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Probabilistic Dual-Hesitant Pythagorean Fuzzy Sets and Their Application in Multi-attribute Group Decision-Making

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Abstract

As modern socioeconomic decision-making problems are becoming more and more complex, it also becomes more and more difficult to appropriately depict decision makers' cognitive information in decision-making process. In addition, in group decision-making problems, decision makers' cognition is usually diverse, which makes it more complicated to express the overall preference information. Recently, the dual-hesitant Pythagorean fuzzy sets (DHPFSs) have been proved to be an effective tool to depict decision makers' evaluation values in multi-attribute group decision-making (MAGDM) procedure. The basic elements of DHPFSs are dual-hesitant Pythagorean fuzzy numbers (DHFNs), which are characterized by some possible membership degrees and non-membership degrees. In a DHFN, all members have the same importance, which indicates that multiple occurrence and appearance of some elements is ignored. Hence, the DHPFSs still have some drawbacks when expressing decision makers' evaluation information in MAGDM problems. This paper aims at proposing a novel tool to describe decision maker's evaluation values and apply it in solving MAGDM problems. This paper extends the traditional DHPFSs to probabilistic dual-hesitant Pythagorean fuzzy sets (PDHPFSs), which consider not only multiple membership and non-membership degrees, but also their probabilistic information. Afterward, we investigate the applications of PDHPFSs in MAGDM process. To this end, we first introduce the concept of DHPFSs as well as some related notions, such as operational rules, score function, accuracy function, comparison method, and distance measure. Second, based on the power average and Hamy mean, some aggregation operators for DHPFSs are presented. Properties of these new operators are also discussed. Third, we put forward a novel MAGDM method under PDHPFSs. A novel MAGDM method is developed, and further, we conduct numerical examples to show the performance and advantages of the new method. Results indicate that our method can effectively handle MAGDM problems in reality. In addition, comparative analysis also reveals the advantages of our method. This paper contributed a novel MAGDM method and numerical examples as well as comparative analysis were provided to show the effectiveness and advantages of our proposed method. Our contributions provide decision makers a new manner to determine the optimal alternative in realistic MAGDM problems.

Keywords Dual-hesitant Pythagorean fuzzy sets · Probabilistic dual-hesitant Pythagorean fuzzy sets · Power average · Hamy mean · Multi-attribute group decision-making

Introduction

The theory of multi-attribute group decision-making (MAGDM) has received extensive attention and quite a few achievements have been reported in the past decades. The final decision result of a MAGDM problem relies on decision makers' (DMs) cognitive information and hence, describing DMs' cognition appropriately and effectively is a precondition of determining the final optimal option. Nevertheless, it is very difficult to depict DMs' cognitive information because of the increasing complexity and uncertainty of realistic socioeconomic environment. In addition, especially in MAGDM procedure, as DMs usually have different expertise, their cognition is usually diverse. Consequently, DMs' cognitive process is full of complexity, vagueness, and uncertainty. In the past decades, some scholars have attempted to depict DMs' cognitive information from the

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perspective of fuzzy sets theory. For instance, the good performance of Yager's Pythagorean fuzzy sets (PFSs) [1] in describing fuzzy and uncertain information has received much attention. In the framework of PFSs, it is necessary to not only depict fuzzy information from both positive and negative aspects but also provide enough freedom for DMs to comprehensively express their evaluation values. Due to the efficiency of PFSs in denoting attribute values that provided by DMs, they have been extensively applied to MAGDM problems. In addition, quite a few new decision-making approaches have been proposed [2]. To accommodate more complicated decision-making situations, some scholars and scientists focused on extending the classical PFSs, in order to more appropriately and accurately depict DMs' preference information over feasible alternatives. For instance, Garg [3] extended the PFSs to the interval-valued PFSs, which employ interval numbers to denote the membership degrees (MDs) and non-membership degrees (NMDs). Peng and Yang [4], Geng et al. [5], and Wei et al. [6] proposed the Pythagorean fuzzy linguistic sets, Pythagorean fuzzy uncertain linguistic sets, and Pythagorean 2-tuple linguistic sets, respectively, which can describe both DMs' quantitative and qualitative decision information. Additionally, based on the interval-valued PFSs, Du et al. [7], Liu et al. [8], and Wang et al. [9] proposed the interval-valued Pythagorean fuzzy linguistic sets, interval-valued Pythagorean fuzzy uncertain linguistic sets, and interval-valued Pythagorean 2-tuple linguistic sets, respectively.

The abovementioned extensions of PFSs have been proved to be efficient to handle DMs' complex cognitive information. However, sometimes the difficulty in establishing MDs and NMDs lies in handling a set of possible values instead of margin errors or some possibility distribution values. In order to deal with such situations, Wei and Lu [10] extended the PFSs to the dual-hesitant Pythagorean fuzzy sets (DHPFSs). Similar to the dual-hesitant fuzzy set (DHFS) [11], the DHPFS is also characterized by two sets of some values in [0, 1], denoting the possible MDs and NMDs, respectively. The DHPFSs absorb the advantages of both PFSs and DHFSs, and they are more powerful and flexible to depict attribute values in MAGDM problems. Afterward, Lu et al. [12] proposed a bidirectional project method of DHPFSs and applied it in performance assessment of new rural construction. Ji et al. [13] generalized the traditional TODIM into DHPFSs to evaluate personal default risk in P2P lending platform. In addition, some scholars investigated MAGDM methods under DHPFSs from the perspective of dual-hesitant Pythagorean fuzzy (DHPF) aggregation operators (AOs). For instance, Tang and Wei [14, 15], Wei et al. [16] proposed the DHPF Bonferroni mean operators, generalized DHPF Bonferroni mean operators, DHPF Heronian mean operators, and DHPF Hamy mean operators, respectively. These AOs have been successfully applied in MAGDM problems and some novel decision-making approaches have been proposed correspondingly.

Nevertheless, the DHPFSs still have a shortcoming when dealing with DMs' evaluation information. They neglect the frequency or multiple occurrences of elements in each DHPF numbers (DHFNs). For example, three professors are invited to evaluate the research ability of a student. The three professors would like to use three sets of values $\{0.5, 0.6\}$, $\{0.6\}$, and $\{0.5, 0.8\}$ to denote the possible MDs of their assessments. Similarly, three sets of values $\{0.3, 0.4, 0.6\}$, $\{0.4\}$, and $\{0.6\}$ are employed by the three professors to denote the NMDs of their evaluations. Then, the overall evaluation value by the decision-making group can be denoted as $d=\{\{0.5, 0.6, 0.8\}, \{0.3, 0.4, 0.6\}\}$, which is a dual-hesitant Pythagorean fuzzy number (DHPFN). However, in the DHPFN d the multiple occurrence and appearance of the MDs 0.5 and 0.6, and the NMDs 0.4 and 0.6 is ignored. In the other word, the DHPFN d cannot accurately express the comprehensive evaluation value of the group. As a matter of fact, to more accurately denote the evaluation information of a group, not only the MDs and NMDs but also their corresponding frequency or probabilistic information should be considered. Similar researches can be found in recent publications. For example, Pang et al. [17] extended the hesitant fuzzy linguistic sets [18] to probabilistic linguistic sets (PLSs) by taking the probabilistic information of each linguistic term into account. In PLSs, not only linguistic terms provided by DMs, but also the frequencies of them are captured. Hence, PLSs can more accurately express DMs' evaluation information. In [19–22], many scholars focused on applications of PLSs in practical MAGDM problems. Similarly, Jiang and Ma [23] generalized the hesitant fuzzy sets into probabilistic hesitant fuzzy sets (PHFSs) which also consider the probabilities of all the MDs. Song et al. [24] extended the interval-valued hesitant fuzzy sets into interval-valued probabilistic hesitant fuzzy sets by capturing the probabilistic information of each interval-valued MDs. Recently, Hao et al. [25] extended dual hesitant fuzzy sets to probabilistic dual hesitant fuzzy sets (PDHFSs).

The shortcomings of DHPFSs in depicting DMs' evaluation values and the high efficiency of PLSs, PHFSs and PDHFSs in representing fuzzy information motivated us to propose an enhanced form of DHPFSs to more appropriately express DMs' preference information. We noticed the that the reason of why PLSs, PHFSs and PDHFSs have good ability of representing evaluation information, is because all of them not only consider each member in an evaluation element but also take into account its corresponding probabilistic information. Hence, based on above analysis, to improve the ability of DHPFSs in expressing DMs' overall evaluation information, it is necessary to extend the classical DHPFSs to probabilistic DHPFSs (PDHPFSs). In the PDHPFSs, both the MDs, NMDs as well as their corresponding probabilistic information are considered. Hence, compared with the DHPFSs, the PDHPFSs can more effectively denote DMs' evaluation values in complicated MAGDM problems. In addition, the PDHPFSs are parallel to PDHFSs but are more powerful. This is because PDHPFSs have laxer constraint, which provides freedom for DMs to comprehensively express their evaluation information. For the applications of PDHPFSs in MAGDM, we further propose operational rules, comparison method, and distance measure of probabilistic dual-hesitant Pythagorean fuzzy elements (PDHPFEs). When considering the AOs of PDHPFEs, we are impressed by the good performance of power Hamy mean (PHM) [26] operator in fusing information. The PHM is a combination of the power average (PA) [27] operator and Hamy mean (HM) [28]. The performance of PHM is impressive as it not only reduces the bad influence of DMs' extreme evaluation values on the results but also considers the interrelationship among multiple connected attributes. Thus, this paper employs PHM to aggerate PDHPFEs and propose novel AOs. Finally, based on the new AOs, we further introduce a novel MAGDM method, wherein attribute values are in the form of PDHPFEs.

The main contributions of this study consist of three aspects. First, we propose a novel information representation tool to depict overall evaluation values of a group, viz., PDHPFSs. Because PPDHPFS considers both MDs, NMDs as well as their frequency or probabilistic information, it is more powerful than PDHPFS. In addition, due to its laxer constraint that the square sum of MD and NMD is less than or equal to one, PDHPFS is also more powerful than Hao et al.'s [25] PDHFSs. Second, a series of AOs for PDHPFSs are presented. Evidence is provided to demonstrate the advantages of superiorities of the proposed AOs in solving MAGDM problems. Finally, a new MAGDM method is put forward. The make our paper more readable, we organize our manuscript as follows. The "Preliminaries" section briefly reviews some basic concepts. The "The Probabilistic Dual-Hesitant Pythagorean Fuzzy Sets" section proposes the notion of PDHPFSs as well as some other related concepts. The "Probabilistic Dual-Hesitant Pythagorean Fuzzy Aggregation Operators" section investigates AOs of PDHPFEs and studies their important properties. The "An Approach to Multiple Attribute Decision Making with Probabilistic Dual-Hesitant Pythagorean Fuzzy Information" section presents a new MAGDM approach based on PDHPFSs. The "Numerical Example" section conducts numerical experiments. The "Conclusions" section summarizes the paper and gives the future research directions.

