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Distance and Aggregation-Based Methodologies for Hesitant Fuzzy Decision Making

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Abstract Hesitant fuzzy set (HFS) as an effective tool to reflect human's hesitancy has received great attention in recent years. The importance weights of possible values in hesitant fuzzy elements (HFEs), which are the basic units of a HFS, have not been taken into account in the existing literature. Thus, the frequently used HFEs cannot deal with the situations where all the possible values are provided by experts with different levels of expertise. Consequently, in this paper, we propose an extension of typical HFS called the ordered weighted hesitant fuzzy set (OWHFS). The basic units of an OWHFS allow the membership of a given element to be defined in terms of several possible values together with their importance weights. Moreover, in order to indicate that the OWHFS has a good performance in decision making, we first present some information measures and several aggregation operators for OWHFSs. Then, we apply them to multi-attribute decision making with ordered weighted hesitant fuzzy information.

Keywords Ordered weighted hesitant fuzzy set (OWHFS) · Distance measure · Similarity measure · Aggregation operator · Multi-attribute decision making

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Introduction

Nowadays, many different extensions of fuzzy sets (FSs) have been made, including: L-fuzzy sets (L-FSs) [18], interval-valued fuzzy sets (IVFSs) [30], vague sets (VSs) [6], intuitionistic fuzzy sets (IFSs) [3], interval-valued intuitionistic fuzzy sets (IVIFSs) [4], linguistic fuzzy sets (LFSs) [36], type-2 fuzzy sets (T2FSs) [26], type-n fuzzy sets (TnFSs) [8], and fuzzy multisets (FMSs) [25].

Recently, Torra and Narukawa [27] introduced hesitant fuzzy sets (HFSs) which are quite suitable for the situation where we have a set of possible values. Later, a number of other extensions of the HFSs have been developed such as dual hesitant fuzzy sets (DHFSs) [12, 45], generalized hesitant fuzzy sets (G-HFSs) [28], hesitant fuzzy linguistic term sets (HFLTSs) [29], and higher order hesitant fuzzy sets (HOHFSs) [11]. Moreover, various applications of HFSs in decision making problems have been discussed in the existing literature, such as Rodriguez et al. [29], Wei et al. [33], Yu [40], Meng and Chen [22], Meng et al. [23], and Tian et al. [43], etc.

However, HFS [27] has its inherent drawbacks, because it expresses the membership degrees of an element to a given set only by possible values without emphasizing the importance of each possible value. In many practical decision making problems, the information provided by decision makers, who are familiar with the area, might often be described by the same preferences. In such situations, the value repeated several times is more important than that appeared only one time. Thus, the importance of possible membership degrees (i.e., their repetition rate) should be considered in improving the definition of HFS. This fact has, as far as we know, rarely been studied. The only work, in which the importance of possible values in HFEs has been considered, was done by Zhang and Wu [42].

They introduced the concept of weighted hesitant fuzzy set (WHFS), and then illustrated the procedure of constructing a WHFE as follows: Suppose that L experts are asked to evaluate the membership degree of the element x in the set ${}^{\omega}H$. l_1 experts provide $h^{\sigma(1)}(x)$, l_2 experts provide $h^{\sigma(2)}(x)$, ..., and l_m experts provide $h^{\sigma(m)}(x)$ such that $\sum_{k=1}^{m} l_k = L$. Keeping in the mind that these L experts cannot persuade each other to change their opinions. In such a situation, the membership degree of the element x in the set ${}^{\omega}H$ has m possible values $h^{\sigma(1)}(x)$, $h^{\sigma(2)}(x)$,..., and $h^{\sigma(m)}(x)$ associated respectively with the weights $w^{\sigma(1)}(x) = \frac{l_1}{L}$, $w^{\sigma(2)}(x) = \frac{l_2}{L}$,..., and $w^{\sigma(m)}(x) = \frac{l_m}{L}$. In this regard, the membership degree of the element x in the set ${}^{\omega}H$ should be represented by a weighted hesitant fuzzy element (WHFE) ${}^{\omega}h(x) = \bigcup_{1 \leq j \leq m} \{\langle h^{\sigma(j)}(x), w^{\sigma(j)}(x)\rangle\}$. On the basis of the above analysis, one can construct a WHFE by the help of these two steps: (i) Collecting different possible membership degrees into a HFE; (ii) Assigning the weights to these different membership degrees.

In this contribution, we will show that Zhang and Wu's definitions of union, intersection, addition, and multiplication operations for WHFSs are not correctly proposed. This motivates us to modify a fault of WHFS proposed by Zhang and Wu [42]. The modified definition of WHFS is acceptable in accordance with the well-known axioms for mathematical operations. It also allows that all information measures are to be defined reasonably. We call the new proposed extension of HFS as the ordered weighted HFS (OWHFS).

Nowadays, a growing number of studies have focused on the distance measure, the similarity measure for HFSs [37] and some extensions of HFS [9–17]. Distance measures are fundamentally important in various fields such as decision making, market prediction, pattern recognition, etc.

Based on the theorem which shows that the similarity and distance measures can be transformed by each other, this article deals mainly with distance measures for OWHFSs.

Besides the measures of HFSs, the aggregation operators for HFSs are one of the most important research topic at present. Many researchers have proposed a variety of aggregation operators for HFSs and investigated their properties. For instance, Xia and Xu [35] developed a series of aggregation operators for hesitant fuzzy information. Wei [31] investigated hesitant fuzzy prioritized operators. Zhu et al. [46] investigated hesitant fuzzy geometric Bonferroni means.

The recent popular attention to this research topic motivates us to develop some aggregation operators for OWHFEs in this contribution.

The present paper is organized as follows: The ordered weighted HFS (OWHFS) is introduced in "Preliminaries". Section "Distance and Similarity Measures for OWHFSs" presents the axioms for distance and similarity measures and gives a variety of distance measures for OWHFSs. Section "Aggregation Operators for OWHFSs" is devoted to the development of some aggregation operators for OWHFEs. In "Multi-Attribute Decision Making Problem Involving OWHFSs", we apply the proposed distance measures and aggregation operators to multi-attribute decision-making. Finally, the conclusion is drawn in "Conclusion".

Preliminaries

This section is devoted to describing the basic definitions and notions of hesitant fuzzy set (HFS) which was originally developed by Torra [27].

Definition 2.1 [35] Let X be the universe of discourse. A hesitant fuzzy set (HFS) on X is symbolized by

$$H = \{\langle x, h(x)\rangle : x \in X\},$$

where $h(x)$, referred to as the hesitant fuzzy element (HFE) [35], is a set of some values in [0, 1] . It denotes the possible membership degree of the element $x \in X$ to the set H.

Example 2.1 [10] If $X = \{x_1, x_2, x_3\}$ is the universe of discourse, $h(x_1) = \{0.2, 0.4, 0.5\}$, $h(x_2) = \{0.3, 0.4\}$ and $h(x_3) = \{0.3, 0.2, 0.5, 0.6\}$ are the HFEs of x_i $(i = 1, 2, 3)$ to a set H, respectively. Then H can be considered as a HFS, i.e.,

$$H = \{\langle x_1, \{0.2, 0.4, 0.5\}\rangle, \langle x_2, \{0.3, 0.4\}\rangle, \langle x_3, \{0.3, 0.2, 0.5.0.6\}\rangle\}.$$

Assumption 2.1 Notice that the number of values in different HFEs may be different. Suppose that $l(h_1(x))$ stands for the number of values in $h_1(x)$. Hereafter, the following assumptions are made: (see [31, 35, 37, 46]) (A1) All the elements in each $h_1(x)$ are arranged in increasing order, and then $h_1^{\sigma(j)}(x)$ is referred to as the jth largest value in $h_1(x)$. (A2) If, for some $x \in X$, $l(h_1(x)) \neq l(h_2(x))$, then $l_x = \max\{l(h_1(x)), l(h_2(x))\}$. To have a correct comparison, the two HFEs $h_1(x)$ and $h_2(x)$ should have the same length l_x. If there are fewer elements in $h_1(x)$ than in $h_2(x)$, then we can extend $h_1(x)$ by repeating its maximum element until it has the same length with $h_2(x)$.

Throughout this paper, we assume that all HFEs have the same length N.

Definition 2.2 Let $h = \bigcup_{1\le j\le N}\{h^{\sigma(j)}\}$, $h_1 = \bigcup_{1\le j\le N}\{h_1^{\sigma(j)}\}$ and $h_2 = \bigcup_{1\le j\le N}\{h_2^{\sigma(j)}\}$ be three HFEs. Then, some operations on the HFEs h, h_1 and h_2 which are also HFEs can be defined as follows (see [35] and [27]):

$$h^c = \bigcup_{1\le j\le N}\{1-h^{\sigma(j)}\}; \tag{1}$$

$$h_1\cup h_2 = \bigcup_{1\le j\le N}\{\max\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}\}; \tag{2}$$

$$h_1\cap h_2 = \bigcup_{1\le j\le N}\{\min\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}\}; \tag{3}$$

$$h^\lambda = \bigcup_{1\le j\le N}\{h^{\sigma(j)^\lambda}\}; \tag{4}$$

$$\lambda h = \bigcup_{1\le j\le N}\{1-(1-h^{\sigma(j)})^\lambda\}; \tag{5}$$

$$h_1\oplus h_2 = \bigcup_{1\le j\le N}\{h_1^{\sigma(j)}+h_2^{\sigma(j)}-h_1^{\sigma(j)}h_2^{\sigma(j)}\}; \tag{6}$$

$$h_1\otimes h_2 = \bigcup_{1\le j\le N}\{h_1^{\sigma(j)}h_2^{\sigma(j)}\}. \tag{7}$$

Note that all the latter definitions are not only possible expressions for these operations. Among the great variety of expressions for the operations of complement, union and intersection the above standard fuzzy operations have certain properties that give them a special significance [21].

As can be seen from Definition 2.1, HFS expresses the membership degrees of an element to a given set only by several real numbers between 0 and 1 with equal importance. In many real-world situations, assigning exact values without importance weights to the membership degrees does not describe properly the imprecise or uncertain decision information. Thus, it seems to be difficult for the decision makers to rely on the present form of HFSs for expressing uncertainty of an element.

