International Journal of Control, Automation, and Systems 22(4) (2024) 1150-1162
http://dx.doi.org/10.1007/s12555-022-1055-y
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Stochastic Consensus for Heterogeneous Multi-agent Networks With Constraints and Communication Noises

Haokun Hu, Lipo Mo*, and Fei Long

Abstract: The mean-square consensus of the discrete-time heterogeneous multi-agent systems (HMASs) with convex position constraints, nonconvex velocity constraints and communication noises is reported in this paper, where the dynamics of HMASs are composed of first-order or second-order difference equations, and the noises are assumed to be martingale difference sequences. Firstly, a new algorithm is designed based on the information from neighbor agents with noises, and the original system is changed into an equivalent one by introducing a coordinate transformation. Secondly, when the communication graph is joint strongly connected, it is proved that mean-square consensus can be achieved by the properties of stochastic matrix, projection operator and martingale, and the position and velocity states of agents stay at the corresponding constraint sets. Specially, the situations of a network containing only first-order agents or second-order agents are considered, respectively. Finally, the correctness of the theoretical results is verified by numerical simulations.

Keywords: Communication noises, heterogeneous multi-agent networks, nonconvex constraints, stochastic consensus.

1. INTRODUCTION

Over the past few years, consensus problem has aroused increasing attention from the control field due to its widespread applications including wireless sensor network [1], formation control [2], satellite cluster [3] and so on. The objective of consensus is to drive all agents to coordinately converge to a static or dynamic point. It has been shown that the states of systems can reach consensus if the graph has a directed spanning tree in [4-7]. However, the aforementioned researches focused on the consensus problem of the homogeneous multi-agent systems (MASs), which means that the system has the same dynamics. It is hard to ensure that all agents have the same dynamic structure in practical applications. For example, there are different commands, control and data collection functions in joint UAV and ground vehicle operations. As a result, many researchers have turned their attention to HMASs.

As a part of the distributed coordination problem, constrained consensus for MASs has also many engineering applications, such as power transmission, formation satellite attitude control. In some physical systems, the position and velocity can not be arbitrarily large due to the limitations of internal and external environment. For example, even if a vehicle is given too much driving force, it might not cause the vehicle to exceed its maximum speed. Especially, the actual velocity should be constrained in certain nonconvex sets due to the existence of a velocity dead zone of a physical object. To solve this problem, a constraint operator was introduced in [8]. It was shown that the states of the agents can reach constrained consensus if the communication graph is jointly strongly connected. Later, this result was extended to the distributed consensus of HMASs with nonconvex input constraints [9-18]. However, all of these works only considered one type of the constraints and few works considered two or more types of constraints. Authors of [10-16] considered consensus with nonconvex input constraints and group consensus with input constraints, respectively. The consensus of second-order with input saturation was considered in [11]. Moreover, some results on heterogeneous consensus of event-triggered control, fractional-order, hybrid and nonlinear MASs have been studied in [19-24], respec-

Manuscript received November 2, 2022; revised March 6, 2023; accepted May 7, 2023. Recommended by Associate Editor Jun Moon under the direction of Senior Editor Yoshito Ohta. This work was supported in part by the National Natural Science Foundation of China under Grant 61973329 and the National Key Technology R&D Program of China under Grant 2021YFD2100605.

Haokun Hu is with the School of Mathematics and Statistics, Beijing Technology and Business University, Beijing, China. Haokun Hu is also with the School of Mathematics and Statistic, Central South University, Changsha, China (e-mail: hhkun0520@csu.edu.cn). Lipo Mo is with the School of Computer and Artificial Intelligence, Beijing Technology and Business University. Lipo Mo is also with Research Centre for Agri-Product Quality Traceability, Beijing Technology and Business University, Beijing, China (e-mail: beihangmlp@126.com). Fei Long is with the School of Artificial Intelligence and Electrical Engineering, Guizhou Institute of Technology, Guiyang, China (e-mail: feilong@git.edu.cn).
* Corresponding author.

tively.

It was assumed that each agent can receive accurate information from their neighbours in the above literatures. In reality, the network is always in a complex and changeable environment, which means that the communication between agents is always affected by stochastic disturbances. Therefore, the stochastic system with communication noises should be considered rather than a deterministic form. It is a very challenging subject for MASs with communication noises to achieve consensus. In [25-27], some distributed algorithms were designed to solve consensus and optimization problem in mean square or with probability one, respectively. Especially, some sufficient and necessary conditions were given to drive all agents to reach mean square consensus in [25,26]. Later, these results were extended to consensus for HMASs. For example, authors of [28] studied mean square consensus for continuous-time HMASs with and without leader. Mean square bounded consensus was considered for discrete-time HMASs over Markov switching graph in [29]. But these results did not considered any constraints.

Motivated by the above results, we focus on stochastic consensus for discrete-time HMASs with state constraints in this paper. The main contributions of this paper are as follows:

I) Compared with existing results [10-24], this paper studies consensus problem with nonconvex velocity constraints and convex position constraints. A novel model based on projection operator and nonconvex contraction operator is built to deal with position and velocity constraints. Due to stochastic noises exist extensively in practical application, it is assumed that the communication between agents is affected by noises. A more general distributed algorithm is proposed to solve constrained consensus problem with the consideration of communication noises.

II) Compared with [4-9,25-27], where the distributed consensus problems were studied for first-order or second-order MASs, this paper extends these results to HAMSs with first-order and second-order dynamics, which brings us more difficulties due to the differences of each agent's dynamics.

III) The position and velocity constraints are considered simultaneously, which make the results of [28,29] be a special case of this paper. Different from [29], our algorithm can guarantee that the states of all agents can reach consensus rather than boundedness consensus in mean square, and the position states can stay inside the corresponding convex constraint sets. Meanwhile, we also obtain the same results for the situation of a network containing only first-order agents or second-order agents, respectively.

Notations: Let $\mathbb{R}^n$ be the vector space with dimension n. Let $\mathbf{0}$ be zero vector with corresponding dimension. Given the vector $z \in \mathbb{R}^n$ and matrix A. Let A^T be its transposed matrix, $\|A\|$ and $\|z\|$ be the 2-norm. Let $P_Z(z) = \arg\min\{\|z-y\| | y \in Z\}$ be projection of the vector $z \in \mathbb{R}^n$ on the convex set $Z \in \mathbb{R}^n$. Given a random variable x. Denote $E[x]$, $Var[x]$ and $E[x|\mathcal{F}(k)]$ as its mathematical expectation, variance and conditional expectation on the σ-algebra $\mathcal{F}(k)$, respectively. $\otimes$ represents kronecker product. Define $S_{V_i}(\cdot)$ to be a contract operator with $S_{V_i}(y) = \frac{y}{\|y\|} \max_{0\le\ell\le\|y\|} \left\{\ell | \frac{\ell\vartheta y}{\|y\|} \in V_i, \forall 0 \le \vartheta \le \ell\right\}$, if $y \neq 0$; otherwise, $S_{V_i}(y) = 0$.

2. PRELIMINARIES AND PROBLEM STATEMENT

The communication topology of HMASs can be modeled as a directed graph $\mathcal{G}(k) = (\mathcal{I}_{m+n}, \mathcal{E}(k), \mathcal{A}(k))$, $\mathcal{I}_{m+n} = \{1, 2, \cdots, m+n\}$ represents the sets of agents, $\mathcal{E}(k) \subseteq \mathcal{I} \times \mathcal{I}$ represents the sets of edges and $\mathcal{A}(k) = [a_{ij}(k)]$ represents a weighted adjacency matrix. $(i, j) \in \mathcal{E}(k)$ represents that agent j can receive information from agent i. $a_{ij}(k) > 0$ if $(j,i) \in \mathcal{E}(k)$, and $a_{ij}(k) = 0$ if $(j,i) \notin \mathcal{E}(k)$. Let $\mathcal{N}_i(k) = \{j \in \mathcal{I}_{m+n} \mid (j,i) \in \mathcal{E}(k)\}$ be the neighbor set of agent i. The Laplacian matrix is defined as $\mathcal{L}(k) = \mathcal{D}(k) - \mathcal{A}(k)$, where $\mathcal{D}(k) = diag\{d_1, \cdots, d_{m+n}\}$ is called as the degree matrix and $d_i = \sum_{j=1}^{m+n} a_{ij}(k)$. For a directed graph, if there exists at least one directed path between any two different nodes, then the graph is strongly connected. The graph is joint strongly connected if there exists a integer $B > 0$, such that the union graph $\bigcup_{j=1}^{B} \mathcal{G}(k+j)$ is strongly connected for any k, where $\bigcup_{j=1}^{B} \mathcal{G}(k+j) = \mathcal{G}(k+1) \bigcup \mathcal{G}(k+2) \bigcup \cdots \bigcup \mathcal{G}(k+B)$.

In this paper, we consider a network with m second-order agents and n first-order agents. The dynamics of second-order agents

$$\begin{aligned} x_i(k+1) &= P_{X_i}[x_i(k) + v_i(k)T], \\ v_i(k+1) &= S_{V_i}[v_i(k) + u_i(k)T], \; i \in \mathcal{I}_m, \end{aligned} \tag{1}$$

where $x_i(k)$, $v_i(k)$, $u_i(k) \in \mathbb{R}^n$ represent position, velocity and input of the ith agent, respectively. $X_i \subset \mathbb{R}^n$ represents a bounded closed convex set, $0 \in V_i \subset \mathbb{R}^n$ represents a bounded set which may be nonconvex. T is sampling time.

The dynamics of first-order agents can be expressed as

$$x_i(k+1) = P_{X_i}[x_i(k) + S_{V_i}(u_i(k))T], \; i \in \mathcal{I}_{m+n} - \mathcal{I}_m, \tag{2}$$

where $x_i(k)$, $u_i(k) \in \mathbb{R}^n$ represent position and input of the ith agent, respectively. $X_i \subset \mathbb{R}^n$ represents a bounded closed convex set, $0 \in V_i \subset \mathbb{R}^n$ represents a bounded set which may be nonconvex. T is a sampling time.