Preliminaries

In this section, we briefly review some basic notions, which will be used in the following sections.

The Dual-Hesitant Pythagorean Fuzzy Set

Definition 1 [10] Let X be an ordinary set, then a dual hesitant Pythagorean fuzzy set (DHPFS) A defined on X is expressed as

$$A = \left\{ \langle x, h_A(x), g_A(x) \rangle \mid x \in X \right\}, \tag{1}$$

where $h_A(x)$ and $g_A(x)$ are two sets of some interval values, denoting the possible MDs and NMDs of the element $x \in X$ to the set A, such that.

$$0 \le \gamma, \eta \le 1, \ \left(\gamma^+\right)^2 + \left(\eta^+\right)^2 \le 1, \tag{2}$$

where $\gamma \in h_A(x), \eta \in g_A(x)$, $\gamma^+ = \cup_{\gamma \in h_A(x)} \max\{\gamma\}$ and $\eta^+ = \cup_{\eta \in g_A(x)} \max\{\eta\}$. For convenience, the ordered pair $\alpha_A(x) = \left(h_A(x), g_A(x)\right)$ is called a DHPFN, which can be denoted as $\alpha = (h, g)$ for simplicity.

To rank any two DHPFNs, Wei and Lu [10] proposed a comparison method for DHPFNs.

Definition 2 [10] Let $\alpha = (h, g)$ be a DHPFN, the score function of α is defined as

$$S(\alpha) = \frac{1}{2}\left(1 + \frac{1}{\#h}\sum\nolimits_{\gamma \in h} \gamma^2 - \frac{1}{\#g}\sum\nolimits_{\eta \in g} \eta^2\right), \tag{3}$$

and the accuracy function of α is expressed as

$$H(\alpha) = \frac{1}{\#h}\sum\nolimits_{\gamma \in h} \gamma^2 + \frac{1}{\#g}\sum\nolimits_{\eta \in g} \eta^2, \tag{4}$$

where $\#h$ and $\#g$ denote the numbers of values in h and g, respectively. For any two DHPFNs, $\alpha_1 = \left(h_1, g_1\right)$ and $\alpha_2 = \left(h_2, g_2\right)$,

1. If $S\left(\alpha_1\right) > S\left(\alpha_2\right)$, then $\alpha_1 > \alpha_2$;
2. If $S\left(\alpha_1\right) = S\left(\alpha_2\right)$, then

if $H\left(\alpha_1\right) > H\left(\alpha_2\right)$, then $\alpha_1 > \alpha_2$;
if $H\left(\alpha_1\right) = H\left(\alpha_2\right)$, then $\alpha_1 = \alpha_2$.

Wei and Lu [10] proposed basic operations rules of DHPFNs.

Definition 3 [10] Let $\alpha_1 = \left(h_1, g_1\right)$, $\alpha_2 = \left(h_2, g_2\right)$ and $\alpha = (h, g)$ be any three DHPFNs, and λ be a positive real number, then

1. $$\alpha_1 \oplus \alpha_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ \left(\gamma_1^2 + \gamma_2^2 - \gamma_1^2\gamma_2^2\right)^{1/2} \right\}, \left\{ \eta_1\eta_2 \right\} \right\};$$
2. $$\alpha_1 \otimes \alpha_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ \gamma_1\gamma_2 \right\}, \left\{ \left(\eta_1^2 + \eta_2^2 - \eta_1^2\eta_2^2\right)^{1/q} \right\} \right\};$$

3. $\lambda\alpha = \cup_{\gamma\in h,\eta\in g}\left\{\left\{\left(1-\left(1-\gamma^2\right)^{\lambda}\right)^{1/2}\right\},\left\{\eta^{\lambda}\right\}\right\}$;

4. $\alpha^{\lambda} = \cup_{\gamma\in h,\eta\in g}\left\{\left\{\eta^{\lambda}\right\},\left\{\left(1-\left(1-\gamma^2\right)^{\lambda}\right)^{1/2}\right\}\right\}$..

Power Average Operator and Hamy Mean

Definition 4 [27] Let $a_i(i = 1, 2, ..., n)$ be a collection of non-negative crisp numbers, then the power average (PA) operator is defined as

$$PA\left(a_1, a_2, ..., a_n\right) = \frac{\sum_{i=1}^{n}\left(1 + T\left(a_i\right)\right)a_i}{\sum_{i=1}^{n}\left(1 + T\left(a_i\right)\right)}, \tag{5}$$

where $T\left(a_i\right) = \sum_{j=1,i\neq j}^{n} Sup\left(a_i, a_j\right)$, $Sup\left(a_i, a_j\right)$ denotes the support for a_i from a_j, satisfying the conditions

1. $0 \le Sup\left(a_i, a_j\right) \le 1$
2. $Sup\left(a_i, a_j\right) = Sup\left(a_j, a_i\right)$;
3. $Sup(a, b) \le Sup(c, d)$, if $|a, b| \ge |c, d|$.

Definition 5 [28] Let $a_i(i = 1, 2, ..., n)$ be a collection of crisp numbers and $k = 1, 2, ..., n$. If

$$HM^{(k)}\left(a_1, a_2, ..., a_n\right) = \frac{1}{C_n^k}\sum_{1\le i_1<....<i_k\le n}\left(\prod_{j=1}^{k} a_{i_j}\right)^{1/k}, \tag{6}$$

then $HM^{(k)}$ is the Hamy mean (HM) operator, where $\left(i_1, i_2, ..., i_k\right)$ traverses all k-tuple combination of $(1, 2, ..., n)$, and C_n^k is the binomial coefficient.

The Probabilistic Dual-Hesitant Pythagorean Fuzzy Sets

In this section, we attempt to propose the concept of PDHPFS. To this end, we first present the motivations to explain why we need PDHPFSs. Then, the definition, operational rules, comparison method, and distance measure of PDHPFSs are presented.

The Motivation of Proposing the PDHPFSs

As discussed above, DHPFSs can effectively depict DMs' fuzzy evaluation values as well as their high hesitation. Nevertheless, it has been pointed out that DHPFSs are still insufficient to handle some practical decision-making situations and some important information is lost. We provide the following example to explain this drawback of DHPFSs.

Example 1 Suppose there are three professors, and they are required to evaluate a thesis of a student under the attribute novelty. In order to comprehensively capture the professors' evaluation information, they are permitted to provide multiple MDs and NMDs. The possible MDs and NMDs provided by the three decision experts are listed in Table 1.

Obviously, in the framework of DHPFSs, the overall evaluation value of provided the three professors can be denoted as d={{0.1, 0.2, 0.3, 0.4, 0.5, 0.6}, {0.1, 0.2, 0.3, 0.4}}, which is a DHPFN. However, it is noted that the multiple occurrences of MD 0.4 and 0.6 and the NMD 0.2, 0.3 and 0.4 are ignored, which indicates that there exits information loss and the DHPFN d fails to fully depict the overall evaluation value of the decision-making group. As a matter of fact, to more accurately depict DMs' evaluation values in hesitant fuzzy environment, not only MDs and NMDs, but also their corresponding probabilistic information should be taken into consideration. In other word, in Example 1, the multiple occurrence and appearance of the MDs 0.4 and 0.6 and the NMDs 0.2, 0.3, and 0.4 should be also accounted. Hence, motivated by PHFSs and PDHFSs, in the following we extend DHPFSs to PDHPFSs.

The Definition of PDHPFSs

Definition 6 Let X be a fixed set, a probabilistic dual hesitant Pythagorean fuzzy set (PDHPFS) D defined on X is expressed as

$$D = \{\langle x, h(x)|p(x), g(x)|t(x)\,|x \in X\rangle\}, \tag{7}$$

where $h_A(x)$ and $g_A(x)$ are two sets of values in the interval [0,1], denoting the possible MDs and NMDs of the element $x \in X$ to the set D. $p(x)$ and $t(x)$ are the probabilistic information for the MDs and NMDs, respectively. In addition,$h(x)$,$g(x)$,$p(x)$ and $t(x)$ satisfy the following conditions:

$$0 \le \gamma, \eta \le 1,\ \left(\gamma^+\right)^2 + \left(\eta^+\right)^2 \le 1,\ 0 \le p_i, t_j \le 1, \\ \sum_{i=1}^{\#h} p_i = 1,\ \sum_{j=1}^{\#g} t_j = 1 \tag{8}$$

where $\gamma \in h_A(x)$,$\eta \in g_A(x)$, $\gamma^+ = \cup_{\gamma\in h_A(x)}\max\{\gamma\}$, $\eta^+ = \cup_{\eta\in g_A(x)}\max\{\eta\}$,$p_i \in p(x)$, $t_j \in t(x)$, $\#h$ and $\#g$ denote the numbers of values in h and g, respectively. For convenience,

Table 1 The possible MDs and NMDs provided by the three professors in Example 1

	The possible MDs	The possible NMDs
The first professor	{0.3, 0.4, 0.5}	{0.2, 0.3}
The second professor	{0.2, 0.6}	{0.1, 0.2, 0.3, 0.4}
The third professor	{0.1, 0.4, 0.6}	{0.3, 0.4}

the ordered pair $d(x) = (h(x)|p(x), g(x)|t(x))$ is called a probabilistic dual hesitant Pythagorean fuzzy element (PDHPFE), which can be denoted as $d = (h|p, g|t)$ for simplification. Notably, we can find out that the proposed PDHPFS is an extension of DHPFS. If we take the corresponding probabilistic information of each element into account in the DHPFS, then the PDHPFS is obtained. In addition, as PDHPFS describes larger information space, PDHPFS can be regarded as a generalized form and PDHFS can be regarded as a special case of PDHPFS. This indicates that PDHPFSs are more powerful and flexible than PDHFS. Moreover, all probabilistic dual hesitant fuzzy elements (PDHFEs) are PDHPFEs and PDHFEs are a special case of PDHPFEs.