To overcome the difficulty associated with the present form of HFSs, Zhang and Wu [42] introduced the concept of weighted hesitant fuzzy set (WHFS). The membership degrees of an element to a WHFS can be expressed by several possible values together with their importance weights.

Definition 2.3 [42] Let X be the universe of discourse. Zhang and Wu's representation of weighted hesitant fuzzy set (WHFS) on X can be defined as:

$$\begin{aligned}{}^{\omega}H &= \{\langle x, {}^{\omega}h(x)\rangle : x\in X\}\\ &= \{\langle x, \bigcup_{\gamma\in {}^{\omega}h(x)}\{(\gamma, w_{x\gamma})\}\rangle : x\in X\},\end{aligned} \tag{8}$$

where ${}^{\omega}h(x)$ is a set of some different values in [0, 1], denoting all possible membership degrees of the element $x\in X$ to the set ${}^{\omega}H$, and $w_{x\gamma}\in[0,1]$ is the weight of γ such that $\sum_{\gamma\in {}^{\omega}h(x)} w_{x\gamma}=1$ for any $x\in X$.

Zhang and Wu [42] called ${}^{\omega}h(x)=\bigcup_{\gamma\in {}^{\omega}h(x)}\{(\gamma, w_{x\gamma})\}$ a weighted hesitant fuzzy element (WHFE). A WHFE is conveniently denoted by ${}^{\omega}h=\bigcup_{\gamma\in {}^{\omega}h}\{(\gamma, w_\gamma)\}$.

Definition 2.4 [42] Let ${}^{\omega}h = \bigcup_{\gamma\in {}^{\omega}h}\{(\gamma, w_\gamma)\}$, ${}^{\omega}h_1 = \bigcup_{\gamma_1\in {}^{\omega}h_1}\{(\gamma_1, w_{\gamma_1})\}$ and ${}^{\omega}h_2=\bigcup_{\gamma_2\in {}^{\omega}h_2}\{(\gamma_2, w_{\gamma_2})\}$ be three WHFEs. Then, some operations on the WHFEs ${}^{\omega}h$, ${}^{\omega}h_1$ and ${}^{\omega}h_2$ were defined by Zhang and Wu as follows:

$${}^{\omega}h^c = \bigcup_{\gamma\in {}^{\omega}h}\{(1-\gamma, w_\gamma)\}; \tag{9}$$

$${}^{\omega}h_1\cup {}^{\omega}h_2 = \bigcup_{\gamma_1\in {}^{\omega}h_1,\gamma_2\in {}^{\omega}h_2}\{(\max\{\gamma_1,\gamma_2\}, w_{\gamma_1}.w_{\gamma_2})\}; \tag{10}$$

$${}^{\omega}h_1\cap {}^{\omega}h_2 = \bigcup_{\gamma_1\in {}^{\omega}h_1,\gamma_2\in {}^{\omega}h_2}\{(\min\{\gamma_1,\gamma_2\}, w_{\gamma_1}.w_{\gamma_2})\}. \tag{11}$$

$${}^{\omega}h^\lambda = \bigcup_{\gamma\in {}^{\omega}h}\{(\gamma^\lambda, w_\gamma)\}; \tag{12}$$

$$\lambda^{\omega}h = \bigcup_{\gamma_1\in {}^{\omega}h_1,\gamma_2\in {}^{\omega}h_2}\{(1-(1-\gamma)^\lambda, w_\gamma)\}; \tag{13}$$

$${}^{\omega}h_1\oplus {}^{\omega}h_2 = \bigcup_{\gamma_1\in {}^{\omega}h_1,\gamma_2\in {}^{\omega}h_2}\{(\gamma_1+\gamma_2-\gamma_1\gamma_2, w_{\gamma_1}.w_{\gamma_2})\}; \tag{14}$$

$${}^{\omega}h_1\otimes {}^{\omega}h_2 = \bigcup_{\gamma_1\in {}^{\omega}h_1,\gamma_2\in {}^{\omega}h_2}\{(\gamma_1.\gamma_2, w_{\gamma_1}.w_{\gamma_2})\}. \tag{15}$$

Notice that Zhang and Wu [42] were careless about their definitions of the above mathematical operations because such definitions inherit some fundamental disadvantages. It is not hard to see that Zhang and Wu's union and intersection operations given by Eqs. 10 and 11 are not idempotent, that is, for any WHFE ${}^{\omega}h = \bigcup_{\gamma\in {}^{\omega}h}\{(\gamma, w_\gamma)\} = \{(\gamma_1, w_{\gamma_1}), ..., (\gamma_l, w_{\gamma_l})\}$

$$\begin{aligned}{}^{\omega}h\cup {}^{\omega}h &= \bigcup_{1\le j\le l}\{(\gamma_j, f_j(w_{\gamma_1}, ..., w_{\gamma_l}))\}\\ &\ne \bigcup_{1\le j\le l}\{(\gamma_j, w_{\gamma_j})\} = {}^{\omega}h;\end{aligned} \tag{16}$$

$$\begin{aligned}{}^{\omega}h\cap {}^{\omega}h &= \bigcup_{1\le j\le l}\{(\gamma_j, g_j(w_{\gamma_1}, ..., w_{\gamma_l}))\}\\ &\ne \bigcup_{1\le j\le l}\{(\gamma_j, w_{\gamma_j})\} = {}^{\omega}h,\end{aligned} \tag{17}$$

where f_j and g_j are real functions of $w_{\gamma_1}, ..., w_{\gamma_l}$ such that $f_j(w_{\gamma_1}, ..., w_{\gamma_l}) \ne w_{\gamma_j}$ and $g_j(w_{\gamma_1}, ..., w_{\gamma_l}) \ne w_{\gamma_j}$ for $1\le j\le l$.

For example, a company wants to classify some different cars. It asks 10 experts to provide their evaluation information of a car with respect to the safety criterion. Six experts express their evaluation information by the value "70 percent" and others by the value "80 percent". Keeping in mind that these 10 experts cannot persuade each other to change their opinions. In such a situation, their evaluation information can be described by a WHFE as ${}^{\omega}h = \{\langle 0.7, \frac{6}{10}\rangle, \langle 0.8, \frac{4}{10}\rangle\}$. If we apply Zhang and Wu's union and intersection definitions given by Eqs. 10 and 11 to ${}^{\omega}h$, it results in

$$
\begin{aligned}
{}^{\omega}h \cup {}^{\omega}h &= \{\langle 0.7, 0.36\rangle, \langle 0.8, 0.64\rangle\};\\
{}^{\omega}h \cap {}^{\omega}h &= \{\langle 0.7, 0.84\rangle, \langle 0.8, 0.16\rangle\}.
\end{aligned}
$$

From ${}^{\omega}h \cup {}^{\omega}h$, one finds that near four experts are confident with "70 %" about the safety of a car, and near six experts are confident with "80 %". But, as observed from the definition of WHFE ${}^{\omega}h = \{\langle 0.7, \frac{6}{10}\rangle, \langle 0.8, \frac{4}{10}\rangle\}$, the number of experts who are confident with "70 %" and "80 %" are 6 and 4, respectively. Such a comparison of confidence level can be made for ${}^{\omega}h \cap {}^{\omega}h$, where near eight experts are confident with "70 %" about the safety of a car, and near two experts are confident with "80 %". These numbers of experts have been already mentioned as 6 and 4 in the WHFE ${}^{\omega}h$.

On the other hand, one can see that applying Zhang and Wu's addition and multiplication definitions given by Eqs. 14 and 15 to any WHFE ${}^{\omega}h$ does not give a reasonable result, that is,

$$
\begin{aligned}
{}^{\omega}h \oplus {}^{\omega}h &= \bigcup_{1\le j\le l} \{(2\gamma_j - \gamma_j^2, f_j(w_{\gamma_1}, ..., w_{\gamma_l}))\}\\
&\neq \bigcup_{1\le j\le l} \{(2\gamma_j - \gamma_j^2, w_{\gamma_j}))\} = 2\,{}^{\omega}h;\\
{}^{\omega}h \otimes {}^{\omega}h &= \bigcup_{1\le j\le l} \{(\gamma_j^2, g_j(w_{\gamma_1}, ..., w_{\gamma_l}))\}\\
&\neq \bigcup_{1\le j\le l} \{(\gamma_j^2, w_{\gamma_j})\} = {}^{\omega}h^2,
\end{aligned}
$$

where f_j and g_j are real functions of $w_{\gamma_1}, ..., w_{\gamma_l}$ such that $f_j(w_{\gamma_1}, ..., w_{\gamma_l}) \neq w_{\gamma_j}$ and $g_j(w_{\gamma_1}, ..., w_{\gamma_l}) \neq w_{\gamma_j}$ for $1 \le j \le l$.

Once again, we consider the WHFE ${}^{\omega}h = \{\langle 0.7, \frac{6}{10}\rangle, \langle 0.8, \frac{4}{10}\rangle\}$, then

$$
\begin{aligned}
{}^{\omega}h \oplus {}^{\omega}h &= \{\langle 0.91, 0.36\rangle, \langle 0.94, 0.48\rangle, \langle 0.96, 0.16\rangle\} \neq 2\,{}^{\omega}h\\
&= \{\langle 0.91, 0.6\rangle, \langle 0.96, 0.4\rangle\};\\
{}^{\omega}h \otimes {}^{\omega}h &= \{\langle 0.49, 0.36\rangle, \langle 0.56, 0.48\rangle, \langle 0.64, 0.16\rangle\} \neq {}^{\omega}h^2\\
&= \{\langle 0.49, 0.6\rangle, \langle 0.64, 0.4\rangle\}.
\end{aligned}
$$

Here, in order to avoid the disadvantages arising from Zhang and Wu's definition of WHFS and mathematical operations on WHFSs, we define the ordered weighted hesitant fuzzy set (OWHFS) as follows:

Definition 2.5 Let X be the universe of discourse. An ordered weighted hesitant fuzzy set (OWHFS) on X is defined as:

$$
\begin{aligned}
{}^{\omega}H &= \{\langle x, {}^{\omega}h(x)\rangle : x \in X\}\\
&= \{\langle x, \bigcup_{1\le j\le l_x} \{\langle h^{\sigma(j)}(x), w^{\sigma(j)}(x)\rangle\}\rangle : x \in X\}, \qquad (18)
\end{aligned}
$$

where ${}^{\omega}h(x)$, referred to as the ordered weighted hesitant fuzzy element (OWHFE), is a set of some different values in [0, 1]. It denotes all possible membership degrees of the element $x \in X$ to the set ${}^{\omega}H$, and $w^{\sigma(j)}(x) \in [0, 1]$ is the weight of $h^{\sigma(j)}(x)$ such that $\sum_{1\le j\le l_x} w^{\sigma(j)}(x) = 1$ for any $x \in X$.

