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Interval-Valued Intuitionistic Fuzzy Power Bonferroni Aggregation Operators and Their Application to Group Decision Making

Peide Liu[1] · **Honggang Li**[1]

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Abstract The power Bonferroni mean (PBM) operator can take the advantages of power operator and Bonferroni mean operator, which can overcome the influence of the unreasonable attribute values and can also consider the interaction between two attributes. However, it cannot be used to process the interval-valued intuitionistic fuzzy numbers (IVIFNs). It is importantly meaningful to extend the PBM operator to IVIFNs. We extend PBM operator to process IVIFNs and propose some new PBM operators for IVIFNs and apply them to solve the multi-attribute group decision-making (MAGDM) problems. Firstly, the definition, properties, score function, and operational rules of IVIFNs are introduced briefly. Then, the power Bonferroni mean (IVIFPBM) operator, the weighted PBM (IVIFWPBM) operator, the power geometric BM (IVIFPGBM) operator, and the weighted power geometric BM (IVIFWPGBM) operator for IVIFNs are proposed. Furthermore, some deserved properties of them are explored, and several special cases are analyzed. The decision-making methods are developed to deal with the MAGDM problems with the information of the IVIFNs based on the proposed operators, and by an illustrative example, the proposed methods are verified, and their advantages are explained by comparing with the other methods. The proposed methods can effectively solve the MAGDM problems with the IVIFNs, and they can consider the interaction between two attributes and overcome the influence of the unreasonable attribute values.

Keywords Power Bonferroni aggregation operators · Interval-valued intuitionistic fuzzy numbers · MAGDM

✉ Peide Liu
peide.liu@gmail.com

[1] School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan, Shandong 250014, China

Introduction

Fuzzy set (FS) theory, firstly proposed by Zadeh [1], has been a hot research topic. Further, in order to express some types of fuzzy information, Atanassov [2, 3] presented the intuitionistic fuzzy set (IFS) by adding a non-membership function based on FS. Furthermore, Atanassov [4] and Atanassov and Gargov [5] extended the IFS to interval-valued IFS (IVIFS) in which the membership and non-membership degrees are described by interval numbers. Then, some operational laws and relations of IVIFS were defined. Liu [6] and Zhang [7] presented some information entropy for IVIFS. Based on the prospect theory, Wang [8] proposed a new score function to overcome the weakness of not comparing two interval-valued intuitionistic fuzzy numbers (IVIFNs). Many researchers also developed some similarity measurements of IVIFS [9–11] to compare two IVIFNs. In addition, Tan and Zhang [12] developed an extended TOPSIS method on the basis of IVIFNs to solve the MADM problems. Hashemi et al. [13] proposed the extended ELECTRE III method for IVIFNs. Wang and Xu [14] provided a fractional programming method to solve the IVIF-MADM problems.

The aggregation operators are a powerful method for the MAGDM problems [15–23]. Particularly, the information aggregation operators on the basis of IVIFS have attracted more and more attentions [19, 24–32]. Yager [33] firstly proposed the power average (PA) operator, which could eliminate the influence of unreasonable data from the biased decision makers. Further, Xu and Yager [34] developed power geometric operator. Bonferroni [35] introduced Bonferroni mean (BM) operator, which could capture the interrelationships of two arguments. Zhu [36] proposed the geometric Bonferroni mean by combining BM and geometric mean operators. He [37, 38] introduced the interaction of BM operator for intuitionistic fuzzy information. To consider the advantages of PA and BM operators together, He et al. [39–41] proposed some power Bonferroni mean (PBM) operators by combining the PA operator and BM operator.

The PBM operator can take the advantages of PA and BM operators. However, up to now, there is no research on how to use PBM operator to aggregate the IVIFNs, so the goal and motivation of this study are to extend the PBM operator to IVIFNs and to propose some MAGDM methods for IVIFNs.

For that, the structure of this paper is shown as follows. In the "Preliminaries" section, we introduce the definition of the IVIFNs, the PBM, and PGBM operators in brief. In the "Some interval-valued intuitionistic fuzzy PBM operators" section, we combine the IVIFNs with PBM aggregation operators and develop some new operators to aggregate the IVIFNs. In the "The MAGDM approach based on IVIFWPBM and IVIFWPGBM operators" section, on the basis of these operators, an effective method is developed for MAGDM problems with the IVIFNs. The "An application example" section presented an application example to verify the feasibility of the novel developed method. In the "Conclusion" section, some concluding remarks are given.

Preliminaries

Interval-Valued Intuitionistic Fuzzy Set

Definition 1 [2]. Let $Z = \{z_1, z_2, \cdots, z_n\}$ be a fixed set, then an IFS named A in Z is expressed as

$$A = \{< z, u_A(z), v_A(z) > | z \in Z\} \tag{1}$$

where $0 \le u_A(z) \le 1$, $0 \le v_A(z) \le 1$ and $0 \le u_A(z) + v_A(z) \le 1$. $u_A(z)$ and $v_A(z)$ represent membership and non-membership degrees of the element z to A, respectively.

In addition, suppose $\pi(z) = 1 - u_A(z) - v_A(z)$, then $\pi(z)$ is named the hesitancy degree of z to A [2, 3]. It is apparent that $0 \le \pi(z) \le 1$ for $\forall z \in Z$.

To element $z \in Z$ from IFS A, the pair $(u_A(z), v_A(z))$ represents an intuitionistic fuzzy value (IFV). For convenience, it can be denoted as $\tilde{a} = (u_{\tilde{a}}, v_{\tilde{a}})$, satisfying that $0 \le u_{\tilde{a}} \le 1$, $0 \le v_{\tilde{a}} \le 1$ and $0 \le u_{\tilde{a}} + v_{\tilde{a}} \le 1$.

Definition 2 [3, 4]. Let $Z = \{z_1, z_2, \cdots, z_n\}$ be a fixed set, and then an IVIFS AL is expressed as

$$AL = \{< z, u_{AL}(z), v_{AL}(z) > | z \in Z\} \tag{2}$$

where the interval numbers $u_{AL}(z) \subseteq [0, 1]$ and $v_{AL}(z) \subseteq [0, 1]$ satisfies $0 \le \sup(u_{AL}(z)) + \sup(v_{AL}(z)) \le 1$. $u_{AL}(z)$ and $v_{AL}(z)$ represent the membership and non-membership degrees of the element z to AL respectively. For simplicity, $al = ([\mu a, \mu b], [vc, vd])$ is called an IVIFN.

Definition 3 [42]. Suppose $al_1 = ([\mu a_1, \mu b_1], [vc_1, vd_1])$ and $al_2 = ([\mu a_2, \mu b_2], [vc_2, vd_2])$ are two IVIFNs, then the Euclidean distance between them is defined as follows:

$$d(al_1, al_2) = \sqrt{\frac{1}{4}\left((\mu a_1 - \mu a_2)^2 + (\mu b_1 - \mu b_2)^2 + (vc_1 - vc_2)^2 + (vd_1 - vd_2)^2\right)} \tag{3}$$

Definition 4 [43]. Suppose $al_1 = ([\mu a_1, \mu b_1], [vc_1, vd_1])$ and $al_2 = ([\mu a_2, \mu b_2], [vc_2, vd_2])$ are two IVIFNs, then the operational laws can be expressed as follows:

$$al_1 \otimes al_2 = ([\mu a_1 \mu a_2, \mu b_1 \mu b_2], [vc_1 + vc_2 - vc_1 vc_2, vd_1 + vd_2 - vd_1 vd_2]), \tag{4}$$

$$al_1 \oplus al_2 = ([\mu a_1 + \mu a_2 - \mu a_1 \mu a_2, \mu b_1 + \mu b_2 - \mu b_1 \mu b_2], [vc_1 vc_2, vd_1 vd_2]), \tag{5}$$

$$n \cdot al_1 = \left([1-(1-\mu a_1)^n, 1-(1-\mu b_1)^n], [vc_1^n, vd_1^n]\right) \quad n > 0, \tag{6}$$

$$al_1^n = \left([\mu a_1^n, \mu b_1^n], [1-(1-vc_1)^n, 1-(1-vd_1)^n]\right) \quad n > 0. \tag{7}$$

Example 1. Suppose $al_1 = ([0.1, 0.3], [0.4, 0.5])$ and $al_2 = ([0.2, 0.4], [0.3, 0.5])$ are two IVIFNs, and $n = 2$, then on the basis of Definition 4, we can get

$al_1 \oplus al_2 = ([0.1 + 0.2 - 0.1 \times 0.2, 0.3 + 0.4 - 0.3 \times 0.4], [0.4 \times 0.3, 0.5 \times 0.5]) = ([0.28, 0.58], [0.12, 0.25])$, $al_1 \otimes al_2 = ([0.1 \times 0.2, 0.3 \times 0.4], [0.4 + 0.3 - 0.4 \times 0.3, 0.5 + 0.5 - 0.5 \times 0.5]) = ([0.02, 0.12], [0.58, 0.75])$, $n \cdot al_1 = 2al_1 = ([1-(1-0.1)^2, 1-(1-0.3)^2], [0.4^2, 0.5^2]) = ([0.19, 0.51], [0.16, 0.25])$, $al_1^n = al_1^2 = ([0.1^2, 0.3^2], [1-(1-0.4)^2, 1-(1-0.5)^2]) = ([0.01, 0.09], [0.64, 0.75])$.