Example 2 If we use PDHPFE to depict the overall evaluation value of the three professors, then it should be denoted as $d=\{\{0.1 \mid 0.125, 0.2 \mid 0.125, 0.3 \mid 0.125, 0.4 \mid 0.250, 0.5 \mid 0.125, 0.6 \mid 0.250\}, \{0.1 \mid 0.125, 0.2 \mid 0.250, 0.3 \mid 0.375, 0.4 \mid 0.250\}\}$. Evidently, d is a PDHPFE. In this evaluation value d, different MDs and NMDs have different probabilities. In addition, the multiple appearance of MDs 0.4 and 0.6 and the NMDs 0.2, 0.3, and 0.4 are considered. The example reveals that PDHPFSs dig more information over DHPFSs, and it is more suitable to employ PDHPFSs to depict DMs' evaluation values in MAGDM procedure. Additionally, it is easy to notice that d is also a PDHFE, and this is because $0.6 + 0.4 \leq 1$. However, if the third professor uses the set of values $\{0.4, 0.5\}$ to denote the NMDs, then the overall evaluation value of group is $d' =\{\{0.1 \mid 0.125, 0.2 \mid 0.125, 0.3 \mid 0.125, 0.4 \mid 0.250, 0.5 \mid 0.125, 0.6 \mid 0.250\}, \{0.1 \mid 0.125, 0.2 \mid 0.250, 0.3 \mid 0.250, 0.4 \mid 0.250, 0.5 \mid 0.125\}\}$. Evidently, d' cannot be handled by PDHFEs as $0.5 + 0.6 \nleq 1$, which indicates that the evaluation value d' does not satisfy the constraint of PDHFSs and PDHFEs. However, according to the constraint of PDHPFS presented in Definition 6, evaluation value d' is still a PDHPFE, as $0.6^2 + 0.5^2 = 0.61 \leq 1$. This example indicates that PDHPFSs are more powerful and flexible than PDHFSs.

Motivated by the operations of DHPFNs and PDHFEs, we can derive the basic operational rules of PDHPFEs.

Definition 7 Let $d_1 = \left(h_1\middle|p_{h_1}, g_1\middle|t_{g_1}\right)$, $d_2 = \left(h_2\middle|p_{h_2}, g_2\middle|t_{g_2}\right)$ and $d = \left(h\middle|p_h, g\middle|t_g\right)$ be any three PDHPFEs, and λ be a possible real number, then

1. $$d_1 \oplus d_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2} \left\{\left\{\left(\gamma_1^2+\gamma_2^2-\gamma_2^2\gamma_2^2\right)^{1/2}\middle|p_{\gamma_1}p_{\gamma_2}\right\}, \left\{\eta_1\eta_2\middle|t_{\eta_1}t_{\eta_2}\right\}\right\};$$
2. $$d_1 \otimes d_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2} \left\{\left\{\gamma_1\gamma_2\middle|p_{\gamma_1}p_{\gamma_2}\right\}, \left\{\left(\eta_1^2+\eta_2^2-\eta_2^2\eta_2^2\right)^{1/2}\middle|t_{\eta_1}t_{\eta_2}\right\}\right\};$$
3. $$\lambda d = \cup_{\gamma\in h,\eta\in g}\left\{\left\{\left(1-\left(1-\gamma^2\right)^{\lambda}\right)^{1/2}\middle|p_\gamma\right\}, \left\{\eta^\lambda\middle|t_\eta\right\}\right\};$$
4. $$d^\lambda = \cup_{\gamma\in h,\eta\in g}\left\{\left\{\gamma^\lambda\middle|p_\gamma\right\}, \left\{\left(1-\left(1-\eta^2\right)^{\lambda}\right)^{1/2}\middle|t_\eta\right\}\right\}.$$

Example 3 Let $d_1 = \{\{0.3|0.5, 0.6|0.5\}, \{0.1|0.5, 0.2|0.2, 0.6|0.3\}\}$, $d_2 = \{\{0.2|0.3, 0.4|0.3, 0.5|0.4\}, \{0.5|0.7, 0.6|0.3\}\}$ be two PDHPFEs, then

1. $$d_1 \oplus d_2 = \left\{\begin{array}{l}\{0.3555|0.15, 0.4854|0.15, 0.5635|0.2, 0.621|0.15, 0.68|0.15, 0.7211|0.2\} \\ \{0.05|0.35, 0.06|0.15, 0.1|0.14, 0.12|0.06, 0.3|0.21, 0.36|0.09\}\end{array}\right\};$$
2. $$d_1 \otimes d_2 = \left\{\begin{array}{l}\{0.06|0.15, 0.12|0.3, 0.15|0.2, 0.24|0.15, 0.3|0.2\} \\ \{0.5074|0.35, 0.6053|0.15, 0.5292|0.14, 0.621|0.06, 0.7211|0.21, 0.7684|0.09\}\end{array}\right\};$$
3. $$3d_1 = \{\{0.4964|0.5, 0.859|0.5\}, \{0.001|0.5, 0.008|0.2, 0.216|0.3\}\};$$
4. $$d_2^3 = \{\{0.008|0.3, 0.064|0.3, 0.125|0.4\}, \{0.7603|0.7, 0.859|0.3\}\}.$$

To rank any two PDHPFEs, we propose a comparison method.

Definition 8 Let $d = \left(h\middle|p_h, g\middle|t_g\right)$ be a PDHPFE, then the score function of d is expressed as

$$S(d) = \sum_{i=1,\gamma\in h}^{\#h} \gamma_i p_i - \sum_{j=1,\eta\in g}^{\#g} \eta_j t_j, \tag{9}$$

And the accuracy function of d is defined as

$$H(d) = \sum_{i=1,\gamma\in h}^{\#h} \gamma_i p_i + \sum_{j=1,\eta\in g}^{\#g} \eta_j t_j, \tag{10}$$

where $\#h$ and $\#g$ denote the numbers of values in h and g, respectively.

Let $d_1 = \left(h_1\middle|p_{h_1}, g_1\middle|t_{g_1}\right)$ and $d_2 = \left(h_2\middle|p_{h_2}, g_2\middle|t_{g_2}\right)$ be any two PDHPFEs,

1. If $S(d_1) > S(d_2)$, then $d_1 > d_2$;
2. If $S(d_1) = S(d_2)$, then

if $H(d_1) > H(d_2)$, then $d_1 > d_2$;
if $H(d_1) = H(d_2)$, then $d_1 = d_2$.

Example 4 Let $d_1 = \{\{0.3|0.5, 0.6|0.5\}, \{0.1|0.5, 0.2|0.2, 0.6|0.3\}\}$, $d_2 = \{\{0.2|0.3, 0.4|0.3, 0.5|0.4\}, \{0.5|0.7, 0.6|0.3\}\}$ be two PDHPFEs, then we have

$S(d_1) = 0.18,\ H(d_1) = 0.72,\ S(d_2) = -0.15,\ H(d_2) = = 0.91.$

According of Definition 8, we can get $d_1 > d_2$.

Distance Between Two PDHPFEs

Definition 9 Let $d_1 = \left(h_1\middle|p_{h_1}, g_1\middle|t_{g_1}\right)$ and $d_2 = \left(h_2\middle|p_{h_2}, g_2\middle|t_{g_2}\right)$ be two PDHPFEs, then the distance between d_1 and d_2 is defined as

$$dis(d_1, d_2) = \frac{1}{\#h + \#g}\left(\sum_{i=1}^{\#h}\left|\left(\gamma_1^{\sigma(i)}\right)^2 p_{\gamma_1^{\sigma(i)}} - \left(\gamma_2^{\sigma(i)}\right)^2 p_{\gamma_2^{\sigma(i)}}\right| + \sum_{j=1}^{\#g}\left|\left(\eta_1^{\sigma(j)}\right)^2 t_{\eta_1^{\sigma(j)}} - \left(\eta_2^{\sigma(j)}\right)^2 t_{\eta_2^{\sigma(j)}}\right|\right), \tag{11}$$

where $\gamma_1^{\sigma(i)} \in h_1, \eta_1^{\sigma(j)} \in g_1, \gamma_2^{\sigma(i)} \in h_2, \eta_2^{\sigma(j)} \in g_2, \gamma_1^{\sigma(i)} < \gamma_1^{\sigma(i+1)}, \eta_1^{\sigma(j)} < \eta_1^{\sigma(j+1)}, \gamma_2^{\sigma(i)} < \gamma_2^{\sigma(i+1)}$ and $\eta_2^{\sigma(j)} < \eta_2^{\sigma(j+1)}$. The symbol $\#h$ denotes the number of values in h_1 and h_2, and $\#g$ represents the number of values in g_1 and g_2.

Remark 1 From Definition 9, we can find out that when calculating the distance between two PDHPFEs, they must have the same numbers of MDs and NMDs. However, this requirement cannot be always met. Hence, to operate correctly, the shorter PDHPFEs should be extended by adding some values until the numbers of the MDs and NMDs of the two PDHPFEs are equal. In the following, we present a principle to extend the short PDHPFEs. Let d_1 and d_2 be any two PDHPFEs, which can be expressed as

$$d_1 = (h_1, g_1) = \left\{\left\{\gamma_1^{\sigma(1)}\middle|p_{\gamma_1^{\sigma(1)}}, \gamma_1^{\sigma(2)}\middle|p_{\gamma_1^{\sigma(2)}}, \ldots, \gamma_1^{\sigma(\#h_1)}\middle|p_{\gamma_1^{\sigma(\#h_1)}}\right\}, \left\{\eta_1^{\sigma(1)}\middle|t_{\eta_1^{\sigma(1)}}, \eta_1^{\sigma(2)}\middle|t_{\eta_1^{\sigma(2)}}, \ldots, \eta_1^{\sigma(\#g_1)}\middle|t_{\eta_1^{\sigma(\#g_1)}}\right\}\right\}, \tag{12}$$

and

$$d_2 = (h_2, g_2) = \left\{\left\{\gamma_2^{\sigma(1)}\middle|p_{\gamma_2^{\sigma(1)}}, \gamma_2^{\sigma(2)}\middle|p_{\lambda_2^{\sigma(2)}}, \ldots, \gamma_2^{\sigma(\#h_2)}\middle|p_{\gamma_2^{\sigma(\#h_2)}}\right\}, \left\{\eta_2^{\sigma(1)}\middle|t_{\eta_2^{\sigma(1)}}, \eta_2^{\sigma(2)}\middle|t_{\eta_2^{\sigma(2)}}, \ldots, \eta_2^{\sigma(\#g_1)}\middle|t_{\eta_2^{\sigma(\#g_1)}}\right\}\right\}, \tag{13}$$

If $\#h_1 < \#h_2$ and $\#g_2 < \#g_1$, then we have two methods to extend d_1 and d_2. First, we assume DMs are optimistic to their evaluations, then we can extend d_1 and d_2 to.