It is interesting to note that if we take $w^{\sigma(1)}(x) = ... = w^{\sigma(l_x)}(x) = \frac{1}{l_x}$ for any $x \in X$, then the OWHFS ${}^{\omega}H$ is reduced to a typical HFS.

Hereafter, for the convenience of representation, we denote the OWHFE ${}^{\omega}h(x)$ by ${}^{\omega}h = \bigcup_{1\le j\le l_x} \{\langle h^{\sigma(j)}, w^{\sigma(j)}\rangle\}$.

Assumption 2.2 Notice that the number of values in different OWHFEs may be different. Suppose that $l({}^{\omega}h_1(x))$ stands for the number of values in ${}^{\omega}h_1(x)$. Hereafter, the following assumptions are made: (A1) All the first components of elements in each ${}^{\omega}h_1(x)$ are arranged in increasing order, and then $h_1^{\sigma(j)}(x)$ is referred to as the jth largest value in ${}^{\omega}h_1(x)$. (A2) If, for some $x \in X$, $l({}^{\omega}h_1(x)) \neq l({}^{\omega}h_2(x))$, then $l_x = \max\{l({}^{\omega}h_1(x)), l({}^{\omega}h_2(x))\}$. To have a correct comparison, the two OWHFEs ${}^{\omega}h_1(x)$ and ${}^{\omega}h_2(x)$ should have the same length l_x. If there are fewer elements in ${}^{\omega}h_1(x)$ than in ${}^{\omega}h_2(x)$, then we can extend ${}^{\omega}h_1(x)$ by repeating the maximum first component of elements associated with zero weight until it has the same length with ${}^{\omega}h_2(x)$. This kind of extension is quite reasonable since the added element with zero weight is meant to be an element that does not really exist.

Throughout this paper, we assume that all OWHFEs have the same length N.

Definition 2.6 Let ${}^{\omega}h = \bigcup_{1\le j\le N} \{\langle h^{\sigma(j)}, w^{\sigma(j)}\rangle\}$, ${}^{\omega}h_1 = \bigcup_{1\le j\le N} \{\langle h_1^{\sigma(j)}, w_1^{\sigma(j)}\rangle\}$ and ${}^{\omega}h_2 = \bigcup_{1\le j\le N} \{\langle h_2^{\sigma(j)}, w_2^{\sigma(j)}\rangle\}$

be three OWHFEs. Then, some operations on the OWHFEs ${}^{\omega}h$, ${}^{\omega}h_1$ and ${}^{\omega}h_2$ are defined as follows:

$$ {}^{\omega}h^c = \bigcup_{1\le j\le N} \{\langle 1-h^{\sigma(j)}, w^{\sigma(j)}\rangle\}; \quad (19)$$

$$ {}^{\omega}h_1 \cup {}^{\omega}h_2 = \bigcup_{1\le j\le N} \{\langle \max\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}, \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}\rangle\}; \quad (20)$$

$$ {}^{\omega}h_1 \cap {}^{\omega}h_2 = \bigcup_{1\le j\le N} \{\langle \min\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}, \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}\rangle\}; \quad (21)$$

$$ {}^{\omega}h^{\lambda} = \bigcup_{1\le j\le N} \{\langle {h^{\sigma(j)}}^{\lambda}, w^{\sigma(j)}\rangle\}; \quad (22)$$

$$ \lambda{}^{\omega}h = \bigcup_{1\le j\le N} \{\langle 1-(1-h^{\sigma(j)})^{\lambda}, w^{\sigma(j)}\rangle\}; \quad (23)$$

$$ {}^{\omega}h_1 \oplus {}^{\omega}h_2 = \bigcup_{1\le j\le N} \{\langle h_1^{\sigma(j)}+h_2^{\sigma(j)}-h_1^{\sigma(j)}h_2^{\sigma(j)}, \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}\rangle\}; \quad (24)$$

$$ {}^{\omega}h_1 \otimes {}^{\omega}h_2 = \bigcup_{1\le j\le N} \{\langle h_1^{\sigma(j)}h_2^{\sigma(j)}, \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}\rangle\}, \quad (25)$$

where $\overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}$ for $1\le j\le N$, referred hereafter to as the normalized weights, are determined in two steps: (i) We first calculate the weight of jth component of the binary operation ${}^{\omega}h_1 \odot {}^{\omega}h_2$ by simply adding the weights $w_1^{\sigma(j)}$ and $w_2^{\sigma(j)}$ for $1\le j\le N$; (ii) After the whole components of ${}^{\omega}h_1 \odot {}^{\omega}h_2$ are obtained, their weights are considered again and then normalized. In this regard, the normalized weights of the above binary operations are defined as follows:

$$ \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})} = \frac{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}{\sum_{k=1}^{N}(w_1^{\sigma(k)}+w_2^{\sigma(k)})}, \quad 1\le j\le N. \quad (26)$$

Remark 2.1 In the case that the associative binary operation $\odot$ is iterated on the finite set of OWHFEs ${}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_m$, i.e., ${}^{\omega}h_1 \odot {}^{\omega}h_2 \odot ... \odot {}^{\omega}h_m = (...(({}^{\omega}h_1 \odot {}^{\omega}h_2) \odot {}^{\omega}h_3)... \odot {}^{\omega}h_m)$, we can construct the normalized weights as:

$$ \overline{(\sum_{i=1}^{m} w_i^{\sigma(j)})} := \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)}+...+w_m^{\sigma(j)})} = \frac{(...((w_1^{\sigma(j)}+w_2^{\sigma(j)})+w_3^{\sigma(j)})+...+w_m^{\sigma(j)})}{\sum_{k=1}^{N}(w_1^{\sigma(k)}+w_2^{\sigma(k)}+...+w_m^{\sigma(k)})}, \quad 1\le j\le N. \quad (27)$$

Example 2.2 Suppose that ${}^{\omega}h_1 = \{\langle 0.2, 0.1\rangle, \langle 0.4, 0.3\rangle, \langle 0.5, 0.6\rangle\}$ and ${}^{\omega}h_2 = \{\langle 0.3, 0.5\rangle, \langle 0.7, 0.5\rangle\}$ are two given OWHFEs. Bearing Assumption 2.2 in mind, ${}^{\omega}h_2$ should be first extended as ${}^{\omega}h_2 = \{\langle 0.3, 0.5\rangle, \langle 0.7, 0.5\rangle, \langle 0.7, 0.0\rangle\}$. Then, we get

$$ {}^{\omega}h_1^c = \{\langle 0.5, 0.6\rangle, \langle 0.6, 0.3\rangle, \langle 0.8, 0.1\rangle\};$$

$$ {}^{\omega}h_1 \cup {}^{\omega}h_2 = \{\langle \max\{0.2, 0.3\}, \frac{(0.1+0.5)}{2}\rangle, \langle 0.7, 0.4\rangle, \langle 0.7, 0.3\rangle\};$$

$$ {}^{\omega}h_1 \cap {}^{\omega}h_2 = \{\langle \min\{0.2, 0.3\}, \frac{(0.1+0.5)}{2}\rangle, \langle 0.4, 0.4\rangle, \langle 0.5, 0.3\rangle\};$$

$$ ({}^{\omega}h_1 \cup {}^{\omega}h_2) \cup {}^{\omega}h_1 = \{\langle \max\{\max\{0.2, 0.3\}, 0.2\}, \frac{((0.1+0.5)+0.1)}{3}\rangle, \langle 0.7, \frac{1.1}{3}\rangle, \langle 0.7, \frac{1.2}{3}\rangle\};$$

$$ ({}^{\omega}h_1 \cap {}^{\omega}h_2) \cap {}^{\omega}h_1 = \{\langle \min\{\min\{0.2, 0.3\}, 0.2\}, \frac{((0.1+0.5)+0.1)}{3}\rangle, \langle 0.4, \frac{1.1}{3}\rangle, \langle 0.5, \frac{1.2}{3}\rangle\};$$

$$ {}^{\omega}h_1^{\lambda} = \{\langle 0.2^{\lambda}, 0.1\rangle, \langle 0.4^{\lambda}, 0.3\rangle, \langle 0.5^{\lambda}, 0.6\rangle\};$$

$$ \lambda{}^{\omega}h_1 = \{\langle 1-0.8^{\lambda}, 0.1\rangle, \langle 1-0.6^{\lambda}, 0.3\rangle, \langle 1-0.5^{\lambda}, 0.6\rangle\};$$

$$ {}^{\omega}h_1 \oplus {}^{\omega}h_2 = \{\langle 0.44, 0.3\rangle, \langle 0.82, 0.4\rangle, \langle 0.85, 0.3\rangle\};$$

$$ {}^{\omega}h_1 \otimes {}^{\omega}h_2 = \{\langle 0.06, 0.3\rangle, \langle 0.28, 0.4\rangle, \langle 0.35, 0.3\rangle\}.$$

