROFPR $R = (r_{ij})_{n\times n}$ reduces to an LPFPR [42].

Definition 4 [26] For an Lq-ROFN $\widetilde{s} = (s_\mu, s_\nu)$, $\mathbb{S}(\widetilde{s}) = I^{-1}\left(\sqrt[q]{\left((2\tau)^q + I^q(s_\mu) - I^q(s_\nu)\right)/2}\right)$ is denoted as the score function, and the accuracy function is given by $\mathbb{H}(\widetilde{s}) = I^{-1}\left(\sqrt[q]{I^q(s_\mu) + I^q(s_\nu)}\right)$. To compare two Lq-ROFNs $\widetilde{s_1} = (s_{\mu_1}, s_{\nu_1})$ and $\widetilde{s_2} = (s_{\mu_2}, s_{\nu_2})$, the following comparison law is proposed:

If $\mathbb{S}(\widetilde{s_1}) > \mathbb{S}(\widetilde{s_2})$, then $\widetilde{s_1} > \widetilde{s_2}$;
If $\mathbb{S}(\widetilde{s_1}) = \mathbb{S}(\widetilde{s_2})$, and

if $\mathbb{H}(\widetilde{s_1}) > \mathbb{H}(\widetilde{s_2})$, then $\widetilde{s_1} > \widetilde{s_2}$;
if $\mathbb{H}(\widetilde{s_1}) = \mathbb{H}(\widetilde{s_2})$, then $\widetilde{s_1} = \widetilde{s_2}$.

Definition 5 Given two Lq-ROFPRs $R^k = (r^k_{ij})_{n\times n}$ with $r^k_{ij} = (s_{\mu^k_{ij}}, s_{\nu^k_{ij}})$ $(k = 1, 2)$, the distance between them is defined as follows:

$$d(R^1,R^2)=\frac{1}{n(n-1)(2\tau)^q}\sum_{1\le i<j\le n}\Big(\big|I^q(s_{\mu_{ij}^1})-I^q(s_{\mu_{ij}^2})\big|+\big|I^q(s_{v_{ij}^1})-I^q(s_{v_{ij}^2})\big|\Big). \tag{1}$$

Additive Consistency Analysis for Incomplete Lq-ROFPR

For any preference relation, consistency is a basic property because a lack of consistency may lead to an unreliable conclusion. In this section, the definition of additive consistency is first introduced, and a mathematical formula is developed to construct an additively consistent Lq-ROFPR from the normalized Lq-ROFPWV. Then, a model is presented to derive the priority weight vector. Finally, two programming models are proposed to address inconsistent and incomplete Lq-ROFPRs.

Definition 6 An Lq-ROFPR $R=(r_{ij})_{n\times n}$ with $r_{ij}=(s_{\mu_{ij}},s_{v_{ij}})$ is called additively consistent if

$$I^q(s_{\mu_{ij}})+I^q(s_{\mu_{jk}})+I^q(s_{v_{ik}})=I^q(s_{v_{ij}})+I^q(s_{v_{jk}})+I^q(s_{\mu_{ik}}),\quad i,j,k=1,2,\cdots,n. \tag{2}$$

Theorem 1 *Let $R=(r_{ij})_{n\times n}$ with $r_{ij}=(s_{\mu_{ij}},s_{v_{ij}})$ be an Lq-ROFPR; then, the following statements are equivalent:*

(a) $I^q(s_{\mu_{ij}})+I^q(s_{\mu_{jk}})+I^q(s_{v_{ik}})=I^q(s_{v_{ij}})+I^q(s_{v_{jk}})+I^q(s_{\mu_{ik}}), i,j,k=1,2,\cdots,n,$

(b) $I^q(s_{\mu_{ij}})+I^q(s_{\mu_{jk}})+I^q(s_{v_{ik}})=I^q(s_{v_{ij}})+I^q(s_{v_{jk}})+I^q(s_{\mu_{ik}}),\quad i,j,k=1,2,\cdots,n,\quad i<j<k.$

Proof This theorem can be easily demonstrated according to the six possible position cases of i,j,k. □

For a decision-making problem with preference relations, deriving the priority weight vector to rank objects is one of the most important tasks. For the Lq-ROFPR, it is reasonable to assume that the priority weights are Lq-ROFNs. Assume $s_\omega=(s_{\omega_1},s_{\omega_2},\cdots,s_{\omega_n})^T$ is an Lq-ROFPWV, where $s_{\omega_i}=(s_{\omega_i^\mu},s_{\omega_i^v})\,(i=1,2,\cdots,n)$ is an Lq-ROFN, then $s_{\omega_i^\mu},s_{\omega_i^v}\in\overline{S}$ and $I^q(s_{\omega_i^\mu})+I^q(s_{\omega_i^v})\le(2\tau)^q$ are satisfied.

Definition 7 Given an Lq-ROFPWV $s_\omega=(s_{\omega_1},s_{\omega_2},\cdots,s_{\omega_n})^T$ with $s_{\omega_i}=(s_{\omega_i^\mu},s_{\omega_i^v})$, $s_{\omega_i^\mu},s_{\omega_i^v}\in\overline{S}$ and $I^q(s_{\omega_i^\mu})+I^q(s_{\omega_i^v})\le(2\tau)^q$, s_ω is said to be normalized if it satisfies the following equations for all $i=1,2,\cdots,n$:

$$\sum_{j=1,j\ne i}^{n}I^q(s_{\omega_j^\mu})\le I^q(s_{\omega_i^v}),$$

$$I^q(s_{\omega_i^\mu})+(n-2)(2\tau)^q\ge\sum_{j=1,j\ne i}^{n}I^q(s_{\omega_j^v}).$$

Using the normalized Lq-ROFPWV $s_\omega=(s_{\omega_1},s_{\omega_2},\cdots,s_{\omega_n})^T$, we can construct an additively consistent Lq-ROFPR.

Theorem 2 *Let $s_\omega=(s_{\omega_1},s_{\omega_2},\cdots,s_{\omega_n})^T$ be a normalized Lq-ROFPWV with $s_{\omega_i}=(s_{\omega_i^\mu},s_{\omega_i^v})$, $s_{\omega_i^\mu},s_{\omega_i^v}\in\overline{S}$, $I^q(s_{\omega_i^\mu})+I^q(s_{\omega_i^v})\le(2\tau)^q$, $\sum_{j=1,j\ne i}^{n}I^q(s_{\omega_j^\mu})\le I^q(s_{\omega_i^v})$ and $I^q(s_{\omega_i^\mu})+(n-2)(2\tau)^q\ge\sum_{j=1,j\ne i}^{n}I^q(s_{\omega_j^v})$ for all $i=1,2,\cdots,n$; then, the preference relation $P=(p_{ij})_{n\times n}$ is an additively consistent Lq-ROFPR, where*

$$p_{ij}=(s_{p_{ij}^\mu},s_{p_{ij}^v})=\begin{cases}\left(s_{2\tau\sqrt[q]{0.5}},s_{2\tau\sqrt[q]{0.5}}\right), & i=j,\\ \left(I^{-1}\left(\sqrt[q]{0.5\left(I^q(s_{\omega_i^\mu})+I^q(s_{\omega_j^v})\right)}\right),\ I^{-1}\left(\sqrt[q]{0.5\left(I^q(s_{\omega_i^v})+I^q(s_{\omega_j^\mu})\right)}\right)\right), & i\ne j.\end{cases} \tag{3}$$

Proof It is obvious that $s_{p_{ij}^\mu}=s_{p_{ji}^v}$, $s_{p_{ij}^v}=s_{p_{ji}^\mu}$, and $s_{p_{ii}^\mu}=s_{p_{ii}^v}=s_{2\tau\sqrt[q]{0.5}}$ for all $i,j=1,2,\cdots,n$. Since $s_{\omega_i^\mu},s_{\omega_i^v}\in\overline{S}$, and $I^q(s_{\omega_i^\mu})+I^q(s_{\omega_i^v})\le(2\tau)^q$, we can easily obtain

$$\sqrt[q]{0.5\left(I^q(s_{\omega_i^\mu})+I^q(s_{\omega_j^v})\right)}\le\sqrt[q]{0.5\left((2\tau)^q+(2\tau)^q\right)}=2\tau\Longrightarrow I^{-1}\left(\sqrt[q]{0.5\left(I^q(s_{\omega_i^\mu})+I^q(s_{\omega_j^v})\right)}\right)\in\overline{S},$$

$$\sqrt[q]{0.5\left(I^q(s_{\omega_i^v})+I^q(s_{\omega_j^\mu})\right)}\le\sqrt[q]{0.5\left((2\tau)^q+(2\tau)^q\right)}=2\tau\Longrightarrow I^{-1}\left(\sqrt[q]{0.5\left(I^q(s_{\omega_i^v})+I^q(s_{\omega_j^\mu})\right)}\right)\in\overline{S},$$

$$\left(\sqrt[q]{0.5\left(I^q(s_{\omega_i^\mu})+I^q(s_{\omega_j^v})\right)}\right)^q+\left(\sqrt[q]{0.5\left(I^q(s_{\omega_i^v})+I^q(s_{\omega_j^\mu})\right)}\right)^q=0.5\left(I^q(s_{\omega_i^\mu})+I^q(s_{\omega_i^v})+I^q(s_{\omega_j^\mu})+I^q(s_{\omega_j^v})\right)\le0.5\left((2\tau)^q+(2\tau)^q\right)=(2\tau)^q,$$

which means that $p_{ij}=(s_{p_{ij}^\mu},s_{p_{ij}^v})$ is an Lq-ROFN. Thus, $P=(p_{ij})_{n\times n}$ is an Lq-ROFPR. Moreover, for $i<j<k$,

$$
\begin{aligned}
&I^q(s_{p^\mu_{ij}}) + I^q(s_{p^\mu_{jk}}) + I^q(s_{p^\nu_{ik}}) \\
&= 0.5\Big(I^q(s_{\omega^\mu_i}) + I^q(s_{\omega^\nu_j}) + I^q(s_{\omega^\mu_j}) \\
&\quad + I^q(s_{\omega^\nu_k}) + I^q(s_{\omega^\nu_i}) + I^q(s_{\omega^\mu_k})\Big), \\
&I^q(s_{p^\nu_{ij}}) + I^q(s_{p^\nu_{jk}}) + I^q(s_{p^\mu_{ik}}) \\
&= 0.5\Big(I^q(s_{\omega^\nu_i}) + I^q(s_{\omega^\mu_j}) + I^q(s_{\omega^\nu_j}) \\
&\quad + I^q(s_{\omega^\mu_k}) + I^q(s_{\omega^\mu_i}) + I^q(s_{\omega^\nu_k})\Big).
\end{aligned}
$$

Clearly, $I^q(s_{p^\mu_{ij}}) + I^q(s_{p^\mu_{jk}}) + I^q(s_{p^\nu_{ik}}) = I^q(s_{p^\nu_{ij}}) + I^q(s_{p^\nu_{jk}}) + I^q(s_{p^\mu_{ik}})$. In accordance with Definition 6 and Theorem 1, Lq-ROFPR $P = (p_{ij})_{n\times n}$ is additively consistent. □

Corollary 1 *For an Lq-ROFPR $R = (r_{ij})_{n\times n}$ with $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$, if there exists a normalized Lq-ROFPWV $s_\omega = (s_{\omega_1}, s_{\omega_2}, \cdots, s_{\omega_n})^T$ with $s_{\omega_i} = (s_{\omega^\mu_i}, s_{\omega^\nu_i})$ such that*

$$
r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}}) = \begin{cases} \left(s_{2\tau\sqrt[q]{0.5}}, s_{2\tau\sqrt[q]{0.5}}\right), \ i = j, \\ \left(I^{-1}\left(\sqrt[q]{0.5\left(I^q(s_{\omega^\mu_i}) + I^q(s_{\omega^\nu_j})\right)}\right),\right. \\ \quad \left. I^{-1}\left(\sqrt[q]{0.5\left(I^q(s_{\omega^\nu_i}) + I^q(s_{\omega^\mu_j})\right)}\right)\right), \ i \neq j, \end{cases} \tag{4}
$$

then $R = (r_{ij})_{n\times n}$ is additively consistent.