$$d_1' = (h_1', g_1') = \left\{\begin{array}{l}\left\{\begin{array}{l}\gamma_1^{\sigma(1)}\middle|p_{\gamma_1^{\sigma(1)}}, \gamma_1^{\sigma(2)}\middle|p_{\gamma_1^{\sigma(2)}}, \ldots, \gamma_1^{\sigma(\#h_1)}\middle|p_{\frac{\gamma_1^{\sigma(\#h_1)}}{\#h_2-\#h_1+1}}, \\ \gamma_1^{\sigma(\#h_1)}\middle|p_{\frac{\gamma_1^{\sigma(\#h_1)}}{\#h_2-\#h_1+1}}, \ldots, \gamma_1^{\sigma(\#h_1)}\middle|p_{\frac{\gamma_1^{\sigma(\#h_1)}}{\#h_2-\#h_1+1}}\end{array}\right\}, \\ \left\{\eta_1^{\sigma(1)}\middle|t_{\eta_1^{\sigma(1)}}, \eta_1^{\sigma(2)}\middle|t_{\eta_1^{\sigma(2)}}, \ldots, \eta_1^{\sigma(\#g_1)}\middle|t_{\eta_1^{\sigma(\#g_1)}}\right\}\end{array}\right\}, \tag{14}$$

and

$$d_2' = (h_2', g_2') = \left\{\begin{array}{l}\left\{\gamma_2^{\sigma(1)}\middle|p_{\gamma_2^{\sigma(1)}}, \gamma_2^{\sigma(2)}\middle|p_{\gamma_2^{\sigma(2)}}, \ldots, \gamma_2^{\sigma(\#h_2)}\middle|p_{\gamma_2^{\sigma(\#h_2)}}\right\}, \\ \left\{\begin{array}{l}\eta_2^{\sigma(1)}\middle|t_{\eta_2^{\sigma(1)}}, \eta_2^{\sigma(2)}\middle|t_{\eta_2^{\sigma(2)}}, \ldots, \eta_2^{\sigma(\#g_1)}\middle|t_{\frac{\eta_2^{\sigma(\#g_1)}}{\#g_1-\#g_2+1}}, \\ \eta_2^{\sigma(\#g_1)}\middle|t_{\frac{\eta_2^{\sigma(\#g_1)}}{\#g_1-\#g_2+1}} \ldots, \eta_2^{\sigma(\#g_1)}\middle|t_{\frac{\eta_2^{\sigma(\#g_1)}}{\#g_1-\#g_2+1}}\end{array}\right\}\end{array}\right\}, \tag{15}$$

respectively, where $\#h_1' = \#h_2' = \#h_2$ and $\#g_1' = \#g_2' = \#g_1$. If DMs are pessimistic to their evaluations, then we can extend d_1 and d_2 to

$$d_1' = (h_1', g_1') = \left\{\begin{array}{l}\left\{\begin{array}{l}\gamma_1^{\sigma(1)}\middle|p_{\frac{\gamma_1^{\sigma(1)}}{\#h_2-\#h_1+1}}, \gamma_1^{\sigma(1)}\middle|p_{\frac{\gamma_1^{\sigma(1)}}{\#h_2-\#h_1+1}}, \ldots., \mu_1^{\sigma(1)}\middle|p_{\frac{\gamma_1^{\sigma(1)}}{\#h_2-\#h_1+1}}, \\ \gamma_1^{\sigma(2)}\middle|p_{\gamma_1^{\sigma(2)}}, \ldots, \gamma_1^{\sigma(\#h_1)}\middle|p_{\frac{\gamma_1^{\sigma(\#h_1)}}{\#h_2-\#h_1+1}}\end{array}\right\}, \\ \left\{\eta_1^{\sigma(1)}\middle|t_{\eta_1^{\sigma(1)}}, \eta_1^{\sigma(2)}\middle|t_{\eta_1^{\sigma(2)}}, \ldots, \eta_1^{\sigma(\#g_1)}\middle|t_{\eta_1^{\sigma(\#g_1)}}\right\}\end{array}\right\}, \tag{16}$$

and

$$d_2' = (h_2', g_2') = \left\{\begin{array}{l}\left\{\gamma_2^{\sigma(1)}\middle|p_{\gamma_2^{\sigma(1)}}, \gamma_2^{\sigma(2)}\middle|p_{\gamma_2^{\sigma(2)}}, \ldots, \gamma_2^{\sigma(\#h_2)}\middle|p_{\gamma_2^{\sigma(\#h_2)}}\right\}, \\ \left\{\begin{array}{l}\eta_2^{\sigma(1)}\middle|t_{\frac{\eta_2^{\sigma(1)}}{\#g_1-\#g_2+1}}, \mu_2^{\sigma(1)}\middle|t_{\frac{\eta_2^{\sigma(1)}}{\#g_1-\#g_2+1}}, \ldots, \eta_2^{\sigma(1)}\middle|t_{\frac{\eta_2^{\sigma(1)}}{\#g_1-\#g_2+1}}, \\ \eta_2^{\sigma(2)}\middle|t_{v_2^{\sigma(2)}}, \ldots, \eta_2^{\sigma(\#g_1)}\middle|t_{\frac{\eta_2^{\sigma(\#g_1)}}{\#g_1-\#g_2+1}}\end{array}\right\}\end{array}\right\}, \tag{17}$$

respectively, where $\#h_1' = \#h_2' = \#h_2$ and $\#g_1' = \#g_2' = \#g_1$. In this paper, we assume DMs are optimistic to their evaluation values. To better demonstrate the distance between two PDHPFEs, we provide the following example. Let $d_1 = \{\{0.3|0.4, 0.6|0.6\}, \{0.1|0.2, 0.2|0.3, 0.5|0.5\}\}$ and $d_2 = \{\{0.5|0.3, 0.6|0.2, 0.7|0.1, 0.8|0.4\}, \{0.1|0.2, 0.3|0.8\}\}$ be two PDHPFEs, when calculating the distance between d_1 and d_2, we should extend d_1 and d_2 to

$d_1{}' = \{\{0.3|0.4, 0.6|0.2, 0.6|0.2, 0.6|0.2\}, \{0.1|0.2, 0.2|0.3, 0.5|0.5\}\}$

$d_2{}' = \{\{0.5|0.3, 0.6|0.2, 0.7|0.1, 0.8|0.4\}, \{0.1|0.2, 0.3|0.4, 0.3|0.4\}\}$

Then

$$d(d_1, d_2) = \frac{1}{4+3}\begin{pmatrix} |0.3^2 \times 0.4 - 0.5^2 \times 0.3| + |0.6^2 \times 0.2 - 0.6^2 \times 0.2| + \\ |0.6^2 \times 0.2 - 0.7^2 \times 0.1| + |0.6^2 \times 0.2 - 0.8^2 \times 0.4| + \\ |0.1^2 \times 0.2 - 0.1^2 \times 0.2| + |0.2^2 \times 0.3 - 0.3^2 \times 0.4| + \\ |0.5^2 \times 0.5 - 0.3^2 \times 0.4| \end{pmatrix} = 0.0513.$$

Probabilistic Dual-Hesitant Pythagorean Fuzzy Aggregation Operators

In this section, we propose some combined AOs for PDHPFEs based on PHM. Properties and special cases of the proposed AOs are also studied in this section.

The Probabilistic Dual-Hesitant Pythagorean Fuzzy Power Hamy Mean Operator

Definition 10 Let $d_i = \left(h_i\middle|p_{h_i}, g_i\middle|t_{g_i}\right)(i = 1, 2, ..., n)$ be a collection of PDHPFEs and $k = 1, 2, ..., n$. The probabilistic dual-hesitant Pythagorean fuzzy power Hamy mean (PDHPFPHM) operator is expressed as

$$PDHPFPHM^{(k)}(d_1, d_2, ..., d_n) = \frac{1}{C_n^k} \underset{1 \le i_1 < .... < i_k \le n}{\oplus} \left(\overset{k}{\underset{j=1}{\otimes}} \left(\frac{n\left(1 + T\left(d_{i_j}\right)\right) d_{i_j}}{\sum_{s=1}^n \left(1 + T(d_s)\right)} \right) \right)^{1/k}, \tag{18}$$

where $(i_1, i_2, ..., i_k)$ traverses all k-tuple combination of $(1, 2, ..., n)$, and C_n^k is the binomial coefficient, where $T(d_i) = \sum_{j=1, i \ne j}^{n} Sup(d_i, d_j)$, $Sup(d_i, d_j)$ denotes the support for d_i from d_j, satisfying the following conditions

1. $0 \le Sup(d_i, d_j) \le 1$
2. $Sup(d_i, d_j) = Sup(d_j, d_i)$;
3. $Sup(a, b) \le Sup(c, d)$, if $|a, b| \ge |c, d|$.

If we assume

$$\xi_i = \frac{1 + T(d_i)}{\sum_{s=1}^n \left(1 + T(d_s)\right)} \tag{19}$$

then Eq. (18) can be transformed into

$$PDHPFPHM^{(k)}(d_1, d_2, ..., d_n) = \frac{1}{C_n^k} \underset{1 \le i_1 < .... < i_k \le n}{\oplus} \left(\overset{k}{\underset{j=1}{\otimes}} \left(n\xi_{i_j} d_{i_j} \right) \right)^{1/k} \tag{20}$$

Theorem 1 *Let $d_i = \left(h_i\middle|p_{h_i}, g_i\middle|t_{g_i}\right)(i = 1, 2, ..., n)$ be a collection of PDHPFEs and $k = 1, 2, ..., n$, the aggregated value by the PDHPFPHM operator is still a PDHPFE and*

$$PDHPFPHM^{(k)}(d_1, d_2, ..., d_n) = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \left(1 - \prod_{1 \le i_1 < .... < i_k \le n} \left(1 - \prod_{j=1}^k \left(1 - \left(1 - \gamma_{i_j}^2\right)^{n\xi_{i_j}} \right)^{1/k} \right)^{\frac{1}{C_n^k}} \right)^{1/2} \middle| \prod_{1 \le i_1 < .... < i_k \le n} \prod_{j=1}^k p_{\gamma_{i_j}} \right\}, \left\{ \prod_{1 \le i_1 < .... < i_k \le n} \left(1 - \prod_{j=1}^k \left(1 - \eta_{i_j}^{2n\xi_{i_j}} \right)^{1/k} \right)^{\frac{1}{2C_n^k}} \middle| \prod_{1 \le i_1 < .... < i_k \le n} \prod_{j=1}^k t_{\eta_{i_j}} \right\} \right\}. \tag{21}$$

Proof According to Definition 7 and Eq. (20), we can obtain

$$n\xi_{i_j} d_{i_j} = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \left(1 - \left(1 - \gamma_{i_j}^2\right)^{n\xi_{i_j}} \right)^{1/2} \middle| p_{\gamma_{i_j}} \right\}, \left\{ \eta_{i_j}^{n\xi_{i_j}} \middle| t_{\eta_{i_j}} \right\} \right\}$$

and

$$\overset{k}{\underset{j=1}{\otimes}}\left(n\xi_{i_j}d_{i_j}\right)=\cup_{\gamma_{i_j}\in h_{i_j},\eta_{i_j}\in g_{i_j}}\left\{\left\{\prod_{j=1}^{k}\left(1-\left(1-\gamma_{i_j}^2\right)^{n\xi_{i_j}}\right)^{1/2}\middle|\prod_{j=1}^{k}p_{\gamma_{i_j}}\right\},\left\{\left(1-\prod_{j=1}^{k}\left(1-\eta_{i_j}^{2n\xi_{i_j}}\right)\right)^{1/2}\middle|\prod_{j=1}^{k}t_{\eta_{i_j}}\right\}\right\}.$$