Theorem 1 [43]. Suppose $al_1 = ([\mu a_1, \mu b_1], [vc_1, vd_1])$ and $al_2 = ([\mu a_2, \mu b_2], [vc_2, vd_2])$ are two IVIFNs, then

$$(1)\ al_1 \oplus al_2 = al_2 \oplus al_1 \tag{8}$$

$$(2)\ al_1 \otimes al_2 = al_2 \otimes al_1 \tag{9}$$

$$(3)\ \eta(al_1 \oplus al_2) = \eta \cdot al_1 \oplus \eta \cdot al_2, \eta \ge 0 \tag{10}$$

$$(4)\ \eta \cdot al_1 \oplus \eta_2 \cdot al_1 = (\eta_1 + \eta_2) al_1, \eta_1, \eta_2 \ge 0 \tag{11}$$

$$(5)\ al_1^{\eta_1} \otimes al_1^{\eta_2} = (al_1)^{\eta_1 + \eta_2}, \eta_1, \eta_2 \ge 0 \tag{12}$$

$$(6)\ al_1^{\eta} \otimes al_2^{\eta} = (al_1 \otimes al_2)^{\eta} \tag{13}$$

Definition 5 [44]. Supposing $al_i = ([\mu a_i, \mu b_i], [vc_i, vd_i])$ is an IVIFN, we can define the score function sf of al_i as follows:

$$sf(al_i) = \frac{\mu a_i + \mu b_i - vc_i - vd_i}{2} \tag{14}$$

Definition 6 [44]. Supposing $al_i = ([\mu a_i, \mu b_i], [vc_i, vd_i])$ is an IVIFN, we can define the accuracy function af of the IVIFN al_i as follows:

$$af(al_i) = \frac{\mu a_i + \mu b_i + vc_i + vd_i}{2} \tag{15}$$

Definition 7 [44]. If $al_1 = ([\mu a_1, \mu b_1], [vc_1, vd_1])$ and $al_2 = ([\mu a_2, \mu b_2], [vc_2, vd_2])$ are two IVIFNs, we can get

(1) If $sf(al_1) > sf(al_2)$, then $al_1 > al_2$;

(2) If $sf(al_1) = sf(al_2)$, then.

If $af(al_1) > af(al_2)$, then $al_1 > al_2$;

If $af(al_1) = af(al_2)$, then $al_1 = al_2$.

Example 2. Supposing $al_1 = ([0.4, 0.5], [0.2, 0.3])$ and $al_2 = ([0.2, 0.5], [0.1, 0.3])$ are two IVIFNs, then based on the Definition 7, we can get the following results:

$$s(al_1) = \frac{0.4 + 0.5 - 0.2 - 0.3}{2} = 0.2, \quad s(al_2) = \frac{0.2 + 0.5 - 0.1 - 0.3}{2} = 0.15.$$

Because $sf(al_1) > sf(al_2)$, we can get $al_1 > al_2$.

If $al_1 = ([0.4, 0.5], [0.2, 0.3])$ and $al_2 = ([0.2, 0.5], [0.1, 0.2])$, then we can get

$$sf(al_1) = \frac{0.4 + 0.5 - 0.2 - 0.3}{2} = 0.2, \quad sf(al_2) = \frac{0.2 + 0.5 - 0.1 - 0.2}{2} = 0.2;$$
$$af(al_1) = \frac{0.4 + 0.5 + 0.2 + 0.3}{2} = 0.7, \quad af(al_2) = \frac{0.2 + 0.5 + 0.1 + 0.2}{2} = 0.5.$$

Because $sf(al_1) = sf(al_2)$ and $af(al_1) > af(al_2)$, we can get $al_1 > al_2$.

The Power Bonferroni Mean Operator and Power Geometric Bonferroni Mean Operator

Definition 8 [41]. Let $ra_k (k = 1, 2, \cdots, n)$ be a set of positive real numbers and $x, y \geq 0$ the aggregation function

$$\mathrm{PBM}^{x,y}(ra_1, ra_2, \cdots, ra_n) = \left(\frac{1}{n^2 - n} \sum_{\substack{g=1,h=1 \\ g \neq h}}^{n} \left(\left(\frac{n(T(ra_g)+1)}{\sum_{t=1}^{n}(T(ra_t)+1)} ra_g \right)^x \otimes \left(\frac{n(T(ra_h)+1)}{\sum_{t=1}^{n}(T(ra_t)+1)} ra_h \right)^y \right) \right)^{\frac{1}{x+y}} \tag{16}$$

is called power Bonferroni mean (PBM) operator.

Definition 9 [41]. Let $ra_k (k = 1, 2, \cdots, n)$ be a set of positive real numbers and $x, y > 0$ the aggregation function

$$\mathrm{PGBM}^{x,y}(ra_1, ra_2, ..., ra_n) = \frac{1}{x+y} \left(\prod_{\substack{g=1,h=1 \\ g \neq h}}^{n} \left(x ra_g^{\frac{n(T(ra_g)+1)}{\sum_{t=1}^{n}(T(ra_t)+1)}} + y ra_h^{\frac{n(T(ra_h)+1)}{\sum_{t=1}^{n}(T(ra_t)+1)}} \right) \right)^{\frac{1}{n^2 - n}} \tag{17}$$

is called power geometric Bonferroni mean (PGBM) operator.

In definitions 8 and 9, $T(ra_g) = \sum_{\substack{h=1 \\ h \neq g}}^{n} Sup(ra_g, ra_h)$, and $Sup(ra_g, ra_h)$ is the support degree for ra_g from ra_h satisfying the properties as

1. $Sup(ra_g, ra_h) = 1 - d(ra_g, ra_h)$, so $Sup(ra_g, ra_h) \in [0, 1]$;
2. $Sup(ra_g, ra_h) = Sup(ra_h, ra_g)$;
3. If $|ra_g - ra_h| < |ra_l - ra_r|$, then $Sup(ra_g, ra_h) > Sup(ra_l, ra_r)$.

where d is Euclidean distance from Definition 3. $T(ra_g)$ can represent the support of ra_g by all the other numbers, and the closer two values are, the bigger the support degree is.

Some Interval-Valued Intuitionistic Fuzzy PBM Operators

On the basis of IVIFNs, the PBM and PGBM operators, we shall propose the weighted PBM (IVIFWPBM) operator of the IVIFNs and the weighted PGBM (IVIFWPGBM) operator of the IVIFNs.

The Interval-Valued Intuitionistic Fuzzy Power Bonferroni Mean Operator

Definition 10 [41]. Suppose $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ is a set of the IVIFNs $(j = 1, 2, \cdots, n)$, then the IVIFPBM operator is defined as

$$\mathrm{IVIFPBM}^{x,y}(al_1, al_2, \cdots, al_n) = \left(\frac{1}{n^2 - n} \sum_{\substack{g=1,h=1 \\ g \neq h}}^{n} \left(\left(\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)} al_g \right)^x \otimes \left(\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)} al_h \right)^y \right) \right)^{\frac{1}{x+y}} \tag{18}$$

where $T(al_g) = \sum_{h=1,h \neq g}^{n} Sup(al_g, al_h)$, $x, y > 0$.

Theorem 2. Based on the IVIFNs $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ $(j = 1, 2, 3, \cdots, n)$, the aggregated result from Definition 10 is expressed by

$$\mathrm{IVIFPBM}^{x,y}(al_1, al_2, \cdots, al_n) = \left(\left[\left(1-\left(\prod_{g=1,h=1,\ g\neq h}^{n}\left(1-\left(1-\left(1-\mu a_g\right)^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(a_{lt})+1)}}\right)^{x}\times\left(1-\left(1-\mu a_h\right)^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right.\right.$$
$$\left.\left(1-\left(\prod_{g=1,h=1,\ g\neq h}^{n}\left(1-\left(1-\left(1-\mu b_g\right)^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{x}\times\left(1-\left(1-\mu b_h\right)^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right],$$
$$\left[1-\left(1-\left(\prod_{g=1,h=1,\ g\neq h}^{n}\left(1-\left(1-vc_g^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{x}\left(1-vc_h^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right.$$
$$\left.\left.1-\left(1-\left(\prod_{g=1,h=1,\ g\neq h}^{n}\left(1-\left(1-vd_g^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{x}\left(1-vd_h^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right]\right) \tag{19}$$

Proof.

Let $\tau_k = \frac{n(T(al_k)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}$ $(k = 1, 2, \cdots, n)$, we can get

$$\mathrm{IVIFPBM}^{x,y}(al_1, al_2, \cdots, al_n) = \left(\frac{1}{n^2-n}\sum_{g=1,h=1,\ g\neq h}^{n}\left((\tau_g \cdot al_g)^x \otimes (\tau_h \cdot al_h)^y\right)\right)^{\frac{1}{x+y}}.$$

Calculate $\tau_g \cdot al_g$ and $\tau_h \cdot al_h$, and we can get

$$\tau_g \cdot al_g = \left(\left[1-\left(1-\mu a_g\right)^{\tau_g}, 1-\left(1-\mu b_g\right)^{\tau_g}\right], \left[vc_g^{\tau_g}, vd_g^{\tau_g}\right]\right),$$
$$\tau_h \cdot al_h = \left(\left[1-\left(1-\mu a_h\right)^{\tau_h}, 1-\left(1-\mu b_h\right)^{\tau_h}\right], \left[vc_h^{\tau_h}, vd_h^{\tau_h}\right]\right).$$

1. Calculate $(\tau_g \cdot al_g)^x$ and $(\tau_h \cdot al_h)^y$, and we can get

$$(\tau_g \cdot al_g)^x = \left(\left[\left(1-\left(1-\mu a_g\right)^{\tau_g}\right)^x, \left(1-\left(1-\mu b_g\right)^{\tau_g}\right)^x\right], \left[1-\left(1-vc_g^{\tau_g}\right)^x, 1-\left(1-vd_g^{\tau_g}\right)^x\right]\right),$$
$$(\tau_h \cdot al_h)^y = \left(\left[\left(1-\left(1-\mu a_h\right)^{\tau_h}\right)^y, \left(1-\left(1-\mu b_h\right)^{\tau_h}\right)^y\right], \left[1-\left(1-vc_h^{\tau_h}\right)^y, 1-\left(1-vd_h^{\tau_h}\right)^y\right]\right).$$

2. Calculate $(\tau_g \cdot al_g)^x \otimes (\tau_h \cdot al_h)^y$, and we can get

$\left(\tau_g \cdot al_g\right)^x \otimes \left(\tau_h \cdot al_h\right)^y = \left(\left[\left(1-\left(1-\mu a_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu a_h\right)^{\tau_h}\right)^y, \left(1-\left(1-\mu b_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu b_h\right)^{\tau_h}\right)^y\right], \left[1-\left(1-vc_g^{\tau_g}\right)^x \times \left(1-vc_h^{\tau_h}\right)^y, 1-\left(1-vd_g^{\tau_g}\right)^x \times \left(1-vd_h^{\tau_h}\right)^y\right]\right)$