According to Corollary 1, for an additively consistent Lq-ROFPR $R = (r_{ij})_{n\times n}$, there exists a normalized Lq-ROFPWV $s_\omega = (s_{\omega_1}, s_{\omega_2}, \cdots, s_{\omega_n})^T$ such that $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$ can be expressed as Eq. (4). Then, for $i \neq j$, we have

$$
\begin{aligned}
I^q(s_{\mu_{ij}}) &= 0.5\Big(I^q(s_{\omega^\mu_i}) + I^q(s_{\omega^\nu_j})\Big), \\
I^q(s_{\nu_{ij}}) &= 0.5\Big(I^q(s_{\omega^\nu_i}) + I^q(s_{\omega^\mu_j})\Big).
\end{aligned}
$$

However, in a real decision-making problem, it is extremely difficult for the expert to give an additively consistent Lq-ROFPR $R = (r_{ij})_{n\times n}$, which means that the above equations may not always hold. Therefore, the following deviation variables can be introduced:

$$
\begin{aligned}
\varepsilon_{ij} &= I^q(s_{\mu_{ij}}) - 0.5\Big(I^q(s_{\omega^\mu_i}) + I^q(s_{\omega^\nu_j})\Big), \ \ i \neq j, \\
\eta_{ij} &= I^q(s_{\nu_{ij}}) - 0.5\Big(I^q(s_{\omega^\nu_i}) + I^q(s_{\omega^\mu_j})\Big), \ \ i \neq j.
\end{aligned}
$$

Clearly, the additive consistency of Lq-ROFPR $R = (r_{ij})_{n\times n}$ is better when the absolute deviations $|\varepsilon_{ij}|$ and $|\eta_{ij}|$ are smaller. Let $\varepsilon^+_{ij} = \frac{|\varepsilon_{ij}|+\varepsilon_{ij}}{2} \geq 0$, $\varepsilon^-_{ij} = \frac{|\varepsilon_{ij}|-\varepsilon_{ij}}{2} \geq 0$, $\eta^+_{ij} = \frac{|\eta_{ij}|+\eta_{ij}}{2} \geq 0$, $\eta^-_{ij} = \frac{|\eta_{ij}|-\eta_{ij}}{2} \geq 0$, then $\varepsilon_{ij} = \varepsilon^+_{ij} - \varepsilon^-_{ij}$, $|\varepsilon_{ij}| = \varepsilon^+_{ij} + \varepsilon^-_{ij}$, $\eta_{ij} = \eta^+_{ij} - \eta^-_{ij}$, $|\eta_{ij}| = \eta^+_{ij} + \eta^-_{ij}$. Since $\varepsilon_{ij} = \eta_{ji}$ and $\eta_{ij} = \varepsilon_{ji}$, we can establish Model 1 to obtain the normalized Lq-ROFPWV $s_\omega = (s_{\omega_1}, s_{\omega_2}, \cdots, s_{\omega_n})^T$.

Model 1

$$
\min J = \sum_{1\leq i<j\leq n} \left(\varepsilon^+_{ij} + \varepsilon^-_{ij} + \eta^+_{ij} + \eta^-_{ij}\right)
$$

$$
s.t.\begin{cases} I^q(s_{\mu_{ij}}) - 0.5\Big(I^q(s_{\omega^\mu_i}) + I^q(s_{\omega^\nu_j})\Big) - \varepsilon^+_{ij} + \varepsilon^-_{ij} = 0, \ \ 1 \leq i < j \leq n, \\ I^q(s_{\nu_{ij}}) - 0.5\Big(I^q(s_{\omega^\nu_i}) + I^q(s_{\omega^\mu_j})\Big) - \eta^+_{ij} + \eta^-_{ij} = 0, \ \ 1 \leq i < j \leq n, \\ I(s_{\omega^\mu_i}), I(s_{\omega^\nu_i}) \in [0, 2\tau], \ I^q(s_{\omega^\mu_i}) + I^q(s_{\omega^\nu_i}) \leq (2\tau)^q, \ \ i = 1, 2, \cdots, n, \\ \sum_{j=1, j\neq i}^n I^q(s_{\omega^\mu_j}) \leq I^q(s_{\omega^\nu_i}), \ I^q(s_{\omega^\mu_i}) + (n-2)(2\tau)^q \geq \sum_{j=1, j\neq i}^n I^q(s_{\omega^\nu_j}), \ \ i = 1, 2, \cdots, n, \\ \varepsilon^+_{ij} \geq 0, \ \varepsilon^-_{ij} \geq 0, \ \eta^+_{ij} \geq 0, \ \eta^-_{ij} \geq 0, \ \ 1 \leq i < j \leq n. \end{cases}
$$

In a practical decision-making process, it is unrealistic for an expert to give an absolutely additively consistent Lq-ROFPR. In this case, there exist i, j and k such that Eq. (2) does not hold. Then, the following definition of the additive consistency index (ACI) for an Lq-ROFPR can be introduced.

Definition 8 Let $R = (r_{ij})_{n\times n}$ with $r_{ij} = (s_{\mu_{ij}}, s_{\nu_{ij}})$ be an Lq-ROFPR, and the additive consistency index of R is defined as follows:

$$
\begin{aligned}
ACI(R) = &\frac{2}{n(n-1)(n-2)(2\tau)^q} \sum_{1\leq i<j<k\leq n} \\
&\times \left|I^q(s_{\mu_{ij}}) + I^q(s_{\mu_{jk}}) + I^q(s_{\nu_{ik}}) \right. \\
&\left. - I^q(s_{\nu_{ij}}) - I^q(s_{\nu_{jk}}) - I^q(s_{\mu_{ik}})\right|,
\end{aligned} \tag{5}
$$

where $q \geq 1$ and $ACI(R) \in [0, 1]$.

Clearly, if $ACI(R) = 0$, the Lq-ROFPR $R = (r_{ij})_{n\times n}$ is absolutely additively consistent. Since the absolute consistency is too strict, we further introduce the concept of acceptably additively consistent Lq-ROFPR.

Definition 9 An Lq-ROFPR $R=(r_{ij})_{n\times n}$ is said to be acceptably additively consistent if $ACI(R)\le\overline{ACI}$, where $\overline{ACI}\in[0,1]$ is the consistency threshold.

Due to the complexities and uncertainties of most practical decision-making problems, experts usually provide unacceptably additively consistent Lq-ROFPRs. Since an inconsistent Lq-ROFPR may lead to unreasonable results, we should first repair the unacceptably additively consistent Lq-ROFPR to be of acceptable consistency, which can be realized via Model 2.

The second restraint condition in Model 2 means that $r'_{ij}=(s_{\mu'_{ij}},s_{\nu'_{ij}})$ is an Lq-ROFN, and the first restraint condition guarantees that $R'=(r'_{ij})_{n\times n}$ with $r'_{ij}=(s_{\mu'_{ij}},s_{\nu'_{ij}})$ is acceptably additively consistent. Moreover, according to Eq. (1), the objective function means that Lq-ROFPR $R'=(r'_{ij})_{n\times n}$ has the smallest distance from $R=(r_{ij})_{n\times n}$. Thus, the adjusted Lq-ROFPR $R'=(r'_{ij})_{n\times n}$ can retain more original cognitive information. In particular, when the consistency threshold $\overline{ACI}=0$, the Lq-ROFPR $R'=(r'_{ij})_{n\times n}$ is absolutely additively consistent.

By introducing some positive slack variables a^+_{ij}, a^-_{ij}, b^+_{ij}, b^-_{ij}, c^+_{ijk}, and c^-_{ijk}, Model 2 can be transformed into Model 3.

Model 2

$$\min f=\sum_{1\le i<j\le n}\left(\left|I^q(s_{\mu_{ij}})-I^q(s_{\mu'_{ij}})\right|+\left|I^q(s_{\nu_{ij}})-I^q(s_{\nu'_{ij}})\right|\right)$$

$$s.t.\begin{cases}\dfrac{2}{n(n-1)(n-2)(2\tau)^q}\displaystyle\sum_{1\le i<j<k\le n}\left|I^q(s_{\mu'_{ij}})+I^q(s_{\mu'_{jk}})+I^q(s_{\nu'_{ik}})-I^q(s_{\nu'_{ij}})-I^q(s_{\nu'_{jk}})-I^q(s_{\mu'_{ik}})\right|\le\overline{ACI},\\ I(s_{\mu'_{ij}}),I(s_{\nu'_{ij}})\in[0,2\tau],\ I^q(s_{\mu'_{ij}})+I^q(s_{\nu'_{ij}})\le(2\tau)^q,\ \ 1\le i<j\le n.\end{cases}$$

Model 3

$$\min f=\sum_{1\le i<j\le n}\left(a^+_{ij}+a^-_{ij}+b^+_{ij}+b^-_{ij}\right)$$

$$s.t.\begin{cases}I^q(s_{\mu_{ij}})-I^q(s_{\mu'_{ij}})-a^+_{ij}+a^-_{ij}=0,\ \ 1\le i<j\le n,\\ I^q(s_{\nu_{ij}})-I^q(s_{\nu'_{ij}})-b^+_{ij}+b^-_{ij}=0,\ \ 1\le i<j\le n,\\ \dfrac{2}{n(n-1)(n-2)(2\tau)^q}\displaystyle\sum_{1\le i<j<k\le n}\left(c^+_{ijk}+c^-_{ijk}\right)\le\overline{ACI},\\ I^q(s_{\mu'_{ij}})+I^q(s_{\mu'_{jk}})+I^q(s_{\nu'_{ik}})-I^q(s_{\nu'_{ij}})-I^q(s_{\nu'_{jk}})-I^q(s_{\mu'_{ik}})-c^+_{ijk}+c^-_{ijk}=0,\ \ 1\le i<j<k\le n,\\ I(s_{\mu'_{ij}}),I(s_{\nu'_{ij}})\in[0,2\tau],\ I^q(s_{\mu'_{ij}})+I^q(s_{\nu'_{ij}})\le(2\tau)^q,\ \ 1\le i<j\le n,\\ a^+_{ij}\ge0,\ a^-_{ij}\ge0,\ b^+_{ij}\ge0,\ b^-_{ij}\ge0,\ \ 1\le i<j\le n,\\ c^+_{ijk}\ge0,\ c^-_{ijk}\ge0,\ \ 1\le i<j<k\le n.\end{cases}$$

Using Lingo software to solve Model 3, an acceptably additively consistent Lq-ROFPR $R' = (r'_{ij})_{n\times n}$ with $r'_{ij} = (s_{\mu'_{ij}}, s_{v'_{ij}})$ can be constructed via the optimal solutions.