Theorem 2.1 *Let* ${}^{\omega}h = \bigcup_{1\le j\le N}\{\langle h^{\sigma(j)}, w^{\sigma(j)}\rangle\}$, ${}^{\omega}h_1 = \bigcup_{1\le j\le N}\{\langle h_1^{\sigma(j)}, w_1^{\sigma(j)}\rangle\}$ *and* ${}^{\omega}h_2 = \bigcup_{1\le j\le N}\{\langle h_2^{\sigma(j)}, w_2^{\sigma(j)}\rangle\}$ *be three OWHFEs. Then, all operations* ${}^{\omega}h_1^c$, ${}^{\omega}h_1 \cup {}^{\omega}h_2$, ${}^{\omega}h_1 \cap {}^{\omega}h_2$, ${}^{\omega}h_1^{\lambda}$, $\lambda{}^{\omega}h_1$, ${}^{\omega}h_1 \oplus {}^{\omega}h_2$, ${}^{\omega}h_1 \otimes {}^{\omega}h_2$ *are also OWHFEs.*

Proof We only prove that ${}^{\omega}h_1 \cup {}^{\omega}h_2$ is also OWHFE. Known by the definition of ${}^{\omega}h_1 \cup {}^{\omega}h_2$ from Eq. 20, i.e., ${}^{\omega}h_1 \cup {}^{\omega}h_2 = \bigcup_{1\le j\le N}\{\langle \max\{h_1^{\sigma(j)}, h_2^{\sigma(j)}\}, \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}\rangle\}$, we need to show that

$$ \sum_{j=1}^{N} \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})} = 1.$$

By definition of the normalized weight $\overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}$, we have

$$ \sum_{j=1}^{N} \overline{(w_1^{\sigma(j)}+w_2^{\sigma(j)})} = \sum_{j=1}^{N} \frac{(w_1^{\sigma(j)}+w_2^{\sigma(j)})}{\sum_{k=1}^{N}(w_1^{\sigma(k)}+w_2^{\sigma(k)})} = \frac{\sum_{j=1}^{N}(w_1^{\sigma(j)}+w_2^{\sigma(j)})}{\sum_{k=1}^{N}(w_1^{\sigma(k)}+w_2^{\sigma(k)})} = 1.$$

This completes the proof. □

Theorem 2.2 *Let* ${}^{\omega}h = \bigcup_{1\le j\le N}\{\langle h^{\sigma(j)}, w^{\sigma(j)}\rangle\}$, ${}^{\omega}h_1 = \bigcup_{1\le j\le N}\{\langle h_1^{\sigma(j)}, w_1^{\sigma(j)}\rangle\}$ *and* ${}^{\omega}h_2 = \bigcup_{1\le j\le N}\{\langle h_2^{\sigma(j)}, w_2^{\sigma(j)}\rangle\}$ *be three OWHFEs. Then,*

$$({}^{\omega}h^c)^{\lambda} = (\lambda{}^{\omega}h)^c; \quad (28)$$
$$({}^{\omega}h^{\lambda})^c = \lambda({}^{\omega}h^c); \quad (29)$$
$$({}^{\omega}h_1 \cup {}^{\omega}h_2)^c = {}^{\omega}h_1^c \cap {}^{\omega}h_2^c; \quad (30)$$
$$({}^{\omega}h_1 \cap {}^{\omega}h_2)^c = {}^{\omega}h_1^c \cup {}^{\omega}h_2^c; \quad (31)$$
$$({}^{\omega}h_1 \otimes {}^{\omega}h_2)^{\lambda} = {}^{\omega}h_1^{\lambda} \otimes {}^{\omega}h_2^{\lambda}; \quad (32)$$
$$\lambda({}^{\omega}h_1 \oplus {}^{\omega}h_2) = \lambda{}^{\omega}h_1 \oplus \lambda{}^{\omega}h_2; \quad (33)$$
$$({}^{\omega}h_1 \oplus {}^{\omega}h_2)^c = {}^{\omega}h_1^c \otimes {}^{\omega}h_2^c; \quad (34)$$
$$({}^{\omega}h_1 \otimes {}^{\omega}h_2)^c = {}^{\omega}h_1^c \oplus {}^{\omega}h_2^c; \quad (35)$$
$${}^{\omega}h_1 \oplus {}^{\omega}h_2 = {}^{\omega}h_2 \oplus {}^{\omega}h_1; \quad (36)$$
$${}^{\omega}h_1 \otimes {}^{\omega}h_2 = {}^{\omega}h_2 \otimes {}^{\omega}h_1; \quad (37)$$
$${}^{\omega}h_1 \cup {}^{\omega}h_2 = {}^{\omega}h_2 \cup {}^{\omega}h_1; \quad (38)$$
$${}^{\omega}h_1 \cap {}^{\omega}h_2 = {}^{\omega}h_2 \cap {}^{\omega}h_1; \quad (39)$$
$${}^{\omega}h \cup ({}^{\omega}h_1 \cup {}^{\omega}h_2) = ({}^{\omega}h \cup {}^{\omega}h_1) \cup {}^{\omega}h_2; \quad (40)$$
$${}^{\omega}h \cap ({}^{\omega}h_1 \cap {}^{\omega}h_2) = ({}^{\omega}h \cap {}^{\omega}h_1) \cap {}^{\omega}h_2; \quad (41)$$
$${}^{\omega}h \cup {}^{\omega}h = {}^{\omega}h; \quad (42)$$
$${}^{\omega}h \cap {}^{\omega}h = {}^{\omega}h; \quad (43)$$
$${}^{\omega}h \oplus {}^{\omega}h = 2\,{}^{\omega}h; \quad (44)$$
$${}^{\omega}h \otimes {}^{\omega}h = {}^{\omega}h^2. \quad (45)$$

It is noteworthy to say that properties given by Eqs. 42–45 show the superiority of OWHFS proposed here over WHFS suggested by Zhang and Wu [42].

For further discussion on the properties of aggregation operators, we need to define a comparison law to compare OWHFEs.

Definition 2.7 Let ${}^{\omega}h = \bigcup_{1\le j\le N}\{\langle h^{\sigma(j)}, w^{\sigma(j)}\rangle\}$ be an OWHFE. Then, we define

$$\triangle({}^{\omega}h) = \sum_{j=1}^{N} h^{\sigma(j)} w^{\sigma(j)}, \quad (46)$$

$$\nabla({}^{\omega}h) = \sum_{j=1}^{N} (\triangle({}^{\omega}h) - h^{\sigma(j)})^2 w^{\sigma(j)}. \quad (47)$$

Now, we are in a position to present a law to compare any two OWHFEs ${}^{\omega}h_1$ and ${}^{\omega}h_2$ as follows:

- If $\triangle({}^{\omega}h_1) > \triangle({}^{\omega}h_2)$, then ${}^{\omega}h_1 > {}^{\omega}h_2$;
- If $\triangle({}^{\omega}h_1) = \triangle({}^{\omega}h_2)$, then
  - if $\nabla({}^{\omega}h_1) > \nabla({}^{\omega}h_2)$, then ${}^{\omega}h_1 < {}^{\omega}h_2$;
  - if $\nabla({}^{\omega}h_1) < \nabla({}^{\omega}h_2)$, then ${}^{\omega}h_1 > {}^{\omega}h_2$;
  - if $\nabla({}^{\omega}h_1) = \nabla({}^{\omega}h_2)$, then ${}^{\omega}h_1 = {}^{\omega}h_2$.

Distance and Similarity Measures for OWHFSs

There are many studies which deal with the distance measures and the similarity measures for FSs [41], IFSs [2] and interval-valued intuitionistic fuzzy sets (IVIFSs) [32]. Little effort has been made to study the similarity measures for T2FSs [24]. There are several similarity measures for interval T2FSs (IT2FSs), such as Gorzalczany's degree of compatibility [19], Bustince's interval-valued normal similarity measure [5], and Wu and Mendel's vector similarity measure [34].

Distance measures are fundamentally important in various fields such as decision making, market prediction, pattern recognition, etc.

A growing number of studies have focused on the distance measure and the similarity measure for HFSs [10, 37] and HOHFS [11]. In this section, we are interested in introducing a class of distance measures and similarity measures for OWHFSs.

In the following, we first give the axiomatic definition of information measures for OWHFSs. First of all, we call ${}^{\omega}0$ the empty OWHFS, where ${}^{\omega}0 = \{\langle x, {}^{\omega}0(x)\rangle : x \in X\} = \{\langle x, \bigcup_{1\le j\le N}\{\langle 0, \frac{1}{N}\rangle\}\rangle : x \in X\}$. We call ${}^{\omega}1$ the full OWHFS, where ${}^{\omega}1 = \{\langle x, {}^{\omega}1(x)\rangle : x \in X\} = \{\langle x, \bigcup_{1\le j\le N}\{\langle 1, \frac{1}{N}\rangle\}\rangle : x \in X\}$.

Definition 3.1 Let ${}^{\omega}H_1 = \{\langle x, {}^{\omega}h_1(x)\rangle : x \in X\}$ and ${}^{\omega}H_2 = \{\langle x, {}^{\omega}h_2(x)\rangle : x \in X\}$ be two OWHFSs on X. Then d is called a distance measure for OWHFSs if it possesses the following properties:

(d0) *Boundary*: $0 \le d({}^{\omega}H_1, {}^{\omega}H_2) \le 1$;
(d1) *Symmetry*: $d({}^{\omega}H_1, {}^{\omega}H_2) = d({}^{\omega}H_2, {}^{\omega}H_1)$;
(d2) *Complementarity*: $d({}^{\omega}H_1, {}^{\omega}H_1^c) = 1$ iff ${}^{\omega}H_1$ is the empty OWHFS ${}^{\omega}0$ or the full OWHFS ${}^{\omega}1$;
(d3) *Reflexivity*: $d({}^{\omega}H_1, {}^{\omega}H_2) = 0$ iff ${}^{\omega}H_1 = {}^{\omega}H_2$;

where ${}^{\omega}H_1^c = \{\langle x, {}^{\omega}h_1^c(x)\rangle : x \in X\}$ is the complement set of OWHFS ${}^{\omega}H_1$.