Then,

$$\left(\overset{k}{\underset{j=1}{\otimes}}\left(n\xi_{i_j}d_{i_j}\right)\right)^{1/k}=\cup_{\gamma_{i_j}\in h_{i_j},\eta_{i_j}\in g_{i_j}}\left\{\left\{\prod_{j=1}^{k}\left(1-\left(1-\gamma_{i_j}^2\right)^{n\xi_{i_j}}\right)^{1/2k}\middle|\prod_{j=1}^{k}p_{\gamma_{i_j}}\right\},\left\{\left(1-\left(\prod_{j=1}^{k}\left(1-\eta_{i_j}^{2n\xi_{i_j}}\right)\right)^{1/k}\right)^{1/2}\middle|\prod_{j=1}^{k}t_{\eta_{i_j}}\right\}\right\}.$$

Therefore,

$$\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(n\xi_{i_j}d_{i_j}\right)\right)^{1/k}=\cup_{\gamma_{i_j}\in h_{i_j},\eta_{i_j}\in g_{i_j}}\left\{\left\{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\gamma_{i_j}^2\right)^{n\xi_{i_j}}\right)^{1/k}\right)\right)^{1/2}\middle|\prod_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k}p_{\gamma_{i_j}}\right\},\right.$$
$$\left.\left\{\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\left(\prod_{j=1}^{k}\left(1-\eta_{i_j}^{2n\xi_{i_j}}\right)\right)^{1/k}\right)^{1/2}\middle|\prod_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k}t_{\eta_{i_j}}\right\}\right\}.$$

Thus,

$$\frac{1}{C_n^k}\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(n\xi_{i_j}d_{i_j}\right)\right)^{1/k}=\cup_{\gamma_{i_j}\in h_{i_j},\eta_{i_j}\in g_{i_j}}\left\{\left\{\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\gamma_{i_j}^2\right)^{n\xi_{i_j}}\right)^{1/k}\right)^{\frac{1}{C_n^k}}\right)^{1/2}\middle|\prod_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k}p_{\gamma_{i_j}}\right\},\right.$$
$$\left.\left\{\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\eta_{i_j}^{2n\xi_{i_j}}\right)^{1/k}\right)^{\frac{1}{2C_n^k}}\middle|\prod_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k}t_{\eta_{i_j}}\right\}\right\}.$$

Theorem 2 *Let* $d_i=\left(h_i\middle|p_{h_i},g_i\middle|t_{g_i}\right)(i=1,2,\ldots,n)$ *be a collection of PDHPFEs and* $k=1,2,\ldots,n$ *if* $d_i=d=\left(h_i\middle|p_{h_i},g_i\middle|t_{g_i}\right)$ *for* $i=1,2,\ldots,n$, *then*

$$PDHPFPHM^{(k)}\left(d_1,d_2,\ldots,d_n\right)=d. \tag{22}$$

Proof Since $d_i=d=\left(h\middle|p_h,g\middle|t_g\right)$ for any i, we can get $Sup\left(d_i,d_j\right)=1$ for $i,j=1,2,\ldots,n$. Thus, $T\left(d_i\right)=\sum_{j=1,i\ne j}^{n}Sup\left(d_i,d_j\right)=n-1$, therefore, $\xi_i=1/n(i=1,2,\ldots,n)$ holds for all $j=1,2,\ldots,n$. Therefore,

$$
\begin{aligned}
&PDHPFPHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) \\
&= \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \left(1 - \prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \gamma^{2/k} \right)^{\frac{1}{C_n^k}} \right)^{1/2} \middle| \prod_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} p_\gamma \right\}, \right. \\
&\qquad \left. \left\{ \prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1-\eta^2\right)^{1/k} \right)^{\frac{1}{2C_n^k}} \middle| \prod_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} t_\eta \right\} \right\} \\
&= \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \left(\gamma^2\right)^{1/2} \middle| \prod_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} p_\gamma \right\}, \left\{ \left(\eta^2\right)^{1/2} \middle| \prod_{1 \le i_1 < \ldots < i_k \le n} \prod_{j=1}^{k} t_\eta \right\} \right\} \\
&= \left(\gamma \middle| p_h, \eta \middle| t_g \right) = d.
\end{aligned}
$$

Now, we can discuss some special cases of the PDHPFPHM operator with respect to the parameter k.

Case 1 When $k = 1$, then the PDHPFPHM operator reduces to the probabilistic dual-hesitant Pythagorean fuzzy power average (PDHPFPA) operator, i.e.,

$$
\begin{aligned}
&PDHPFPHM^{(1)}\left(d_1, d_2, \ldots, d_n\right) \\
&= \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \left(1 - \prod_{i=1}^{n} \left(1-\gamma_i^2\right)^{\xi_i} \right)^{1/2} \middle| \prod_{i=1}^{n} p_{\gamma_i} \right\}, \left\{ \prod_{i=1}^{n} \left(\eta_i^{\xi_i}\right) \middle| \prod_{i=1}^{n} t_{\eta_i} \right\} \right\} \\
&= \bigoplus_{i=1}^{n} \xi_i d_i = PDHPFPA\left(d_1, d_2, \ldots, d_n\right).
\end{aligned} \tag{23}
$$

Case 2 When $k = n$, then the PDHPFPHM operator reduces to the probabilistic dual-hesitant Pythagorean fuzzy power geometric (PDHPFPG) operator, i.e.,

$$
\begin{aligned}
&PDHPFPHM^{(n)}\left(d_1, d_2, \ldots, d_n\right) \\
&= \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \prod_{i=1}^{n} \left(1 - \left(1-\gamma_i^2\right)^{\xi_i} \right)^{1/2} \middle| \prod_{i=1}^{n} p_{\gamma_i} \right\}, \right. \\
&\qquad \left. \left\{ \left(1 - \prod_{i=1}^{n} \left(1-\eta_i^{2n\xi_i}\right)^{1/n} \right)^{1/2} \middle| \prod_{i=1}^{n} t_{\eta_i} \right\} \right\} \\
&= \bigotimes_{i=1}^{n} d_i^{\xi_i} = DHPFPG\left(d_1, d_2, \ldots, d_n\right).
\end{aligned} \tag{24}
$$

The Probabilistic Dual-Hesitant Pythagorean Fuzzy Power Weighted Hamy Mean Operator

Definition 11 Let $d_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right) (i = 1, 2, \ldots, n)$ be a collection of PDHPFEs, $k = 1, 2, \ldots, n$ and $w = \left(w_1, w_2, \ldots, w_n\right)^T$ be the weight vector, such that $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$ The probabilistic dual-hesitant Pythagorean fuzzy power weight Hamy mean (PDHPFPWHM) operator can be defined as.

$$
PDHPFPWHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \frac{1}{C_n^k} \bigoplus_{1 \le i_1 < \ldots < i_k \le n} \left(\bigotimes_{j=1}^{k} \left(\frac{n w_{i_j} \left(1 + T\left(d_{i_j}\right)\right) d_{i_j}}{\sum_{s=1}^{n} w_s \left(1 + T\left(d_s\right)\right)} \right) \right)^{1/k}, \tag{25}
$$

where $T\left(d_j\right) = \sum_{i=1, i \ne j}^{n} Sup\left(d_i, d_j\right)$, $\left(i_1, i_2, \ldots, i_k\right)$ traverses all k-tuple combination of $(1, 2, \ldots, n)$, and C_n^k is the binomial coefficient. $Sup\left(d_i, d_j\right)$ denotes the support for d_i from d_j, satisfying the properties presented in Definition 10. If we assume

$$
\sigma_i = \frac{w_i \left(1 + T\left(d_i\right)\right)}{\sum_{s=1}^{n} w_s \left(1 + T\left(d_s\right)\right)}, \tag{26}
$$

then Eq. (25) can be transformed into

$$
PDHPFPWHM^{(k)}\left(d_1, d_2, \ldots, d_n\right) = \frac{1}{C_n^k} \bigoplus_{1 \le i_1 < \ldots < i_k \le n} \left(\bigotimes_{j=1}^{k} \left(n \sigma_{i_j} d_{i_j} \right) \right)^{1/k}. \tag{27}
$$

Theorem 3 *Let* $d_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, ..., n)$ *be a collection of PDHPFEs and* $k = 1, 2, ..., n$, *the aggregated value by the PDHPFPWHM operator is still a PDHPFE and*

$$\begin{aligned} & PDHPFPWHM^{(k)}\left(d_1, d_2, ..., d_n\right) \\ & = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \left(1 - \prod_{1 \le i_1 < .... < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \gamma_{i_j}^2 \right)^{n\sigma_{i_j}} \right)^{1/k} \right)^{\frac{1}{C_n^k}} \right)^{1/2} \middle| \prod_{1 \le i_1 < .... < i_k \le n} \prod_{j=1}^{k} p_{\gamma_{i_j}} \right\}, \right. \\ & \left. \left\{ \prod_{1 \le i_1 < .... < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \eta_{i_j}^{2n\sigma_{i_j}} \right)^{1/k} \right)^{\frac{1}{2C_n^k}} \middle| \prod_{1 \le i_1 < .... < i_k \le n} \prod_{j=1}^{k} t_{\eta_{i_j}} \right\} \right\} \end{aligned} \tag{28}$$

Theorem 4 *Let* $d_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, ..., n)$ *be a collection of PDHPFEs and* $k = 1, 2, ..., n$ *if* $d_i = d = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)$ *for* $i = 1, 2, ..., n$, *then*

$$PDHPFPWHM^{(k)}\left(d_1, d_2, ..., d_n\right) = d \tag{29}$$

The proof of Theorem 4 is similar to that of Theorem 2, which is omitted here. In the following, we discuss some special cases of the PDHPFPWHM operator with respect to the parameter k.