Calculate $\sum_{g=1,h=1,\, g\neq h}^{n}\left(\left(\tau_g \cdot al_g\right)^x \otimes \left(\tau_h \cdot al_h\right)^y\right)$, and we get

$$\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\tau_g \cdot al_g\right)^x \otimes \left(\tau_h \cdot al_h\right)^y\right) = \left(\left[1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu a_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu a_h\right)^{\tau_h}\right)^y\right), 1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu b_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu b_h\right)^{\tau_h}\right)^y\right)\right], \left[\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vc_g^{\tau_g}\right)^x \times \left(1-vc_h^{\tau_h}\right)^y\right), \prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vd_g^{\tau_g}\right)^x \times \left(1-vd_h^{\tau_h}\right)^y\right)\right]\right).$$

3. Calculate $\frac{1}{n(n-1)}\sum_{g=1,h=1,\, g\neq h}^{n}\left(\left(\tau_g \cdot al_g\right)^x \otimes \left(\tau_h \cdot al_h\right)^y\right)$, and we get

$$\frac{1}{n(n-1)}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\tau_g \cdot al_h\right)^x \otimes \left(\tau_h \cdot al_h\right)^y\right) = \left(\left[1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu a_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu a_h\right)^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}, 1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu b_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu b_h\right)^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}\right], \left[\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vc_g^{\tau_g}\right)^x \times \left(1-vc_h^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}, \prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vd_g^{\tau_g}\right)^x \times \left(1-vd_h^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}\right]\right).$$

4. Calculate $\left(\frac{1}{n(n-1)}\sum_{g=1,h=1,\, g\neq h}^{n}\left(\left(\tau_g \cdot al_g\right)^x \otimes \left(\tau_h \cdot al_h\right)^y\right)\right)^{\frac{1}{x+y}}$, and we get

$$\left(\frac{1}{n(n-1)}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\tau_g \cdot al_g\right)^x \otimes \left(\tau_h \cdot al_h\right)^y\right)\right)^{\frac{1}{x+y}} = \left(\left[\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu a_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu a_h\right)^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}, \left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu b_g\right)^{\tau_g}\right)^x \times \left(1-\left(1-\mu b_h\right)^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right], \left[1-\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vc_g^{\tau_g}\right)^x \times \left(1-vc_h^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}, 1-\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vd_g^{\tau_g}\right)^x \times \left(1-vd_h^{\tau_h}\right)^y\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right]\right).$$

5. Replace $\tau_k = \frac{n(T(al_k)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}$, and we get

$$\left(\frac{1}{n(n-1)}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\left(T\left(al_g\right)+1\right)}{\sum\limits_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_g\right)^{x}\otimes\left(\frac{n\left(T\left(al_h\right)+1\right)}{\sum\limits_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_h\right)^{y}\right)\right)^{\frac{1}{x+y}}=$$

$$\left(\left[\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu a_g\right)^{\frac{n\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{x}\times\left(1-\left(1-\mu a_h\right)^{\frac{n\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right.\right.$$

$$\left.\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu b_g\right)^{\frac{n\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{x}\times\left(1-\left(1-\mu b_h\right)^{\frac{n\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right],\tag{20}$$

$$\left[1-\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vc_g^{\frac{n\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{x}\left(1-vc_h^{\frac{n\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right.$$

$$\left.\left.1-\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vd_g^{\frac{n\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{x}\left(1-vd_h^{\frac{n\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}}\right)^{y}\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right]\right).$$

The proof ends.

Now, we will give an example to demonstrate the aggregation process.

Example 3. Suppose that there are two IVIFNs $al_1 = ([0.1, 0.3], [0.4, 0.5])$ and $al_2 = ([0.2, 0.4], [0.3, 0.5])$, and let $x = 1$, $y = 2$, then we can derive the following results:

Calculate $\tau_k = \frac{n(T(al_k)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}$, we can get $\tau_1 = \frac{2(T(al_1)+1)}{\sum_{t=1}^{2}(T(al_t)+1)} = 1$ and $\tau_2 = \frac{2(T(al_2)+1)}{\sum_{t=1}^{2}(T(al_t)+1)} = 1$. So, $\mathrm{IVIFPBM}^{1,2}(al_1, al_2) = \left(\left[\left(1-\left(\left(1-\left(1-(1-0.1)^1\right)^1\otimes\left(1-(1-0.2)^1\right)^2\right)\otimes(1-(1-(1-0.2)^1)^1\otimes\left(1-(1-0.1)^1\right)^2)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, (1-((1-(1-(1-0.3)^1)^1\otimes\left(1-(1-0.4)^1\right)^2)\otimes\left(1-\left(1-(1-0.4)^1\right)^1\otimes\left(1-(1-0.3)^1\right)^2\right))^{\frac{1}{2}})^{\frac{1}{3}}\right], \left[1-\left(1-\left(\left(1-(1-0.4^1)^1(1-0.3^1)^2\right)\otimes\left(1-(1-0.3^1)^1(1-0.4^1)^2\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, 1-\left(1-\left(\left(1-(1-0.5^1)^1(1-0.5^1)^2\right)\otimes\left(1-(1-0.5^1)^1(1-0.5^1)^2\right)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\right]\right) = ([0.1442, 0.6490], [0.3510, 0.5])$.

By the operations of IVIFNs, several properties of the IVIFPBM operator shall be proved.

Theorem 3 (idempotency). Suppose $al_k = al = ([ua, ub], [vc, vd])(k = 1, 2, \ldots, n)$, then

$$\mathrm{IVIFPBM}(al_1, al_2, ..., al_n) = al.$$

Proof.

Since $al_k = al(k = 1, 2, \ldots, n)$, then according to Definition 10,

$$\mathrm{IVIFPBM}^{x,y}(al_1, al_2, \ldots, al_n) = \left(\frac{1}{n^2-n}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_g\right)^x \otimes \left(\frac{n\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_h\right)^y\right)\right)^{\frac{1}{x+y}} =$$

$$\left(\frac{1}{n^2-n}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\left(T(al)+1\right)}{\sum_{t=1}^{n}\left(T(al)+1\right)}al\right)^x \otimes \left(\frac{n\left(T(al)+1\right)}{\sum_{t=1}^{n}\left(T(al)+1\right)}al\right)^y\right)\right)^{\frac{1}{x+y}} = \left(\frac{1}{n^2-n}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n} al^{(x+y)}\right)^{\frac{1}{x+y}} = al.$$

Theorem 4 (commutativity). Suppose $al_k^{'}$ is any permutation of $al_k (k = 1, 2, \ldots, n)$, then

$$\mathrm{IVIFPBM}\left(al_1^{'}, al_2^{'}, \ldots, al_n^{'}\right) = \mathrm{IVIFPBM}(al_1, al_2, \ldots, al_n).$$

Proof.
Based on Definition 10, we get

$$\mathrm{IVIFPBM}^{x,y}\left(al_1^{'}, al_2^{'}, \ldots, al_n^{'}\right) = \left(\frac{1}{n^2-n}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\left(T\left(al_g^{'}\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t^{'}\right)+1\right)}al_i^{'}\right)^x \otimes \left(\frac{n\left(T\left(al_h^{'}\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t^{'}\right)+1\right)}al_j^{'}\right)^y\right)\right)^{\frac{1}{x+y}}, \text{ (and)}$$

$$\mathrm{IVIFPBM}^{x,y}(al_1, al_2, \ldots, al_n) = \left(\frac{1}{n^2-n}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_g\right)^x \otimes \left(\frac{n\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_h\right)^y\right)\right)^{\frac{1}{x+y}}$$

Because

$$\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\left(T\left(al_g^{'}\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t^{'}\right)+1\right)}al_g^{'}\right)^x \otimes \left(\frac{n\left(T\left(al_h^{'}\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t^{'}\right)+1\right)}al_h^{'}\right)^y\right)$$

$$= \sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_g\right)^x \otimes \left(\frac{n\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\left(T\left(al_t\right)+1\right)}al_h\right)^y\right),$$

Thus, $\mathrm{IVIFPBM}\left(al_1^{'}, al_2^{'}, ..., al_n^{'}\right) = \mathrm{IVIFPBM}(al_1, al_2, ..., al_n)$.

In the IVIFPBM operator, it is noted that we only consider the power weight vector and the interrelationship among input arguments and do not take the importance of the input arguments into account. In what follows, the IVIFWPBM operator shall be proposed to overcome the shortcoming.

Definition 11. Suppose $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ is a set of the IVIFNs ($j = 1, 2, \cdots, n$), then the IVIFWPBM operator is defined as

$$\mathrm{IVIFWPBM}^{x,y}(al_1, al_2, \cdots, al_n) = \left(\frac{1}{n^2-n}\sum_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(\left(\frac{n\omega_g\left(T\left(al_g\right)+1\right)al_g}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}\right)^x\left(\frac{n\omega_h(T(al_h)+1)al_h}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}\right)^y\right)\right)^{\frac{1}{x+y}} \tag{21}$$

where $T\left(al_g\right) = \sum_{h=1,h\neq g}^{n} Sup\left(al_g, al_h\right)$, $x, y > 0$. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of the IVIFNs, $0 \le \omega_k \le 1$ $(k = 1, 2, ..., n)$ and $\sum_{k=1}^{n}\omega_k = 1$.