Definition 3.2 Let ${}^{\omega}H_1 = \{\langle x, {}^{\omega}h_1(x)\rangle : x \in X\}$ and ${}^{\omega}H_2 = \{\langle x, {}^{\omega}h_2(x)\rangle : x \in X\}$ be two OWHFSs on X. Then S is called a similarity measure for OWHFSs if it possesses the following properties:

(S0) *Boundary*: $0 \le S({}^{\omega}H_1, {}^{\omega}H_2) \le 1$;
(S1) *Symmetry*: $S({}^{\omega}H_1, {}^{\omega}H_2) = S({}^{\omega}H_2, {}^{\omega}H_1)$;
(S2) *Complementarity*: $S({}^{\omega}H_1, {}^{\omega}H_1^c) = 0$ if ${}^{\omega}H_1 = {}^{\omega 0}$ or ${}^{\omega}H_1 = {}^{\omega 1}$;
(S3) *Reflexivity*: $S({}^{\omega}H_1, {}^{\omega}H_2) = 1$ iff ${}^{\omega}H_1 = {}^{\omega}H_2$.

Theorem 3.1 *Let* $Z : [0,1] \to [0,1]$ *be a strictly monotone decreasing real function, and d be a distance between*

OWHFSs. Then, for any OWHFSs ${}^{\omega}H_1$ and ${}^{\omega}H_2$ on X

$$S_d({}^{\omega}H_1, {}^{\omega}H_2) = \frac{Z(d({}^{\omega}H_1, {}^{\omega}H_2)) - Z(1)}{Z(0) - Z(1)},$$

is a similarity measure for OWHFSs based on the corresponding distance d.

Proof From Definition 3.1 and the property of $Z(.)$, it is evident that S_d meets all the requirements listed in Definition 3.2.

By Theorem 3.1, different formulas can be developed to calculate the similarity measures between OWHFSs using different strictly monotone decreasing functions $Z : [0, 1] \to [0, 1]$, for instance, (1) $Z(t) = 1 - t$; (2) $Z(t) = \frac{1-t}{1+t}$; (3) $Z(t) = 1 - te^{t-1}$; (4) $Z(t) = 1 - t^2$.

For more information regarding the relationship between the distance measure and the similarity measure for HFSs (as a special case of OWHFSs) based on their axiomatic definitions, please refer to [10]. Here, we mainly discuss the distance measures for OWHFSs, and the corresponding similarity measures can be obtained easily.

The definitions of distance measures for OWHFSs are based on those for their OWHFEs. Among numerous distance measures for HFSs, the most widely used distance measures for two HFSs $H_1 = \{\langle x, h_1(x)\rangle : x \in X\} = \{\langle x, \bigcup_{1\le j\le l_x}\{h_1^{\sigma(j)}(x)\}\rangle : x \in X\}$ and $H_2 = \{\langle x, h_2(x)\rangle : x \in X\} = \{\langle x, \bigcup_{1\le j\le l_x}\{h_2^{\sigma(j)}(x)\}\rangle : x \in X\}$ on $X = \{x_1, ..., x_n\}$ are as follows (see [37]):

- The generalized hesitant normalized distance:

$$d_{ghn}(H_1, H_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}; \tag{48}$$

- The generalized hesitant normalized Hausdorff distance:

$$d_{ghnh}(H_1, H_2) = [\frac{1}{n}\sum_{i=1}^{n}\max_{1\le j\le l_{x_i}}\{|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda}\}]^{\frac{1}{\lambda}}; \tag{49}$$

- The generalized hybrid hesitant normalized distance:

$$d_{ghhn}(H_1, H_2) = [\frac{1}{2n}\sum_{i=1}^{n}(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda} + \max_{1\le j\le l_{x_i}}\{|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda}\})]^{\frac{1}{\lambda}}, \tag{50}$$

where $\lambda > 0$.

If the weight of the element $x_i \in X$ is ξ_i $(i = 1, ..., n)$ with $\xi_i \in [0, 1]$ and $\sum_{i=1}^{n}\xi_i = 1$, then we get the weighted form of distance measures as follows:

- The generalized hesitant weighted normalized distance:

$$d_{ghwn}(H_1, H_2) = \left[\sum_{i=1}^{n}\xi_i\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda}\right)\right]^{\frac{1}{\lambda}}; \tag{51}$$

- The generalized hesitant weighted normalized Hausdorff distance:

$$d_{ghwnh}(H_1, H_2) = \left[\sum_{i=1}^{n}\xi_i\max_{1\le j\le l_{x_i}}\left\{|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda}\right\}\right]^{\frac{1}{\lambda}}; \tag{52}$$

- The generalized hybrid hesitant weighted normalized distance:

$$d_{ghhwn}(H_1, H_2) = \left[\frac{1}{2}\sum_{i=1}^{n}\xi_i\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda} + \max_{1\le j\le l_{x_i}}\{|h_1^{\sigma(j)}(x_i) - h_2^{\sigma(j)}(x_i)|^{\lambda}\}\right)\right]^{\frac{1}{\lambda}}, \tag{53}$$

where $\lambda > 0$.

Before formulating distance measures for two OWHFSs, let us make a convention: if ${}^{\omega}h_1 = \bigcup_{1\le j\le N}\{\langle h_1^{\sigma(j)}, w_1^{\sigma(j)}\rangle\}$ and ${}^{\omega}h_2 = \bigcup_{1\le j\le N}\{\langle h_2^{\sigma(j)}, w_2^{\sigma(j)}\rangle\}$ are two OWHFEs, then

$$||\langle h_1^{\sigma(j)}, w_1^{\sigma(j)}\rangle - \langle h_2^{\sigma(j)}, w_2^{\sigma(j)}\rangle|| = (|h_1^{\sigma(j)} - h_2^{\sigma(j)}|^2 + |w_1^{\sigma(j)} - w_2^{\sigma(j)}|^2)^{\frac{1}{2}}, \tag{54}$$

for any $1 \le j \le N$.

By the help of the latter convention, we can formulate some distance measures for two OWHFSs ${}^{\omega}H_1 = \{\langle x, {}^{\omega}h_1(x)\rangle : x \in X\} = \{\langle x, \bigcup_{1\le j\le N}\{\langle h_1^{\sigma(j)}(x), w_1^{\sigma(j)}(x)\rangle\}\rangle : x \in X\}$ and ${}^{\omega}H_2 = \{\langle x, {}^{\omega}h_2(x)\rangle : x \in X\} = \{\langle x, \bigcup_{1\le j\le N}\{\langle h_2^{\sigma(j)}(x), w_2^{\sigma(j)}(x)\rangle\}\rangle : x \in X\}$ on $X = \{x_1, ..., x_n\}$ as follows:

- The generalized ordered weighted hesitant normalized distance:

$$d_{gowhn}({}^{\omega}H_1, {}^{\omega}H_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{N}\sum_{j=1}^{N}||\left\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\right\rangle - \left\langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\right\rangle||^{\lambda}\right)\right]^{\frac{1}{\lambda}}; \tag{55}$$

- The generalized ordered weighted hesitant normalized Hausdorff distance:

$$d_{gowhnh}({}^{\omega}H_1, {}^{\omega}H_2) = \left[\frac{1}{n}\sum_{i=1}^{n} \max_{1\le j\le N}\{||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\rangle||^{\lambda}\}\right]^{\frac{1}{\lambda}}; \quad (56)$$

- The generalized hybrid ordered weighted hesitant normalized distance:

$$d_{ghowhn}({}^{\omega}H_1, {}^{\omega}H_2) = [\frac{1}{2n}\sum_{i=1}^{n}(\frac{1}{N}\sum_{j=1}^{N}||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\rangle||^{\lambda} + \max_{1\le j\le N}\{||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\rangle||^{\lambda}\})]^{\frac{1}{\lambda}}, \quad (57)$$

where $\lambda > 0$.

If the weight of the element $x_i \in X$ is ξ_i $(i = 1, ..., n)$ with $\xi_i \in [0, 1]$ and $\sum_{i=1}^{n} \xi_i = 1$, then we get the weighted form of distance measures as follows:

- The generalized ordered weighted hesitant weighted normalized distance:

$$d_{gowhwn}({}^{\omega}H_1, {}^{\omega}H_2) = [\sum_{i=1}^{n}\xi_i(\frac{1}{N}\sum_{j=1}^{N}||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\rangle||^{\lambda})]^{\frac{1}{\lambda}}; \quad (58)$$

- The generalized ordered weighted hesitant weighted normalized Hausdorff distance:

$$d_{gowhwnh}({}^{\omega}H_1, {}^{\omega}H_2) = [\sum_{i=1}^{n}\xi_i \max_{1\le j\le N}\{||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\rangle||^{\lambda}\}]^{\frac{1}{\lambda}}; \quad (59)$$

- The generalized hybrid ordered weighted hesitant weighted normalized distance:

$$d_{ghowhwn}({}^{\omega}H_1, {}^{\omega}H_2) = [\frac{1}{2}\sum_{i=1}^{n}\xi_i(\frac{1}{N}\sum_{j=1}^{N}||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\rangle||^{\lambda} + \max_{1\le j\le N}\{||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle h_2^{\sigma(j)}(x_i), w_2^{\sigma(j)}(x_i)\rangle||^{\lambda}\})]^{\frac{1}{\lambda}}, \quad (60)$$

where $\lambda > 0$. □

Theorem 3.2 *All measure functions* $d_{gowhn}, d_{gowhnh}, d_{ghowhn}, d_{gowhwn}, d_{gowhwnh}, d_{ghowhwn}$ *given respectively by* Eqs. 55–60 *are distance measures for OWHFSs.*

Proof It is necessary to show that each measure function satisfies the requirements (d0)–(d3) listed in Definition 3.1. The proofs of (d0), (d1), and (d3) for $d_{ghowhwn}$ given by Eq. 60 are straightforward and we prove only (d2). Let

$$d_{ghowhwn}({}^{\omega}H_1, {}^{\omega}H_1^c) = [\frac{1}{2}\sum_{i=1}^{n}\xi_i(\frac{1}{N}\sum_{j=1}^{N}||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle 1 - h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle||^{\lambda} + \max_{1\le j\le N}\{||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle 1 - h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle||^{\lambda}\})]^{\frac{1}{\lambda}} = 1,$$

if and only if

$$||\langle h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle - \langle 1 - h_1^{\sigma(j)}(x_i), w_1^{\sigma(j)}(x_i)\rangle|| = 1, \quad for\ any\ 1 \le i \le n,\ and\ 1 \le j \le N,$$

if and only if

$$|h_1^{\sigma(j)}(x_i) - (1 - h_1^{\sigma(j)}(x_i))| = 1, \quad |w_1^{\sigma(j)}(x_i) - w_1^{\sigma(j)}(x_i)| = 0, \quad for\ any\ 1 \leq i \leq n\ and\ 1 \leq j \leq N,$$

if and only if

$$h_1^{\sigma(j)}(x_i) = 0, \quad or \quad h_1^{\sigma(j)}(x_i) = 1, \quad for\ any\ 1 \leq i \leq n\ and\ 1 \leq j \leq N.$$

This implies that ${}^{\omega}H_1$ is the empty OWHFS ${}^{\omega}0$ or the full OWHFS ${}^{\omega}1$. □

Aggregation Operators for OWHFSs

Definition 4.1 Let ${}^{\omega}E = \{{}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n\}$ be a set of n OWHFEs, and Θ be a function on ${}^{\omega}E$. Then

$$\Theta_{{}^{\omega}E} = \bigcup_{1 \leq j \leq N} \left\{ \left\langle \left(h_1^{\sigma(j)}, ..., h_n^{\sigma(j)} \right), \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}} \right) \right\rangle \right\}, \quad (61)$$

where $(h_1^{\sigma(j)}, ..., h_n^{\sigma(j)}) \in {}^{\omega}h_1 \times ... \times {}^{\omega}h_n$, and $(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}})$ is calculated using Eq. 27 in Remark 2.1.