Case 3 When $k = 1$, then the PDHPFPWHM operator reduces to probabilistic dual hesitant Pythagorean fuzzy power weighted average (PDHPFPWA) operator, i.e.,

$$\begin{aligned} PDHPFPWHM^{(1)}\left(d_1, d_2, ..., d_n\right) &= \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \left(1 - \prod_{i=1}^{n} \left(1 - \gamma_i^2 \right)^{\sigma_i} \right)^{1/2} \middle| \prod_{i=1}^{n} p_{\gamma_i} \right\}, \left\{ \prod_{i=1}^{n} \eta_i^{\sigma_i} \middle| \prod_{i=1}^{n} t_{\eta_i} \right\} \right\} \\ &= \mathop{\oplus}_{i=1}^{n} \sigma_i d_i = PDHPFPWA\left(d_1, d_2, ..., d_n\right). \end{aligned} \tag{30}$$

Case 4 When $k = n$, then PDHPFPWHM operator reduces to probabilistic dual-hesitant Pythagorean fuzzy power weighted geometric (PDHPFPWG) operator, i.e.,

$$\begin{aligned} & PDHPFPWHM^{(n)}\left(d_1, d_2, ..., d_n\right) \\ & = \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \prod_{i=1}^{n} \left(1 - \left(1 - \gamma_i^2 \right)^{\sigma_i} \right)^{1/2} \middle| \prod_{i=1}^{n} p_{\gamma_i} \right\}, \left\{ \left(1 - \prod_{i=1}^{n} \left(1 - \eta_i^{2n\sigma_i} \right)^{1/n} \right)^{1/2} \middle| \prod_{i=1}^{n} t_{\eta_i} \right\} \right\} \\ & = \mathop{\otimes}_{i=1}^{n} d_i^{\sigma_i} = DHPFPWG\left(d_1, d_2, ..., d_n\right). \end{aligned} \tag{31}$$

An Approach to Multiple Attribute Decision Making with Probabilistic Dual-Hesitant Pythagorean Fuzzy Information

In this section, we utilize the proposed aggregation operators to handle MAGDM problems with probabilistic dual-hesitant Pythagorean fuzzy information. Let $A = \left\{A_1, A_2, ..., A_m\right\}$ be m alternatives, and $C = \left\{C_1, C_2, ..., C_n\right\}$ be n attributes, $w = \left(w_1, w_2, ..., w_n\right)^T$ be the weight vector, satisfying $\sum_{j=1}^{n} w_j = 1$ and $0 \le w_j \le 1$. If the DMs provide several values for the alternative A_i under the attribute C_j with anonymity and each value has the precise probabilistic information based on some rules, these values can be considered as a PDHPFE $d_{ij} = \left(h_{ij} \middle| p_{h_{ij}}, g_{ij} \middle| t_{g_{ij}}\right)$. Suppose that the decision matrix $D = \left(d_{ij}\right)_{m \times n}$ is the probabilistic dual hesitant Pythagorean fuzzy decision matrix, where $d_{ij} (i = 1, 2, ..., m, j = 1, 2, ..., n)$ are in the form of PDHPFEs. In the following, we apply the PDHPFPWHM operator to the MAGDM problems for potential evaluation.

Step 1. Standardize the decision matrices according to the following equation,

$$d_{ij} = \begin{cases} \left(h_{ij}\middle|p_{h_{ij}}, g_{ij}\middle|t_{g_{ij}}\right) & C_J \in I_1 \\ \left(g_{ij}\middle|t_{g_{ij}}, h_{ij}\middle|p_{h_{ij}}\right) & C_J \in I_2 \end{cases}, \tag{32}$$

where I_1 and I_2 represent the positive type and the negative type respectively.

Step 2. Calculate the support $Sup\left(d_{ik}, d_{ir}\right)$ according to the following equation,

$$Sup\left(d_{ik}, d_{id}\right) = 1 - d\left(d_{ik}, d_{id}\right), \tag{33}$$

where $k, r = 1, 2, ..., n; k \neq r$ and $d\left(d_{ik}, d_{ir}\right)$ is the distance between d_{ik} and d_{ir},which is proposed in Definition 9.

Step 3. Calculate $T\left(d_{ij}\right)$ by

$$T\left(d_{ij}\right) = \sum_{k=1,k\neq r}^{n} Sup\left(d_{ik}, d_{ir}\right), \tag{34}$$

where $k, r = 1, 2, ..., n; i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Step 4. Compute the power weights δ_{ij} associated with the PDHPFE d_{ij} by

$$\delta_{ij} = \frac{w_j\left(1 + T\left(d_{ij}\right)\right)}{\sum_{s=1}^{n} w_s\left(1 + T\left(d_{is}\right)\right)}, \tag{35}$$

where $i = 1, 2, ..., m; j = 1, 2, ..., n; \delta_{ij} > 0$ and $\sum_{j=1}^{n} \delta_{ij} = 1$.

Step 5. For alternative $A_i (i = 1, 2, ..., m)$, utilize the PDHPFPWHM operator

$$d_i = PDHPFPWHM^{(k)}\left(d_{i1}, d_{i2}, ..., d_{in}\right), \tag{36}$$

to aggregate attributes, and an overall evaluation value can be obtained.

Step 6. Rank the overall evaluation values $d_i (i = 1, 2, ..., m)$ according to Definition 8.

Step 7. Rank alternatives according to the rank of the overall values, and choose the best alternative.

To better illustrate the steps of our MAGDM method, we provide the following flowchart (see Fig. 1).

Numerical Example

Example 5 We illustrate our novel method by using an example, which demonstrates how to assign limited extravascular membrane oxygenation (ECMO) to patients with acute respiratory distress syndrome (ARDS), under uncertain medical situations.

ECMO is a first-aid device using extracorporeal respiratory circulation to assist patients with severe cardiopulmonary failure. It works by pumping blood from a vein, passing it through the membrane of the lung to oxygenate hemoglobin and removing carbon dioxide, and then transferring the blood back to the patient after the gas exchange [29]. ARDS is an acute and diffuse inflammatory lung injury which is one of the common respiratory critical diseases threatening human health. The mortality of ARDS is greater than 40% which remains high [30]. Approximately 80% of all deaths in adult ARDS patients occur within 2–3 weeks after the onset of the syndrome [31]. The characteristics of ARDS are the acute onset of hypoxemia and bilateral pulmonary infiltrates, which are consistent with pulmonary edema. These characteristics are in line with the indications for ECMO. Some patients can use ECMO for adjuvant therapy to improve the cure rate in some situations. However, ECMO is costly which means that the number of such devices in a normal hospital is small [29]. Therefore, it is not available for every patient with ARDS to use ECMO equipment. In addition, this treatment method will cause irreversible damage to patients' lungs, and some serious complications will occur in the use process [30]. Therefore, reasonable arrangement of using ECMO equipment will be conducive to improving the cure rate and saving lives. The real clinical environment is always very complex. The vital signs are different from patient to patient, and conventional treatment for them may be different. It's difficult for doctors to determine which patient is more suitable for ECMO treatment. In this paper, we proposed a method based on probabilistic dual-hesitant Pythagorean fuzzy aggregation operators which has a powerful ability to deal with the information with high degree of ambiguity and uncertainty.

Suppose that there are four patients $A_i (i = 1, 2, 3, 4)$ with ARDS in the same ICU, and all of them show four conditions $C_j (j = 1, 2, 3, 4)$ of different degrees: cardio-pulmonary function (C_1); hepatorenal function (C_2); complication risk (C_3), and total vital signs (C_4). The weighted vector of attributes is $w = (0.32, 0.26, 0.18, 0.24)^T$, satisfying $\sum_{j=1}^{n} w_j = 1$ and $0 \leq w_j \leq 1$. In order to evaluate the most suitable patient, the doctor performing ECMO is invited to evaluate the situation of four patients from four conditions respectively and the decision matrices $D = \left(d_{ij}\right)_{4\times4}$ is shown in Table 2.

The Decision-Making Process

In this subsection, we use the method introduced in the "Probabilistic Dual-Hesitant Pythagorean Fuzzy Aggregation Operators" section to determine the optimal alternative. The decision-making steps are presented as follows.

Step 1. As the attributes C_3 (complication risk) is negative, we need to standardize the decision matrices according to Eq. (32). Then, the normalized matrix D_2 can be obtained, which is shown in Table 3.

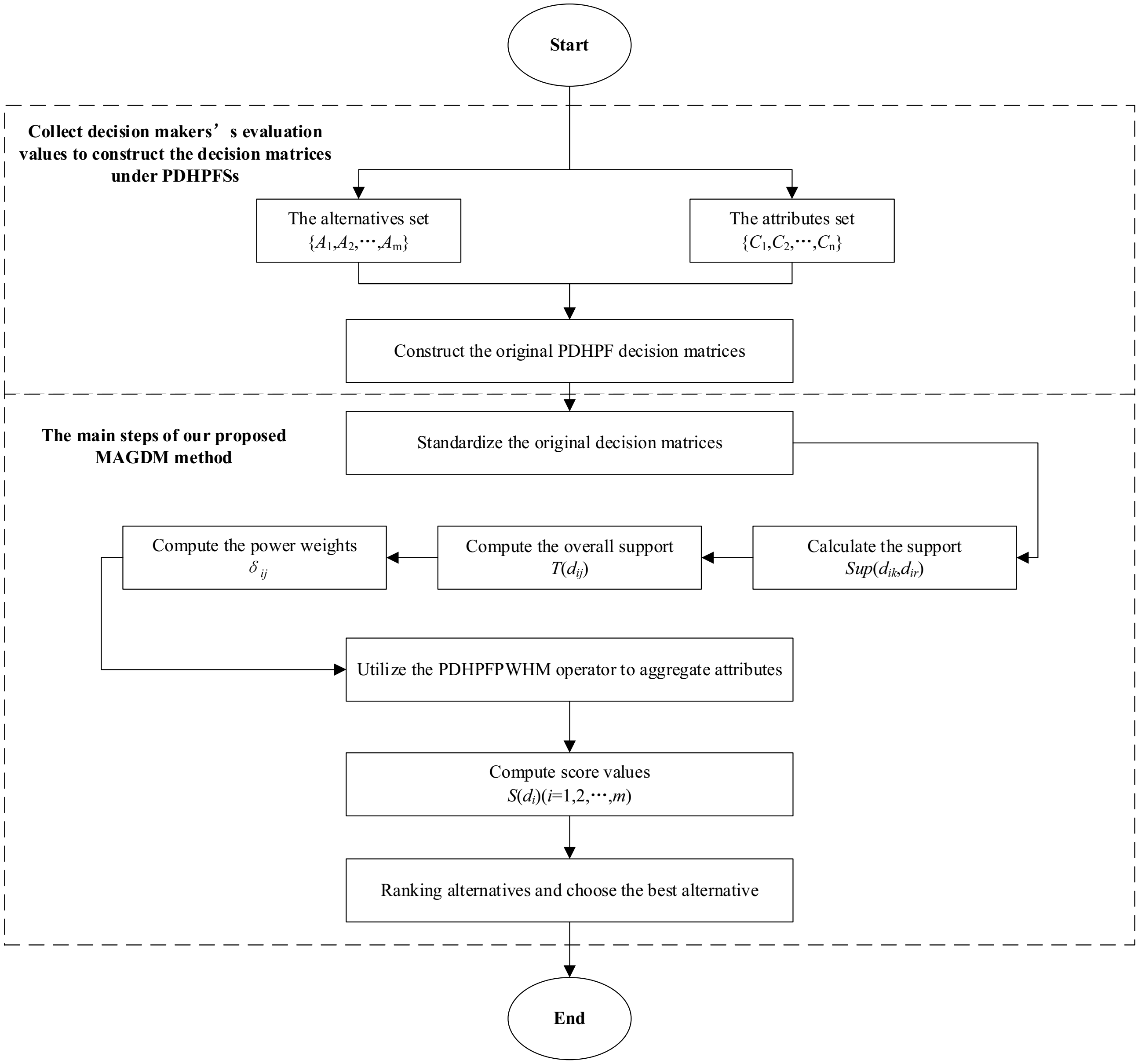

Fig. 1 The flowchart of our proposed MAGDM method

Step 2. Calculate the support $Sup\left(d_{ik}, d_{ir}\right)$ according to Eq. (33). For convenience, we utilize the symbol S^{kr} to represent the support between d_{ik} and d_{ir} satisfying $k, r = 1, 2, 3, 4$ and $k \neq r$. Hence, we obtain the following results.