Consider an Lq-ROFPR $R = (r_{ij})_{n\times n}$ with $r_{ij} = (s_{\mu_{ij}}, s_{v_{ij}})$; if some judgments of R are unknown, then R is called an incomplete Lq-ROFPR. Let $Z_\mu = \{(i,j) \mid s_{\mu_{ij}} \textit{ is unknown}, 1 \le i < j \le n\}$, $Z_v = \{(i,j) \mid s_{v_{ij}} \textit{ is unknown}, 1 \le i < j \le n\}$, since the estimated values should make the additive consistency index $ACI(R)$ smaller, we can build Model 4 to estimate the unknown values.

Model 4 $\quad \min\ \delta$

$$s.t.\begin{cases} \dfrac{2}{n(n-1)(n-2)(2\tau)^q}\displaystyle\sum_{1\le i<j<k\le n}\left|I^q(s_{\mu_{ij}})+I^q(s_{\mu_{jk}})+I^q(s_{v_{ik}})-I^q(s_{v_{ij}})-I^q(s_{v_{jk}})-I^q(s_{\mu_{ik}})\right|\le\delta,\\ I(s_{\mu_{ij}})\in[0,2\tau],\ I^q(s_{\mu_{ij}})+I^q(s_{v_{ij}})\le(2\tau)^q,\ \ (i,j)\in Z_\mu,\ (i,j)\notin Z_v,\\ I(s_{v_{ij}})\in[0,2\tau],\ I^q(s_{\mu_{ij}})+I^q(s_{v_{ij}})\le(2\tau)^q,\ \ (i,j)\notin Z_\mu,\ (i,j)\in Z_v,\\ I(s_{\mu_{ij}}),I(s_{v_{ij}})\in[0,2\tau],\ I^q(s_{\mu_{ij}})+I^q(s_{v_{ij}})\le(2\tau)^q,\ \ (i,j)\in Z_\mu,\ (i,j)\in Z_v. \end{cases}$$

By deleting the absolute value symbols, Model 4 can be transformed into Model 5.

Model 5 $\quad \min\ \delta$

$$s.t.\begin{cases} \dfrac{2}{n(n-1)(n-2)(2\tau)^q}\displaystyle\sum_{1\le i<j<k\le n}\left(\theta^+_{ijk}+\theta^-_{ijk}\right)\le\delta,\\ I^q(s_{\mu_{ij}})+I^q(s_{\mu_{jk}})+I^q(s_{v_{ik}})-I^q(s_{v_{ij}})-I^q(s_{v_{jk}})-I^q(s_{\mu_{ik}})-\theta^+_{ijk}+\theta^-_{ijk}=0,\ \ 1\le i<j<k\le n,\\ I(s_{\mu_{ij}})\in[0,2\tau],\ I^q(s_{\mu_{ij}})+I^q(s_{v_{ij}})\le(2\tau)^q,\ \ (i,j)\in Z_\mu,\ (i,j)\notin Z_v,\\ I(s_{v_{ij}})\in[0,2\tau],\ I^q(s_{\mu_{ij}})+I^q(s_{v_{ij}})\le(2\tau)^q,\ \ (i,j)\notin Z_\mu,\ (i,j)\in Z_v,\\ I(s_{\mu_{ij}}),I(s_{v_{ij}})\in[0,2\tau],\ I^q(s_{\mu_{ij}})+I^q(s_{v_{ij}})\le(2\tau)^q,\ \ (i,j)\in Z_\mu,\ (i,j)\in Z_v,\\ \theta^+_{ijk}\ge0,\ \theta^-_{ijk}\ge0,\ \ 1\le i<j<k\le n. \end{cases}$$

Solving Model 5, we can obtain the optimal objective value δ^* and the optimal solutions $I(s_{\mu^*_{ij}})\big((i,j)\in Z_\mu\big)$ and $I(s_{v^*_{ij}})\big((i,j)\in Z_v\big)$. Using these optimal solutions, a complete Lq-ROFPR $R = (r_{ij})_{n\times n}$ can be built. Four cases may exist for the upper triangular elements: (a) $r_{ij} = (s_{\mu_{ij}}, s_{v_{ij}})$, if $(i,j)\notin Z_\mu,\ (i,j)\notin Z_v$; (b) $r_{ij} = \left(I^{-1}\left(I(s_{\mu^*_{ij}})\right), s_{v_{ij}}\right)$, if $(i,j)\in Z_\mu$, $(i,j)\notin Z_v$; (c) $r_{ij} = \left(s_{\mu_{ij}}, I^{-1}\left(I(s_{v^*_{ij}})\right)\right)$, if $(i,j)\notin Z_\mu,\ (i,j)\in Z_v$; and (d) $r_{ij} = \left(I^{-1}\left(I(s_{\mu^*_{ij}})\right), I^{-1}\left(I(s_{v^*_{ij}})\right)\right)$, if $(i,j)\in Z_\mu,\ (i,j)\in Z_v$. Moreover, if the optimal objective value $\delta^* \le \overline{ACI}$, the complete Lq-ROFPR $R = (r_{ij})_{n\times n}$ derived from Model 5 is acceptably additively consistent.

Group Decision-Making with Incomplete Lq-ROFPRs

Consider a GDM problem, the set of alternatives is $X = \{x_1, x_2, \cdots, x_n\}$, the set of experts is $D = \{d_1, d_2, \cdots, d_m\}$, and the decision-maker weight vector is $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}^T$ such that $\lambda_l > 0$ and $\sum_{l=1}^m \lambda_l = 1$. The expert d_l ($l = 1, 2, \cdots, m$) can provide a complete or incomplete Lq-ROFPR $R_l = (r_{l,ij})_{n\times n}$ with $r_{l,ij} = (s_{l\mu_{ij}}, s_{lv_{ij}})$ by making pairwise comparisons for all alternatives. If the Lq-ROFPR R_l is incomplete, then we should first use Model 5 to obtain the complete Lq-ROFPR, which is still denoted as R_l. If the individual Lq-ROFPR R_l is of unacceptable consistency, then Model 3 should be used to obtain an acceptably additively consistent Lq-ROFPR $R'_l = (r'_{l,ij})_{n\times n}$ with $r'_{l,ij} = (s_{l\mu'_{ij}}, s_{lv'_{ij}})$.

In real decision-making problems, different experts may have different ranking orders. Thus, it is very important to find an acceptable solution that is agreed upon by all decision-makers, so consensus analysis should be considered. In the following, the concept of collective Lq-ROFPR is first introduced.

Definition 10 Let $R'_l = (r'_{l,ij})_{n\times n}$ with $r'_{l,ij} = (s_{l\mu'_{ij}}, s_{lv'_{ij}})$ ($l = 1, 2, \cdots, m$) be individual Lq-ROFPRs and $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}^T$ be the weight vector of decision-makers such that $\lambda_l > 0$ and $\sum_{l=1}^m \lambda_l = 1$; then, the collective Lq-ROFPR $R_c = (r_{c,ij})_{n\times n}$ is defined as follows:

$$r_{c,ij} = (s_{c\mu_{ij}}, s_{cv_{ij}}) = \left(I^{-1}\left(\sqrt[q]{\textstyle\sum_{l=1}^m \lambda_l I^q(s_{l\mu'_{ij}})}\right), I^{-1}\left(\sqrt[q]{\textstyle\sum_{l=1}^m \lambda_l I^q(s_{lv'_{ij}})}\right)\right). \tag{6}$$

Based on Eq. (6), we can easily prove that $R_c = (r_{c,ij})_{n\times n}$ is also an Lq-ROFPR. Moreover, the additive consistency of $R_c = (r_{c,ij})_{n\times n}$ can be considered by the following theorem.

Theorem 3 *The collective Lq-ROFPR $R_c = (r_{c,ij})_{n\times n}$ is acceptably additively consistent if all individual Lq-ROFPRs $R'_l = (r'_{l,ij})_{n\times n}$ $(l = 1, 2, \cdots, m)$ are acceptably additively consistent.*

Proof Since Lq-ROFPR $R'_l = (r'_{l,ij})_{n\times n}$ with $r'_{l,ij} = (s_{l\mu'_{ij}}, s_{l\nu'_{ij}})$ is of acceptable consistency, according to Definitions 8 and 9, we have

$$ACI(R'_l) = \frac{2}{n(n-1)(n-2)(2\tau)^q}\sum_{1\le i<j<k\le n} \times \left|I^q(s_{l\mu'_{ij}}) + I^q(s_{l\mu'_{jk}}) + I^q(s_{l\nu'_{ik}}) - I^q(s_{l\nu'_{ij}}) - I^q(s_{l\nu'_{jk}}) - I^q(s_{l\mu'_{ik}})\right| \le \overline{ACI}.$$

Using Eq. (6), we can obtain

$$\begin{aligned}
ACI(R_c) &= \frac{2}{n(n-1)(n-2)(2\tau)^q}\sum_{1\le i<j<k\le n} \times \left|I^q(s_{c\mu_{ij}}) + I^q(s_{c\mu_{jk}}) + I^q(s_{c\nu_{ik}}) - I^q(s_{c\nu_{ij}}) - I^q(s_{c\nu_{jk}}) - I^q(s_{c\mu_{ik}})\right| \\
&= \frac{2}{n(n-1)(n-2)(2\tau)^q}\sum_{1\le i<j<k\le n} \times \left|\sum_{l=1}^m \lambda_l I^q(s_{l\mu'_{ij}}) + \sum_{l=1}^m \lambda_l I^q(s_{l\mu'_{jk}}) + \sum_{l=1}^m \lambda_l I^q(s_{l\nu'_{ik}}) - \sum_{l=1}^m \lambda_l I^q(s_{l\nu'_{ij}}) - \sum_{l=1}^m \lambda_l I^q(s_{l\nu'_{jk}}) - \sum_{l=1}^m \lambda_l I^q(s_{l\mu'_{ik}})\right| \\
&= \frac{2}{n(n-1)(n-2)(2\tau)^q}\sum_{1\le i<j<k\le n} \times \left|\sum_{l=1}^m \lambda_l \left(I^q(s_{l\mu'_{ij}}) + I^q(s_{l\mu'_{jk}}) + I^q(s_{l\nu'_{ik}}) - I^q(s_{l\nu'_{ij}}) - I^q(s_{l\nu'_{jk}}) - I^q(s_{l\mu'_{ik}})\right)\right| \\
&\le \sum_{l=1}^m \lambda_l \frac{2}{n(n-1)(n-2)(2\tau)^q}\sum_{1\le i<j<k\le n} \times \left|I^q(s_{l\mu'_{ij}}) + I^q(s_{l\mu'_{jk}}) + I^q(s_{l\nu'_{ik}}) - I^q(s_{l\nu'_{ij}}) - I^q(s_{l\nu'_{jk}}) - I^q(s_{l\mu'_{ik}})\right| \\
&= \sum_{l=1}^m \lambda_l ACI(R'_l) \le \sum_{l=1}^m \lambda_l \overline{ACI} = \overline{ACI}.
\end{aligned}$$

Thus, the collective Lq-ROFPR $R_c = (r_{c,ij})_{n\times n}$ is acceptably additively consistent, which completes the proof. □

In the above theorem, if the consistency threshold $\overline{ACI} = 0$, indicating that all the individual Lq-ROFPRs $R'_l = (r'_{l,ij})_{n\times n}$ are absolutely additively consistent, then the collective Lq-ROFPR $R_c = (r_{c,ij})_{n\times n}$ is also absolutely additively consistent.