Remark 1: In the most of existing works [10-16], the dynamics of the HMASs was usually assumed to be the following form: $x_i(k+1) = x_i(k) + v_i(k)T$, $v_i(k+1) =$

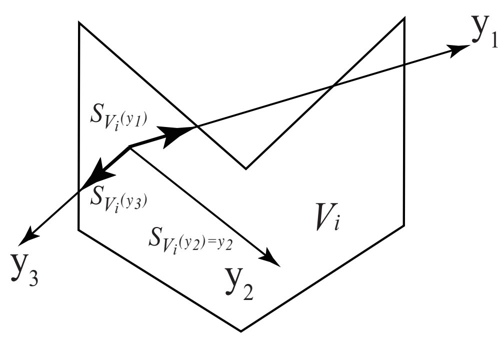

Fig. 1. Three examples of the nonconvex constraint operator $S_{V_i}(\cdot)$.

$v_i(k)+u_i(k)T$, for all $i \in \mathcal{I}_m$; $x_i(k+1)=x_i(k)+u_i(k)T$, for all $i \in \mathcal{I}_{m+n}-\mathcal{I}_m$, where $x_i(k)$, $v_i(k)$, $u_i(k) \in \mathbb{R}^n$ are position, velocity and control input of agent i respectively. In the actual engineering applications, the position and velocity are usually subjected to irregular constraints. For example, the drive force of each flight vehicle, such as quadrotors, have different constraints in different directions. On this basis, it is assumed here that $v_i(k) \in V_i$ for all $i \in \mathcal{I}_m$, where V_i is a nonconvex set. For the first-order agent, the control input $u_i(k)$ can usually be viewed as velocity. Hence, it is also assumed that $u_i(k) \in V_i$ for all $i \in \mathcal{I}_{m+n}-\mathcal{I}_m$. Moreover, if the position of a vehicle is subjected to a certain area, its position may be restricted when the vehicle touches the boundary. In particular, there are different functions for different agents, hence, the constraint sets may be different. It is usually assumed that $x_i(k) \in X_i$ for all $i \in \mathcal{I}_{m+n}$, where X_i is a convex set.

To describe the nonconvex and convex contraint sets, we first introduce two assumptions proposed in [5,8], respectively.

Assumption 1: Let $0 \in V_i \subset \mathbb{R}^n$ be a bounded closed nonconvex sets, which satisfy $\max_{y\in V_i}\|S_{V_i}(y)\| = \overline{\hbar}$, $\min_{y\notin V_i}\|S_{V_i}(y)\| = \underline{\hbar}$ and $0 \in V_i$ for all $i \in \mathcal{I}_{m+n}$, where $\overline{\hbar}$ and $\underline{\hbar}$ are two positive constants.

Assumption 2: Let X_i be a bounded closed convex set. The intersection X of all convex sets X_i is assumed to be nonempty, i.e., $X=\bigcap_{i=1}^{m+n} X_i \neq \emptyset$.

Remark 2: Under Assumptions 1 and 2, it is indicated that the velocities of all second-order agents and control input of the first-order agents in system (1), (2) can not be arbitrarily large. It is easy to see that the constraint operator only changes the magnitude of the velocity, not its direction from the definition of S_{V_i}, such that $cS_{V_i}(\cdot) \subseteq V_i$ for all $c \in [0,\,1]$. (See Fig. 1 for some specific examples) The major difference between this operator and the projection operator $P_{X_i}(\cdot)$ is that the projection operator will not only change its size but also its direction.

The considered system with disturbance is a stochastic process due the noises, the states of agents can not reach consensus exactly. Hence, it is necessary to study consensus problem in the statistical sense. Therefore, the mean square consensus of HMASs with communication noises is considered. The definition of mean square consensus is given in the following.

Definition 1 [28]**:** All agents are said to reach mean square consensus if for any $x_i(0) \in X_i \subset \mathbb{R}^n$ and $v_i(0) \in V_i \subset \mathbb{R}^n$, there exists a random variable x^*, such that $\lim_{k\to\infty} E[\|x_i(k)-x^*\|^2]=0$, $\lim_{k\to\infty} E[\|v_j(k)\|^2]=0$, and $Var(x^*)<\infty$ for all $i \in \mathcal{I}_{m+n}$, $j \in \mathcal{I}_m$.

Lemma 1 [5]**:** Let Ω be a nonempty closed convex set in $\mathbb{R}^n$, for any $x, y \in \mathbb{R}^n$, $z \in \Omega \subset \mathbb{R}^n$, $a_i \geq 0$ and $\sum_{i=1}^n a_i = 1$. Then

i) $\|P_\Omega(x)-z\|^2 \leq \|x-z\|^2+\|x-P_\Omega(x)\|^2$;
ii) $\|P_\Omega(x)-P_\Omega(y)\| \leq \|x-y\|$;
iii) $\|\sum_{i=1}^n a_i x_i - P_\Omega(\sum_{i=1}^n a_i x_i)\| \leq \sum_{i=1}^n a_i \|x_i - P_\Omega(x_i)\|$.

Lemma 2 [30]**:** Let $\{S(k)\}$, $\{B(k)\}$ and $\{Q(k)\}$ be nonnegative random variable sequences and let $\zeta(k)$ be a deterministic nonnegative scalar sequence. Let $\mathcal{F}(k)$ be the σ-algebra generated by $S(1),\cdots,S(k),B(1),\cdots,B(k)$, $Q(1),\cdots,Q(k)$. Suppose $\sum_{k=1}^\infty \zeta(k)<\infty$,

$$E[S(k+1)|\mathcal{F}(k)] \leq (1+\zeta(k))S(k)-B(k)+Q(k),$$

and $\sum_{k=1}^\infty Q(k)<\infty$ almost surely. Then, the sequence $\{S(k)\}$ almost surely converges to a nonnegative random variable and $\sum_{k=1}^\infty B(k)<\infty$.

The main purpose of this paper is to design a distributed control protocol such that all agents can reach mean square consensus, while the position states of all agents keep in the corresponding convex sets, and the velocity states of all second-order agents keep in the corresponding nonconvex sets, i.e., $x_i(k) \in X_i$, $v_j(k) \in V_j$, $\forall\, i \in \mathcal{I}_{m+n}$, $j \in \mathcal{I}_m$, k.

3. MAIN RESULT

The convergence analysis of constrained consensus in mean square of HMASs is given in this section. For systems (1) and (2) over jointly strongly connected graph, design the following distributed algorithm

$$u_i(k)=\begin{cases} -p_i v_i(k)+\pi_i(k), & i\in\mathcal{I}_m,\\ \pi_i(k), & i\in\mathcal{I}_{m+n}-\mathcal{I}_m,\end{cases} \tag{3}$$

where $\pi_i(k)=\gamma(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)(x_j(k)-x_i(k)+\xi_{ji}(k))$, $\gamma(k)$ is a diminishing step size and $\xi_{ji}(k)$ is stochastic noise, $p_i>0$ is the feedback gain.

Assumption 3: The step size $\gamma(k)$ satisfies the following conditions

$$\sum_{k=0}^\infty \gamma(k)=\infty,\ \sum_{k=0}^\infty \gamma^2(k)<\infty,\ \gamma(k+1)<\gamma(k).$$

Assumption 4: The stochastic noise $\{\xi_{ji}(k)\}$ is assumed to be a martingale difference sequence with $E[\xi_{ji}(k)|\mathcal{F}(k-1)] = 0$, $E[\|\xi_{ji}(k)\|^2|\mathcal{F}(k-1)] \leq \sigma_\xi^2$, where $\sigma_\xi > 0$ and $\mathcal{F}(k)$ is a σ-algebra generated by the entire history information from 0 to k, namely, $\mathcal{F}(k) = \{x_i(0),\ v_i(0),\ \xi_{ji}(s),\ u_i(s),\ \forall 0 \leq s \leq k-1,\ i,\ j \in \mathcal{I}_{m+n}\}$, $\mathcal{F}(0) = \{x_i(0), v_i(0), \xi_{ji}(0), \forall i,\ j \in \mathcal{I}_{m+n}\}$.

To facilitate the convergence analysis of the system, the coordination transformation is needed to be introduced. For $i \in \mathcal{I}_m$, let $\iota_i(k) = \frac{\|S_{V_i}[v_i(k)+u_i(k)T]\|}{\|v_i(k)+u_i(k)T\|}$, if $v_i(k)+u_i(k)T \neq 0$; otherwise, $\iota_i(k) = 1$. For $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$, let $\iota_i(k) = \frac{\|S_{V_i}[u_i(k)]\|}{\|u_i(k)\|}$, if $u_i(k) \neq 0$; otherwise, $\iota_i(k) = 1$. Obviously, $0 < \iota_i(k) \leq 1$. Therefore,

$$\begin{aligned} v_i(k+1) &= S_{V_i}[v_i(k)+u_i(k)T] \\ &= S_{V_i}[(1-p_iT)v_i(k)+\pi_i(k)T] \\ &= \iota_i(k)v_i(k) - \iota_i(k)(v_i(k)p_i - \pi_i(k))T \\ &= v_i(k) - \tau_i(k)v_i(k)T + \iota_i(k)\pi_i(k)T, \end{aligned}$$

where $\tau_i(k) = \frac{1-\iota_i(k)(1-p_iT)}{T}$. Let $y_i(k) = x_i(k)+v_i(k)$, for $i \in \mathcal{I}_m$. Then,

$$\begin{aligned} x_i(k+1) &= P_{X_i}[x_i(k)+v_i(k)T] \\ &= (1-T)x_i(k)+Ty_i(k)+\rho_i(k), \end{aligned}$$

where $\rho_i(k) = P_{X_i}[(1-T)x_i(k)+Ty_i(k)] - [(1-T)x_i(k)+Ty_i(k)]$, and

$$\begin{aligned} y_i(k+1) &= x_i(k+1)+v_i(k+1) \\ &= (1-T)x_i(k)+Ty_i(k)+\rho_i(k)+v_i(k) \\ &\quad - \tau_i(k)v_i(k)T + \iota_i(k)\pi_i(k)T \\ &= (\tau_i(k)T - T)x_i(k) + \iota_i(k)\pi_i(k)T \\ &\quad + (1-\tau_i(k)T+T)y_i(k)+\rho_i(k). \end{aligned}$$