$$S^{12} = S^{21} = (0.9606, 0.7950, 0.9356, 0.9734),$$
$$S^{13} = S^{31} = (0.9213, 0.8083, 0.6413, 0.7154)$$

$$S^{14} = S^{41} = (0.9578, 0.9250, 0.8968, 0.9546),$$
$$S^{23} = S^{32} = (0.7638, 0.7250, 0.6232, 0.7856)$$

$$S^{24} = S^{42} = (0.9692, 0.8000, 0.8848, 0.9800),$$
$$S^{34} = S^{43} = (0.8550, 0.8583, 0.8640, 0.5313)$$

Step 3. Calculate the support $T\left(d_{ij}\right)$ according to Eq. (34). For convenience, we use the symbol T_{ij} to represent the values $T\left(d_{ij}\right)(i,j = 1, 2, 3, 4)$, and we can obtain the following matrix:

$$T = \begin{bmatrix} 2.8396 & 2.6935 & 2.5400 & 2.7819 \\ 2.5283 & 2.3200 & 2.3917 & 2.5833 \\ 2.4736 & 2.4436 & 2.1285 & 2.6456 \\ 2.6434 & 2.7391 & 2.0323 & 2.4658 \end{bmatrix}$$

Table 2 The probabilistic dual hesitant Pythagorean fuzzy decision matrix D given by doctor

	C_1	C_2
A_1	$\{\{0.7\|0.3, 0.6\|0.3, 0.5\|0.4\}, \{0.2\|1\}\}$	$\{\{0.7\|1\}, \{0.25\|1\}\}$
A_2	$\{\{0.1\|1\}, \{0.4\|1\}\}$	$\{\{0.3\|1\}, \{0.7\|1\}\}$
A_3	$\{\{0.6\|1\}, \{0.35\|1\}\}$	$\{\{0.56\|1\}, \{0.2\|1\}\}$
A_4	$\{\{0.05\|0.7, 0.2\|0.3\}, \{0.5\|1\}\}$	$\{\{0.3\|0.5, 0.2\|0.5\}, \{0.6\|0.5, 0.5\|0.5\}\}$
	C_3	C_4
A_1	$\{\{0.2\|1\}, \{0.2\|1\}\}$	$\{\{0.7\|0.5, 0.6\|0.5\}, \{0.3\|1\}\}$
A_2	$\{\{0.3\|0.5, 0.2\|0.5\}, \{0.7\|1\}\}$	$\{\{0.3\|1\}, \{0.3\|1\}\}$
A_3	$\{\{0.7\|1\}, \{0.1\|1\}\}$	$\{\{0.2\|0.6, 0.4\|0.4\}, \{0.4\|1\}\}$
A_4	$\{\{0.15\|1\}, \{0.8\|1\}\}$	$\{\{0.2\|1\}, \{0.6\|1\}\}$

Step 4. Calculate the power weight δ_{ij} associated with the PDHPFS d_{ij} according to Eq. (35), and we have

$$\delta=\begin{bmatrix} 0.3291 & 0.2572 & 0.1707 & 0.2431 \\ 0.3261 & 0.2493 & 0.1763 & 0.2484 \\ 0.3227 & 0.2599 & 0.1635 & 0.2540 \\ 0.3316 & 0.2765 & 0.1553 & 0.2366 \end{bmatrix}$$

Step 5. For the patient $A_i(i = 1, 2, 3, 4)$, utilize the PDHPFPWHM operator to calculate the overall evaluation $d_i(i = 1, 2, 3, 4)$. Without the loss of generality, let the parameter $k = 3$, and the overall evaluation values (only the first elements of each membership and non-membership degrees is shown) are shown as follows:

$$d_1 = \{\{0.5414|0.034, \ldots\}, \{0.2636|1\}\},$$
$$d_2 = \{\{0.2945|1\}, \{0.4698|0.1250, \ldots\}\}$$

$$d_3 = \{\{0.3068|0.2160, \ldots\}, \{0.4822|1\}\},$$
$$d_4 = \{\{0.2447|0.0429, \ldots\}, \{0.4941|0.1250, \ldots\}\}$$

Step 6. Calculate the score values $S(d_i)(i = 1, 2, 3, 4)$ of the overall evaluation values, and we can get

$$S(d_1) = 0.2400,\ S(d_2) = -0.1630,$$
$$S(d_3) = -0.1490,\ S(d_4) = -0.2231$$

Step 7. According to the score values, the ranking orders of the alternatives can be determined, that is $A_1 \succ A_3 \succ A_2 \succ A_4$. Therefore, A_1 is the most suitable patient to use ECMO equipment.

Analysis of the Influence of the Parameter *k*

It is worth pointing out that the parameter k has great impact on the decision results. In the following, we calculate the score functions of alternatives with different values of the parameter k. The results are shown in Table 4 and depicted in Fig. 2.

From Fig. 2, we can observe that the parameter k has a certain impact on the final comprehensive score value and ranking orders. The score values of alternatives obtained by the PDHPFPWHM operator become smaller with the increase of the parameter k. However, the best alternative is always A_1 and the worst alternative is always A_4. For $k = 1, 2, 3$, although the score values are different, we can get the same ranking of the four alternatives. On the contrary, for $k = 4$, A_2 is considered as the suboptimal solution, whereas the third best alternative is identified for other three values of parameter k. This is because the method ($k = 4$) considers the interrelationship among the four attributes. This also illustrates that our method is very flexible and practical and can deal with MAGDM problems where the

Table 3 The normalized decision matrix D_2

	C_1	C_2
A_1	$\{\{0.7\|0.3, 0.6\|0.3, 0.5\|0.4\}, \{0.2\|1\}\}$	$\{\{0.7\|1\}, \{0.25\|1\}\}$
A_2	$\{\{0.1\|1\}, \{0.4\|1\}\}$	$\{\{0.3\|1\}, \{0.7\|1\}\}$
A_3	$\{\{0.6\|1\}, \{0.35\|1\}\}$	$\{\{0.56\|1\}, \{0.2\|1\}\}$
A_4	$\{\{0.05\|0.7, 0.2\|0.3\}, \{0.5\|1\}\}$	$\{\{0.3\|0.5, 0.2\|0.5\}, \{0.6\|0.5, 0.5\|0.5\}\}$
	C_3	C_4
A_1	$\{\{0.2\|1\}, \{0.2\|1\}\}$	$\{\{0.7\|0.5, 0.6\|0.5\}, \{0.3\|1\}\}$
A_2	$\{\{0.7\|1\}, \{0.3\|0.5, 0.2\|0.5\}\}$	$\{\{0.3\|1\}, \{0.3\|1\}\}$
A_3	$\{\{0.1\|1\}, \{0.7\|1\}\}$	$\{\{0.2\|0.6, 0.4\|0.4\}, \{0.4\|1\}\}$
A_4	$\{\{0.8\|1\}, \{0.15\|1\}\}$	$\{\{0.2\|1\}, \{0.6\|1\}\}$

Table 4 Scores and ranking orders of Example 5 with different k in the PDHPFPWHM operator

k	Score function $S(\alpha_i)(k=1,2,3,4)$	Ranking orders
$k=1$	$S(\alpha_1)=0.3742$ $S(\alpha_2)=0.0013$ $S(\alpha_3)=0.1335$ $S(\alpha_4)=-0.0268$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$k=2$	$S(\alpha_1)=0.2889$ $S(\alpha_2)=-0.1139$ $S(\alpha_3)=-0.0417$ $S(\alpha_4)=-0.1811$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$k=3$	$S(\alpha_1)=0.2400$ $S(\alpha_2)=-0.1630$ $S(\alpha_3)=-0.1490$ $S(\alpha_4)=-0.2231$	$A_1 \succ A_3 \succ A_2 \succ A_4$
$k=4$	$S(\alpha_1)=0.2046$ $S(\alpha_2)=-0.1933$ $S(\alpha_3)=-0.2188$ $S(\alpha_4)=-0.2464$	$A_1 \succ A_2 \succ A_3 \succ A_4$

interrelationship exists among all attributes. In practice, we can assign different values to k according to actual needs.

Validity Test

In this subsection, we try to show the validity of our proposed method. In order to do this, we use our method and that developed by Hao et al. [25] based on the probabilistic dual-hesitant fuzzy weighted average (PDHFWA) operator, and that proposed by Garg and Kaur [32] based on the probabilistic dual-hesitant fuzzy weighted Einstein average (PDHFWEA) operator to solve Example 5 and compare their decision results. The results derived by the three methods are presented in Table 5. It is noted that the final ranking result of alternatives derived by our developed method is same as those obtained by Hao et al.'s [25] and Garg and Kaur's [32] decision-making methods, which implies the validity of our method.

Advantages and Superiorities of Our Method

To better illustrate the advantages and superiorities of our proposed method, we utilize our proposed method and some existing MAGDM methods to solve some practical decision-making problems and conduct comparative analysis.

It Can Express DMs' Evaluations Flexibly

As an extension of the classical PDHFSs, the PDHPFSs also satisfy the constraint that the square sum of MD and NMD is less than or equal to one. Evidently, the constraint of PDHPFSs is laxer than that of PDHFSs. Consequently, PDHPFSs can deal with some cases that PDHFSs are powerless to handle. In other word, our proposed method gives DMs more freedom to express their evaluation information. In order to better explain this advantage, we provide the following example.

Example 6 In realistic MAGDM problems, DMs provide their evaluation values according to their own cognition. Hence, sometimes DMs may provide evaluation values where the sum of MD and NMD is greater than one. To better demonstrate this situation, we replace the evaluation value of C_4 of alternative A_1 with $\{\{0.7|0.5, 0.6|0.5\}, \{0.7|1\}\}$ in Example 5. We use our proposed method and Hao et al.'s [25] and Garg's methods [32] to deal with Example 6 and the decision results are presented in Table 6.