Theorem 5. Suppose $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ is a set of the IVIFNs ($j = 1, 2\cdots, n$) and $x, y > 0$, then the aggregated result from Definition 11 is expressed by

$$\begin{aligned}
&\mathrm{IVIFWPBM}^{x,y}(al_1, al_2, \cdots, al_n) = \\
&\left(\left[\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu a_g\right)^{\frac{n\omega_g\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x\times\left(1-\left(1-\mu a_h\right)^{\frac{n\omega_h\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right.\right. \\
&\left.\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\left(1-\mu b_g\right)^{\frac{n\omega_g\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x\times\left(1-\left(1-\mu b_h\right)^{\frac{n\omega_h\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right], \\
&\left[1-\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vc_g^{\frac{n\omega_g\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x\left(1-vc_h^{\frac{n\omega_h\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right. \\
&\left.\left.1-\left(1-\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-vd_g^{\frac{n\omega_g\left(T\left(al_g\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x\left(1-vd_h^{\frac{n\omega_h\left(T\left(al_h\right)+1\right)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right]\right).
\end{aligned} \tag{22}$$

The Interval-Valued Intuitionistic Fuzzy Power Geometric Bonferroni Mean Operator

Definition 12 [41]. Suppose $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ is a set of the IVIFNs $(j = 1, 2\cdots, n)$, then the IVIFPGBM operator is defined as

$$\mathrm{IVIFPGBM}^{x,y}(al_1, al_2, \cdots, al_n) = \frac{1}{x+y}\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(x al_g^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}} + y al_h^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)\right)^{\frac{1}{n(n-1)}} \tag{23}$$

where $T(al_g) = \sum_{h=1, h\neq g}^{n} Sup(al_g, al_h),\ x, y > 0.$

Theorem 8. Suppose $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ is a set of IVIFNs $(j = 1, 2\cdots, n)$, then the aggregated result according to Definition 12 is expressed by

$$\begin{aligned}
&\mathrm{IVIFPGBM}^{x,y}(al_1, al_2, \cdots, al_n) = \\
&\left(\left[\left(1-\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\mu a_g^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{x}\times\left(1-\mu a_h^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right.\right.\right., \\
&\left.1-\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\mu b_g^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{x}\times\left(1-\mu b_h^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right], \\
&\left[\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-(1-vc_g)^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{x}\times\left(1-(1-vc_h)^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right., \\
&\left.\left.\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-(1-vd_g)^{\frac{n(T(al_g)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{x}\times\left(1-(1-vd_h)^{\frac{n(T(al_h)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}}\right)^{y}\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right]\right).
\end{aligned} \tag{24}$$

Similar to Theorem 2, the proof of Theorem 8 is omitted.

Now, we will give an example to demonstrate the aggregation process.

Example 4. Suppose that $al_1 = ([0.1, 0.3], [0.4, 0.5])$ and $al_2 = ([0.2, 0.4], [0.3, 0.5])$ are two IVIFNs, and let $x = 1$, $y = 2$, then we can derive the following results.

Calculate $\tau_k = \frac{n(T(al_k)+1)}{\sum_{t=1}^{n}(T(al_t)+1)}$, we can get $\tau_1 = \frac{2(T(al_1)+1)}{\sum_{t=1}^{2}(T(al_t)+1)} = 1$, $\tau_2 = \frac{2(T(al_2)+1)}{\sum_{t=1}^{2}(T(al_t)+1)} = 1$. So,

$$\text{IVIFPGBM}^{1,2}(al_1, al_2) = \Big(\Big[\big(1-\big(1-\big(1-\big(1-0.1^1\big)^1 \times \big(1-0.2^1\big)^2\big)^{\frac{1}{2}} \times \big(1-\big(1-0.2^1\big)^1 \times \big(1-0.1^1\big)^2\big)^{\frac{1}{2}}\big)^{\frac{1}{3}}, \big(1-\big(1-\big(1-\big(1-0.3^1\big)^1 \times \big(1-0.4^1\big)^2\big)^{\frac{1}{2}} \times \big(1-\big(1-0.4^1\big)^1 \times \big(1-0.3^1\big)^2\big)^{\frac{1}{2}}\big)^{\frac{1}{3}}\Big], [(1-(1-(1-(1-0.4)^1)^1 \times (1-(1-0.3)^1)^2)^{\frac{1}{2}} \times (1-(1-(1-0.3)^1)^1 \times (1-(1-0.4)^1)^2)^{\frac{1}{2}})^{\frac{1}{3}}, [(1-(1-(1-(1-0.4)^1)^1 \times (1-(1-0.3)^1)^2)^{\frac{1}{2}} \times (1-(1-(1-0.3)^1)^1 \times (1-(1-0.4)^1)^2)^{\frac{1}{2}})^{\frac{1}{3}}]\Big) = ([0.1502, 0.3510], [0.3477, 0.5]).$$

By the operations of IVIFNs, several properties of the IVIFPGBM operator shall be proved.

Theorem 9 (idempotency). Suppose $al_k = al = ([ua, ub],[vc, vd])(k = 1, 2, \ldots, n)$, then

$$\text{IVIFPGBM}(al_1, al_2, \ldots, al_n) = al.$$

Similar to Theorem 3, the proof of Theorem 9 is omitted.

Theorem 10 (commutativity). Let al'_k be any permutation of $al_k(k = 1, 2, \ldots, n)$, then

$$\text{IVIFPGBM}(al_1, al_2, \ldots, al_n) = \text{IVIFPGBM}\left(al'_1, al'_2, \ldots, al'_n\right).$$

Similar to Theorem 4, the proof of Theorem 10 is omitted.

Similar to the IVIFWPBM operator, the IVIFWPGBM operator shall be given to overcome the shortcoming of the IVIFPGBM operator.

Definition 13. Suppose $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ is a set of the IVIFNs $(j = 1, 2\cdots, n)$, IVIFWPGBM: $\Omega^n \to \Omega$, then

$$\text{IVIFWPGBM}^{x,y}(al_1, al_2, \cdots, al_n) = \frac{1}{x+y}\left(\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(x al_g^{\frac{n\omega_g(T(al_g)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}} + y al_h^{\frac{n\omega_h(T(al_h)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)\right)^{\frac{1}{n^2-n}} \tag{25}$$

where Ω is the set of all IVIFNs, and $T(al_g) = \sum_{h=1,h\neq g}^{n} Sup(al_g, al_h)$, $x, y > 0$. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of the IVIFNs, $0 \leq \omega_k \leq 1$ $(k = 1, 2, \ldots, n)$ and $\sum_{k=1}^{n}\omega_k = 1$.

Theorem 11. Let $al_j = ([\mu a_j, \mu b_j], [vc_j, vd_j])$ be a set of the IVIFNs $(j = 1, 2\cdots, n)$, $x, y > 0$, the result aggregated based on Definition 13 is expressed by

$$\begin{aligned}
&\text{IVIFWPGBM}^{x,y}(al_1, al_2, \cdots, al_n) = \\
&\left(\left[\left(1-\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\mu a_g^{\frac{n\omega_g(T(al_g)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x \times \left(1-\mu a_h^{\frac{n\omega_h(T(al_h)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right.\right. \\
&\left.1-\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-\mu b_g^{\frac{n\omega_g(T(al_g)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x \times \left(1-\mu b_h^{\frac{n\omega_h(T(al_h)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}}\right], \\
&\left[\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-(1-vc_g)^{\frac{n\omega_g(T(al_g)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x \times \left(1-(1-vc_h)^{\frac{n\omega_h(T(al_h)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right. \\
&\left.\left.\left(1-\prod_{\substack{g=1,h=1\\ g\neq h}}^{n}\left(1-\left(1-(1-vd_g)^{\frac{n\omega_g(T(al_g)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^x \times \left(1-(1-vd_h)^{\frac{n\omega_h(T(al_h)+1)}{\sum_{t=1}^{n}\omega_t(T(al_t)+1)}}\right)^y\right)^{\frac{1}{n^2-n}}\right)^{\frac{1}{x+y}},\right]\right).
\end{aligned} \tag{26}$$

The MAGDM Approach Based on Interval-Valued Intuitionistic Fuzzy Power Bonferroni Mean and Interval-Valued Intuitionistic Fuzzy Power Geometric Bonferroni Mean Operators

For a MAGDM problem with IVIFNs, in which the attributes' and experts' weights are known, let $Z = \{z_1, z_2, \cdots, z_m\}$ be the set of all alternatives, $A = \{a_1, a_2, \cdots, a_n\}$ be the set of attributes, and $E = \{e_1, e_2, \cdots, e_t\}$ be the set of all experts. Assume that $\tilde{a}^k_{gh} = \left(\left[a^k_{gh}, b^k_{gh}\right], \left[c^k_{gh}, d^k_{gh}\right]\right)$ is the attribute evaluation value given by the expert e_k for the alternative z_g about the attribute a_h. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weight vector of $\{a_1, a_2, \cdots, a_n\}$ satisfying with $\omega_h \in [0, 1]$, $\sum_{h=1}^{n} \omega_h = 1$. $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_t)$ is the weight vector of $\{e_1, e_2, \cdots, e_t\}$, and $\gamma_k \in [0, 1]$, $\sum_{k=1}^{t} \gamma_k = 1 (k = 1, 2, \cdots, t)$, then the goal of this MAGDM problem is to rank the alternatives.

The Decision-Making Steps Based on Interval-Valued Intuitionistic Fuzzy Weighted Power Bonferroni Mean and Interval-Valued Intuitionistic Fuzzy Weighted Power Geometric Bonferroni Mean Operators

Step 1. Normalize the decision matrix.

Generally, if there are the different types in attributes, we need to convert them to the same type. For convenience, we need to convert the cost type to the benefit type by the following method:

$$\tilde{r}^k_{gh} = \left(\left[\underline{u}^k_{gh}, \overline{u}^k_{gh}\right], \left[\underline{f}^k_{gh}, \overline{f}^k_{gh}\right]\right) = \begin{cases} \left(\left[a^k_{gh}, b^k_{gh}\right], \left[c^k_{gh}, d^k_{gh}\right]\right) \text{for benefit attribute } a_h \\ \left(\left[c^k_{gh}, d^k_{gh}\right], \left[a^k_{gh}, b^k_{gh}\right]\right) \text{ for cost attribute } a_h \end{cases} \quad (27)$$

So, the decision matrices $\tilde{A} = \left[\tilde{a}^k_{gh}\right]_{m\times n}$ can be converted to matrices $\tilde{R} = \left[\tilde{r}^k_{gh}\right]_{m\times n}$.

Step 2. Calculate the supports $Sup\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right) (g = 1, 2, \cdots, m; k = 1, 2 \cdots, t; h, l = 1, 2, \cdots, n)$ by

$$Sup\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right) = 1 - d\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right) \quad (28)$$

where $d\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right)$ is the Euclidean distance between two IVIFNs $\tilde{r}^k_{gh}$ and $\tilde{r}^k_{gl}$, which is from Definition 3.

Step 3. Calculate $T\left(\tilde{r}^k_{gh}\right)$ by

$$T\left(\tilde{r}^k_{gh}\right) = \sum_{\substack{l=1 \\ l\neq h}}^{n} Sup\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right) \ (g = 1, 2, \cdots, m; k = 1, 2 \cdots, t; h = 1, 2, \cdots, n) \quad (29)$$

Step 4. Calculate $\tau^k_{gh} = \frac{n\omega_h\left(1+T\left(\tilde{r}^k_{gh}\right)\right)}{\sum_{t=1}^{n}\omega_t\left(1+T\left(\tilde{r}^k_{gt}\right)\right)}$ $(g = 1, 2, \cdots, m; k = 1, 2 \cdots, t; h = 1, 2, \cdots, n)$.