For $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$, we have

$$\begin{aligned} x_i(k+1) &= P_{X_i}[x_i(k)+S_{V_i}(u_i(k))T] \\ &= x_i(k)+\iota_i(k)\pi_i(k)T+\rho_i(k), \end{aligned}$$

where $\rho_i(k) = P_{X_i}[x_i(k) + \iota_i(k)\pi_i(k)T] - [x_i(k) + \iota_i(k)\pi_i(k)T]$. Define $Z(k) = [Z_1^T(k), \cdots, Z_{2m+n}^T(k)]^T$, where $[Z_1^T(k),\ \cdots,\ Z_m^T(k)] = [x_1^T(k),\ \cdots,\ x_m^T(k)]^T$, $[Z_{m+1}^T(k),\ \cdots,\ Z_{2m}^T(k)] = [y_1^T(k), \cdots, y_m^T(k)]^T$, $[Z_{2m+1}^T(k), \cdots, Z_{2m+n}^T(k)] = [x_{m+1}^T(k),\ \cdots,\ x_{m+n}^T(k)]^T$. Define $\rho(k) = [\bar{\rho}_1^T(k),\ \cdots,\ \bar{\rho}_{2m+n}^T(k)]^T = [\rho_1^T(k),\ \cdots,\ \rho_m^T(k),\ \rho_1^T(k),\ \cdots,\ \rho_m^T(k),\ \rho_{m+1}^T(k), \cdots, \rho_{m+n}^T(k)]^T$,

$$\Phi(k) = \begin{bmatrix} \Phi_1(k) & \Phi_2(k) & \mathbf{0} \\ \Phi_3(k)\gamma(k)T & \Phi_7(k) & \Phi_4(k)\gamma(k)T \\ \Phi_5(k)\gamma(k)T & \mathbf{0} & \Phi_6(k)\gamma(k)T \end{bmatrix},$$

where $\Phi_1(k) = diag\{1-T, \cdots, 1-T\}$, $\Phi_2(k) = diag\{T, \cdots, T\}$,

$$\Phi_3(k) = \begin{bmatrix} \Phi_{31}(k) & \iota_2(k)a_{12}(k) & \cdots & \iota_1(k)a_{1m}(k) \\ \iota_2(k)a_{21}(k) & \Phi_{32}(k) & \cdots & \iota_2(k)a_{2m}(k) \\ \vdots & \vdots & \vdots & \vdots \\ \iota_m(k)a_{m1}(k) & \cdots & \cdots & \Phi_{3m}(k) \end{bmatrix},$$

$\Phi_{31}(k) = -\iota_1(k)\sum_{j\in\mathcal{N}_1(k)} a_{1j}(k) + \frac{\tau_1(k)-1}{\gamma(k)}$, $\Phi_{32}(k) = -\iota_2(k)\sum_{j\in\mathcal{N}_2(k)} a_{2j}(k) + \frac{\tau_2(k)-1}{\gamma(k)}$, $\cdots$, $\Phi_{3m}(k) = \frac{\tau_m(k)-1}{\gamma(k)} - \iota_m(k)\sum_{j\in\mathcal{N}_m(k)} a_{mj}(k)$, $[\Phi_4(k)]_{ij} = [\iota_i(k)a_{ij}(k)]$, $\forall i \in \mathcal{I}_m$, $j \in \mathcal{I}_{m+n} - \mathcal{I}_m$, $[\Phi_5(k)]_{ij} = [\iota_i(k)a_{ij}(k)]$, $\forall i \in \mathcal{I}_{m+n} - \mathcal{I}_m$, $j \in \mathcal{I}_m$,

$$\begin{aligned} &\Phi_6(k) \\ &= \Phi_{6s}(k)\begin{bmatrix} \Phi_{61}(k) & a_{m+1m+2}(k) & \cdots & a_{m+1m+n}(k) \\ a_{m+2m+1}(k) & \Phi_{62}(k) & \cdots & a_{m+2m+n}(k) \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+nm+1}(k) & \cdots & \cdots & \Phi_{6n}(k) \end{bmatrix}, \end{aligned}$$

$\Phi_{6s}(k) = diag\{\iota_{m+1}(k),\ \cdots,\ \iota_{m+n}(k)\}$, $\Phi_{61}(k) = \frac{\frac{1}{\iota_{m+1}(k)}}{T\gamma(k)} - \sum_{j\in\mathcal{N}_{m+1}(k)} a_{m+1j}(k)$, $\Phi_{62}(k) = \frac{\frac{1}{\iota_{m+2}(k)}}{T\gamma(k)} - \sum_{j\in\mathcal{N}_{m+2}(k)} a_{m+2j}(k)$, $\cdots$, $\Phi_{6n}(k) = \frac{\frac{1}{\iota_{m+n}(k)}}{T\gamma(k)} - \sum_{j\in\mathcal{N}_{m+n}(k)} a_{m+nj}(k)T$, $\Phi_7(k) = diag\{1-\tau_1(k)T+T,\ \cdots,\ 1-\tau_m(k)T+T\}$. Let $\xi(k) = [\bar{\xi}_1^T(k),\ \bar{\xi}_2^T(k),\ \cdots,\ \bar{\xi}_{2m+n}^T(k)]^T = [0,\ \cdots,\ 0,\ \xi_1^T(k),\ \cdots,\ \xi_{m+n}^T(k)]^T$,

$$\Xi(k) = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \Xi_{11}(k)T & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & \Xi_{N1}(k)T \end{bmatrix},$$

where $\xi_i(k) = [\xi_{1i}(k),\ \cdots,\ \xi_{m+ni}(k)]^T$, $\Xi_{i1}(k) = \iota_i(k)[a_{i1}(k),\ \cdots,\ a_{iN}(k)]$. Then the dynamics of HMASs (1) and (2) with algorithm (3) can be rewritten as the following compact form

$$Z(k+1) = [\Phi(k)\otimes I_n]Z(k)+\rho(k)+\gamma(k)\Xi(k)\xi(k). \tag{4}$$

To further analyze the convergence of algorithm, the following assumptions need to be satisfied.

Assumption 5: Suppose that there exists $0 < \mu_1 < 1$, such that $a_{ij}(k) \geq \mu_1$ if and only if $a_{ij}(k) > 0$, and $\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k) \leq 1-\mu_1$, for all $i,\ j \in \mathcal{I}_{m+n}$. $1 < p_i < \frac{1}{T}$, $T < 1$ for all $i \in \mathcal{I}_m$.

Assumption 6: Supposed that there exists a sequence $0 = k_0 < k_1 < k_2 < \cdots$ such that the union graph of all subgraphs $\mathcal{G}(k_t)$, $\mathcal{G}(k_t+1)$, $\cdots$, $\mathcal{G}(k_{t+1}-1)$ is strongly connected, and $k_t - k_{t-1} \leq B$ for all t.

Remark 3: Assumption 5 implies that information from neighbor agent exists persistently. If the assumption is absent, it may lead to the loss of information from certain agents. The conditions are easy to be satisfied and the illustration is given in the following simulation example. Assumption 6 shows that there exists a directed path between any two agents in a limited communication interval.

Lemma 3: Under Assumptions 1-6, let $Y \subset X$ be arbitrary nonempty closed convex set, the following inequalities hold for any sample path.

1) For all $i \in \mathcal{I}_m$, $\|x_i(k+1) - P_Y(x_i(k+1))\| \leq (1-T)\|x_i(k) - P_Y(x_i(k))\| + T\|y_i(k) - P_Y(y_i(k))\|$, $\|\rho_i(k)\|^2 \leq [(1-T)\|x_i(k) - P_Y(x_i(k))\| + T\|y_i(k) - P_Y(y_i(k))\|]^2 - \|x_i(k+1) - P_Y(x_i(k+1))\|^2$.
2) For all $i \in \mathcal{I}_m$, there exist $\varphi_{ix}(k)$, $\varphi_{iy}(k)$, $\sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k)$, $\underline{k}_1 \geq 0$ and $\varphi_{ix}(k) + \varphi_{iy}(k) + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k) = 1$, such that $\|y_i(k+1) - P_Y(y_i(k+1))\| \leq \varphi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\| + \|\gamma(k)\delta_{i\xi}(k)\| + \varphi_{iy}(k)\|y_i(k) - P_Y[y_i(k)]\|$, where $\underline{k}_1$ satisfies $\gamma(k) \leq \frac{(p_i-1)^2}{p_i(1-\mu_1)}$ for all $k \geq \underline{k}_1$.
3) For all $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$, $\|x_i(k+1) - P_Y[x_i(k+1)]\| \leq \chi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| + \|\gamma(k)\chi_{i\xi}(k)\| + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\|$, $\|\rho_i(k)\|^2 \leq [\|\gamma(k)\chi_{i\xi}(k)\| + \chi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\|]^2 - \|x_i(k+1) - P_Y[x_i(k+1)]\|^2$, where $\chi_{ix}(k) = 1 - \iota_i(k)\gamma(k)T\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)$, $\chi_{i\xi}(k) = \iota_i(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)\xi_{ji}(k)T$, $\chi_{ij}(k) = \iota_i(k)\gamma(k)Ta_{ij}(k)$.

Lemma 4: Under Assumptions 1-6, $x_i(k)$, $y_i(k)$ are bounded almost surely, and there exists a constant $\beta_0 > 0$, such that $\beta_0 \leq \iota_i(k) \leq 1$ almost surely, where $\beta_0 = \min\{\frac{\hbar_i}{L_1}, \frac{\hbar_j}{L_2} \mid i \in \mathcal{I}_m, j \in \mathcal{I}_{m+n} - \mathcal{I}_m\}$.

The proofs of Lemmas 3 and 4 can be found in Appendices A and B, resepctively.