Definition 11 Let $R'_l = (r'_{l,ij})_{n\times n}$ with $r'_{l,ij} = (s_{l\mu'_{ij}}, s_{l\nu'_{ij}})$ $(l = 1, 2, \cdots, m)$ be individual Lq-ROFPRs and $R_c = (r_{c,ij})_{n\times n}$ with $r_{c,ij} = (s_{c\mu_{ij}}, s_{c\nu_{ij}})$ be the collective Lq-ROFPR; then, the group consensus index (GCI) of R'_l is defined as follows:

$$GCI(R'_l) = \frac{1}{n(n-1)(2\tau)^q}\sum_{1\le i<j\le n}\left(\left|I^q(s_{l\mu'_{ij}}) - I^q(s_{c\mu_{ij}})\right| + \left|I^q(s_{l\nu'_{ij}}) - I^q(s_{c\nu_{ij}})\right|\right), \tag{7}$$

where $q \ge 1$ and $GCI(R'_l) \in [0, 1]$.

If all individual Lq-ROFPRs are the same, i.e., $R'_l = R'_p$ for any $l, p = 1, 2, \cdots, m$, then the group reaches full consensus. Clearly, in this case, $GCI(R'_l) = 0$. However, achieving a full consensus is a utopian goal, thus we should further propose the concept of acceptable consensus.

Definition 12 An individual Lq-ROFPR R'_l $(l = 1, 2, \cdots, m)$ is said to be of acceptable consensus if $GCI(R'_l) \le \overline{GCI}$, where $\overline{GCI} \in [0, 1]$ is the consensus threshold.

When all individual Lq-ROFPRs R'_l are of acceptable consensus, i.e., $GCI(R'_l) \le \overline{GCI}$ for all $l = 1, 2, \cdots, m$, then the group reaches acceptable consensus.

In the process of generating collective Lq-ROFPR $R_c = (r_{c,ij})_{n\times n}$, the weights of experts are used. Sometimes, the decision-maker weight vector is unknown. Therefore, we should first obtain the weights. According to Definition 11, when we calculate $GCI(R'_l)$, for each pair of (i, j), we have

$$\begin{aligned}
&\left|I^q(s_{l\mu'_{ij}}) - I^q(s_{c\mu_{ij}})\right| + \left|I^q(s_{l\nu'_{ij}}) - I^q(s_{c\nu_{ij}})\right| \\
&= \left|I^q(s_{l\mu'_{ij}}) - \sum_{t=1}^m \lambda_t I^q(s_{t\mu'_{ij}})\right| + \left|I^q(s_{l\nu'_{ij}}) - \sum_{t=1}^m \lambda_t I^q(s_{t\nu'_{ij}})\right|.
\end{aligned} \tag{8}$$

Clearly, a smaller value of Eq. (8) denotes a better consensus level. Hence, Model 6 can be built to determine the weights of decision-makers.

Model 6
$$\min\ g = \sum_{l=1}^m \sum_{1\le i<j\le n}\left(\left|I^q(s_{l\mu'_{ij}}) - \sum_{t=1}^m \lambda_t I^q(s_{t\mu'_{ij}})\right| + \left|I^q(s_{l\nu'_{ij}}) - \sum_{t=1}^m \lambda_t I^q(s_{t\nu'_{ij}})\right|\right)$$
$$s.t.\begin{cases} \lambda_l \in (0, 1),\quad l = 1, 2, \cdots, m, \\ \sum_{l=1}^m \lambda_l = 1. \end{cases}$$

By introducing some positive slack variables ϑ^+_{lij}, ϑ^-_{lij}, χ^+_{lij}, and χ^-_{lij}, Model 6 can be transformed into Model 7.

Model 7 $\min\ g=\sum_{l=1}^{m}\sum_{1\le i<j\le n}\left(\vartheta_{lij}^{+}+\vartheta_{lij}^{-}+\chi_{lij}^{+}+\chi_{lij}^{-}\right)$

$$s.t.\begin{cases} I^q(s_{l\mu'_{ij}})-\sum_{t=1}^{m}\lambda_t I^q(s_{t\mu'_{ij}})-\vartheta_{lij}^{+}+\vartheta_{lij}^{-}=0,\ \ 1\le i<j\le n,\ l=1,2,\cdots,m,\\ I^q(s_{l\nu'_{ij}})-\sum_{t=1}^{m}\lambda_t I^q(s_{t\nu'_{ij}})-\chi_{lij}^{+}+\chi_{lij}^{-}=0,\ \ 1\le i<j\le n,\ l=1,2,\cdots,m,\\ \lambda_l\in(0,1),\ \ l=1,2,\cdots,m,\\ \sum_{l=1}^{m}\lambda_l=1,\\ \vartheta_{lij}^{+}\ge0,\ \vartheta_{lij}^{-}\ge0,\ \chi_{lij}^{+}\ge0,\ \chi_{lij}^{-}\ge0,\ \ 1\le i<j\le n,\ l=1,2,\cdots,m. \end{cases}$$

Using Lingo software to solve Model 7, the decision-maker weight vector $\lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_m\}^T$ can be obtained.

When Lq-ROFPR $R'_l=(r'_{l,ij})_{n\times n}$ with $r'_{l,ij}=(s_{l\mu'_{ij}},s_{l\nu'_{ij}})$ is of unacceptable consensus, i.e., $GCI(R'_l)>\overline{GCI}$, then we need to modify it to increase the consensus level. Let $R''_l=(r''_{l,ij})_{n\times n}$ with $r''_{l,ij}=(s_{l\mu''_{ij}},s_{l\nu''_{ij}})$ be an adjusted Lq-ROFPR, where

$$\begin{cases} s_{l\mu''_{ij}}=I^{-1}\left(\sqrt[q]{(1-\beta)I^q(s_{l\mu'_{ij}})+\beta I^q(s_{c\mu_{ij}})}\right),\\ s_{l\nu''_{ij}}=I^{-1}\left(\sqrt[q]{(1-\beta)I^q(s_{l\nu'_{ij}})+\beta I^q(s_{c\nu_{ij}})}\right), \end{cases} \tag{9}$$

and $\beta\in[0,1]$ is the adjustment parameter. From Theorem 3, we know that the modified Lq-ROFPR R''_l is still acceptably additively consistent. Moreover, the adjusted Lq-ROFPR R''_l should have an acceptable consensus level, and the adjustment parameter β should be as small as possible to maintain more original information. Therefore, we can establish Model 8.

Model 8 $\min\ \beta$

$$s.t.\begin{cases} \frac{1}{n(n-1)(2\tau)^q}\sum_{1\le i<j\le n}\left(\left|(1-\lambda_l)\left((1-\beta)I^q(s_{l\mu'_{ij}})+\beta I^q(s_{c\mu_{ij}})\right)-\sum_{k\ne l}\lambda_k I^q(s_{k\mu'_{ij}})\right|\right.\\ \left.+\left|(1-\lambda_l)\left((1-\beta)I^q(s_{l\nu'_{ij}})+\beta I^q(s_{c\nu_{ij}})\right)-\sum_{k\ne l}\lambda_k I^q(s_{k\nu'_{ij}})\right|\right)\le\overline{GCI},\\ \beta\in[0,1]. \end{cases}$$

By deleting the absolute value symbols, Model 8 can be transformed into Model 9.

Model 9 $\min\ \beta$

$$s.t.\begin{cases} \frac{1}{n(n-1)(2\tau)^q}\sum_{1\le i<j\le n}\left(\zeta_{ij}^{+}+\zeta_{ij}^{-}+\xi_{ij}^{+}+\xi_{ij}^{-}\right)\le\overline{GCI},\\ (1-\lambda_l)\left((1-\beta)I^q(s_{l\mu'_{ij}})+\beta I^q(s_{c\mu_{ij}})\right)-\sum_{k\ne l}\lambda_k I^q(s_{k\mu'_{ij}})-\zeta_{ij}^{+}+\zeta_{ij}^{-}=0,\ \ 1\le i<j\le n,\\ (1-\lambda_l)\left((1-\beta)I^q(s_{l\nu'_{ij}})+\beta I^q(s_{c\nu_{ij}})\right)-\sum_{k\ne l}\lambda_k I^q(s_{k\nu'_{ij}})-\xi_{ij}^{+}+\xi_{ij}^{-}=0,\ \ 1\le i<j\le n,\\ \beta\in[0,1],\\ \zeta_{ij}^{+}\ge0,\ \zeta_{ij}^{-}\ge0,\ \xi_{ij}^{+}\ge0,\ \xi_{ij}^{-}\ge0,\ \ 1\le i<j\le n. \end{cases}$$

Solving Model 9, we can obtain the optimal objective value β^*. Then, taking β^* into Eq. (9), a modified Lq-ROFPR $R''_l=(r''_{l,ij})_{n\times n}$ with acceptable consensus is derived.

In the following, the concrete steps of the GDM method with incomplete Lq-ROFPRs are provided.

Step 1. According to a predefined linguistic set, decision-maker $d_l\ (l=1,2,\cdots,m)$ constructs an individual Lq-ROFPR $R_l=(r_{l,ij})_{n\times n}$ by making pairwise comparisons for all alternatives $X=\{x_1,x_2,\cdots,x_n\}$. If all Lq-ROFPRs R_l are complete, go to Step 2. If R_l is incomplete, then plug R_l into Model 5 to generate a complete Lq-ROFPR, which is still denoted as R_l. This process is repeated until all Lq-ROFPRs $R_l=(r_{l,ij})_{n\times n}\ (l=1,2,\cdots,m)$ are complete.