As seen in Table 6, the methods introduced by Hao et al. [25] and Garg and Kaur [32] fails to handle Example 6, whereas our

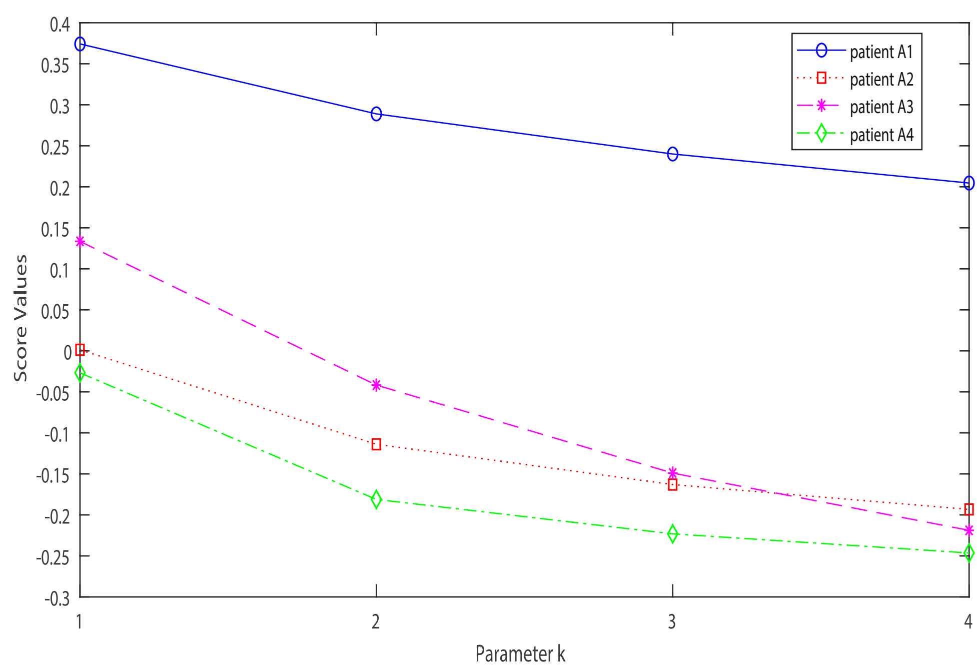

Fig. 2 The score values of alternatives when $k \in [1, 4]$

Table 5 The decision results of Example 5 by different methods

Method	Score function	Ranking orders
Hao et al.'s [25] method based on PDHFWA operator [25]	$S(d_1) = 0.3586$ $S(d_2) = -0.0469$ $S(d_3) = 0.1006$ $S(d_4) = -0.0677$	$A_1 \succ A_3 \succ A_2 \succ A_4$
Garg and Kaur's [32] method based on PDHFWEA operator	$S(d_1) = 0.3496$ $S(d_2) = -0.0737$ $S(d_3) = 0.0800$ $S(d_4) = -0.1115$	$A_1 \succ A_3 \succ A_2 \succ A_4$
The proposed method based on the PDHPFPWHM operator when $k = 1$	$S(d_1) = 0.3742$ $S(d_2) = 0.0013$ $S(d_3) = 0.1335$ $S(d_4) = -0.0268$	$A_1 \succ A_3 \succ A_2 \succ A_4$

method produces the ranking results $A_1 \succ A_3 \succ A_2 \succ A_4$. The reason is that the evaluation values α_{14} is cannot be handled by PDHFSs, as $0.7 + 0.7 = 1.4 > 1$. In addition, we notice $0.7^2 + 0.7^2 = 0.98 < 1$, and hence our proposed method can still handle Example 6. Therefore, our method is more flexible and can handle more complicated MAGDM problems.

It Can Consider the Interrelationship Among Attributes

In most realistic MAGDM problems, it is widely realized that attributes are usually interactive. In other word, when considering to compute the overall evaluation values, not only the attribute values, but also the interrelationship between attributes should be taken into account. The HM operator is a powerful tool to deal with the MAGDM problems because it can capture the interrelationship among the multi-input arguments. Therefore, our proposed method also considers the interrelationship among attributes. In most real MAGDM problems, there are some interrelationship among attributes. Our proposed method can solve this problem by allowing the DMs to set the value of the parameter k. In order to better demonstrate this feature, we give the following example.

Example 7 Assuming that all attributes are dependent in Example 5, then we utilize some existing MAGDM methods and the method we proposed in this paper to solve this example and present the result in Table 7.

It is seen from Table 7 that Hao et al.'s [25] and Garg and Kaur's [32] MAGDM methods produce the same ranking order, i.e., $A_1 \succ A_3 \succ A_2 \succ A_4$. In addition, when $k = 1$, our proposed method also obtains the same ranking result. This is because Hao et al.'s [25] method and Garg and Kaur's [32] approach are based on the simple weighted average operator, which do not reflect the interrelationship among attributes. Similarly, when $k = 1$, our proposed method does not consider the interrelationship among attributes, either. Moreover, we can find out that when $k = 4$, our method produces different ranking results as Hao et al.'s [25] and Garg and Kaur's [32] MAGDM methods do. This is because when $k = 4$, our method takes the interrelationship among all the four attributes into consideration. In addition, we should point out that when $k = 2$ and 3, our method captures the interrelationship among any two and any three attributes, respectively. In realistic MAGDM problems, there exits interrelationship among attributes and usually such interrelationship is usually complicated. Hence, compared with Hao et al.'s [25] and Garg and Kaur's [32] methods, our proposed method is more powerful and flexible to solve actual MAGDM problems.

It Effectively Handles Extreme Evaluation Values

In modern decision-making situations, DMs usually have different background and preference and they sometime provide prejudiced evaluation values. In addition, as practical MAGDM problems are becoming more and more complicated, DMs can hardly get all information related the decision-making problems and they probably provide unduly high or low evaluation values. It is realized that the negative influence of such kind of extreme evaluation values should be eliminated in order to obtain fair final decision results.

Table 6 The decision results on Example 6 by different methods

Method	Score function	Ranking orders
Hao et al.'s [25] method based on PDHFWA operator	Cannot be calculated	None
Garg and Kaur's [32] method based on PDHFWEA operator	Cannot be calculated	None
The proposed method based on the PDHPFPWHM operator when $k = 1$	$S(d_1) = 0.3243$ $S(d_2) = -0.0260$ $S(d_3) = 0.1335$ $S(d_4) = -0.0268$	$A_1 \succ A_3 \succ A_2 \succ A_4$

Table 7 The decision results of Example 6 by different methods

Method	Score function	Ranking orders
Hao et al.'s [25] method based on PDHFWA operator	$S(d_1) = 0.3586$ $S(d_2) = -0.0469$ $S(d_3) = 0.1006$ $S(d_4) = -0.0677$	$A_1 \succ A_3 \succ A_2 \succ A_4$
Garg and Kaur's [32] method based on PDHFWEA operator	$S(d_1) = 0.3496$ $S(d_2) = -0.0737$ $S(d_3) = 0.0800$ $S(d_4) = -0.1115$	$A_1 \succ A_3 \succ A_2 \succ A_4$
The proposed method based on the PDHPFPWHM operator when $k = 1$	$S(d_1) = 0.3742$ $S(d_2) = 0.0013$ $S(d_3) = 0.1335$ $S(d_4) = -0.0268$	$A_1 \succ A_3 \succ A_2 \succ A_4$
The proposed method based on the PDHPFPWHM operator when $k = 4$	$S(\alpha_1) = 0.2046$ $S(\alpha_2) = -0.1933$ $S(\alpha_3) = -0.2188$ $S(\alpha_4) = -0.2464$	$A_1 \succ A_2 \succ A_3 \succ A_4$

Table 8 The decision results on Example 7 by different methods

Method	Score function	Ranking orders
Hao et al.'s [25] method based on PDHFWA operator	$S(d_1) = 0.3586$ $S(d_2) = -0.0912$ $S(d_3) = 0.1006$ $S(d_4) = -0.0677$	$A_1 \succ A_3 \succ A_4 \succ A_2$
Garg and Kaur's [32] method based on PDHFWEA operator	$S(d_1) = 0.3496$ $S(d_2) = -0.1258$ $S(d_3) = 0.0800$ $S(d_4) = -0.1115$	$A_1 \succ A_3 \succ A_4 \succ A_2$
The proposed method based on the PDHPFPWHM operator when $k = 1$	$S(d_1) = 0.3742$ $S(d_2) = -0.0260$ $S(d_3) = 0.1335$ $S(d_4) = -0.0268$	$A_1 \succ A_3 \succ A_2 \succ A_4$

In other word, if such kind of bad influence of unreasonable evaluations is not reduced or eliminated, the results may be not reliable and reasonable. Our method is based the PA operator, and hence, it has the ability of reducing the bad influence of DMs' extreme evaluation values on the final decision results, making the ranking order of alternatives more reliable. In order to demonstrate such advantage, we provide the following example.

Example 8 Doctors need to make a quick decision about whether to use ECMO equipment because their patients with ARDS disease may deteriorate rapidly. But different doctors focus on different issues, some of them may not be able to give reasonable evaluation values in a short period of time. Therefore, some evaluation values may be higher or lower than normal levels. Assume the doctor gives a low evaluation value $\{\{0.179|1\}, \{0.821|1\}\}$ of the patient A_2 under the attributes C_2. The other evaluation values are the same as decision matrices shown in Table 2. We utilize different methods to solve Example 8 and present the results in Table 8.

From Table 8, we can find that the ranking result obtained by our method based on PDHPFPWHM operator has not changed, which is $A_1 \succ A_3 \succ A_2 \succ A_4$. Whereas Hao et al.'s [25] method based on PDHFWA operator and Garg and Kaur's method [32] based on PDHFWEA operator produce a different ranking result, i.e., $A_1 \succ A_3 \succ A_4 \succ A_2$. The reason is that the PDHFWA and PDHFWEA operator only analyze the whole data simply but not avoid the unreasonable influence. However, our proposed approach based on PDHPFPWHM is constructed on the PA operator, which can reduce the impact of unreasonable data on ranking results by assigning smaller weights to data that are too high or too low. Therefore, our proposed method can eliminate the influence of unreasonable data on the results and is more reasonable and powerful than other methods.

Conclusions

The traditional DHPFSs, which absorb the advantages of DHFSs and PFSs, have the ability of describing fuzzy decision-making information in MAGDM procedure. However, DHPFSs still have limitations or drawbacks when depicting fuzzy decision-making information, i.e., they do not consider the possibilistic information of possible MDs and NMDs. This paper presented the PDHPFSs, which take corresponding probabilistic values into consideration in DHPFSs. For the sake of usage of PDHPFSs in MAGDM, we further put forward a collection of probabilistic dual hesitant fuzzy AOs. As seen in numerical examples, comparative analysis demonstrated the advantages and superiorities of our proposed AOs. Our paper contributed a novel MAGDM method and provided DMs a new manner to determine the best alternative.

In further works, we will continue our study from three aspects. First, we will study applications of our proposed method in realistic MAGDM problems. Second, we shall study more AOs for PDHPFEs and propose some new

MAGDM methods. Third, we are considering to extend DHPFSs to an improved form. For example, considering DMs would like to use interval numbers instead of crisp numbers to present probabilistic information, we shall extend PDHPFSs to interval-valued PDHPFSs and continue to investigate MAGDM method based on interval-valued PDHPFSs.

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Declarations

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of Interest The authors declare that they have no conflict of interest.

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