Remark 4: It is shown in Lemma 4 that the sequences $\{x_i(k)\}$ and $\{y_j(k)\}$ almost surely converge to a nonnegative variable for $i \in \mathcal{I}_{m+n}$, $j \in \mathcal{I}_m$. Take the expectation of both side of the inequality (B.1)-(B.3), it follows that $E[\|x_i(k) - s_l\|^2]$ and $E[\|y_i(k) - s_l\|^2]$ are bounded by the definition of convergence of a sequence, which means that $x_i(k)$ and $y_j(k)$ are also bounded in mean square, hence, almost surely. Lemma 4 plays a very significant role in the subsequent convergence analysis.

Lemma 5: Under the Assumptions 1-6, there exists a constant $\underline{k}_2$ such that $\Phi(k)$ and $\Psi(k,s)$ are stochastic matrices for all $k \geq s > \underline{k}_2 \geq \underline{k}_1$. All nonzero elements of $\Phi(k)$ and $\Psi(k,s)$ have a positive lower bound. And there exist $\theta_j(s) > 0$ and $0 < \beta_1 < \frac{1}{2m+n}$, such that $\|[\Psi(k,s)]_{ij} - \theta_j(s)\| \leq 2C_\mu(1-2\beta_1)^{\frac{k-s}{B_0}}$ almost surely, for all $i, j \in \mathcal{I}_{m+n}$, where $B_0 = (N-1)B = (2m+n-1)B$, $C_\mu = \frac{1}{1-2\beta_1}$ $\Psi(k,s) = \Phi(k)\Phi(k-1)\cdots\Phi(s)$, for all $k \geq s > \underline{k}_2 \geq \underline{k}_1$.

Proof: From $\tau_i(k) = \frac{1-\iota_i(k)(1-p_iT)}{T}$ and $1 < p_i < \frac{1}{T}$, it follows that $1 < p_i \leq \tau_i(k) < \frac{1}{T}$. Note that $\lim_{k\to\infty}\gamma(k) = 0$ and $0 < \iota_i(k) \leq 1$ for all $i \in \mathcal{I}_{m+n}$, then there exists $\underline{k}_2$, such that $\gamma(k) < \min\{\frac{p_i-1}{2\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)}, \frac{1}{2\beta_0 T\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)}\}$, for all $k > \underline{k}_2$.

Hence, for all $i \in \mathcal{I}_m$, $\tau_i(k)T - T - \iota_i(k)\gamma(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)T \geq (p_i-1)T - \iota_i(k)\gamma(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)T \geq \frac{(p_i-1)T}{2} > 0$ and $\iota_i(k)\gamma(k)a_{ij}(k)T > \beta_0\gamma(k)\mu_1 T > 0$, for $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$, $1 - \iota_i(k)\gamma(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)T \geq \frac{1}{2} > 0$ almost surely. Therefore, all nonzero elements of $\Phi(k)$ and $\Psi(k,s)$ have a positive lower bounded and it is easy to verify that $\Phi(k)\mathbf{1}_{2m+n} = \mathbf{1}$ and $\Psi(k,s)\mathbf{1}_{2m+n} = \mathbf{1}$. Therefore, $\Phi(k)$ and $\Psi(k,s)$ are stochastic matrices for all $k \geq s > \underline{k}_2 \geq \underline{k}_1$. □

In the following, it is shown in Lemma 6 that the states of system can converge to the interior of the intersection of all position constraints in mean square. Then it is proved in Theorem 1 that all position and velocity states of system can reach mean-square consensus. the convergence analysis is given in Corollary 1 that when considering the case of a network containing only first-order agents and second-order agents, respectively.

Lemma 6: Under Assumptions 1-6, $\lim_{k\to\infty} E[\|Z_i(k) - P_X[Z_i(k)]\|^2] = 0$, for all $i \in \mathcal{I}_{m+n}$.

The proof of Lemma 6 can be found in Appendix C.

Theorem 1: Under Assumptions 1-6, there exists a random variable $x^* \in X$ such that $\lim_{k\to\infty} E[\|x_i(k) - x^*\|^2] = 0$, for all $i, j \in \mathcal{I}$, $\lim_{k\to\infty} E[\|v_j(k)\|^2] = 0$, for all $i \in \mathcal{I}_{m+n}$, $j \in \mathcal{I}_m$, and $Var(x^*) < \infty$.

Proof: Let $x^* = \frac{1}{2m+n}\sum_{i=1}^{2m+n} Z_i(k+1)$. From Lemma 3 to Lemma 6, $\lim_{k\to\infty} E[\|\rho_i(k)\|]^2 = 0$. Note that $\lim_{k\to\infty}\gamma(k) = 0$. Therefore, for any $\varpi > 0$, there exist $K > 0$ such that $E[\|\rho_i(k)\|]^2 < \varpi$, $16[\gamma(k)(2m+n)(1-\mu)T\sigma_\xi]^2 < \varpi$ for all $k > K$. From the above analysis, we have

$$
\begin{aligned}
&E\left[\|Z_i(k+1) - x^*\|^2\right] \\
&\leq 12\left[\sum_{j=1}^{2m+n} 4C_\mu(1-2\beta_1)^{\frac{k}{B_0}}\right]^2 \|Z_j(s)\|^2 \\
&\quad + 12\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n} 4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2 \varpi + 28\varpi \\
&\quad + 8\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n} 4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2 \\
&\quad \times [(2m+n)(1-\mu)T\sigma_\xi]^2.
\end{aligned}
$$

Hence, $\lim_{k\to\infty} E\left[\|Z_i(k+1) - x^*\|^2\right] \leq 7\varpi$. According to the arbitrariness of ϖ, we have $\lim_{k\to\infty} E[\|x_i(k) - x^*\|^2] = 0$, $\lim_{k\to\infty} E[\|y_i(k) - x^*\|^2] = 0$ for all $i \in \mathcal{I}_{m+n}$. Note that $y_i(k) = x_i(k) + v_i(k)$, then $\lim_{k\to\infty} E[\|v_i(k)\|^2] = 0$, for all $i \in \mathcal{I}_{m+n}$.

Morover, $E(x^*) = \sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}[\Psi(k,r)]_{ij}\bar{\rho}_j(r-1) + \sum_{j=1}^{2m+n}[\Psi(k,s)]_{ij}Z_j(s)+\bar{\rho}_i(k)$, $Var(x^*) = E[\|\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}[\Psi(k,r)]_{ij}\gamma(r-1)\sum_{u=1}^{2m+n}[\Xi(r-1)]_{ju}\bar{\xi}_u^T(r-1) + \gamma(k)\sum_{j=1}^{2m+n}[\Xi(k)]_{ij}\bar{\xi}_j^T(k)\|^2] < \infty$. Therefore, all states of the agents can reach consensus in mean square. Since $x_i(k) \in X_i$ and X_i is a bounded closed convex set, it means that $x^* \in X = \bigcap_{i=1}^{m+n} X_i$. □

Remark 5: From the above analysis, it is easy to obtain that constrained consensus can also be achieved in mean square by alogrithm (3) for first-order or second-order agents, i.e., $n = 0$ or $m = 0$.

Corollary 1: For second-order agent systems, i.e., $n = 0$, there exists a random variable x^* such that $\lim_{k\to\infty} E[\|x_i(k) - x^*\|^2] = 0$, $\lim_{k\to\infty} E[\|v_i(k)\|^2] = 0$, $i = 1, 2, \cdots, m$. For first-order agent systems, i.e., $m = 0$, $\lim_{k\to\infty} E[\|x_i(k) - x^*\|^2] = 0$, $i = 1, 2, \cdots, n$.

Proof: For $n = 0$, Let $y_i(k) = x_i(k) + v_i(k)$, then $x_i(k+1) = P_{X_i}[x_i(k) + v_i(k)T] = (1-T)x_i(k) + Ty_i(k) + \rho_i(k)$, and $y_i(k+1) = (\tau_i(k)T - T)x_i(k) + \iota_i(k)\pi_i(k)T + (1-\tau_i(k)T + T)y_i(k) + \rho_i(k)$, where $\rho_i(k) = P_{X_i}[(1-T)x_i(k) + Ty_i(k)] - [(1-T)x_i(k) + Ty_i(k)]$. Define $Z(k) = [Z_1^T(k), \cdots, Z_{2m}^T(k)]^T = [x_1^T(k), \cdots, x_m^T(k), y_1^T(k), \cdots, y_m^T(k)]^T$, $\rho(k) = [\bar{\rho}_1^T(k), \cdots, \bar{\rho}_{2m}^T(k)]^T = [\rho_1^T(k), \cdots, \rho_m^T(k), \rho_1^T(k), \cdots, \rho_m^T(k)]^T$,

$$\Phi(k) = \begin{bmatrix} \Phi_1(k) & \Phi_2(k) \\ \Phi_3(k) & \Phi_4(k) \end{bmatrix},$$

where $\Phi_1(k) = diag\{1-T, \cdots, 1-T\}$, $\Phi_2(k) = diag\{T, \cdots, T\}$,

$$\Phi_3(k) = \Phi_{3s}(k)\begin{bmatrix} \Phi_{31}(k) & a_{12}(k)\gamma(k)T & \cdots & a_{1m}(k)\gamma(k)T \\ a_{21}(k)\gamma(k)T & \Phi_{32}(k) & \cdots & a_{2m}(k)\gamma(k)T \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}(k)\gamma(k)T & \cdots & \cdots & \Phi_{3m}(k) \end{bmatrix},$$

$\Phi_{3s}(k) = diag\{\iota_1(k), \cdots, \iota_m(k)\}$, $\Phi_{31}(k) = \frac{\tau_1(k)T-T}{\iota_1(k)} - \gamma(k)\sum_{j\in\mathcal{N}_1(k)} a_{1j}(k)T$, $\Phi_{32}(k) = \frac{\tau_2(k)-T}{\iota_2(k)} - \gamma(k)\sum_{j\in\mathcal{N}_2(k)} a_{2j}(k)T, \cdots, \Phi_{3m}(k) = \frac{\tau_m(k)-T}{\iota_m(k)} - \gamma(k)\sum_{j\in\mathcal{N}_m(k)} a_{mj}(k) T$, $\Phi_4(k) = diag\{1-\tau_1(k)T+T, \cdots, 1-\tau_m(k)T+T\}$. Then the dynamics of second-order MASs with algorithm (3) can be rewritten as a compact form

$$Z(k+1) = [\Phi(k)\otimes I_n]Z(k) + \rho(k) + \gamma(k)\Xi(k)\xi(k),$$

where $\Xi(k) = diag\{0, \cdots, 0, \Xi_1^T(k), \cdots, \Xi_m^T(k)\}$, $\Xi_i(k) = [a_{i1}(k), \cdots, a_{im}(k)]$, $\xi(k) = [\bar{\xi}_1^T(k), \cdots, \bar{\xi}_{2m}^T(k)]^T = [0, \cdots, 0, \xi_1^T(k), \cdots, \xi_m^T(k)]^T$, $\xi_i(k) = [\xi_{i1}(k), \cdots, \xi_{im}(k)]^T$.