Step 2. Compute the additive consistency indices $ACI(R_l)$ $(l=1,2,\cdots,m)$ based on Eq. (5). $\overline{ACI}$ is set as the consistency threshold, and Definition 9 is used to check whether R_l is acceptably additively consistent or not. If all individual Lq-ROFPRs R_l

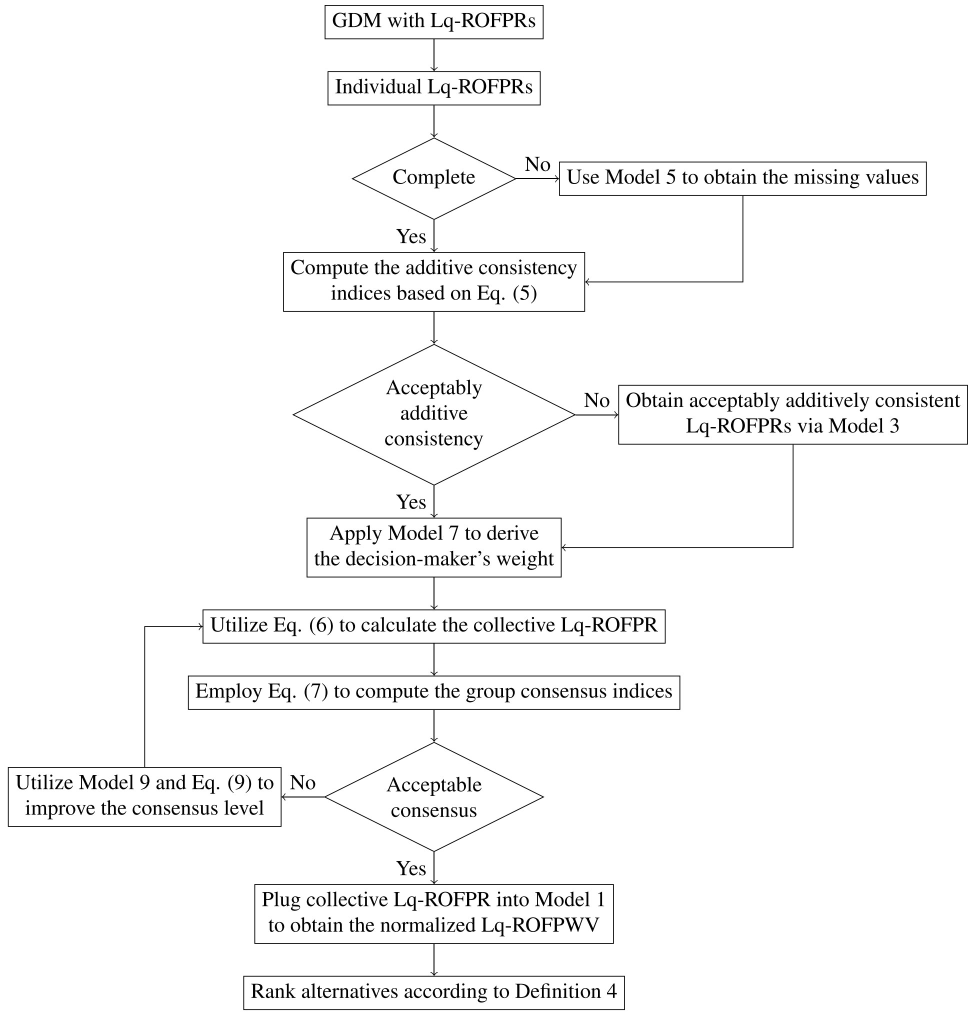

Fig. 1 The frame diagram of the proposed method

$(l = 1, 2, \cdots, m)$ are of acceptable consistency, let $R'_l = R_l$, and go to Step 4. Otherwise, go to Step 3.

Step 3. Plug an Lq-ROFPR R_l with unacceptable consistency into Model 3, and solve this model to obtain an acceptably additively consistent Lq-ROFPR $R'_l = (r'_{l,ij})_{n\times n}$. This process is repeated until all individual Lq-ROFPRs are acceptably consistent. For the Lq-ROFPR R_l with acceptable additive consistency, we set $R'_l = R_l$.

Step 4. Take all acceptably additively consistent Lq-ROFPRs R'_l $(l = 1, 2, \cdots, m)$ into Model 7 to derive the decision-maker weight vector $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}^T$.

Step 5. According to Eq. (6), we calculate the collective Lq-ROFPR $R_c = (r_{c,ij})_{n\times n}$. Moreover, Eq. (7) is applied to compute the group consensus index $GCI(R'_l)$ $(l = 1, 2, \cdots, m)$.

Step 6. Given a consensus threshold $\overline{GCI}$, we utilize Definition 12 to check whether the individual Lq-ROFPR R'_l is of acceptable consensus or not. If all consensus indices $GCI(R'_l) \le \overline{GCI}$ $(l = 1, 2, \cdots, m)$, let $R''_l = R'_l$, and go to Step 8. Otherwise, let $GCI(R'_k) = \max_{1\le l\le m}\{GCI(R'_l)\}$, and go to Step 7.

Step 7. Plug R'_k into Model 9 and solve this model to obtain an optimal adjustment parameter β^*, and Eq. (9) is then applied to generate a modified

Table 1 Individual Lq-ROFPR $R_1 = (r_{1,ij})_{4\times 4}$ given by the first expert

	Taobao	Tmall	Jingdong	Pinduoduo
Taobao	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	$(s_5, -)$	(s_3, s_8)	(s_9, s_2)
Tmall	$(-, s_5)$	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	(s_6, s_8)	$(-, s_4)$
Jingdong	(s_8, s_3)	(s_8, s_6)	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	(s_8, s_5)
Pinduoduo	(s_2, s_9)	$(s_4, -)$	(s_5, s_8)	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$

Table 2 Individual Lq-ROFPR $R_2 = (r_{2,ij})_{4\times4}$ given by the second expert

	Taobao	Tmall	Jingdong	Pinduoduo
Taobao	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	(s_5, s_7)	(s_4, s_6)	$(-, -)$
Tmall	(s_7, s_5)	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	(s_6, s_4)	(s_7, s_3)
Jingdong	(s_6, s_4)	(s_4, s_6)	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	(s_9, s_6)
Pinduoduo	$(-, -)$	(s_3, s_7)	(s_6, s_9)	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$

Lq-ROFPR $R''_k = (r''_{k,ij})_{n\times n}$ with acceptable consensus. Then, we set $R'_k = R''_k$ and return to Step 5.

Step 8. For the individual Lq-ROFPRs R''_l $(l = 1, 2, \cdots, m)$, which are of acceptable additive consistency and acceptable consensus, Eq. (6) is used to obtain the collective Lq-ROFPR R^*_c. Taking R^*_c into Model 1 and solving this model, a normalized Lq-ROFPWV $s_\omega = (s_{\omega_1}, s_{\omega_2}, \cdots, s_{\omega_n})^T$ can be derived.

Step 9. On the basis of Definition 4, we compute the score function and accuracy function of s_{ω_i} $(i = 1, 2, \cdots, n)$. Then, the ranking order of alternatives can be gained by applying the comparison law for Lq-ROFNs.

For the convenience of application, a frame diagram of the proposed method is presented in Fig. 1.

This paper provides a GDM method with incomplete Lq-ROFPRs through the conducting of consistency and consensus analysis. The novel findings are presented as follows: (*i*) a new preference relation called Lq-ROFPR is defined to describe the qualitative cognitive information of experts; (*ii*) additive consistency analysis is conducted, and a function is established to transform the normalized Lq-ROFPWV into a consistent Lq-ROFPR; (*iii*) the missing values of incomplete Lq-ROFPR are ascertained; (*iv*) the consistency checking and consistency repairing process are performed; (*v*) the decision-maker weight vector is objectively obtained from an optimization model; (*vi*) the adjusted Lq-ROFPRs can maintain more original information during the consensus improving procedure, and the repaired Lq-ROFPRs are still acceptably additively consistent; and (*vii*) the ranking order is derived from acceptably additively consistent Lq-ROFPRs with an acceptable consensus level.

A Case Study and Comparison Analysis

In the past decade, with the improvement in living standards, online shopping has become very common in China. According to the 49th Statistical Report on China's Internet Development published by the China Internet Network Information Center, up until December 2021, the user size of online shopping in China had reached 842 million, which was up 59.68 million from December 2020 and accounted for 81.6% of all internet users. At present, four main successful online shopping platforms exist in China, including x_1: Taobao, x_2: Tmall, x_3: Jingdong, and x_4: Pinduoduo. To gain more business information, shopping platforms mainly consider four measures, including product price, product quality, delivery time, and service level. Three experts were invited to assess these four shopping platforms to obtain a comprehensive ranking order. To fully express the evaluations of experts, such experts are allowed to use linguistic variables defined in the following linguistic set:

$$\begin{aligned} S = \{ & s_0: \textit{extremely bad}, s_1: \textit{very bad}, s_2: \textit{bad}, \\ & s_3: \textit{relatively bad}, s_4: \textit{a little bad}, s_5: \textit{fair}, \\ & s_6: \textit{a little good}, s_7: \textit{relatively good}, \\ & s_8: \textit{good}, s_9: \textit{very good}, s_{10}: \textit{extremely good}\}. \end{aligned}$$

Considering the judgments for qualitative preferred and nonpreferred degrees, the decision-makers established three individual Lq-ROFPRs by making pairwise comparisons for the four shopping platforms. Furthermore, when the experts were unable or unwilling to provide judgments, missing values are permitted. Suppose $q = 3$; in this case, three individual Lq-ROFPRs are constructed as shown in Tables 1, 2, and 3.

Table 3 Individual Lq-ROFPR $R_3 = (r_{3,ij})_{4\times4}$ given by the third expert

	Taobao	Tmall	Jingdong	Pinduoduo
Taobao	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	$(-, s_6)$	(s_7, s_8)	(s_6, s_5)
Tmall	$(s_6, -)$	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	$(-, s_7)$	(s_8, s_4)
Jingdong	(s_8, s_7)	$(s_7, -)$	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$	(s_9, s_2)
Pinduoduo	(s_5, s_6)	(s_4, s_8)	(s_2, s_9)	$(s_{10\sqrt[3]{0.5}}, s_{10\sqrt[3]{0.5}})$

To rank these four shopping platforms, the following decision-making steps are conducted.

Step 1. With respect to each incomplete individual Lq-ROFPR, Model 5 is applied to obtain the missing linguistic variables, and three complete individual Lq-ROFPRs R_1, R_2, and R_3 can be derived.

$$R_1=\begin{pmatrix}(s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_5,s_{3.4656}) & (s_3,s_8) & (s_9,s_2)\\ (s_{3.4656},s_5) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_6,s_8) & (s_{7.4364},s_4)\\ (s_8,s_3) & (s_8,s_6) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_8,s_5)\\ (s_2,s_9) & (s_4,s_{7.4364}) & (s_5,s_8) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}})\end{pmatrix}.$$

$$R_2=\begin{pmatrix}(s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_5,s_7) & (s_4,s_6) & (s_{8.3378},s_{6.0243})\\ (s_7,s_5) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_6,s_4) & (s_7,s_3)\\ (s_6,s_4) & (s_4,s_6) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_9,s_6)\\ (s_{6.0243},s_{8.3378}) & (s_3,s_7) & (s_6,s_9) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}})\end{pmatrix}.$$

$$R_3=\begin{pmatrix}(s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{2.6558},s_6) & (s_7,s_8) & (s_6,s_5)\\ (s_6,s_{2.6558}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{6.0358},s_7) & (s_8,s_4)\\ (s_8,s_7) & (s_7,s_{6.0358}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_9,s_2)\\ (s_5,s_6) & (s_4,s_8) & (s_2,s_9) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}})\end{pmatrix}.$$

Step 2. According to Eq. (5), we can obtain $ACI(R_1) = 0.1365$, $ACI(R_2) = 0.0582$, and $ACI(R_3) = 0.0768$. Without loss of generality, we assume that the consistency threshold $\overline{ACI} = 0.1$. Clearly, Lq-ROFPR R_1 is unacceptably additively consistent.

Step 3. By plugging Lq-ROFPR R_1 into Model 3, a repaired Lq-ROFPR R_1' with acceptable additive consistency can be obtained.

$$R_1'=\begin{pmatrix}(s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{5.0000},s_{3.4656}) & (s_{3.0273},s_{6.7621}) & (s_{8.9385},s_{2.0506})\\ (s_{3.4656},s_{5.0000}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{6.0000},s_{8.0000}) & (s_{7.4364},s_{4.0000})\\ (s_{6.7621},s_{3.0273}) & (s_{8.0000},s_{6.0000}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{8.0000},s_{5.0000})\\ (s_{2.0506},s_{8.9385}) & (s_{4.0000},s_{7.4364}) & (s_{5.0000},s_{8.0000}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}})\end{pmatrix}.$$

Using Eq. (5), we can check that $ACI(R_1') = 0.1$, which indicates that R_1' is of acceptable consistency.