Step 5. Utilize the IVIFWPBM or IVIFWPGBM operator.

$$\tilde{r}^k_g = \left(\left[\underline{u}^k_g, \overline{u}^k_g\right], \left[\underline{f}^k_g, \overline{f}^k_g\right]\right) = \text{IVIFWPBM}\left(\tilde{r}^k_{g1}, \tilde{r}^k_{g2}, \cdots, \tilde{r}^k_{gn}\right) \text{ or IVIFWPGBM}\left(\tilde{r}^k_{g1}, \tilde{r}^k_{g2}, \cdots, \tilde{r}^k_{gn}\right) \quad (30)$$

to determine the overall IVIFNs $\tilde{r}^k{}_g$ $(g = 1, 2, \cdots, m; k = 1, 2 \cdots, t)$.

Step 6. Calculate the supports $Sup\left(\tilde{r}^k{}_g, \tilde{r}^l{}_g\right)$ $(g = 1, 2, \cdots, m; k, l = 1, 2, \cdots, t)$ by

$$Sup\left(\tilde{r}^k{}_g, \tilde{r}^l{}_g\right) = 1 - d\left(\tilde{r}^k{}_g, \tilde{r}^l{}_g\right), \quad (31)$$

where $d\left(\tilde{r}^k{}_g, \tilde{r}^l{}_g\right)$ is the Euclidean distance between two IVIFNs $\tilde{r}^k{}_g$ and $\tilde{r}^l{}_g$, which is from Definition 3.

Step 7. Calculate $T\left(\tilde{r}^k_g\right)$ by

$$T\left(\tilde{r}^k_g\right) = \sum_{\substack{l=1 \\ l\neq g}}^{t} Sup\left(\tilde{r}^k_g, \tilde{r}^l_g\right) \ (g = 1, 2, \cdots, m; k = 1, 2, \cdots, t) \quad (32)$$

Step 8. Calculate $\tau^k_g = \frac{t\gamma_k\left(1+T\left(\tilde{r}^k_g\right)\right)}{\sum_{k=1}^{t}\gamma_k\left(1+T\left(\tilde{r}^k_g\right)\right)}$ $(g = 1, 2, \cdots, m; k = 1, 2 \cdots, t)$.

Step 9: Use IVIFWPBM or IVIFWPGBM operators to get the collective IVIFNs $\tilde{r}_g (g = 1, 2, ..., m)$.

```
For zg=1 to m          ; read original data
  For zh=1 to n
    For zk=1 to t
        A[zg][zh][zk]=ã^{zk}_{zg zh} ; read the data of decision matrices A to array A[m][n][t].
For zh=1 to n                ; Normalize the data
 If Cz_h is cost type, then R[zg][zh][zk]= Neg(A[zg][zh][zk]);
   else R[zg][zh][zk]= A[zg][zh][zk].
For zg=1 to m          ; calculate supports
  For zh=1 to n
     For zl=1 to n
      For zk=1 to t
       Do {(1) calculate D[zg][zh][zl][zk];
           (2) calculate SUP[zg][zh][zl][zk];
         }
For zg=1 to m          ; calculate synthetic weights
  For zh=1 to n
    For zk=1 to t
       Do {(1) calculate T[zg][zh][zk];
           (2) calculate  τ [zg][zh][zk];
         }
For zg=1 to m          ; calculate overall preference values
  For zk=1 to t
       Do {calculate R1[zg][zk];
            }
For zg=1 to m          ; calculate supports
  For zl=1 to t
      For zk=1 to t
       Do {(1) calculate D1[zg] [zl][zk];
           (2) calculate SUP1[zg][zl][zk];
         }
For zg=1 to m          ; calculate synthetic weights
  For zk=1 to t
       Do {(1) calculate T1[zg][zk];
           (2) calculate  τ1 [zg][zk];
         }
For zg=1 to m          ; calculate collective overall preference values
     Do {(1) calculate R2[zg];
         (2)calculate score function values S[zg];
            }
For zg=1 to m          ; rank alternatives.
     Do {rank S[zg];
         }
```

Table 1 Air quality data from station e_1

	a_1	a_2	a_3
z_1	([0.220, 0.310], [0.230, 0.540])	([0.130, 0.530], [0.200, 0.360])	([0.120, 0.370], [0.400, 0.560])
z_2	([0.280, 0.410], [0.330, 0.490])	([0.330, 0.530], [0.200, 0.360])	([0.120, 0.370], [0.300, 0.460])
z_3	([0.320, 0.410], [0.230, 0.440])	([0.430, 0.530], [0.160, 0.250])	([0.230, 0.450], [0.210, 0.370])
z_4	([0.390, 0.470], [0.180, 0.360])	([0.390, 0.530], [0.270, 0.320])	([0.280, 0.340], [0.110, 0.230])

$$\tilde{r}_g = \left(\left[\underline{u}_g, \overline{u}_g\right], \left[\underline{f}_g, \overline{f}_g\right]\right) = \text{IVIFWPBM}\left(\overset{1}{\tilde{r}}_g, \overset{2}{\tilde{r}}_g, \cdots, \overset{t}{\tilde{r}}_g\right) \quad (33)$$
$$\text{or IVIFWPGBM}\left(\overset{1}{\tilde{r}}_g, \overset{2}{\tilde{r}}_g, \cdots, \overset{t}{\tilde{r}}_g\right)$$

Step 10: Calculate the score function $sf\left(\tilde{r}_g\right)$ and accuracy function $af\left(\tilde{r}_g\right)$ of the collective IVIFNs$\tilde{r}_g(g = 1, 2, ..., m)$.

Step 11: Rank all the alternatives $\{z_1, z_2, \cdots, z_m\}$ by comparison method of IVIFNs, and opt for the most eligible alternative(s).

Step 12: End.

In order to easily perform the steps, we can give some pseudo codes as follows:

An application example

This example is adapted from Liu [19]. Suppose that four alternatives (z_1, z_2, z_3, z_4) representing the air quality of 2006, 2007, 2008, and 2009 are evaluated (the air quality of Guangzhou). Three attributes are taken into consideration, including the SO2 (a_1), the NO2 (a_2), and the PM10 (a_3). The weight vector about criteria is provided by $(0.40, 0.20, 0.40)^T$. The possible alternatives $z_g(g = 1, 2, 3, 4)$ are assessed by three air-quality monitoring stations regarded as experts (e_1, e_2, e_3). The weight vector about experts is provided by$(0.314, 0.355, 0.331)^T$. The assessment values are represented by the IVIFNs, which are listed in Tables 1, 2, and 3.

Rank the Alternatives by the Proposed Method Based on the Interval-Valued Intuitionistic Fuzzy Power Bonferroni Mean Operator

Step 1: Transform the decision matrix $\tilde{A}^k = \left[\tilde{a}^k{}_{gh}\right]_{m\times n}$ into the normalized matrix $\tilde{R}^k = \left[\tilde{r}^k{}_{gh}\right]_{m\times n}$.

Because all the attributes are the same type, they do not need to be normalized.

Step 2: Calculate the supports$Sup\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right)$.

By formula (28), calculate the supports $Sup\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right)$ (for simplicity, we denote $Sup\left(\tilde{r}^k_{gh}, \tilde{r}^k_{gl}\right)$ with $S^k_{gh,gl}(h, l = 1, 2, 3; g = 1, 2, 3, 4; k = 1, 2, 3.)$). We can get

$$S^1{}_{11,12} = S^1{}_{12,11} = 0.8502, S^1{}_{12,13} = S^1{}_{13,12} = 0.8374, S^1{}_{11,13} = S^1{}_{13,11} = 0.8964$$
$$S^1{}_{21,22} = S^1{}_{22,21} = 0.8874, S^1{}_{22,23} = S^1{}_{23,22} = 0.8503, S^1{}_{21,23} = S^1{}_{23,21} = 0.9149$$
$$S^1{}_{31,32} = S^1{}_{32,31} = 0.8701, S^1{}_{32,33} = S^1{}_{33,32} = 0.8742, S^1{}_{31,33} = S^1{}_{33,31} = 0.9388$$
$$S^1{}_{41,42} = S^1{}_{42,41} = 0.9423, S^1{}_{42,43} = S^1{}_{43,42} = 0.8569, S^1{}_{41,43} = S^1{}_{43,41} = 0.8873$$
$$S^2{}_{11,12} = S^2{}_{12,11} = 0.9178, S^2{}_{12,13} = S^2{}_{13,12} = 0.8188, S^2{}_{11,13} = S^2{}_{13,11} = 0.7655$$
$$S^2{}_{21,22} = S^2{}_{22,21} = 0.8280, S^2{}_{22,23} = S^2{}_{23,22} = 0.7916, S^2{}_{21,23} = S^2{}_{23,21} = 0.8402$$
$$S^2{}_{31,32} = S^2{}_{32,31} = 0.8504, S^2{}_{32,33} = S^2{}_{33,32} = 0.9150, S^2{}_{31,33} = S^2{}_{33,31} = 0.9209$$
$$S^2{}_{41,42} = S^2{}_{42,41} = 0.9390, S^2{}_{42,43} = S^2{}_{43,42} = 0.8184, S^2{}_{41,43} = S^2{}_{43,41} = 0.8642$$
$$S^3{}_{11,12} = S^3{}_{12,11} = 0.9573, S^3{}_{12,13} = S^3{}_{13,12} = 0.9348, S^3{}_{11,13} = S^3{}_{13,11} = 0.9441$$
$$S^3{}_{21,22} = S^3{}_{22,21} = 0.8818, S^3{}_{22,23} = S^3{}_{23,22} = 0.7967, S^3{}_{21,23} = S^3{}_{23,21} = 0.8725$$
$$S^3{}_{31,32} = S^3{}_{32,31} = 0.8972, S^3{}_{32,33} = S^3{}_{33,32} = 0.9333, S^3{}_{31,33} = S^3{}_{33,31} = 0.8948$$
$$S^3{}_{41,42} = S^3{}_{42,41} = 0.8364, S^3{}_{42,43} = S^3{}_{43,42} = 0.8249, S^3{}_{41,43} = S^3{}_{43,41} = 0.8042$$

Step 3: Calculate$T\left(\tilde{r}^k_{gh}\right)$ $(h = 1, 2, 3; g = 1, 2, 3, 4; k = 1, 2, 3)$ by formula (29) (for simplicity, we denote $T\left(\tilde{r}^k{}_{gh}\right)$ with$T_{gh}{}^k$).