For $m = 0$, $x_i(k+1) = P_{X_i}[x_i(k) + S_{V_i}(u_i(k))T] = x_i(k) + \iota_i(k)\pi_i(k)T + \rho_i(k)$, where $\rho_i(k) = P_{X_i}[x_i(k) + \iota_i(k)\pi_i(k)T] - [x_i(k) + \iota_i(k)\pi_i(k)T]$. Define $Z(k) = [x_1^T(k), \cdots, x_n^T(k)]^T$, $\rho(k) = [\rho_1^T(k), \cdots, \rho_n^T(k)]^T$,

$$\Phi(k) = \begin{bmatrix} \Phi_1(k) & \iota_1(k)\gamma(k)a_{12}(k)T & \cdots & \iota_1(k)\gamma(k)a_{1n}(k)T \\ \iota_2(k)\gamma(k)a_{21}(k)T & \Phi_2(k) & \cdots & \iota_2(k)\gamma(k)a_{2n}(k)T \\ \vdots & \vdots & \vdots & \vdots \\ \iota_n(k)\gamma(k)a_{n1}(k)T & \cdots & \cdots & \Phi_n(k) \end{bmatrix},$$

$\Phi_1(k) = 1 - \iota_1(k)\gamma(k)\sum_{j\in\mathcal{N}_1(k)} a_{1j}(k)T$, $\Phi_2(k) = 1 - \iota_2(k)\gamma(k)\sum_{j\in\mathcal{N}_2(k)} a_{2j}(k)T$, $\cdots$, $\Phi_n(k) = 1 - \iota_n(k)\gamma(k)\sum_{j\in\mathcal{N}_n(k)} a_{nj}(k)T$. Then the dynamics of first-order MASs with algorithm (3) can be rewritten as a compact form

$$Z(k+1) = [\Phi(k)\otimes I_n]Z(k) + \rho(k) + \gamma(k)\Xi(k)\xi(k),$$

where $\Xi(k) = diag\{\Xi_1^T(k), \cdots, \Xi_n^T(k)\}$, $\Xi_i(k) = [a_{i1}(k), \cdots, a_{in}(k)]$, $\xi(k) = [\xi_1^T(k), \cdots, \xi_n^T(k)]^T$, $\xi_i(k) = [\xi_{i1}(k), \cdots, \xi_{in}(k)]^T$. Hence, by Lemmas 3-6, we obtain the same result as Theorem 1, i.e., when the networks contains only first-order agents, $\lim_{k\to\infty} E[\|x_i(k) - x^*\|^2] = 0, \forall i = 1, 2, \cdots, n$; when the networks contains only second-order agents, $\lim_{k\to\infty} E[\|x_i(k) - x^*\|^2] = 0$, $\lim_{k\to\infty} E[\|v_i(k)\|^2] = 0, \forall i = 1, 2, \cdots, m$. □

4. SIMULATION

In this section, a numerical simulation example is given to verify the correctness of theoretical results. The communication graph of HMASs switches among three subgraphs shown in Fig. 2, which includes four second-order agents (1)-(4) and two first-order agents (5) and (6). It is obvious that each digraph does not have a spanning tree, but the union graph is strongly connected. Take $T = 0.85$, $p_1 = 1.12$, $p_2 = 1.14$, $p_3 = 1.16$, $p_4 = 1.15$. It is easy to verify that Assumptions 5 and 6 are satisfied. Suppose that nonconvex constraint sets $V_i = \{x|\|x\| \leq 0.8,$

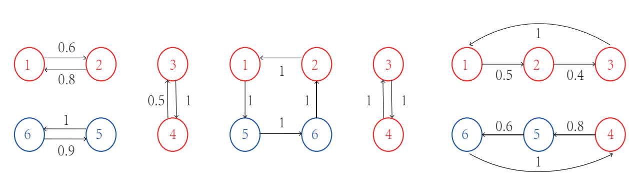

Fig. 2. Communication graph.

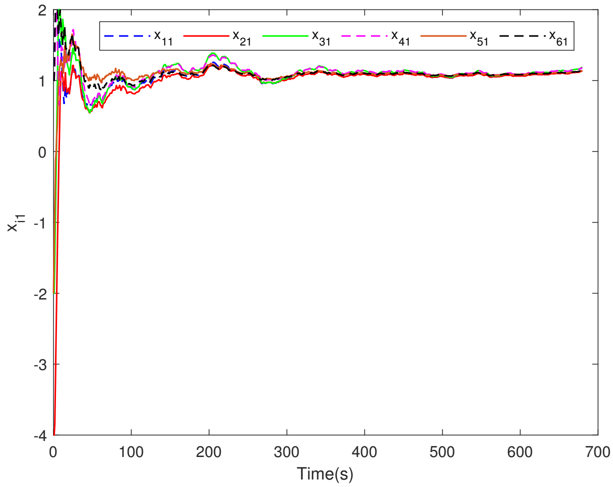

Fig. 3. Trajectories of the first component of position.

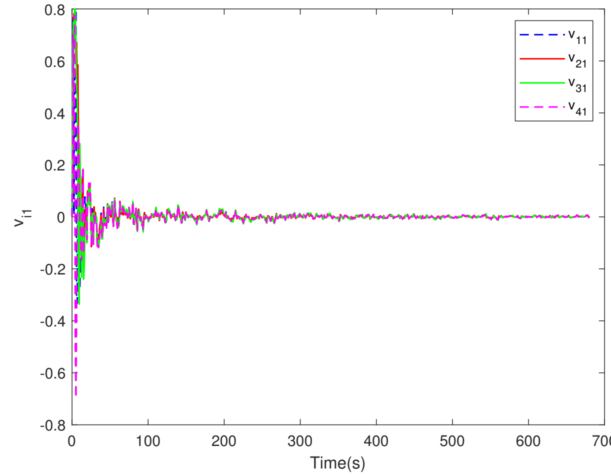

Fig. 5. Trajectories of the first component of velocity.

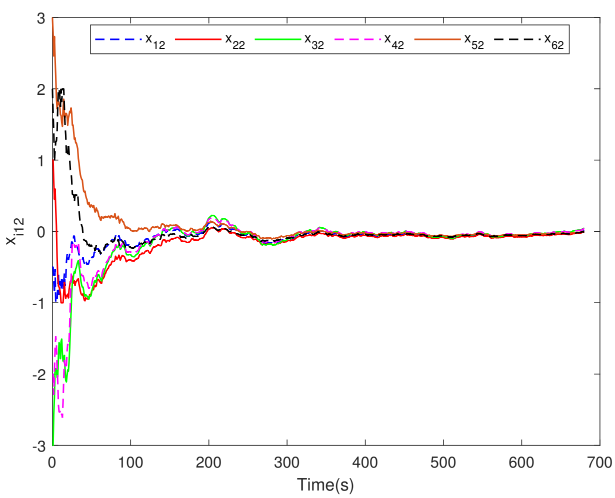

Fig. 4. Trajectories of the second component of position.

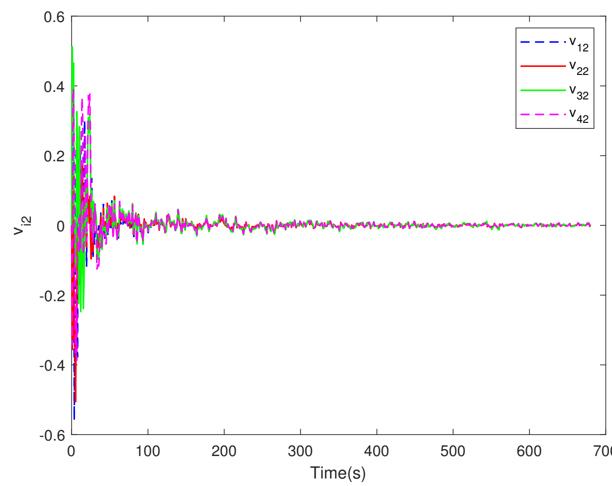

Fig. 6. Trajectories of the second component of velocity.