Step 4. Taking three acceptably additively consistent Lq-ROFPRs R_1', $R_2' = R_2$, and $R_3' = R_3$ into Model 7, we can obtain the decision-maker weight vector $\lambda = \{0.4598, 0.3701, 0.1701\}^T$.

Step 5. According to Eq. (6), the collective Lq-ROFPR R_c can be derived.

$$R_c=\begin{pmatrix}(s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{4.7463},s_{5.6756}) & (s_{4.5595},s_{6.7621}) & (s_{8.3378},s_{4.7348})\\ (s_{5.6756},s_{4.7463}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{6.0061},s_{6.8217}) & (s_{7.3872},s_{3.6915})\\ (s_{6.7621},s_{4.5595}) & (s_{6.8217},s_{6.0061}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{8.5691},s_{5.1773})\\ (s_{4.7348},s_{8.3378}) & (s_{3.6915},s_{7.3872}) & (s_{5.1773},s_{8.5691}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}})\end{pmatrix}.$$

Based on Eq. (7), we have $GCI(R_1') = 0.0672$, $GCI(R_2') = 0.0774$ and $GCI(R_3') = 0.1114$.

Step 6. Assume a consensus threshold of $\overline{GCI} = 0.1$; then, according to Definition 12, the individual Lq-ROFPR R_3' is of unacceptable consensus, which means that the consensus level of R_3' needs to be improved.

Step 7. Plugging R_3' into Model 9 and solving this model, we can obtain an optimal adjustment parameter $\beta^* = 0.1233$. Then, Eq. (9) is applied to generate a modified Lq-ROFPR R_3'' with acceptable consensus.

$$R_3''=\begin{pmatrix}(s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{3.0936},s_{5.9619}) & (s_{6.7853},s_{7.8676}) & (s_{6.3893},s_{4.9688})\\ (s_{5.9619},s_{3.0936}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{6.0322},s_{6.9785}) & (s_{7.9295},s_{3.9645})\\ (s_{7.8676},s_{6.7853}) & (s_{6.9785},s_{6.0322}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{8.9491},s_{2.8895})\\ (s_{4.9688},s_{6.3893}) & (s_{3.9645},s_{7.9295}) & (s_{2.8895},s_{8.9491}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}})\end{pmatrix}.$$

Furthermore, the new collective Lq-ROFPR $R_c^{(1)}$ can be derived.

$$R_c^{(1)}=\begin{pmatrix}(s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{4.7735},s_{5.6684}) & (s_{4.4745},s_{6.7310}) & (s_{8.3742},s_{4.7289})\\ (s_{5.6684},s_{4.7735}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{6.0055},s_{6.8178}) & (s_{7.3732},s_{3.6845})\\ (s_{6.7310},s_{4.4745}) & (s_{6.8178},s_{6.0055}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}}) & (s_{8.5596},s_{5.2112})\\ (s_{4.7289},s_{8.3742}) & (s_{3.6845},s_{7.3732}) & (s_{5.2112},s_{8.5596}) & (s_{10\sqrt[3]{0.5}},s_{10\sqrt[3]{0.5}})\end{pmatrix}.$$

Based on Eq. (7), we have $GCI(R_1') = 0.0665$, $GCI(R_2') = 0.0769$ and $GCI(R_3'') = 0.1$.

Step 8. Now, we have three individual Lq-ROFPRs $R_1'' = R_1'$, $R_2'' = R_2' = R_2$, and R_3'' with an acceptable consensus level. Moreover, according to Theorem 3, R_3'' is also acceptably additively consistent. In fact, we can check that $ACI(R_3'') = 0.0633$ via Eq. (5). Thus, R_1'', R_2'', and R_3'' are all acceptably additively consistent. Taking the collective Lq-ROFPR $R_c^* = R_c^{(1)}$ into Model 1 and solving this model, a normalized Lq-ROFPWV can be derived as follows:

$$\begin{aligned}s_\omega &= (s_{\omega_1}, s_{\omega_2}, s_{\omega_3}, s_{\omega_4})^T\\ &= \big((s_0, s_{6.8734}), (s_{3.4067}, s_{6.5823}),\\ &\quad (s_{6.5823}, s_{6.5657}), (s_0, s_{9.8958})\big)^T.\end{aligned}$$

Step 9. Through calculating the score functions of the linguistic q-rung orthopair fuzzy priority weights

s_{ω_i} $(i = 1, 2, 3, 4)$, we derive $\mathbb{S}(s_{\omega_1}) = s_{6.9633}$, $\mathbb{S}(s_{\omega_2}) = s_{7.2252}$, $\mathbb{S}(s_{\omega_3}) = s_{7.9427}$, and $\mathbb{S}(s_{\omega_4}) = s_{2.4916}$. Using the ranking method in Definition 4, we obtain $s_{\omega_3} > s_{\omega_2} > s_{\omega_1} > s_{\omega_4}$, which means that the four online shopping platforms can be ranked as Jingdong, Tmall, Taobao, and Pinduoduo.

For a GDM with complete Lq-ROFPRs, although the linguistic q-rung orthopair fuzzy aggregation operators can be used to derive the final ranking order, consistency analysis and consensus analysis are not considered. To the best of our knowledge, no previous study has addressed the consistency and consensus of Lq-ROFPR. In the following, to provide a comparative analysis based on consistency and consensus, we use our approach to solve two GDM problems with different preference relations [41, 42].

Example 1 [41]: A city is affected by fog-haze, and there are four influence factors, including x_1: geographical conditions, x_2: concentration of gaseous pollutants, x_3: meteorological conditions, and x_4: mass concentration of PM2.5. Three experts are invited to evaluate the importance degrees of these factors, and the weight vector of experts is $\lambda = \{0.3, 0.4, 0.3\}^T$. Each decision-maker conducts pairwise comparisons for the four factors by using the following linguistic set:

$$S = \{s_0 : \textit{extremely low}, s_1 : \textit{very low}, s_2 : \textit{low}, s_3 : \textit{sightly low}, s_4 : \textit{medium}, s_5 : \textit{sightly high}, s_6 : \textit{high}, s_7 : \textit{very high}, s_8 : \textit{extremely high}\}.$$

Three LIFPRs $R_k^{Jin} = (r_{k,ij}^{Jin})_{4\times4}$ $(k = 1, 2, 3)$ are constructed by experts and listed in Example 4 of [41].

Assuming $\overline{ACI} = 0.1$ and $\overline{GCI} = 0.1$, by using our proposed method (when $q = 1$), we can obtain three LIFPRs $R_1^{\prime Jin}$, $R_2^{\prime Jin} = R_2^{Jin}$, and $R_3^{\prime Jin}$, where each LIFPR $R_k^{\prime Jin}$ $(k = 1, 2, 3)$ is acceptably additively consistent and the group reaches acceptable consensus level (the detailed steps are omitted for brevity).

$$R_1^{\prime Jin} = \begin{pmatrix} (s_4, s_4) & (s_{2.7878}, s_{4.3386}) & (s_{2.0810}, s_{4.8784}) & (s_{4.5851}, s_{2.4149}) \\ (s_{4.3386}, s_{2.7878}) & (s_4, s_4) & (s_{2.3338}, s_{3.8331}) & (s_{3.9190}, s_{2.0810}) \\ (s_{4.8784}, s_{2.0810}) & (s_{3.8331}, s_{2.3338}) & (s_4, s_4) & (s_{5.5851}, s_{1.4149}) \\ (s_{2.4149}, s_{4.5851}) & (s_{2.0810}, s_{3.9190}) & (s_{1.4149}, s_{5.5851}) & (s_4, s_4) \end{pmatrix}.$$

$$R_3^{\prime Jin} = \begin{pmatrix} (s_4, s_4) & (s_{2.9632}, s_{4.3133}) & (s_{3.0323}, s_{3.4515}) & (s_{2.3824}, s_{4.6176}) \\ (s_{4.3133}, s_{2.9632}) & (s_4, s_4) & (s_{3.5853}, s_{3.2074}) & (s_{2.9677}, s_{3.0323}) \\ (s_{3.4515}, s_{3.0323}) & (s_{3.2074}, s_{3.5853}) & (s_4, s_4) & (s_{3.3824}, s_{3.6176}) \\ (s_{4.6176}, s_{2.3824}) & (s_{3.0323}, s_{2.9677}) & (s_{3.6176}, s_{3.3824}) & (s_4, s_4) \end{pmatrix}.$$

Using Eq. (6), the collective LIFPR R_c^{*Jin} can be derived.

$$R_c^{*Jin} = \begin{pmatrix} (s_4, s_4) & (s_{2.1523}, s_{5.3956}) & (s_{2.3340}, s_{4.4990}) & (s_{3.2903}, s_{3.7097}) \\ (s_{5.3956}, s_{2.1523}) & (s_4, s_4) & (s_{3.3757}, s_{3.3121}) & (s_{3.6660}, s_{2.3340}) \\ (s_{4.4990}, s_{2.3340}) & (s_{3.3121}, s_{3.3757}) & (s_4, s_4) & (s_{4.2903}, s_{2.7098}) \\ (s_{3.7097}, s_{3.2903}) & (s_{2.3340}, s_{3.6660}) & (s_{2.7098}, s_{4.2903}) & (s_4, s_4) \end{pmatrix}.$$

We can check that $ACI(R_1^{\prime Jin}) = 0.0392$, $ACI(R_2^{\prime Jin}) = 0.0625$, $ACI(R_3^{\prime Jin}) = 0.0329$, $GCI(R_1^{\prime Jin}) = 0.1$, $GCI(R_2^{\prime Jin}) = 0.0659$, and $GCI(R_3^{\prime Jin}) = 0.0938$. Taking the collective LIFPR R_c^{*Jin} into Model 1, a normalized linguistic intuitionistic fuzzy priority weight vector can be derived as follows:

$$s_\omega^{Jin} = (s_{\omega_1}^{Jin}, s_{\omega_2}^{Jin}, s_{\omega_3}^{Jin}, s_{\omega_4}^{Jin})^T = \big((s_{0.0408}, s_{6.9572}), (s_{2.1242}, s_{4.2638}), (s_{2.0408}, s_{4.6272}), (s_{0.4622}, s_{6.5398})\big)^T.$$

The score functions are $\mathbb{S}(s_{\omega_1}^{Jin}) = s_{0.5418}$, $\mathbb{S}(s_{\omega_2}^{Jin}) = s_{2.9302}$, $\mathbb{S}(s_{\omega_3}^{Jin}) = s_{2.7068}$, and $\mathbb{S}(s_{\omega_4}^{Jin}) = s_{0.9612}$. Thus, we obtain $s_{\omega_2}^{Jin} > s_{\omega_3}^{Jin} > s_{\omega_4}^{Jin} > s_{\omega_1}^{Jin}$, which means that the ranking of these four influence factors is $x_2 > x_3 > x_4 > x_1$, and the ranking order is the same as that shown in [41].