Table 2 Air quality data from station e_2

	a_1	a_2	a_3
z_1	([0.040, 0.210], [0.350,0.460])	([0.100, 0.340], [0.270, 0.450])	([0.320, 0.370], [0.130, 0.200])
z_2	([0.320, 0.390], [0.270,0.390])	([0.030, 0.570], [0.300, 0.360])	([0.160, 0.250], [0.140, 0.190])
z_3	([0.260, 0.370], [0.210,0.400])	([0.230, 0.430], [0.060, 0.150])	([0.210, 0.350], [0.110, 0.290])
z_4	([0.300, 0.430], [0.190,0.350])	([0.280, 0.430], [0.310, 0.340])	([0.390, 0.460], [0.010, 0.170])

$$
\begin{aligned}
&T^1{}_{11} = 1.7466,\ T^1{}_{12} = 1.6876,\ T^1{}_{13} = 1.7339, T^1{}_{21} = 1.8023,\ T^1{}_{22} = 1.7377,\ T^1{}_{23} = 1.7651\\
&T^1{}_{31} = 1.8089,\ T^1{}_{32} = 1.7443,\ T^1{}_{33} = 1.8130, T^1{}_{41} = 1.8296,\ T^1{}_{42} = 1.7992,\ T^1{}_{43} = 1.7442\\
&T^2{}_{11} = 1.6833,\ T^2{}_{12} = 1.7366,\ T^2{}_{13} = 1.5842, T^2{}_{21} = 1.6683,\ T^2{}_{22} = 1.6196,\ T^2{}_{23} = 1.6318\\
&T^2{}_{31} = 1.7714,\ T^2{}_{32} = 1.7654,\ T^2{}_{33} = 1.8359, T^2{}_{41} = 1.8031,\ T^2{}_{42} = 1.7574,\ T^2{}_{43} = 1.6826\\
&T^3{}_{11} = 1.9014,\ T^3{}_{12} = 1.8921,\ T^3{}_{13} = 1.8789, T^3{}_{21} = 1.7543,\ T^3{}_{22} = 1.6785,\ T^3{}_{23} = 1.6692\\
&T^3{}_{31} = 1.7919,\ T^3{}_{32} = 1.8305,\ T^3{}_{33} = 1.8281, T^3{}_{41} = 1.6406,\ T^3{}_{42} = 1.6613,\ T^3{}_{43} = 1.6292
\end{aligned}
$$

Step 4: Calculate$\tau^k{}_{gh}(g = 1, 2, 3, 4; h = 1, 2, 3; k = 1, 2, 3.)$, we get

$$
\begin{aligned}
&\tau^1{}_{11} = 1.2074,\ \tau^1{}_{12} = 0.5907,\ \tau^1{}_{13} = 1.2018, \tau^1{}_{21} = 1.2120,\ \tau^1{}_{22} = 0.5920,\ \tau^1{}_{23} = 1.1959\\
&\tau^1{}_{31} = 1.2048,\ \tau^1{}_{32} = 0.5886,\ \tau^1{}_{33} = 1.2066, \tau^1{}_{41} = 1.2173,\ \tau^1{}_{42} = 0.6021,\ \tau^1{}_{43} = 1.1806\\
&\tau^2{}_{11} = 1.2131,\ \tau^2{}_{12} = 0.6186,\ \tau^2{}_{13} = 1.1683, \tau^2{}_{21} = 1.2110,\ \tau^2{}_{22} = 0.5945,\ \tau^2{}_{23} = 1.1945\\
&\tau^2{}_{31} = 1.1894,\ \tau^2{}_{32} = 0.5934,\ \tau^2{}_{33} = 1.2171, \tau^2{}_{41} = 1.2251,\ \tau^2{}_{42} = 0.6025,\ \tau^2{}_{43} = 1.1724\\
&\tau^3{}_{11} = 1.2045,\ \tau^3{}_{12} = 0.6003,\ \tau^3{}_{13} = 1.1952, \tau^3{}_{21} = 1.2218,\ \tau^3{}_{22} = 0.5941,\ \tau^3{}_{23} = 1.1841\\
&\tau^3{}_{31} = 1.1906,\ \tau^3{}_{32} = 0.6035,\ \tau^3{}_{33} = 1.2060, \tau^1{}_{41} = 1.2002,\ \tau^1{}_{42} = 0.6048,\ \tau^1{}_{43} = 1.1950
\end{aligned}
$$

Step 5: Utilize the IVIFWPBM operator to determine the overall IVIFNs $\tilde{r}^k{}_g$, which is listed in Table 4 (suppose x , y = 1).

Step 6: Calculate the supports $Sup\left(\tilde{r}^k_g, \tilde{r}^l_g\right)$ based on formula (31) (for simplicity, we denote $Sup\left(\tilde{r}^k_g, \tilde{r}^l_g\right)$ with $S^{k,l}_g(g = 1, 2, 3, 4; k, l = 1, 2, 3)$). We can get

$$
\begin{aligned}
&S_1{}^{1,2} = S_1{}^{2,1} = 0.9192, S_1{}^{2,3} = S_1{}^{3,2} = 0.9489, S_1{}^{1,3} = S_1{}^{3,1} = 0.9027\\
&S_2{}^{1,2} = S_2{}^{2,1} = 0.9165, S_2{}^{2,3} = S_2{}^{3,2} = 0.9281, S_2{}^{1,3} = S_2{}^{3,1} = 0.9128\\
&S_3{}^{1,2} = S_3{}^{2,1} = 0.9190, S_3{}^{2,3} = S_3{}^{3,2} = 0.9222, S_3{}^{1,3} = S_3{}^{3,1} = 0.8527\\
&S_4{}^{1,2} = S_4{}^{2,1} = 0.9809, S_4{}^{2,3} = S_4{}^{3,2} = 0.9168, S_4{}^{1,3} = S_4{}^{3,1} = 0.9022
\end{aligned}
$$

Step 7: Calculate $T\left(\tilde{r}^k_g\right)$ $(g = 1, 2, 3, 4; k = 1, 2, 3.)$ based on formula (32) (for simplicity, we denote $T\left(\tilde{r}^k{}_g\right)$ with$T_g{}^k$).

$$
\begin{aligned}
&T^1{}_1 = 1.8219,\ T^2{}_1 = 1.8681,\ T^3{}_1 = 1.8516, T^1{}_2 = 1.8293,\ T^2{}_2 = 1.8446,\ T^3{}_2 = 1.8409\\
&T^1{}_3 = 1.7717,\ T^2{}_3 = 1.8413,\ T^3{}_3 = 1.7750, T^1{}_4 = 1.8831,\ T^2{}_4 = 1.8977,\ T^3{}_4 = 1.8190
\end{aligned}
$$

Table 3 Air quality data from station e_3

	a_1	a_2	a_3
z_1	([0.250, 0.270], [0.230, 0.400])	([0.170, 0.270], [0.260, 0.400])	([0.210, 0.300], [0.170, 0.320])
z_2	([0.250, 0.290], [0.330, 0.390])	([0.180, 0.460], [0.430, 0.500])	([0.060, 0.210], [0.280, 0.300])
z_3	([0.220, 0.270], [0.270, 0.310])	([0.130, 0.370], [0.160, 0.200])	([0.110, 0.240], [0.140, 0.190])
z_4	([0.300, 0.480], [0.090, 0.450])	([0.080, 0.530], [0.200, 0.240])	([0.320, 0.610], [0.010, 0.090])

Table 4 the overall IVIFNs $\tilde{r}^k{}_g$ from three monitoring stations (e_1, e_2, e_3)

	e_1	e_2	e_3
z_1	([0.1514, 0.3818], [0.2967, 0.5070])	([0.1205, 0.2917], [0.2712, 0.3893])	([0.2078, 0.2736], [0.2392, 0.3897])
z_2	([0.2208, 0.4182], [0.2942, 0.4551])	([0.1650, 0.3762], [0.2567, 0.3348])	([0.1437, 0.2961], [0.3379, 0.4087])
z_3	([0.3064, 0.4460], [0.2186, 0.3723])	([0.2270, 0.3694], [0.1364, 0.2942])	([0.1471, 0.2785], [0.2094, 0.2527])
z_4	([0.3394, 0.4276], [0.2033, 0.3224])	([0.3168, 0.4290], [0.1842, 0.3064])	([0.2369, 0.5244], [0.1114, 0.2903])

Step 8: Calculate$\tau^k{}_g (g = 1, 2, 3, 4; k = 1, 2, 3.)$, we get

$$\tau^1{}_1 = 0.9333,\ \ \tau^2{}_1 = 1.0725,\ \ \tau^3{}_1 = 0.9942, \tau^1{}_2 = 0.9389,\ \ \tau^2{}_2 = 1.0673,\ \ \tau^3{}_2 = 0.9938$$

$$\tau^1{}_3 = 0.9333,\ \ \tau^2{}_3 = 1.0817,\ \ \tau^3{}_3 = 0.9850, \tau^1{}_4 = 0.9473,\ \ \tau^2{}_4 = 1.0764,\ \ \tau^3{}_4 = 0.9764$$

Step 9: Utilize the IVIFWPBM operator to determine the collective IVIFNs$\tilde{r}_g$ which is listed in Table 5 (suppose $x, y =$ 1).

Step 10: Calculate the score function$sf(\tilde{r}_g)$, we get

$$sf(\tilde{r}_1) = -0.1143, sf(\tilde{r}_2) = -0.0809, sf(\tilde{r}_3) = 0.0433, sf(\tilde{r}_4) = 0.1410$$

Step 11: Rank all the alternatives.

According to $sf(\tilde{r}_g)$, we rank the alternatives $\{z_1, z_2, z_3, z_4\}$ shown as follows:

$$z_4 \succ z_3 \succ z_2 \succ z_1.$$

Rank the Alternatives by the Proposed Method Based on the Interval-Valued Intuitionistic Fuzzy Weighted Power Geometric Bonferroni Mean Operator

Step 1 to *Step 4* is the same as those in the "Rank the alternatives by the proposed method based on the IVIFWPBM operator" section.