$\frac{|x[0\ 1]^T|}{|x[1\ 0]^T|} \le 1\} \cup \{x| \|x\| \le 0.4,\ \frac{|x[0\ 1]^T|}{|x[1\ 0]^T|} \ge 1\}$ and convex constraint sets $X_1 = X_6 = \{(x_1,x_2)^T \mid 0 \le x_1 \le 2,\ -1 \le x_2 \le 2\}$, $X_3 = X_4 = \{(x_1,x_2)^T \mid -2 \le x_1 \le 2,\ -3 \le x_2 \le 1\}$, $X_2 = X_5 = \{(x_1,x_2)^T \mid -5 \le x_1 \le 3,\ -1 \le x_2 \le 1\} \in \mathbb{R}^2$. Take initial values $x_1(0) = [1, -0.5]^T \in X_1$, $x_2(0) = [-4, 1]^T \in X_2$, $x_3(0) = [-2, -3]^T \in X_3$, $x_4(0) = [1, -2]^T \in X_4$, $x_5(0) = [-2,\ 3]^T \in X_5$, $x_6(0) = [1,\ 2]^T \in X_6$, $v_1(0) = [0,\ 0]^T$, $v_2(0) = [0,\ 0]^T$, $v_3(0) = [0,\ 0]^T$, $v_4(0) = [0,\ 0]^T$, $\gamma(k) = 1/k$. Suppose that the noise sequences $\{\xi_{ji}(k)\}$ is white noise with zero mean and bounded variance. By taking algorithm (3), the position and velocity trajectories are given in Figs. 3-6, from which it is easy to see that the position states of the agents converge to a point and the velocity states of second-order agents converge to zero. Moreover, the velocity states of the second-order agents and the input states of first-order agents in the phase plane are given in Fig. 7, from which we can see that these states are constrained in the nonconvex sets of two concentric circles and straight lines. The position states of all agents

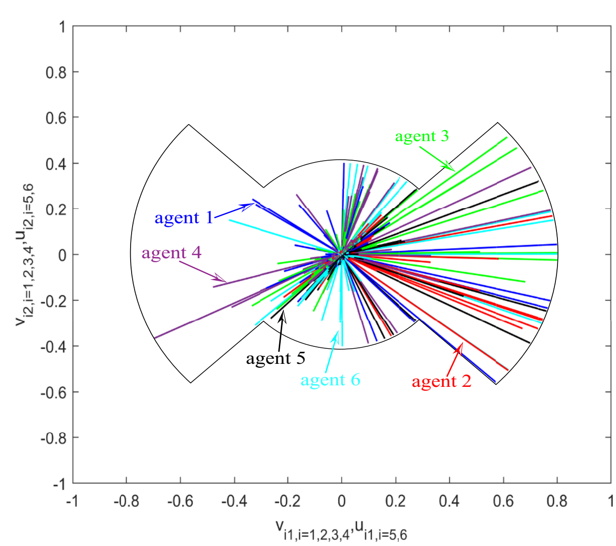

Fig. 7. The velocity states of all second-order agents and input states of all first-order agents in phase plane.

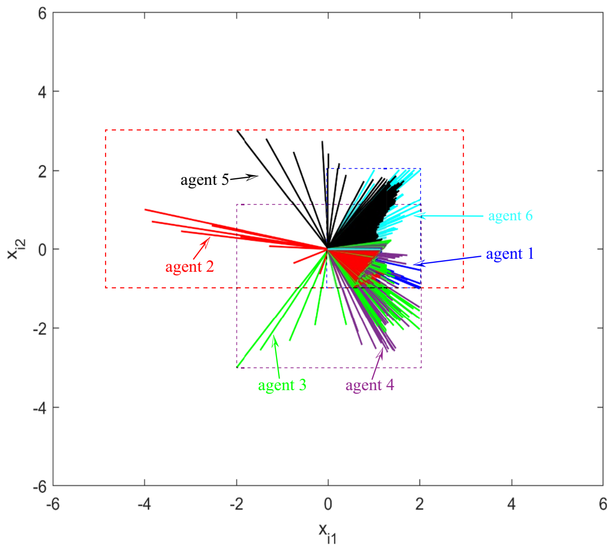

Fig. 8. The position states of all agents in phase plane.

in the phase are given in Fig. 8. It is easy to see that the position states of different agents are constrained in different convex sets. It reveals that our algorithm can solve constrained consensus problems with communication noises.

5. CONCLUSION

In this paper, stochastic consensus problems with bounded convex position and nonconvex constraints have been considered over jointly strongly connected graph. The dynamical models of HMASs were built based on the projection operator and nonconvex constraint operator. A novel algorithm with diminishing step size was designed under communication noises. By control theory and stochastic analysis, it was proved that the proposed algorithm can guarantee that mean-square consensus was achieved. Moreover, the same results have been obtained when considering the case of a network containing only first-order agents or second-order agents, respectively. The simulation example was verified the correctness of theoretical results.

APPENDIX A: PROOFS OF LEMMA 3

The proof of Lemma 3 is given as follows:

Proof: 1) By Lemma 1, we have

$$\begin{aligned}
&\|x_i(k+1)-P_Y(x_i(k+1))\|^2\\
&=\|P_{X_i}[(1-T)x_i(k)\\
&\quad+Ty_i(k)]-P_Y\left(P_{X_i}[(1-T)x_i(k)+Ty_i(k)]\right)\|^2\\
&\le\|(1-T)x_i(k)+Ty_i(k)-P_Y[(1-T)x_i(k)\\
&\quad+Ty_i(k)]\|^2-\|\rho_i(k)\|^2\\
&\le[(1-T)\|x_i(k)-P_Y(x_i(k))\|+T\|y_i(k)\\
&\quad-P_Y(y_i(k))\|]^2-\|\rho_i(k)\|^2.
\end{aligned}$$

Therefore, $\|x_i(k+1)-P_Y(x_i(k+1))\|\le(1-T)\|x_i(k)-P_Y(x_i(k))\|+T\|y_i(k)-P_Y(y_i(k))\|$, $\|\rho_i(k)\|^2\le-\|x_i(k+1)-P_Y(x_i(k+1))\|^2+[(1-T)\|x_i(k)-P_Y(x_i(k))\|+T\|y_i(k)-P_Y(y_i(k))\|]^2, \forall i\in\mathcal{I}_m$.

2) Let $\sigma_{ic}(k)=(1-T)x_i(k)+Ty_i(k)=x_i(k)+T(y_i(k)-x_i(k))$, $\sigma_{ip}(k)=P_{X_i}[\sigma_{ic}(k)]$, $\sigma_{iq}(k)=x_i(k)+Tq_i(k)(y_i(k)-x_i(k))$, $q_i(k)=\max\{q\in[0,\ 1]\mid x_i(k)+Tq(y_i(k)-x_i(k))\in X_i\}$, $\sigma_i(k)=\sigma_{ip}(k)-\sigma_{iq}(k)$. First, consider the situation of $q_i(k)<1$. Obviously, $\rho_i(k)=\sigma_{ip}(k)-\sigma_{ic}(k)=P_{X_i}[\sigma_{ic}(k)]-\sigma_{ic}(k)=\sigma_i(k)-(1-q_i(k))T(y_i(k)-x_i(k))$.

$$\begin{aligned}
y_i(k+1)&=\delta_{ii}(k)x_i(k)+\delta_{iy}(k)y_i(k)\\
&\quad+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)x_j(k)+\sigma_i(k)+\gamma(k)\delta_{i\xi}(k)\\
&=[\delta_{ii}(k)+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)]x_i(k)+\delta_{iy}(k)y_i(k)\\
&\quad+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)x_j(k)-\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)x_i(k)\\
&\quad+\gamma(k)\delta_{i\xi}(k)+\sigma_i(k)\\
&=\delta_{ic}(k)\delta_{is}(k)+(1-\delta_{is}(k))\delta_{iy}(k)y_i(k)\\
&\quad+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)x_j(k)+\sigma_i(k)+\gamma(k)\delta_{i\xi}(k)\\
&=\delta_{is}(k)\delta_{ip}(k)+(1-\delta_{is}(k))\delta_{iy}(k)y_i(k)\\
&\quad+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)x_j(k)+\gamma(k)\delta_{i\xi}(k),
\end{aligned}$$

where $\delta_{ii}(k)=\tau_i(k)T-\iota_i(k)\gamma(k)\sum_{j\in\mathcal{N}_i(k)}a_{ij}(k)T-Tq_i(k)$, $\delta_{iy}(k)=1-\tau_i(k)T+Tq_i(k)$, $\delta_{ij}(k)=\iota_i(k)\gamma(k)a_{ij}(k)T$, $\delta_{i\xi}(k)=\iota_i(k)\sum_{j\in\mathcal{N}_i(k)}a_{ij}(k)\xi_{ji}(k)T$, $\delta_{ic}(k)=\left[\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)+\delta_{ii}(k)\right]x_i(k)+\delta_{iy}(k)y_i(k)$, $\delta_{ip}(k)=\delta_{ic}(k)+\frac{\sigma_i(k)}{\delta_{is}(k)}$, $\delta_{is}(k)=\frac{\delta_{ii}(k)}{\delta_{ii}(k)+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)}$. Note that $\delta_{ii}(k)+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)=\tau_i(k)T-Tq_i(k)$. Hence, $\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)+(1-\delta_{is}(k))\delta_{iy}(k)+\delta_{is}(k)=1$ and $0\le\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)$, $(1-\delta_{is}(k))\delta_{iy}(k)$, $\delta_{is}(k)\le 1$. By Lemma 1, we have

$$\begin{aligned}
&\|y_i(k+1)-P_Y(y_i(k+1))\|\\
&\le\delta_{is}(k)\|\delta_{ip}(k)-P_Y[\delta_{ip}(k)]\\
&\quad+(1-\delta_{is}(k))\delta_{iy}(k)\|y_i(k)-P_Y[y_i(k)]\|\\
&\quad+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)\|x_j(k)-P_Y[x_j(k)]\|+\|\gamma(k)\delta_{i\xi}(k)\|.
\end{aligned}$$