To show the effect of parameter q, we use Table 4 to provide the ranking results with respect to q from 1 to 5. Table 4 demonstrates that the ranking of these four influence factors is always $x_2 > x_3 > x_4 > x_1$, and the ranking order does not vary when parameter q changes. When $q = 1$, we obtain four linguistic intuitionistic fuzzy priority weights as follows: $(s_{0.0408}, s_{6.9572})$, $(s_{2.1242}, s_{4.2638})$, $(s_{2.0408}, s_{4.6272})$, and $(s_{0.4622}, s_{6.5398})$. When $q = 2$, we derive four linguistic Pythagorean fuzzy priority weights as follows: $(s_0, s_{5.6453})$, $(s_{2.3222}, s_{3.5402})$, $(s_{3.1934}, s_{4.3497})$, and $(s_0, s_{5.0227})$. When $q = 3$, we obtain four linguistic q-rung orthopair fuzzy priority weights as follows: $(s_0, s_{5.5198})$, $(s_{1.7960}, s_{3.5445})$, $(s_{3.2144}, s_{4.4187})$, and $(s_{0.6827}, s_{4.6999})$. In addition, with increasing parameter q, the lower index difference between $\mathbb{S}(s_{\omega_i}^{Jin})$ and $\mathbb{S}(s_{\omega_j}^{Jin})$ ($i = 2, j = 3$ or $i = 3, j = 4$ or $i = 4, j = 1$) decreases. Moreover, before aggregating the four acceptably additively

Table 4 The score functions and ranking orders with respect to different parameter q

Parameter	Four score functions	Ranking order
$q = 1$	$\mathbb{S}(s_{\omega_1}^{Jin}) = s_{0.5418}$, $\mathbb{S}(s_{\omega_2}^{Jin}) = s_{2.9302}$, $\mathbb{S}(s_{\omega_3}^{Jin}) = s_{2.7068}$, $\mathbb{S}(s_{\omega_4}^{Jin}) = s_{0.9612}$	$x_2 > x_3 > x_4 > x_1$
$q = 2$	$\mathbb{S}(s_{\omega_1}^{Jin}) = s_{4.0082}$, $\mathbb{S}(s_{\omega_2}^{Jin}) = s_{5.3320}$, $\mathbb{S}(s_{\omega_3}^{Jin}) = s_{5.2573}$, $\mathbb{S}(s_{\omega_4}^{Jin}) = s_{4.4030}$	$x_2 > x_3 > x_4 > x_1$
$q = 3$	$\mathbb{S}(s_{\omega_1}^{Jin}) = s_{5.5603}$, $\mathbb{S}(s_{\omega_2}^{Jin}) = s_{6.1852}$, $\mathbb{S}(s_{\omega_3}^{Jin}) = s_{6.1222}$, $\mathbb{S}(s_{\omega_4}^{Jin}) = s_{5.8892}$	$x_2 > x_3 > x_4 > x_1$
$q = 4$	$\mathbb{S}(s_{\omega_1}^{Jin}) = s_{6.2902}$, $\mathbb{S}(s_{\omega_2}^{Jin}) = s_{6.6537}$, $\mathbb{S}(s_{\omega_3}^{Jin}) = s_{6.6104}$, $\mathbb{S}(s_{\omega_4}^{Jin}) = s_{6.5506}$	$x_2 > x_3 > x_4 > x_1$
$q = 5$	$\mathbb{S}(s_{\omega_1}^{Jin}) = s_{6.7391}$, $\mathbb{S}(s_{\omega_2}^{Jin}) = s_{6.9328}$, $\mathbb{S}(s_{\omega_3}^{Jin}) = s_{6.9111}$, $\mathbb{S}(s_{\omega_4}^{Jin}) = s_{6.8979}$	$x_2 > x_3 > x_4 > x_1$

consistent LIFPRs, which are also of acceptable levels of consensus, the original LIFPRs R_1^{Jin} and R_3^{Jin} are adjusted for $q = 1$ and $q = 2$, and only R_3^{Jin} is adjusted for $q = 3$, while none of R_k^{Jin} $(k = 1, 2, 3)$ is changed for $q = 4$ and $q = 5$. In general, decision-makers can flexibly determine the value of q based on the practical situation.

Example 2 [42]: With development of the economy, people want to find the significant factors that influence the sustainable development of innovative companies. There are four influence factors, including x_1: the sustainable development of the economy and society, x_2: innovative talents, x_3: capacity for continuous innovation, and x_4: enterprise culture. Four experts are invited to evaluate the importance degrees of these factors, and the weight vector of experts is $\lambda = \{0.2, 0.3, 0.3, 0.2\}^T$. Each decision-maker conducts pairwise comparisons for the four factors by using the linguistic set given in Example 1. Four LPFPRs $R_k^{Liu} = (r_{k,ij}^{Liu})_{4\times4}$ $(k = 1, 2, 3, 4)$ are constructed by the experts and listed in subsection 6.1 of [42].

Assuming $\overline{ACI} = 0.1$ and $\overline{GCI} = 0.1$, by using our proposed method (when $q = 2$), we can obtain four LPFPRs $R_1'^{Liu}$, $R_2'^{Liu}$, $R_3'^{Liu}$, and $R_4'^{Liu} = R_4^{Liu}$, where each LPFPR $R_k'^{Liu}$ $(k = 1, 2, 3, 4)$ is acceptably additively consistent and the group reaches acceptable consensus level (the detailed steps are omitted for brevity).

$$R_1'^{Liu} = \begin{pmatrix} (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{5.2915}, s_6) & (s_2, s_5) & (s_3, s_5) \\ (s_6, s_{5.2915}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_4, s_6) & (s_{4.9598}, s_6) \\ (s_5, s_2) & (s_6, s_4) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_7, s_3) \\ (s_5, s_3) & (s_6, s_{4.9598}) & (s_3, s_7) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) \end{pmatrix}.$$

$$R_2'^{Liu} = \begin{pmatrix} (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{5.0433}, s_{5.7974}) & (s_{2.5740}, s_{5.7742}) & (s_{2.8277}, s_{3.4172}) \\ (s_{5.7974}, s_{5.0433}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{3.1289}, s_{5.1498}) & (s_{3.2503}, s_{5.9004}) \\ (s_{5.7742}, s_{2.5740}) & (s_{5.1498}, s_{3.1289}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{2.4326}, s_{3.4008}) \\ (s_{3.4172}, s_{2.8277}) & (s_{5.9004}, s_{3.2503}) & (s_{3.4008}, s_{2.4326}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) \end{pmatrix}.$$

$$R_3'^{Liu} = \begin{pmatrix} (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{4.7958}, s_3) & (s_4, s_5) & (s_3, s_{4.9396}) \\ (s_3, s_{4.7958}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{3.6056}, s_6) & (s_5, s_4) \\ (s_5, s_4) & (s_6, s_{3.6056}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_6, s_2) \\ (s_{4.9396}, s_3) & (s_4, s_5) & (s_2, s_6) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) \end{pmatrix}.$$

Using Eq. (6), the collective LPFPR R_c^{*Liu} can be derived.

$$R_c^{*Liu} = \begin{pmatrix} (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{5.2278}, s_{4.8149}) & (s_{3.5479}, s_{5.4500}) & (s_{2.7747}, s_{4.5632}) \\ (s_{4.8149}, s_{5.2278}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{3.6383}, s_{5.7581}) & (s_{4.1700}, s_{5.4447}) \\ (s_{5.4500}, s_{3.5479}) & (s_{5.7581}, s_{3.6383}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) & (s_{5.2321}, s_{3.1096}) \\ (s_{4.5632}, s_{2.7747}) & (s_{5.4447}, s_{4.1700}) & (s_{3.1096}, s_{5.2321}) & (s_{4\sqrt{2}}, s_{4\sqrt{2}}) \end{pmatrix}.$$

We can check that $ACI(R_1'^{Liu}) = 0.1$, $ACI(R_2'^{Liu}) = 0.0796$, $ACI(R_3'^{Liu}) = 0.1$, $ACI(R_4'^{Liu}) = 0.0781$, $GCI(R_1'^{Liu}) = 0.0960$, $GCI(R_2'^{Liu}) = 0.1$, $GCI(R_3'^{Liu}) = 0.0913$, and $GCI(R_4'^{Liu}) = 0.0931$. Taking the collective LPFPR R_c^{*Liu} into Model 1, a normalized linguistic Pythagorean fuzzy priority weight vector can be derived as follows:

$$\begin{aligned} s_\omega^{Liu} &= (s_{\omega_1}^{Liu}, s_{\omega_2}^{Liu}, s_{\omega_3}^{Liu}, s_{\omega_4}^{Liu})^T \\ &= \big((s_{0.9259}, s_{6.3915}), (s_{2.3485}, s_{7.3350}), \\ &\quad (s_{4.3075}, s_{4.5781}), (s_{0.8915}, s_{5.4094})\big)^T. \end{aligned}$$

The score functions are $\mathbb{S}(s_{\omega_1}^{Liu}) = s_{3.4645}$, $\mathbb{S}(s_{\omega_2}^{Liu}) = s_{2.8030}$, $\mathbb{S}(s_{\omega_3}^{Liu}) = s_{5.5496}$ and $\mathbb{S}(s_{\omega_4}^{Liu}) = s_{4.2150}$. Thus, we obtain $s_{\omega_3}^{Liu} > s_{\omega_4}^{Liu} > s_{\omega_1}^{Liu} > s_{\omega_2}^{Liu}$, which means that the ranking of these four influence factors is $x_3 > x_4 > x_1 > x_2$, and the ranking order is the same as that shown in [42].

Compared with the methods shown in [41, 42], our approach has the following advantages:

1. Jin et al.'s method [41] can only deal with GDM with LIFPRs, while Liu et al. [42] only provide a GDM method based on LPFPRs. However, the constraints of membership and nonmembership degrees in this paper are $I^q(s_{\mu_{ij}}) + I^q(s_{\nu_{ij}}) \le (2\tau)^q$ $(q = 1, 2, \cdots)$, which means that LIFPR and LPFPR can be regarded as special cases of Lq-ROFPR. Thus, Lq-ROFPR increases the scope of application and has a stronger ability to measure cognitive preference information.
2. The methods [41, 42] are only used to analyze complete preference relations; when the preference values are unknown, these two methods are inapplicable. In our proposed algorithm, we consider this issue and establish a programming model to estimate the unknown elements from incomplete Lq-ROFPR in a reasonable way.
3. The weights of experts are predefined in [41, 42], which means that the experts' objective weights cannot be derived. In our method, by taking into account the group consensus level, Model 7 is constructed to generate the decision-maker weight vector.
4. The methods [41, 42] are used to repair the unacceptably consistent preference relation to be of acceptable consistency via an iterative method, which may take considerable time and effort. In contrast, our method utilizes Model 3 to achieve this goal, which is only based on a single adjustment. Moreover, our adjusted Lq-ROFPR has the smallest distance from the original Lq-ROFPR.
5. In the procedure of consensus analysis, the iterative method is still used, and the adjustment parameter is also predetermined [41, 42]. However, our proposed method improves the consensus level only by solving a programming model, which is time-saving. Furthermore, to retain more original information, our adjustment parameter derived from Model 9 is the smallest.