Step 5: Utilize the IVIFWPGBM operator to determine the overall IVIFNs $\tilde{r}^k{}_g$, which is listed in Table 6 (suppose$x, y =$ 1).

Step 6: Calculate the supports $Sup(\tilde{r}^k_g, \tilde{r}^l_g)$ based on formula (31) (for simplicity, we denote $Sup(\tilde{r}^k_g, \tilde{r}^l_g)$ with $S^{k,l}_g$ $(k, l = 1, 2, 3; g = 1, 2, 3, 4.)$). We can get

$$S_1{}^{1,2} = S_1{}^{2,1} = 0.9232, S_1{}^{2,3} = S_1{}^{3,2} = 0.9526, S_1{}^{1,3} = S_1{}^{3,1} = 0.9106$$
$$S_2{}^{1,2} = S_2{}^{2,1} = 0.9280, S_2{}^{2,3} = S_2{}^{3,2} = 0.9312, S_2{}^{1,3} = S_2{}^{3,1} = 0.9211$$
$$S_3{}^{1,2} = S_3{}^{2,1} = 0.9209, S_3{}^{2,3} = S_3{}^{3,2} = 0.9263, S_3{}^{1,3} = S_3{}^{3,1} = 0.8698$$
$$S_4{}^{1,2} = S_4{}^{2,1} = 0.9831, S_4{}^{2,3} = S_4{}^{3,2} = 0.9275, S_4{}^{1,3} = S_4{}^{3,1} = 0.9143$$

Step 7: Calculate $T(\tilde{r}^k_g)$ $(g = 1, 2, 3, 4; k = 1, 2, 3)$ based on formula (32) (for simplicity, we denote $T(\tilde{r}^k{}_g)$ with$T_g{}^k$).

$$T^1{}_1 = 1.8338,\ \ T^2{}_1 = 1.8758,\ \ T^3{}_1 = 1.8632, T^1{}_2 = 1.8490,\ \ T^2{}_2 = 1.8592,\ \ T^3{}_2 = 1.8523$$
$$T^1{}_3 = 1.7907,\ \ T^2{}_3 = 1.8472,\ \ T^3{}_3 = 1.7961, T^1{}_4 = 1.8975,\ \ T^2{}_4 = 1.9106,\ \ T^3{}_4 = 1.8418,,$$

Step 8: Calculate$\tau^k{}_g (g = 1, 2, 3, 4; k = 1, 2, 3.)$, we get

Table 5 The collective IVIFNs $\tilde{r}_g$ for four alternatives

z_1	z_2	z_3	z_4
([0.1609, 0.3134], [0.2697, 0.4296])	([0.1746, 0.3617], [0.2972, 0.4007])	([0.2221, 0.3615], [0.1900, 0.3071])	([0.2961, 0.4590], [0.1661, 0.3070])

Table 6 The overall IVIFNs $\tilde{r}^k{}_g$ from three monitoring stations (e_1, e_2, e_3) by IVIFWPGBM operator

	e_1	e_2	e_3
z_1	([0.1760, 0.4132], [0.2687, 0.4811])	([0.1729, 0.3260], [0.2311, 0.3449])	([0.2277, 0.2986], [0.2095, 0.3606])
z_2	([0.2632, 0.4475], [0.2751, 0.4294])	([0.1605, 0.4115], [0.2195, 0.2932])	([0.1873, 0.3307], [0.3021, 0.3768])
z_3	([0.3414, 0.4751], [0.1983, 0.3501])	([0.2535, 0.3985], [0.1240, 0.2791])	([0.1721, 0.3088], [0.1827, 0.2267])
z_4	([0.3702, 0.4590], [0.1702, 0.2910])	([0.3422, 0.4554], [0.1307, 0.2680])	([0.2363, 0.5547], [0.0738, 0.2238])

Table 7 The collective IVIFNs$\tilde{r}_g$ for four alternatives by IVIFWPGBM operator

z_1	z_2	z_3	z_4
([0.1609, 0.3165], [0.2682, 0.4265])	([0.1770, 0.3642], [0.2949, 0.3975])	([0.2267, 0.3656], [0.1853, 0.3040])	([0.2982, 0.4613], [0.1641, 0.3059])

$$\tau^1{}_1 = 0.9339,\ \ \tau^2{}_1 = 1.0715,\ \ \tau^3{}_1 = 0.9947, \tau^1{}_2 = 0.9405,\ \ \tau^2{}_2 = 1.0670,\ \ \tau^3{}_2 = 0.9925$$
$$\tau^1{}_3 = 0.9347,\ \ \tau^2{}_3 = 1.0781,\ \ \tau^3{}_3 = 0.9872, \tau^1{}_4 = 0.9465,\ \ \tau^2{}_4 = 1.0749,\ \ \tau^3{}_4 = 0.9786$$

Step 9: Utilize the IVIFWPGBM operator to determine the IVIFNs $\tilde{r}_g(g = 1, 2, 3, 4)$, which is listed in Table 7 (suppose$x , y = 1$).

Step 10: Calculate the score function$sf\left(\tilde{r}_g\right)$, we get

$$sf\left(\tilde{r}_1\right) = -0.1087, sf\left(\tilde{r}_2\right) = -0.0756, sf\left(\tilde{r}_3\right) = 0.0514, sf\left(\tilde{r}_4\right) = 0.1448$$

Step 11: Rank the alternatives.

According to $sf\left(\tilde{r}_g\right)$, we rank the alternatives $\{z_1, z_2, z_3, z_4\}$ shown as follows:

$z_4 \succ z_3 \succ z_2 \succ z_1$.

The Influence of the Parameters x , y on the Decision-Making Result

To observe the influence of parameters x , y on decision making, we set the different values x , y in Step 5 and Step 9, then to rank $\{z_1, z_2, z_3, z_4\}$. The results are listed in Tables 8 and 9.

As we can see from Tables 8 and 9, the aggregation results based on IVIFWPBM operator or IVIFWPGBM operator are different, but the orderings are the same. Furthermore, orderings produced by the different parameters x , y are the same. So, the proposed method is practical and effective. In general, we set the parameter $x = y = 1$.

Table 8 Ordering of the alternatives based on *IVIFWPBM* by using the different x , y

x , y	Score functions $sf\left(\tilde{r}_g\right)$	Ranking
$x = 1 , y = 1$	$sf(\tilde{r}_1) = -0.1143, sf(\tilde{r}_2) = -0.0809\ \ sf(\tilde{r}_3) = 0.0433,\ \ sf(\tilde{r}_4) = 0.1410$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x = 1 , y = 0$	$sf(\tilde{r}_1) = -0.0687, sf(\tilde{r}_2) = -0.0353\ \ sf(\tilde{r}_3) = 0.0583, sf(\tilde{r}_4) = 0.2327$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x = 2 , y = 0$	$sf(\tilde{r}_1) = -0.0313, sf(\tilde{r}_2) = 0.0020\ \ sf(\tilde{r}_3) = 0.0874, sf(\tilde{r}_4) = 0.2537$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x = 10 , y = 0$	$sf(\tilde{r}_1) = 0.1125, sf(\tilde{r}_2) = 0.1363\ \ sf(\tilde{r}_3) = 0.1636, sf(\tilde{r}_4) = 0.3633$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x = 2 , y = 1$	$sf(\tilde{r}_1) = -0.0964, sf(\tilde{r}_2) = -0.0611\ \ sf(\tilde{r}_3) = 0.0537, sf(\tilde{r}_4) = 0.1620$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x = 10 , y = 1$	$sf(\tilde{r}_1) = 0.0576, sf(\tilde{r}_2) = 0.0891\ \ sf(\tilde{r}_3) = 0.1420, sf(\tilde{r}_4) = 0.3111$	$z_4 \succ z_3 \succ z_2 \succ z_1$

Table 9 Ordering of the alternatives based on *IVIFWPGBM* by using the different x , y

x , y	Score functions $sf(\tilde{r}_g)$	Ranking
$x=1$, $y=1$	$sf(\tilde{r}_1)=-0.1087, sf(\tilde{r}_2)=-0.0756\ sf(\tilde{r}_3)=0.0514,\ sf(\tilde{r}_4)=0.1448$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x=1$, $y=0$	$sf(\tilde{r}_1)=-0.0687, sf(\tilde{r}_2)=-0.0353\ sf(\tilde{r}_3)=0.0583, sf(\tilde{r}_4)=0.2204$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x=2$, $y=0$	$sf(\tilde{r}_1)=-0.0489, sf(\tilde{r}_2)=-0.0163\ sf(\tilde{r}_3)=0.0618, sf(\tilde{r}_4)=0.2322$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x=10$, $y=0$	$sf(\tilde{r}_1)=-0.0028, sf(\tilde{r}_2)=0.0287\ sf(\tilde{r}_3)=0.0723, sf(\tilde{r}_4)=0.2784$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x=2$, $y=1$	$sf(\tilde{r}_1)=-0.0955, sf(\tilde{r}_2)=-0.0603\ sf(\tilde{r}_3)=0.0536,\ sf(\tilde{r}_4)=0.1616$	$z_4 \succ z_3 \succ z_2 \succ z_1$
$x=10$, $y=1$	$sf(\tilde{r}_1)=-0.0168, sf(\tilde{r}_2)=0.0156\ sf(\tilde{r}_3)=0.0685,\ sf(\tilde{r}_4)=0.2574$	$z_4 \succ z_3 \succ z_2 \succ z_1$

Table 10 Comparisons of ranking results for different methods

Aggregation operator	Score functions	Ranking
Xu's method [27] based on IVIFWA	$sf(\tilde{r}_1)=-0.0574, sf(\tilde{r}_2)=-0.0246\ sf(\tilde{r}_3)=0.0746, sf(\tilde{r}_4)=0.2338$	$z_4 \succ z_3 \succ z_2 \succ z_1$
He's method [45] based on IVIFWPA	$sf(\tilde{r}_1)=-0.0172,\ sf(\tilde{r}_2)=0.0142\ sf(\tilde{r}_3)=0.1026, sf(\tilde{r}_4)=0.2840$	$z_4 \succ z_3 \succ z_2 \succ z_1$
Xu's method [46] based on IVIFWBM	$sf(\tilde{r}_1)=0.9078, sf(\tilde{r}_2)=0.9123\ sf(\tilde{r}_3)=0.9246, sf(\tilde{r}_4)=0.9393$	$z_4 \succ z_3 \succ z_2 \succ z_1$
Proposed method based on IVIFWPBM	$sf(\tilde{r}_1)=-0.1143, sf(\tilde{r}_2)=-0.0809\ sf(\tilde{r}_3)=0.0433, sf(\tilde{r}_4)=0.1410$	$z_4 \succ z_3 \succ z_2 \succ z_1$

Comparison with Other Methods

To further demonstrate the validity of the proposed methods in this paper, we solve the same illustrative example [19] by using the three existing MAGDM methods, which are the IVIFWA operator-based approach proposed by Xu [27], the IVIFWPA operator-based approach proposed by He [45], and the IVIFWBM operator-based approach proposed by Xu [46]. The final orders of the alternatives obtained by the above three methods are listed in Table 10.