It follows that $\delta_{ic}(k)=[\delta_{ii}(k)+\sum_{j\in\mathcal{N}_i(k)}\delta_{ij}(k)]x_i(k)+\delta_{iy}(k)y_i(k)=(\tau_i(k)T-Tq_i(k))x_i(k)+(1-\tau_i(k)T+Tq_i(k))y_i(k)=x_i(k)+[1-\tau_i(k)T+Tq_i(k)](y_i(k)-x_i(k))>Tq_i(k)$ for $q_i(k)<1$. Hence $\delta_{ic}(k)\notin X_i$. Let $\delta_{iy}(k)=\frac{\sigma_{ic}(k)-\sigma_{iq}(k)}{\delta_{is}(k)}+\delta_{ic}(k)$. Note that $\delta_{ip}(k)=\delta_{ic}(k)+\frac{\sigma_{ip}(k)-\sigma_{iq}(k)}{\delta_{is}(k)}$. Therefore, triangles $(\delta_{iy}(k),\delta_{ic}(k),\delta_{ip}(k))$

and $(\sigma_{ic}(k), \sigma_{iq}(k), \sigma_{ip}(k))$ is similar, and vectors $\delta_{ip}(k)-\delta_{ic}(k)$ and $\sigma_{ip}(k)-\sigma_{iq}(k)$ have same the direction. Therefore, the angle between $P_Y[\delta_{iy}(k)]-\delta_{ip}(k)$ and $\delta_{iy}(k)-\delta_{ip}(k)$ belongs to the interval $\left[\frac{\pi}{2},\ \pi\right]$. Hence, $\|\delta_{ip}(k)-P_Y[\delta_{iy}(k)]\| \le \|\delta_{iy}(k)-P_Y[\delta_{iy}(k)]\|$ and $\delta_{iy}(k) = \delta_{ic}(k) + \frac{\sigma_{ic}(k)-\sigma_{iq}(k)}{\delta_{is}(k)} = x_i(k) + [1-\tau_i(k)T + Tq_i(k)](y_i(k-x_i(k)) + \frac{\sigma_{ic}(k)-\sigma_{iq}(k)}{\delta_{is}(k)} = \bar{\delta}_{ic}(k)x_i(k) + (1-\bar{\delta}_{ic}(k))y_i(k)$, where $\bar{\delta}_{ic}(k) = \tau_i(k)T - Tq_i(k) - \frac{T(1-q_i(k))}{\delta_{is}(k)}$. From $1 < p_i < \frac{1}{T}$ and $\tau_i(k) = \frac{1-\iota_i(k)(1-p_iT)}{T}$, it follows that $1 < p_i \le \tau_i(k) < \frac{1}{T}$, and by Assumptions 3 and 5, there exists a constant $\underline{k}_1 > 0$ such that $\gamma(k) \le \frac{(p_i-1)^2}{p_i(1-\mu_1)}$ for all $k \ge \underline{k}_1$. Then, we have $\frac{1}{p_i} < \delta_{is}(k) \le 1$. Hence, $\bar{\delta}_{ic}(k) \le \tau_i(k)T - Tq_i(k) \le 1 - Tq_i(k) < 1$ and $\bar{\delta}_{ic}(k) \ge \tau_i(k)T - Tq_i(k) - \frac{T(1-q_i(k))}{1/p_i} \ge (p_i-1)T > 0$. Therefore, $\|\delta_{ip}(k) - P_Y[\delta_{ip}(k)]\| \le \|\delta_{ip}(k) - P_Y[\delta_{iy}(k)]\| \le \|\delta_{iy}(k) - P_Y[\delta_{iy}(k)]\| \le \bar{\delta}_{ic}(k)\|x_i(k) - P_Y[x_i(k)]\| + (1-\bar{\delta}_{ic}(k))\|y_i(k) - P_Y[y_i(k)]\|$. Then,

$$
\begin{aligned}
&\|y_i(k+1) - P_Y(y_i(k+1))\| \\
&\le \delta_{is}(k)\|\delta_{ip}(k) - P_Y[\delta_{ip}(k)]\| \\
&\quad + (1-\delta_{is}(k))\delta_{iy}(k)\|y_i(k) - P_Y[y_i(k)]\| \\
&\quad + \sum_{j\in\mathcal{N}_i(k)} \delta_{ij}(k)\|x_i(k) - P_Y[x_i(k)]\| + \|\gamma(k)\delta_{i\xi}(k)\| \\
&\le \varphi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| + \varphi_{iy}(k)\|y_i(k) \\
&\quad - P_Y[y_i(k)]\| + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\| \\
&\quad + \|\gamma(k)\delta_{i\xi}(k)\|,
\end{aligned}
$$

where $\varphi_{ix}(k) = \delta_{is}(k)\bar{\delta}_{ic}(k)$, $\varphi_{iy}(k) = \delta_{is}(k)(1-\bar{\delta}_{ic}(k)) + (1-\delta_{is}(k))\delta_{iy}(k)$, $\varphi_{ij}(k) = \delta_{ij}(k)$. It is easy to obtain that $0 \le \varphi_{ix}(k),\ \varphi_{iy}(k),\ \varphi_{ij}(k) \le 1$ and $\varphi_{ix}(k) + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k) + \varphi_{iy}(k) = 1$. Next, let us consider the situation of $q_i(k) = 1$, then $\sigma_{ic}(k) \in X_i$ and $\sigma(k) = 0$. Take $\varphi_{ix}(k) = \tau_i(k)T - T - \iota_i(k)\gamma(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)$, $\varphi_{ij}(k) = \iota_i(k)\gamma(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)$, $\varphi_{iy}(k) = 1 - \tau_i(k) + T$, then $\|y_i(k+1) - P_Y(y_i(k+1))\| \le \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\| + \|\gamma(k)\delta_{i\xi}(k)\| + \varphi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| + \varphi_{iy}(k)\|y_i(k) - P_Y[y_i(k)]\|$, where $\varphi_{ix}(k)$, $\varphi_{iy}(k)$ and $\varphi_{ij}(k)$ are nonegative and $\varphi_{ix}(k) + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k) + \varphi_{iy}(k) = 1$.

3) Let $\chi_{i\xi}(k) = \iota_i(k)\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)\xi_{ji}(k)T$, $\chi_{ij}(k) = \iota_i(k)\gamma(k)Ta_{ij}(k)$, $\chi_{ix}(k) = 1 - \iota_i(k)\gamma(k)T\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)$, for all $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$. Note that $\sum_{j\in\mathcal{N}_i(k)} a_{ij}(k) \le 1 - \mu_1$, $\chi_{ix}(k) + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k) = 1$, and $0 < \mu_1 < 1$, we have $\chi_{ix}(k) \ge 1 - T(1-\mu_1) > 0$. Then, $\chi_{ix}(k) > 0$,

$$
\begin{aligned}
&\|x_i(k+1) - P_Y[x_i(k+1)]\|^2 \\
&= \|P_{X_i}[\chi_{ix}(k)x_i(k) + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)x_j(k) + \gamma(k)\chi_{i\xi}(k)] \\
&\quad - P_Y\big[P_{X_i}[\chi_{ix}(k)x_i(k) + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)x_j(k) \\
&\quad + \gamma(k)\chi_{i\xi}(k)]\big]\|^2 \\
&\le \|P_{X_i}[\chi_{ix}(k)x_i(k) + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)x_j(k) + \gamma(k)\chi_{i\xi}(k)] \\
&\quad - P_Y[\chi_{ix}(k)x_i(k) + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)x_j(k) \\
&\quad + \gamma(k)\chi_{i\xi}(k)]\|^2 \\
&\le \|P_Y[\chi_{ix}(k)x_i(k) + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)x_j(k) + \gamma(k)\chi_{i\xi}(k)] \\
&\quad - \chi_{ix}(k)x_i(k) + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)x_j(k) + \gamma(k)\chi_{i\xi}(k)\|^2 \\
&\quad - \|\rho_i(k)\|^2 \\
&\le \big[\chi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| \\
&\quad + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\| \\
&\quad + \|\gamma(k)\chi_{i\xi}(k)\|\big]^2 - \|\rho_i(k)\|^2.
\end{aligned}
$$

Hence, $\|x_i(k+1) - P_Y[x_i(k+1)]\| \le \|\gamma(k)\chi_{i\xi}(k)\| + \chi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\|$, $\|\rho_i(k)\|^2 \le \big[\chi_{ix}(k)\|x_i(k) - P_Y[x_i(k)]\| + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)\|x_j(k) - P_Y[x_j(k)]\| + \|\gamma(k)\chi_{i\xi}(k)\|\big]^2 - \|x_i(k+1) - P_Y[x_i(k+1)]\|^2$. The proof is completed. □

APPENDIX B: PROOF OF LEMMA 4

The proof of Lemma 4 is given as follows:

Proof: Define $V(k) = \max_{i\in\mathcal{I}_{m+n}} \{\|Z_i(k) - s_l\|^2\}$, where $s_l \in X$. For $i \in \mathcal{I}_m$,

$$
\begin{aligned}
&E[\|x_i(k+1) - s_l\|^2|\mathcal{F}(k)] \\
&\le (1-T)\|x_i(k) - s_l\|^2 + T\|y_i(k) - s_l\|^2 \\
&\le \|x_i(k) - s_l\|^2.
\end{aligned} \tag{B.1}
$$

By Lemma 3, there exist $\varphi_{ix}(k)$, $\varphi_{iy}(k)$, $\sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k) \ge 0$ and $\varphi_{ix}(k) + \varphi_{iy}(k) + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k) = 1$, such that $\|y_i(k+1) - s_l\| \le \varphi_{ix}(k)\|x_i(k) - s_l\| + \varphi_{iy}(k)\|y_i(k) - s_l\| + \|\gamma(k)\delta_{i\xi}(k)\| + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k)\|x_j (k) - s_l\|$. Then

$$
\begin{aligned}
&E[\|y_i(k+1) - s_l\|^2|\mathcal{F}(k)] \\
&\le \varphi_{ix}(k)\|x_i(k) - s_l\|^2 + \varphi_{iy}(k)\|y_i(k) - s_l\|^2 \\
&\quad + \sum_{j\in\mathcal{N}_i(k)} \varphi_{ij}(k)\|x_j(k) - s_l\|^2 \\
&\quad + E[\|\gamma(k)\delta_{i\xi}(k)\|^2|\mathcal{F}(k)] \\
&\le \|y_i(k) - s_l\|^2 + \gamma^2(k)\sigma_\xi^2.
\end{aligned} \tag{B.2}
$$

For $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$, $\|x_i(k+1) - s_l\| \le \chi_{ix}(k)\|x_i(k) - s_l\| + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)\|x_j(k) - s_l\| + \|\gamma(k)\chi_{i\xi}(k)\|$, then

$$
\begin{aligned}
&E[\|x_i(k+1) - s_l\|^2|\mathcal{F}(k)] \\
&\le \chi_{ix}(k)\|x_i(k) - s_l\|^2 + \sum_{j\in\mathcal{N}_i(k)} \chi_{ij}(k)\|x_j(k) - s_l\|^2
\end{aligned}
$$

$$+E[\|\gamma(k)\chi_{i\xi}(k)\|^2|\mathcal{F}(k)] \leq \|x_i(k)-s_l\|^2+\gamma^2(k)\sigma_\xi^2. \tag{B.3}$$