By taking Definition 2.6 and Definition 4.1 into account, we define some aggregation operators for OWHFEs. Let ${}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n$ be n OWHFEs. Then, we define

- The ordered weighted hesitant fuzzy weighted averaging (OWHFWA) operator:

$$OWHFWA({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \bigoplus_{i=1}^{n} (\varpi_i {}^{\omega}h_i)$$
$$= \bigcup_{1 \leq j \leq N} \left\{ \left\langle 1 - \prod_{i=1}^{n} (1 - h_i^{\sigma(j)})^{\varpi_i}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}} \right) \right\rangle \right\}; \quad (62)$$

- The ordered weighted hesitant fuzzy weighted geometric (OWHFWG) operator:

$$OWHFWG({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \bigotimes_{i=1}^{n} ({}^{\omega}h_i^{\varpi_i})$$
$$= \bigcup_{1 \leq j \leq N} \left\{ \left\langle \prod_{i=1}^{n} (h_i^{\sigma(j)})^{\varpi_i}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}} \right) \right\rangle \right\}; \quad (63)$$

- The ordered weighted generalized hesitant fuzzy weighted averaging (OWGHFWA) operator:

$$OWGHFWA_{\lambda}({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \left[\bigoplus_{i=1}^{n} (\varpi_i {}^{\omega}h_i^{\lambda}) \right]^{\frac{1}{\lambda}}$$
$$= \bigcup_{1 \leq j \leq N} \left\{ \left\langle \left[1 - \prod_{i=1}^{n} (1 - (h_i^{\sigma(j)})^{\lambda})^{\varpi_i} \right]^{\frac{1}{\lambda}}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}} \right) \right\rangle \right\}; \quad (64)$$

- The ordered weighted generalized hesitant fuzzy weighted geometric (OWGHFWG) operator:

$$OWGHFWG_{\lambda}({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \frac{1}{\lambda} \left[\bigotimes_{i=1}^{n} (\lambda\ {}^{\omega}h_i^{\varpi_i}) \right]$$
$$= \bigcup_{1 \leq j \leq N} \left\{ \left\langle 1 - \left[1 - \prod_{i=1}^{n} (1 - (1 - h_i^{\sigma(j)})^{\frac{1}{\lambda}})^{\varpi_i}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}} \right) \right]^{\frac{1}{\lambda}} \right\rangle \right\}; \quad (65)$$

where $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)$ is the weight vector of ${}^{\omega}h_i$ $(i = 1, 2, ..., n)$, with $\varpi_i \in [0, 1]$ and $\sum_{i=1}^{n} \varpi_i = 1$. Moreover, $\lambda > 0$.

Theorem 4.1 *Suppose that ${}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n$ are n OWHFEs, and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)$ is the weight vector of ${}^{\omega}h_i$ $(i = 1, 2, ..., n)$ with $\varpi_i \in [0, 1]$ and $\sum_{i=1}^{n} \varpi_i = 1$. Then*

$$OWHFWG({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) \leq OWHFWA({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n); \quad (66)$$
$$OWHFWG({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) \leq OWGHFWA_{\lambda}({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n); \quad (67)$$
$$OWGHFWG_{\lambda}({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) \leq OWHFWA({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n). \quad (68)$$

Proof We only prove the assertion (66), and the assertions (67) and (68) can be proven similarly. For any $h_i^{\sigma(j)} \in {}^{\omega}h_i$ $(i = 1, 2, ..., n)$ and $(\varpi_1, \varpi_2, ..., \varpi_n)$, with $\varpi_i \in [0, 1]$ and $\sum_{i=1}^{n} \varpi_i = 1$, Xia and Xu [35] verified that

$$\prod_{i=1}^{n}(h_i^{\sigma(j)})^{\varpi_i} \leq 1 - \prod_{i=1}^{n}(1 - h_i^{\sigma(j)})^{\varpi_i}. \tag{69}$$

On the other hand, from Eqs. 62 and 63, one observes that the importance weights of the aggregation operators $OWHFWA$ and $OWHFWG$ are equal to $(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}})$. Thus, by applying the comparison law presented in Definition 2.7 together with the relation (69), we get

$$\triangle(OWHFWG({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n)) \leq \triangle(OWHFWA({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n)),$$

which completes the proof.

All the above-mentioned aggregation operators can be extended to the operators being used in a situation where the ordering of OWHFEs is important.

Suppose that ${}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n$ are n OWHFEs, and ${}^{\omega}h_{\delta(i)}$ is the i-th largest of them. Let $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)$ be the aggregation-associated vector such that $\varpi_i \in [0, 1]$ and $\sum_{i=1}^{n} \varpi_i = 1$. Then, motivated by the idea of the OWA operator [38], we define

- The ordered weighted hesitant fuzzy ordered weighted averaging (OWHFOWA) operator:

$$OWHFOWA({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \bigoplus_{i=1}^{n}(\varpi_i {}^{\omega}h_{\delta(i)})$$
$$= \bigcup_{1\leq j\leq N}\left\{\left\langle 1 - \prod_{i=1}^{n}\left(1 - h_{\delta(i)}^{\sigma(j)}\right)^{\varpi_i}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}}\right)\right\rangle\right\}; \tag{70}$$

- The ordered weighted hesitant fuzzy ordered weighted geometric (OWHFOWG) operator:

$$OWHFOWG({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \bigotimes_{i=1}^{n}({}^{\omega}h_{\delta(i)}^{\varpi_i})$$
$$= \bigcup_{1\leq j\leq N}\left\{\left\langle \prod_{i=1}^{n}(h_{\delta(i)}^{\sigma(j)})^{\varpi_i}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}}\right)\right\rangle\right\}; \tag{71}$$

- The ordered weighted generalized hesitant fuzzy ordered weighted averaging (OWGHFOWA) operator:

$$OWGHFOWA_{\lambda}({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \left[\bigoplus_{i=1}^{n}(\varpi_i {}^{\omega}h_{\delta(i)}^{\lambda})\right]^{\frac{1}{\lambda}}$$
$$= \bigcup_{1\leq j\leq N}\left\{\left\langle\left[1 - \prod_{i=1}^{n}(1 - (h_{\delta(i)}^{\sigma(j)})^{\lambda})^{\varpi_i}\right]^{\frac{1}{\lambda}}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}}\right)\right\rangle\right\}; \tag{72}$$

- The ordered weighted generalized hesitant fuzzy ordered weighted geometric (OWGHFOWG) operator:

$$OWGHFOWG_{\lambda}({}^{\omega}h_1, {}^{\omega}h_2, ..., {}^{\omega}h_n) = \frac{1}{\lambda}\left[\bigotimes_{i=1}^{n}(\lambda\, {}^{\omega}h_{\delta(i)}^{\varpi_i})\right]$$
$$= \bigcup_{1\leq j\leq N}\left\{\left\langle 1 - \left[1 - \prod_{i=1}^{n}(1 - (1 - h_{\delta(i)}^{\sigma(j)})^{\frac{1}{\lambda}})^{\varpi_i}, \left(\overline{\sum_{i=1}^{n} w_i^{\sigma(j)}}\right)\right]^{\frac{1}{\lambda}}\right\rangle\right\}; \tag{73}$$

where $\lambda > 0$. □

Multi-Attribute Decision Making Problem Involving OWHFSs

This section is divided into two parts, one is devoted to a distance-based algorithm for ordered weighted hesitant fuzzy multi-attribute decision making (OWHFMADM), and the other is devoted to an aggregation-based algorithm for OWHFMADM.

The Distance-Based Algorithm for OWHFMADM

In what follows, we first apply the proposed distance measures to solve the hesitant fuzzy multi-attribute decision making problems, which can be described below:

Example 5.1.1 (Remodeled and adopted from [20, 37]). Energy is an indispensable factor for the socioeconomic development of societies. Thus, the correct energy policy affects economic development and environment, and so on, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) A_i $(i = 1, 2, 3, 4, 5)$ to be invested, and four attributes to be considered: P_1: technological; P_2: environmental; P_3: socio-political; P_4: economic. The attribute weight vector is $\xi = (0.15, 0.3, 0.2, 0.35)$. Six decision makers D_l $(l = 1, 2, 3, 4, 5, 6)$ are invited to evaluate the performances of the five alternatives. For an alternative under an attribute, all of the decision makers provide anonymously their evaluated values. As an example, for the alternative A_1 under the attribute P_1, the evaluation value provided by the decision makers D_1, D_3, and D_5 is 0.7; D_2 and D_4's evaluation value is 0.4; and D_6's evaluation value is 0.5. In this regard, the evaluation of the alternative A_1 under the attribute P_1 can be represented by an OWHFE as ${}^{\omega}h(A_1, P_1) := {}^{\omega}h_{11} = \{\langle 0.4, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.7, \frac{3}{6}\rangle\}$. Note that the characteristics of the alternative A_1 under the attributes P_j $(j = 2, 3, 4)$, denoted respectively by the OWHFEs ${}^{\omega}h_{1j}$ $(j = 2, 3, 4)$, form the OWHFS ${}^{\omega}H_1$ which is indicated in the first row of Table 1. The results evaluated for other alternatives under the attributes are contained in a weighted hesitant fuzzy decision matrix, shown in Table 1.