Conclusion

Based on the consistency and consensus analysis, this paper presents a new GDM approach using incomplete Lq-ROFPRs. The main advantages and limitations of the proposed method are as follows. Advantages: (1) a new linguistic preference relation is introduced in this paper to provide qualitative cognitive information, and it is called Lq-ROFPR; (2) the additive consistency of Lq-ROFPR is defined, and two models are designed to ascertain the missing values and repair the unacceptably additively consistent Lq-ROFPR; (3) the normalized Lq-ROFPWV of Lq-ROFPR is derived by constructing a consistency-based model; (4) the objective weights of decision-makers are obtained; and (5) the consensus checking and consensus reaching process are considered. Limitations: (1) the consistency and consensus thresholds are predefined, and the effect of these two thresholds is not discussed in this paper; and (2) the number of alternatives and the number of decision-makers are small, and Lq-ROFPRs in a large scale GDM environment are not addressed in this paper.

In the future, the proposed GDM method can be applied to other practical fields, such as cognitive computation [45], ERP selection [46], social cognition [47], medical diagnosis [48], and pattern recognition [49]. Moreover, investigating the multiplicative consistency of Lq-ROFPR or developing a similar method in the interval-valued linguistic environment [50–52] is also an interesting and meaningful research direction.

Funding This work is supported by the National Social Science Foundation of China (No.19CGL045).

Data Availability No data was used for the research described in the article.

Declarations

Ethical Approval This article does not contain any studies with human participants performed by any of the authors.

Conflict of Interest The authors declare no competing interests.

References

1. Frith CD, Singer T. The role of social cognition in decision making. Phil Trans R Soc B. 2008;363:3875–86.
2. Behimehr S, Jamali HR. Relations between cognitive biases and some concepts of information behavior. Data Inf Manag. 2020;4(2):109–18.
3. Schunk DH, Dibenedetto MK. Motivation and social cognitive theory. Contemp Educ Psychol. 2020;60: 101832.
4. Atanassov K. Intuitionistic fuzzy sets Fuzzy Set Syst. 1986;20: 87–96.
5. Yager RR. Pythagorean membership grades in multicriteria decision making. IEEE T Fuzzy Syst. 2014;22(4):958–65.
6. Yager RR. Generalized orthopair fuzzy sets. IEEE T Fuzzy Syst. 2017;25(5):1222–30.
7. Xin XW, Sun JB, Xue ZA, Song JH, Peng WM. A novel intuitionistic fuzzy three-way decision model based on an intuitionistic fuzzy incomplete information system. Int J Mach Learn Cyb. 2022;13:907–27.
8. Zhou F, Chen TY. A hybrid approach combining AHP with TODIM for blockchain technology provider selection under the Pythagorean fuzzy scenario. Artif Intell Rev. 2022;55:5411–43.
9. Krishankumar R, Nimmagadda SS, Rani P, Mishra AR, Ravichandran KS, Gandomi AH. Solving renewable energy source selection problems using a q-rung orthopair fuzzy-based integrated decision-making approach. J Clean Prod. 2021;279: 123329.
10. Garg H, Rani D. An efficient intuitionistic fuzzy MULTIMOORA approach based on novel aggregation operators for the assessment of solid waste management techniques. Appl Intell. 2022;52:4330–63.
11. Wang L, Garg H. Algorithm for multiple attribute decision-making with interactive Archimedean norm operations under Pythagorean fuzzy uncertainty. Int J Comput Int Sys. 2021;14(1):503–27.
12. Tang GL, Yang YX, Gu XW, Chiclana F, Liu PD, Wang FB. A new integrated multi-attribute decision-making approach for mobile medical app evaluation under q-rung orthopair fuzzy environment. Expert Syst Appl. 2022;200(15): 117034.
13. Yang ZY, Zhang LY, Li T. Group decision making with incomplete interval-valued q-rung orthopair fuzzy preference relations. Int J Intell Syst. 2021;36:7274–308.
14. Liu PD, Wang YM. Multiple attribute group decision making methods based on intuitionistic linguistic power generalized aggregation operators. Appl Soft Comput. 2014;17:90–104.
15. Liu PD, Gao H, Fujita H. The new extension of the MULTIMOORA method for sustainable supplier selection with intuitionistic linguistic rough numbers. Appl Soft Comput. 2021;99: 106893.
16. Deng X, Wang J, Wei G, Wei C. Multiple attribute decision making based on Muirhead mean operators with 2-tuple linguistic Pythagorean fuzzy information. Sci Iran E. 2021;28(4):2294–322.
17. Mandal P, Samanta S, Pal M. Large-scale group decision-making based on Pythagorean linguistic preference relations using experts clustering and consensus measure with non-cooperative behavior analysis of clusters. Complex Intell Syst. 2022;8:819–33.
18. Zhao HM, Zhang RT, Zhang A, Zhu XM. Multi-attribute group decision making method with unknown attribute weights based on the q-rung orthopair uncertain linguistic power Muirhead mean operators. Int J Comput Commun. 2021;16(3):4214.
19. Yang ZL, Garg H. Interaction power partitioned Maclaurin symmetric mean operators under q-rung orthopair uncertain linguistic information. Int J Fuzzy Syst. 2022;24:1079–97.
20. Chen ZC, Liu PH, Pei Z. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. Int J Comput Int Sys. 2015;8(4):747–60.
21. Liu JB, Mai JX, Li HX, Huang B, Liu YJ. On three perspectives for deriving three-way decision with linguistic intuitionistic fuzzy information. Informa Sciences. 2022;588:350–80.
22. Garg H. Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process. Int J Intell Syst. 2018;33(6):1234–63.
23. Ping YJ, Liu R, Wang ZL, Liu HC. New approach for quality function deployment with an extended alternative queuing method under linguistic Pythagorean fuzzy environment. Eur J Ind Eng. 2022;16(3):349–70.
24. Liu PD, Liu WQ. Multiple-attribute group decision-making based on power Bonferroni operators of linguistic q-rung orthopair fuzzy numbers. Int J Intell Syst. 2019;34(4):652–89.
25. Liu PD, Liu WQ. Multiple-attribute group decision-making method of linguistic q-rung orthopair fuzzy power Muirhead

mean operators based on entropy weight. Int J Intell Syst. 2019;34(8):1755–94.
26. Lin MW, Li XM, Chen LF. Linguistic q-rung orthopair fuzzy sets and their interactional partitioned Heronian mean aggregation operators. Int J Intell Syst. 2020;35(2):217–49.
27. Ling J, Li XM, Lin MW. Medical waste treatment station selection based on linguistic q-rung orthopair fuzzy numbers. CMES-Comp Model Eng. 2021;129(1):117–48.
28. Liu PD, Naz S, Akram M, Muzammal M. Group decision-making analysis based on linguistic q-rung orthopair fuzzy generalized point weighted aggregation operators. Int J Mach Learn Cyb. 2022;13:883–906.
29. Akram M, Naz S, Edalatpanah SA, Mehreen R. Group decision-making framework under linguistic q-rung orthopair fuzzy Einstein models. Soft Comput. 2021;25:10309–34.
30. Liu DH, Liu YY, Wang LZ. The reference ideal TOPSIS method for linguistic q-rung orthopair fuzzy decision making based on linguistic scale function. J Intell Fuzzy Syst. 2020;39(3):4111–31.
31. Peng D, Wang J, Liu DH, Liu ZM. The similarity measures for linguistic q-rung orthopair fuzzy multi-criteria group decision making using projection method. IEEE Access. 2019;7:176732–45.
32. Verma R. Generalized similarity measures under linguistic q-rung orthopair fuzzy environment with application to multiple attribute decision-making. Granul Comput. 2022;7:253–75.
33. Meng FY, Tang J, Xu ZS. Exploiting the priority weights from interval linguistic fuzzy preference relations. Soft Comput. 2019;23:583–97.
34. Meng FY, Tang J, Zhang YL. Programming model-based group decision making with multiplicative linguistic intuitionistic fuzzy preference relations. Comput Ind Eng. 2019;136:212–24.
35. Wu P, Liu JP, Zhou LG, Chen HY. Algorithm for improving additive consistency of linguistic preference relations with an integer optimization model. Appl Soft Comput. 2020;86: 105955.
36. Ren PJ, Xu ZS, Wang XX, Zeng XJ. Group decision making with hesitant fuzzy linguistic preference relations based on modified extent measurement. Expert Syst Appl. 2021;171: 114235.
37. Xue M, Fu C, Yang SL. A comparative analysis of probabilistic linguistic preference relations and distributed preference relations for decision making. Fuzzy Optim Decis Ma. 2022;21:71–97.
38. Pei LD, Jin FF, Ni ZW, Chen HY, Tao ZF. An automatic iterative decision-making method for intuitionistic fuzzy linguistic preference relations. Int J Syst Sci. 2017;48(13):2779–93.
39. Meng FY, Tang J, Hamido F. Linguistic intuitionistic fuzzy preference relations and their application to multi-criteria decision making. Inform Fusion. 2019;46:77–90.
40. Zhang LY, Liang CL, Li T, Yang WT. A two-stage EDM method based on KUCBR with the incomplete linguistic intuitionistic fuzzy preference relations. Comput Ind Eng. 2022;172: 108552.
41. Jin FF, Ni ZW, Pei LD, Chen HY, Li YP, Zhu XH, Ni LP. A decision support model for group decision making with intuitionistic fuzzy linguistic preferences relations. Neural Comput Appl. 2019;31:1103–24.
42. Liu JP, Fang MD, Jin FF, Tao ZF, Chen HY, Du PC. Pythagorean fuzzy linguistic decision support model based on consistency-adjustment strategy and consensus reaching process. Soft Comput. 2021;25:8205–21.
43. Herrera F, Herrera-Viedma E, Verdegay JL. A model of consensus in group decision making under linguistic assessments. Fuzzy Set Syst. 1996;78:73–87.
44. Xu ZS. EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations. Int J Uncertain Fuzz. 2004;12:791–810.
45. Hu HZ, Tang YB, Xie YQ, Dai YH, Dai WH. Cognitive computation on consumer's decision making of internet financial products based on neural activity data. Comput Sci Inf Syst. 2020;17(2):689–704.
46. Carpitella S, Certa A, Izquierdo J, Cascia ML. Multi-criteria decision-making approach for modular enterprise resource planning sorting problems. J Multi-Criteria Dec. 2021;28:234–47.
47. Ho IK, Lawrence JS. The role of social cognition in medical decision making with Asian American patients. J Racial Ethn Health. 2021;8:1112–8.
48. Chai JS, Selvachandran G, Smarandache F, Gerogiannis VC, Son LH, Bui QT, Vo B. New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. Complex Intell Syst. 2021;7:703–23.
49. Gohain B, Chutia R, Dutta P. Distance measure on intuitionistic fuzzy sets and its application in decision-making, pattern recognition, and clustering problems. Int J Intell Syst. 2022;37(3):2458–501.
50. Tang J, Meng FY, Cabrerizo FJ, Herrera-Viedma E. A procedure for group decision making with interval-valued intuitionistic linguistic fuzzy preference relations. Fuzzy Optim Decis Ma. 2019;18:493–527.
51. Verma1 R, Agarwal N. Multiple attribute group decision-making based on generalized aggregation operators under linguistic interval-valued Pythagorean fuzzy environment. Granul Comput. 2022;7:591-632.
52. Khan MSA, Khan AS, Khan IA, Mashwani WK, Hussain F. Linguistic interval-valued q-rung orthopair fuzzy TOPSIS method for decision making problem with incomplete weight. J Intell Fuzzy Syst. 2021;40(3):4223–35.