From Table 10, the methods proposed in [27, 45, 46] have the same ranking results with the proposed method. This can verify the proposed method. In the following, we give some characteristic comparisons of our proposed method and the aforementioned three methods, which are listed in Table 11.

Table 11 Characteristic comparisons of different operators

Methods	Aggregation operators	Whether captures interrelationship of two arguments	Whether allows input arguments support each other
Xu [27]	IVIFWA	No	No
He [45]	IVIFWPA	No	Yes
Xu [46]	IVIFWBM	Yes	No
Proposed method	IVIFWPBM	Yes	Yes

Conclusion

In this paper, we propose several PBM aggregation operators for IVIFNs, such as IVIFPBM operator, IVIFWPBM operator, IVIFPGBM operator, and IVIFWPGBM operator, and then we discussed several properties and special cases of the proposed operators. Obviously, these operators can take the advantages of power operator and Bonferroni mean operator, i.e., they can overcome the influence of the unreasonable attribute values and can also consider the interaction between two attributes. In addition, we utilized these operators to solve the MAGDM problem with IVIFNs, and an example is provided to illustrate the validity and advantages of the proposed methods by comparing with three existing methods.

In further researches, we will develop some real applications of these proposed operators in other areas, such as supplier selection evaluation, product scheme selection evaluation, fuzzy cluster analysis, and so on. In addition, we can also extend the PBM operators to some new fuzzy information, such as Pythagorean fuzzy set, linguistic interval hesitant fuzzy set, neutrosophic set, and so on.

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Research Involving Human Participants and/or Animals This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh LA. Fuzzy sets. Inf Control. 1965;8:338–56.
2. Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986;20(1):87–96.
3. Atanassov KT. More on intuitionistic fuzzy sets. Fuzzy Sets Syst. 1989;33(1):37–46.
4. Atanassov KT. Operators over interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst. 1994;64:159–74.
5. Atanassov KT, Gargov G. Interval-valued intuitionistic fuzzy sets. Fuzzy sets and systems 1989; 31: (3)343–349.
6. Liu XD, Zheng SH, Xiong FI. Entropy and subsethood for general interval-valued intuitionistic fuzzy sets. Fuzzy systems and knowledge discovery. 2005;42-52
7. Zhang HY, Zhang WX, Mei CL. Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure. Knowl-Based Syst. 2009;22(3):449–54.
8. Wang JQ, Li KJ, Zhang HY. Interval-valued intuitionistic fuzzy multi-criteria decision-making approach based on prospect score function. Knowl-Based Syst. 2012;27(3):119–25.
9. Du B, Zhang L. Target detection based on a dynamic subspace. Pattern Recogn. 2014;47(1):344–58.
10. Du B, Zhang L. A discriminative metric learning based anomaly detection method. IEEE Trans Geosci Remote Sens. 2014;52(11):6844–57.
11. Xu ZS, Yager RR. Intuitionistic and interval-valued intutionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group. Fuzzy Optim Decis Making. 2009;8:123–39.
12. Tan C, Zhang Q. Fuzzy multiple attribute decision making based on interval valued intuitionistic fuzzy sets. In Proceedings of the 2006 I.E. international conference on system, man, and cybernetics, Taipei, Taiwan. Republic of China. 2006;2:1404–7.
13. Hashemi SS, Razavi Hajiagha SH, Zavadskas EK, Mahdiraji HA. Multicriteria group decision making with ELECTRE III method based on interval-valued intuitionistic fuzzy information. Appl Math Model. 2016;40(2):1554–64.
14. Wang Z, Xu J. A fractional programming method for interval-valued intuitionistic fuzzy multi-attribute decision making. In Proceedings of the 49th IEEE international conference on decision and control 2010; 636–641.
15. Czubenko M, Kowalczuk Z, Ordys A. Autonomous driver based on an intelligent system of decision-making. Cogn Comput. 2015;7(5):569–81.
16. Li Y, Liu PD. Some Heronian mean operators with 2-tuple linguistic information and their application to multiple attribute group decision making. Technol Econ Dev Econ. 2015;21(5):797–814.
17. Li YH, Liu PD, Chen YB. Some single valued neutrosophic number Heronian mean operators and their application in multiple attribute group decision making.Informatica 2016; 27(1): 85–110.
18. Liu PD. Some generalized dependent aggregation operators with intuitionistic linguistic numbers and their application to group decision making. J Comput Syst Sci. 2013;79(1):131–43.
19. Liu PD. Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. IEEE Trans Fuzzy Syst. 2014;22(1):83–97.
20. Liu PD, Chen YB, Chu YC. Intuitionistic uncertain linguistic weighted Bonferroni OWA operator and its application to multiple attribute decision making. Cybern Syst. 2014;45(5):418–38.
21. Liu PD, Jin F. Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making. Information Sciences 2012; 205 (2)(2012) 58–71.
22. Liu PD, Li Y, Antuchevičienė J. A multi-criteria decision-making method based on intuitionistic trapezoidal fuzzy prioritized OWA operator. Technol Econ Dev Econ. 2016;22(3):453–69.
23. Liu PD, Liu ZM, Zhang X. Some intuitionistic uncertain linguistic heronian mean operators and their application to group decision making. Appl Math Comput. 2014;230:570–86.
24. Liu PD, Tang GL. Multi-criteria group decision-making based on interval neutrosophic uncertain linguistic variables and Choquet integral. Cogn Comput. 2016;8(6):1036–56.
25. Meng FY, Wang C, Chen XH. Linguistic interval hesitant fuzzy sets and their application in decision making. Cogn Comput. 2016;8(1):52–68.
26. Wei GW, Yi WD. Induced interval-valued intuitionistic fuzzy OWG operator. In Proceedings of fifth international conference on fuzzy systems and knowledge discovery 2008; 605–609.
27. Xu ZS. Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. Control and Decision. 2007;22(2):215–9.
28. Xu ZS, Chen J. On geometric aggregation over interval-valued intuitionistic fuzzy information. In Proceedings of the fourth international conference on fuzzy systems and knowledge discovery 2007; 466–471.
29. Xu ZS, Chen J. Approach to group decision making based on interval-valued intuitionistic judgment matrices. Systems Engineering – Theory and Practice. 2007;27(4):126–33.
30. Yu DJ, Wu YY, Lu T. Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making. Knowl-Based Syst. 2012;30(5):57–66.
31. Zhang HY, Ji P, Wang JQ, et al. A neutrosophic normal cloud and its application in decision-making. Cogn Comput. 2016;8:1–21.
32. Zhao H, Xu ZS, Ni MF, Liu S. Generalized aggregation operators for intuitionistic fuzzy sets. Int J Intell Syst. 2010;25(2):1–30.
33. Yager RR. The power average operator. IEEE Transactions on Systems Man and Cybernetics Part A-Systems and Humans. 2001;31(6):724–31.
34. Xu ZS, Yager RR. Power-geometric operators and their use in group decision making. IEEE Trans Fuzzy Syst. 2010;18(1):94–105.
35. Bonferroni C. Sulle medie multiple di potenze. Bolletino Matematica Italiana. 1950;5:267–70.
36. Zhu B, Xu ZS, Xia MM. Hesitant fuzzy geometric Bonferroni means. Inf Sci. 2010;205(1):72–85.

37. He YD, He Z. Extensions of Atanassov's intuitionistic fuzzy interaction Bonferroni means and their application to multiple attribute decision making. IEEE Trans Fuzzy Syst. 2016;24(3):558–73.
38. He YD, He Z, Chen HY. Intuitionistic fuzzy interaction Bonferroni means and its application to multiple attribute decision making. IEEE Transactions on Cybernetics. 2015;45(1):116–28.
39. He YD, He Z, Chao J, Chen HY. Intuitionistic fuzzy power geometric Bonferroni means and their application to multiple attribute group decision making. International Journal of Uncertainty Fuzziness and Knowledge-Based Systems. 2015;23:285–315.
40. He YD, He Z, Deng YJ, Zhou PP. IFPBMs and their application to multiple attribute group decision making. J Oper Res Soc. 2016;67(1):127–47.
41. He YD, He Z, Wang G, et al. Hesitant fuzzy power Bonferroni means and their application to multiple attribute decision making. IEEE Trans Fuzzy Syst. 2015;23(3):1655–68.
42. Xu ZS. Models for multiple attribute decision-making with intuitionistic fuzzy information. International Journal of Uncertainty, Fuzziness and Knowledge Based System. 2007;15(3):285–97.
43. Wang WZ, Liu XW. Intuitionistic fuzzy information aggregation using Einstein operations. IEEE Trans Fuzzy Syst. 2012;20(5):923–38.
44. Chen SM, Lee LW, Liu HC, Yang SW. Multiattribute decision making based on interval-valued intuitionistic fuzzy values. Expert Syst Appl. 2012;39(3):10343–51.
45. He YD, Chen H, Zhou L, Liu J, Tao Z. Generalized interval-valued atanassovs intuitionistic fuzzy power operators and their application to group decision making. International Journal of Fuzzy Systems. 2013;15(4):401–11.
46. Xu ZS, Chen Q. A multi-criteria decision making procedure based on interval-valued intuitionistic fuzzy Bonferroni means. J Syst Sci Syst Eng. 2011;20(2):217–28.