Table 1 Ordered weighted hesitant fuzzy decision matrix

		P_1	P_2	P_3	P_4
${}^{\omega}H_1$	A_1	$\{\langle 0.4, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.7, \frac{3}{6}\rangle\}$	$\{\langle 0.3, \frac{2}{6}\rangle, \langle 0.6, \frac{1}{6}\rangle, \langle 0.7, \frac{3}{6}\rangle\}$	$\{\langle 0.1, \frac{2}{6}\rangle, \langle 0.2, \frac{1}{6}\rangle, \langle 0.3, \frac{3}{6}\rangle\}$	$\{\langle 0.3, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.8, \frac{3}{6}\rangle\}$
${}^{\omega}H_2$	A_2	$\{\langle 0.3, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.6, \frac{3}{6}\rangle\}$	$\{\langle 0.2, \frac{2}{6}\rangle, \langle 0.4, \frac{1}{6}\rangle, \langle 0.5, \frac{3}{6}\rangle\}$	$\{\langle 0.1, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.6, \frac{3}{6}\rangle\}$	$\{\langle 0.1, \frac{2}{6}\rangle, \langle 0.6, \frac{1}{6}\rangle, \langle 0.8, \frac{3}{6}\rangle\}$
${}^{\omega}H_3$	A_3	$\{\langle 0.1, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.8, \frac{3}{6}\rangle\}$	$\{\langle 0.3, \frac{2}{6}\rangle, \langle 0.4, \frac{1}{6}\rangle, \langle 0.7, \frac{3}{6}\rangle\}$	$\{\langle 0.1, \frac{2}{6}\rangle, \langle 0.2, \frac{1}{6}\rangle, \langle 0.5, \frac{3}{6}\rangle\}$	$\{\langle 0.2, \frac{2}{6}\rangle, \langle 0.3, \frac{1}{6}\rangle, \langle 0.8, \frac{3}{6}\rangle\}$
${}^{\omega}H_4$	A_4	$\{\langle 0.3, \frac{2}{6}\rangle, \langle 0.4, \frac{1}{6}\rangle, \langle 0.5, \frac{3}{6}\rangle\}$	$\{\langle 0.1, \frac{2}{6}\rangle, \langle 0.3, \frac{1}{6}\rangle, \langle 0.5, \frac{3}{6}\rangle\}$	$\{\langle 0.4, \frac{2}{6}\rangle, \langle 0.6, \frac{1}{6}\rangle, \langle 0.7, \frac{3}{6}\rangle\}$	$\{\langle 0.3, \frac{2}{6}\rangle, \langle 0.4, \frac{1}{6}\rangle, \langle 0.6, \frac{3}{6}\rangle\}$
${}^{\omega}H_5$	A_5	$\{\langle 0.2, \frac{2}{6}\rangle, \langle 0.3, \frac{1}{6}\rangle, \langle 0.7, \frac{3}{6}\rangle\}$	$\{\langle 0.4, \frac{2}{6}\rangle, \langle 0.7, \frac{1}{6}\rangle, \langle 0.8, \frac{3}{6}\rangle\}$	$\{\langle 0.2, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.6, \frac{3}{6}\rangle\}$	$\{\langle 0.3, \frac{2}{6}\rangle, \langle 0.5, \frac{1}{6}\rangle, \langle 0.8, \frac{3}{6}\rangle\}$

We let the full OWHFS

$$\begin{aligned} {}^{\omega}1 &= \{\langle x, {}^{\omega}1(x)\rangle : x \in X = \{P_1, ..., P_4\}\} \\ &= \{\langle x, \bigcup_{1\le j\le 3} \{\langle 1, \frac{1}{3}\rangle\}\rangle : x \in X\} \\ &= \{\langle\ P_j, \{\langle 1, \frac{1}{3}\rangle, \langle 1, \frac{1}{3}\rangle, \langle 1, \frac{1}{3}\rangle\}\ \rangle : j = 1, 2, 3, 4\}, \end{aligned}$$

be the representative of ideal alternative. By using Eq. 58 to calculate the deviations between each alternative and the ideal alternative, the ranking of all alternatives can be obtained. For example, the deviation between the alternative A_1 (correspondingly, ${}^{\omega}H_1$) and the ideal alternative (correspondingly, ${}^{\omega}1$) is calculated as follows:

$$d_{gowhwn}({}^{\omega}H_1, {}^{\omega}1) = \left[\sum_{i=1}^{4} \xi_i \left(\frac{1}{3}\sum_{j=1}^{3} ||\langle h_1^{\sigma(j)}(P_i), w_1^{\sigma(j)}(P_i)\rangle - \left\langle 1, \frac{1}{3}\right\rangle ||^{\lambda}\right)\right]^{\frac{1}{\lambda}}.$$

Taking $\lambda = 1$ and $\xi = (0.15, 0.3, 0.2, 0.35)$ gives rise to

$$\begin{aligned} d_{gowhwn}({}^{\omega}H_1, {}^{\omega}1) = 0.15(&\frac{\sqrt{|0.4-1|^2+|\frac{2}{6}-\frac{1}{3}|^2}+\sqrt{|0.5-1|^2+|\frac{1}{6}-\frac{1}{3}|^2}+\sqrt{|0.7-1|^2+|\frac{3}{6}-\frac{1}{3}|^2}}{3}) \\ +0.3(&\frac{\sqrt{|0.3-1|^2+|\frac{2}{6}-\frac{1}{3}|^2}+\sqrt{|0.6-1|^2+|\frac{1}{6}-\frac{1}{3}|^2}+\sqrt{|0.7-1|^2+|\frac{3}{6}-\frac{1}{3}|^2}}{3}) \\ +0.2(&\frac{\sqrt{|0.1-1|^2+|\frac{2}{6}-\frac{1}{3}|^2}+\sqrt{|0.2-1|^2+|\frac{1}{6}-\frac{1}{3}|^2}+\sqrt{|0.3-1|^2+|\frac{3}{6}-\frac{1}{3}|^2}}{3}) \\ +0.35(&\frac{\sqrt{|0.3-1|^2+|\frac{2}{6}-\frac{1}{3}|^2}+\sqrt{|0.5-1|^2+|\frac{1}{6}-\frac{1}{3}|^2}+\sqrt{|0.8-1|^2+|\frac{3}{6}-\frac{1}{3}|^2}}{3}) = 0.5667. \end{aligned}$$

The deviation between the other OWHFSs ${}^{\omega}H_i$ $(i = 2, 3, 4, 5)$ and the representative of ideal alternative ${}^{\omega}1$ are obtained as:

$$\begin{aligned} d_{gowhwn}({}^{\omega}H_2, {}^{\omega}1) = 0.5995, \quad d_{gowhwn}({}^{\omega}H_3, {}^{\omega}1) = 0.6119, \\ d_{gowhwn}({}^{\omega}H_4, {}^{\omega}1) = 0.6150, \quad d_{gowhwn}({}^{\omega}H_5, {}^{\omega}1) = 0.5138. \end{aligned}$$

Corresponding to the ranking of the OWHFSs ${}^{\omega}H_i$ $(i = 1, 2, 3, 4, 5)$, we get the ranking of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ as:

$$A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4.$$

The deviation between ${}^{\omega}H_i$ $(i = 1, 2, 3, 4, 5)$ and the ideal ${}^{\omega}1$ for several values of λ and the corresponding ranking orders are all shown in Table 2.

The Aggregation-Based Algorithm for OWHFMADM

In some practical problems such as in a presidential election, it is required to protect the decision makers' privacy or avoid influencing each other. For instance, in the case where two decision makers provide their preference information over an attribute by the same value, then the value emerges only once in the HFE. Meanwhile, the OWHFE allows us to conserve all opinions without ignoring the repeated opinions, and really, this is the main point of the current work.

In this section, we apply the ordered weighted hesitant fuzzy aggregation operators to multi-attribute decision making with anonymity.

In what follows, we present the aggregation-based decision making method:

Table 2 Results obtained by the generalized ordered weighted hesitant weighted normalized distance for OWHFSs

	A_1	A_2	A_3	A_4	A_5	Rankings
$\lambda = 0.1$	0.5216	0.5561	0.5505	0.5930	0.4664	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
$\lambda = 0.2$	0.5271	0.5614	0.5582	0.5956	0.4721	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
$\lambda = 0.5$	0.5428	0.5766	0.5799	0.6030	0.4886	$A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$
$\lambda = 1$	0.5667	0.5995	0.6119	0.6150	0.5138	$A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$
$\lambda = 2$	0.6053	0.6373	0.6602	0.6368	0.5550	$A_5 \succ A_1 \succ A_4 \succ A_2 \succ A_3$
$\lambda = 5$	0.6756	0.7117	0.7324	0.6881	0.6267	$A_5 \succ A_1 \succ A_4 \succ A_2 \succ A_3$
$\lambda = 10$	0.7391	0.7762	0.7804	0.7440	0.6812	$A_5 \succ A_1 \succ A_4 \succ A_2 \succ A_3$

Step 1. Suppose that the decision maker's evaluation for the alternative A_i under the attribute P_j can be represented by an OWHFE as ${}^{\omega}h_{ij} := {}^{\omega}h(A_i, P_j)$ $(i = 1, ..., m;\ j = 1, ..., n)$.

Step 2. Employ the proposed aggregation operators to obtain the collective OWHFEs ${}^{\omega}h_i$ $(i = 1, 2, ..., m)$ for the alternatives A_i $(i = 1, 2, ..., m)$ such that

$$ {}^{\omega}h_i = AGG({}^{\omega}h_{i1}, {}^{\omega}h_{i2}, ..., {}^{\omega}h_{in}), \tag{74}$$

where AGG is chosen from the set of the proposed aggregation operators for OWHFEs.

Step 3. Determine the rank ordering of ${}^{\omega}h_i$ $(i = 1, 2, ..., m)$ by the use of comparison law presented in Definition 2.7.

Step 4. Specify the priority of the alternatives A_i $(i = 1, 2, ..., m)$ according to the rank ordering of ${}^{\omega}h_i$ $(i = 1, 2, ..., m)$.

Once again, we consider that Example 5.1.1, where the decision maker's evaluation of the alternative A_i under the attribute P_j is in the form of an OWHFE ${}^{\omega}h_{ij} := {}^{\omega}h(A_i, P_j)$ $(i = 1, ..., 5;\ j = 1, ..., 4)$. The corresponding ordered weighted hesitant fuzzy decision matrix is shown in Table 1.

Now, we employ the OWGHFWA operator given by Eq. 64 to get the collective OWHFEs ${}^{\omega}h_i$ $(i = 1, 2, ..., 5)$. For example, let $\lambda = 1$, then by taking into account $\xi = (0.15, 0.3, 0.2, 0.35)$ as the weight vector of ${}^{\omega}h_{ij}$ $(j = 1, 2, 3, 4)$, which is denoted hereafter by $(\varpi_1, \varpi_2, \varpi_3, \varpi_4)$, we get

$$\begin{aligned} {}^{\omega}h_1 &= OWGHFWA_1({}^{\omega}h_{11}, {}^{\omega}h_{12}, {}^{\omega}h_{13}, {}^{\omega}h_{14}) \\ &= OWGHFWA_1\Bigg(\Big\{\Big\langle 0.4, \frac{2}{6}\Big\rangle, \Big\langle 0.5, \frac{1}{6}\Big\rangle, \Big\langle 0.7, \frac{3}{6}\Big\rangle\Big\}, \Big\{\Big\langle 0.3, \frac{2}{6}\Big\rangle, \Big\langle 0.6, \frac{1}{6}\Big\rangle, \Big\langle 0.7, \frac{3}{6}\Big\rangle\Big\}, \\ &\qquad \Big\{\Big\langle 0.1, \frac{2}{6}\Big\rangle, \Big\langle 0.2, \frac{1}{6}\Big\rangle, \Big\langle 0.3, \frac{3}{6}\Big\rangle\Big\}, \Big\{\Big\langle 0.3, \frac{2}{6}\Big\rangle, \Big\langle 0.5, \frac{1}{6}\Big\rangle, \Big\langle 0.8, \frac{3}{6}\Big\rangle\Big\}\Bigg) \\ &= \left[\bigoplus_{k=1}^{4}(\varpi_k {}^{\omega}h_{1k})\right] = \bigcup_{1\le j\le 3}\left\{\left\langle \left[1 - \prod_{k=1}^{4}(1 - (h_{1k}^{\sigma(j)}))^{\varpi_k}\right], \left(\sum_{k=1}^{n} w_k^{\sigma(j)}\right)\right\rangle\right\} \\ &= \Big\{\Big\langle 0.2807, \frac{2}{6}\Big\rangle, \Big\langle 0.4863, \frac{1}{6}\Big\rangle, \Big\langle 0.6916, \frac{3}{6}\Big\rangle\Big\}. \end{aligned}$$

Now, the calculation of the function $\triangle$ introduced in Definition 2.7 for ${}^{\omega}h_1$ results in

$$\begin{aligned} \triangle({}^{\omega}h_1) &= \sum_{j=1}^{3} h_1^{\sigma(j)} w_1^{\sigma(j)} = 0.2807 \times \frac{2}{6} + 0.4863 \times \frac{1}{6} \\ &\quad + 0.6916 \times \frac{3}{6} = 0.5204. \end{aligned}$$

For the other OWHFEs ${}^{\omega}h_i$ $(i = 2, 3, 4, 5)$, the values of the function $\triangle$ are obtained as:

$$\triangle({}^{\omega}h_2) = 0.4719, \quad \triangle({}^{\omega}h_3) = 0.4886,$$
$$\triangle({}^{\omega}h_4) = 0.4507, \quad \triangle({}^{\omega}h_5) = 0.5693.$$

Thus, known by the above values and using the comparison law presented in Definition 2.7, we find that the priority of the alternatives A_i $(i = 1, 2, ..., 5)$ is as follows:

$$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4.$$

Table 3 The values of the function $\triangle$ for the $OWGHFWA_\lambda$ aggregated OWHFEs, and the priority of alternatives

	A_1	A_2	A_3	A_4	A_5	Rankings
$\lambda = 1$	0.5204	0.4719	0.4886	0.4507	0.5693	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
$\lambda = 2$	0.5317	0.4809	0.4968	0.4598	0.5747	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
$\lambda = 5$	0.5546	0.5076	0.5179	0.4826	0.5908	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
$\lambda = 10$	0.5750	0.5381	0.5387	0.5100	0.6097	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$

Table 4 Ordered weighted hesitant fuzzy decision matrix

		P_1	P_2	P_3
${}^{\omega}H_1$	A_1	$\{\langle 0.6, 0.3\rangle, \langle 0.5, 0.3\rangle, \langle 0.4, 0.4\rangle\}$	$\{\langle 0.6, 0.8\rangle, \langle 0.4, 0.2\rangle\}$	$\{\langle 0.5, 0.3\rangle, \langle 0.3, 0.7\rangle\}$
${}^{\omega}H_2$	A_2	$\{\langle 0.4, 0.6\rangle, \langle 0.3, 0.4\rangle\}$	$\{\langle 0.8, 1\rangle\}$	$\{\langle 0.4, 0.2\rangle, \langle 0.3, 0.3\rangle, \langle 0.2, 0.5\rangle\}$
${}^{\omega}H_3$	A_3	$\{\langle 0.8, 1\rangle\}$	$\{\langle 0.7, 0.1\rangle, \langle 0.6, 0.3\rangle, \langle 0.5, 0.6\rangle\}$	$\{\langle 0.2, 0.5\rangle, \langle 0.1, 0.5\rangle\}$

The priority of the alternatives A_i $(i = 1, 2, ..., 5)$ with respect to several values of λ and the corresponding ranking orders are all shown in Table 3.

Comparison of the Proposed Method with Zhang and Wu's Method

Let us now resolve the problem discussed by Zhang and Wu in [42] using the method explained here. This problem was adapted from [44, 46].

Example 5.2.1 A factory intends to select a new site for new buildings. In this regard, there are three alternatives A_i $(i = 1, 2, 3)$ to be invested, and three attributes are considered to decide which site to choose: P_1: price; P_2: location; P_3: environment. The attribute weight vector is $\xi = (0.3, 0.2, 0.5)$. Let the characteristics of the alternatives A_i $(i = 1, 2, 3)$ with respect to the attributes P_j $(j = 1, 2, 3)$ be denoted by the OWHFSs in Table 4.

It is noticeable that all of the attributes P_j $(j = 1, 2, 3)$ are of the benefit type and therefore the performance values of the alternatives A_i $(i = 1, 2, 3)$ do not require any normalization.

Zhang and Wu [42] implemented the WHFHWA operator with $\theta = 1$ (Equation (21) in [42]) to aggregate all the preference values. In this case, the operator WHFHWA is reduced here to the OWGHFWA operator with $\lambda = 1$ in Eq. 24. As applied before in the pervious part of this paper, it is necessary to calculate OWGHFWA_1 to get the collective OWHFSs ${}^{\omega}h_i$ $(i = 1, 2, 3)$.

To do so, we get

$$
\begin{aligned}
{}^{\omega}h_1 &= \{\langle 0.5528, 0.4667\rangle, \langle 0.3864, 0.4000\rangle, \langle 0.3519, 0.1333\rangle\},\\
{}^{\omega}h_2 &= \{\langle 0.5184, 0.6000\rangle, \langle 0.4551, 0.2333\rangle, \langle 0.4175, 0.1667\rangle\},\\
{}^{\omega}h_3 &= \{\langle 0.5662, 0.5333\rangle, \langle 0.5126, 0.2667\rangle, \langle 0.4904, 0.2000\rangle\}.
\end{aligned}
$$

Now, the calculation of the function $\triangle$ introduced in Definition 2.7 for ${}^{\omega}h_i$ $(i = 1, 2, 3)$ results in

$$\triangle({}^{\omega}h_1) = 0.4595, \quad \triangle({}^{\omega}h_2) = 0.4868, \quad \triangle({}^{\omega}h_3) = 0.5368.$$

Thus, known by the above values and using the comparison law presented in Definition 2.7, we find that the priority of the alternatives A_i $(i = 1, 2, 3)$ is as follows:

$$A_3 \succ A_2 \succ A_1,$$

and the best alternative is A_3. This priority of the alternatives was obtained exactly by Zhang and Wu [42].

Once again, it should be mentioned that although the result of Zhang and Wu [42] is similar to the obtained one, the proposed method does not inherit the shortcomings of Zhang and Wu's [42] method.

Conclusion

Recently, the researchers have been challenged with multiple acts of decision making, and it is necessary to use the cognitive information during the decision making process [1, 7, 39]. This article has introduced an extension of HFS, which is referred to as OWHFS. The membership degree of an element to a OWHFS is expressed by several possible values together with their importance weights. By introducing OWHFS, we have modified a fault of WHFS proposed by Zhang and Wu [42]. The OWHFS is acceptable in accordance with the well-known axioms for mathematical operations and also allows that all information measures are to be defined reasonably. Then, we have developed a series of information measures and aggregation operators for OWHFSs and employed them to solve the hesitant fuzzy decision making problems. As future work, we consider the study of score functions of OWHFSs for handling multi-attribute decision making with ordered weighted hesitant fuzzy information. Moreover, the application potentials of OWHFSs are diverse and can be investigated in clustering, pattern recognition, image processing, etc.

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Compliance with Ethical Standards

Conflict of interests Authors declare that they have no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors.

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