By Lemma 2, the sequence $\{\|Z_i(k)-s_l\|\}$ almost surely converges to a nonnegative random variable, i.e., $x_i(k)$, $y_j(k)$ are bounded almost surely for all $i\in\mathcal{I}_{m+n}$, $j\in\mathcal{I}_m$. Hence, $v_i(k)+u_i(k)T$ is bounded almost surely, $\forall i\in\mathcal{I}_m$. Let its boundary be $L_1>\underline{\hbar}_i$. Note that $\|S_{V_i}(y)\|\geq\underline{\hbar}_i$ for all $y\notin V_i$, we have $\|S_{V_i}(y)\|=\underline{\hbar}_i$ for all $\|y\|\leq\underline{\hbar}_i$ and $\|S_{V_i}(y)\|\geq\underline{\hbar}_i$ for all $\|y\|\geq\underline{\hbar}_i$. Hence, for all $i\in\mathcal{I}_m$, $\iota_i(k)=1$, if $v_i(k)+u_i(k)T\leq\underline{\hbar}_i$; $\iota_i(k)\geq\frac{\underline{\hbar}_i}{L_1}$ if $v_i(k)+u_i(k)T\geq\underline{\hbar}_i$. Similarly, for all $i\in\mathcal{I}_{m+n}-i\in\mathcal{I}_m$, there exist $L_2>\underline{\hbar}_i$ such that $\iota_i(k)=1$ if $u_i(k)T\leq\underline{\hbar}_i$ and $\iota_i(k)=\frac{\underline{\hbar}_i}{L_2}$ if $u_i(k)T\geq\underline{\hbar}_i$. Let $\beta_0=\min\{\frac{\underline{\hbar}_i}{L_1},\frac{\underline{\hbar}_j}{L_2}|i\in\mathcal{I}_m, j\in\mathcal{I}_{m+n}-\mathcal{I}_m\}$. Then $0<\beta_0<1$ and $\beta_0\leq\iota_i(k)\leq 1$ for $i\in\mathcal{I}_{m+n}$, k, almost surely.□

APPENDIX C: PROOF OF LEMMA 6

The proof of Lemma 6 is given as follows:

Proof: By Lemma 4, $x_i(k)$ and $y_i(k)$ are bounded. Hence, $E[\|Z_i(k)-P_X[Z_i(k)]\|^2]$ is bounded. If Lemma 6 is not true, there exist a sequence $\{\hat{k}_t\}$ and $M>0$, $\limsup_{t\to\infty}\max_i\{E[\|Z_i(\hat{k}_t)-P_X[Z_i(\hat{k}_t)]\|^2]\}=M$. Moreover, $\lim_{k\to\infty}\gamma(k)E[\|\delta_{i\xi}(k)\|^2]=0$. Then, for any $\epsilon>0$, there exist i_0, $\bar{k}$, l_0 $N>(2m+n-1)B+1$, such that $M-\epsilon<E\left[\|Z_{i_0}(\bar{k})-P_X[Z_{i_0}(\bar{k})]\|^2\right]<M+\epsilon$ for all $\bar{k}-l_0>N$, $E\left[\|Z_i(\hat{k}_t)-P_X[Z_i(\hat{k}_t)]\|^2\right]<M+\epsilon$, $E\left[[\omega_i(\hat{k}_t)]^T[\omega_i(\hat{k}_t)]\right]<\epsilon$, for all $i\in\mathcal{I}_{m+n}$, $\hat{k}_t>l_0$. Define $\psi(k)=[\psi_1(k),\cdots,\psi_m(k),\psi_{m+1}(k),\cdots,\psi_{m+n}(k)]^T=[\|Z_1(k)-P_X[Z_1(k)]\|,\cdots,\|Z_{2m+n}(k)-P_X[Z_{2m+n}(k)]\|]^T$. Let $\mu(k)$ be a $(2m+n)\times(2m+n)$ matrix.

$$\mu(k)=\begin{bmatrix}\mu_1(k) & \mu_2(k) & \mathbf{0}\\ \mu_3(k) & \mu_4(k) & \mu_5(k)\\ \mu_6(k) & \mathbf{0} & \mu_7(k)\end{bmatrix},$$

where $\mu_1(k)=diag\{1-T,\cdots,1-T\}$, $\mu_2(k)=diag\{T,\cdots,T\}$,

$$\mu_3(k)=\begin{bmatrix}\varphi_{1x}(k)+\varphi_{11}(k) & \varphi_{12}(k) & \cdots & \varphi_{1m}(k)\\ \varphi_{21}(k) & \varphi_{22}(k) & \cdots & \varphi_{2m}(k)\\ \vdots & \vdots & \vdots & \vdots\\ \varphi_{m1}(k) & \varphi_{m2}(k) & \cdots & \varphi_{mx}(k)+\varphi_{mm}(k)\end{bmatrix},$$

$\mu_4(k)=diag\{\varphi_{1y}(k),\cdots,\varphi_{my}(k)\}$, $[\mu_5(k)]_{ij}=\varphi_{ij}(k), \forall i\in\mathcal{I}_m, j\in\mathcal{I}_{m+n}-\mathcal{I}_m$, $[\mu_6(k)]_{ij}=\chi_{ij}(k), \forall i\in\mathcal{I}_{m+n}-\mathcal{I}_m, j\in\mathcal{I}_m$,

$$\mu_7(k)=\begin{bmatrix}\mu_{71}(k) & \chi_{m+1m+2}(k) & \cdots & \chi_{m+1m+n}(k)\\ \chi_{m+2m+1}(k) & \mu_{72}(k) & \cdots & \varphi_{m+2m+n}(k)\\ \vdots & \vdots & \vdots & \vdots\\ \chi_{m+nm+1}(k) & \chi_{m+nm+2}(k) & \cdots & \mu_{7n}(k)\end{bmatrix},$$

$\mu_{71}(k)=\chi_{m+1x}(k)+\chi_{m+1m+1}(k)$, $\mu_{72}(k)=\chi_{m+2x}(k)+\chi_{m+2m+2}(k)$, $\cdots$, $\mu_{7n}(k)=\chi_{m+nx}(k)+\chi_{m+nm+n}(k)$. Define $\omega(k)=[\omega_1(k),\cdots,\omega_{2m+n}(k)]^T=\gamma(k)[0,\cdots,0,\|\delta_{1\xi}(k)\|,\cdots,\|\delta_{m\xi}(k)\|,\|\chi_{m+1\xi}(k)\|,\cdots,\|\chi_{m+n\xi}(k)\|]^T$. By Lemma 5, we have $\psi_i(k+1)\leq\sum_{j=1}^{2m+n}[\mu(k)]_{ij}\psi_i(k)+\omega_i(k)$, $\mu(k)$ and $\Theta(k,s)$ are stochastic matrices, where $\Theta(k,s)=\mu(k)\mu(k-1)\cdots\mu(s)$ for all $k\geq s>\underline{k}_2$. Hence, we have $\psi_i(\bar{k}+1)\leq\sum_{j=1}^{2m+n}[\Theta(\bar{k},\hat{k}_t)]_{ij}\psi_j(\hat{k}_t)+\sum_{r=\hat{k}_t+1}^{\bar{k}}\sum_{j=1}^{2m+n}[\Theta(\bar{k},r)]_{ij}\omega_j(r)+\omega_i(\bar{k})$, for all $\bar{k}>\hat{k}_t>l_0$. Note that $E[\psi_i(\hat{k}_t)^T\psi_i(\hat{k}_t)]<M+\epsilon$ if $\hat{k}_t>l_0$, we have

$$\begin{aligned}&E[\psi_{i_0}(\bar{k})^T\psi_{i_0}(\bar{k})]\\ &\leq E\big[[(1-[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i})\psi_j(\hat{k}_t)\\ &\quad+[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}\psi_i(\hat{k}_t)]^T[(1-[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i})\\ &\quad\times\psi_j(\hat{k}_t)+[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}\psi_i(\hat{k}_t)]\big]+5\epsilon\\ &\leq(1-[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i})E[\|\psi_j(\hat{k}_t)\|^2]\\ &\quad+[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}E[\|\psi_i(\hat{k}_t)\|^2]+5\epsilon\\ &\leq(1-[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i})(M+\epsilon)\\ &\quad+[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}E[\|\psi_i(\hat{k}_t)\|^2]+5\epsilon,\end{aligned}$$

for any $\bar{k}>\hat{k}_t$. Similar to Lemma 5 in [31], there exists $\bar{\mu}>0$, such that $\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}>\bar{\mu}$ for all $l_0<\hat{k}_t<\bar{k}-1-(2m+n-1)B$ and $i\in\mathcal{I}_{m+n}$. Note that $M-\epsilon<E[\psi_{i_0}(\bar{k})^T\psi_{i_0}(\bar{k})]$, we have

$$\begin{aligned}&E[\|\psi_i(\hat{k}_t)\|^2]\\ &\geq\frac{E[\|\psi_{i_0}(\bar{k})\|^2]-5\epsilon}{[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}}-\frac{(1-[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i})(M+\epsilon)}{[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}}\\ &\geq\frac{M-\epsilon-(1-[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i})(M+\epsilon)-5\epsilon}{[\Theta(\bar{k}-1,\hat{k}_t)]_{i_0i}}\\ &\geq(M+\epsilon)-\frac{7\epsilon}{\bar{\mu}}.\end{aligned}$$

For all $i\in\mathcal{I}_m$ and $l_0<\hat{k}_t<\bar{k}-1-(2m+n-1)B$, we have

$$\begin{aligned}E[\|\rho_i(\hat{k}_t)\|^2]&\leq E\big[[(1-\frac{p_iT}{2})\|x_i(\hat{k}_t)-P_Y(x_i(\hat{k}_t))\|\\ &\quad+\frac{p_iT}{2}\|y_i(\hat{k}_t)-P_Y(y_i(\hat{k}_t))\|]^2\big]\\ &\quad-E\big[\|x_i(\hat{k}_t+1)-P_Y(x_i(\hat{k}_t+1))\|^2\big]\\ &\leq(M+\epsilon)-[(M+\epsilon)-\frac{7\epsilon}{\bar{\mu}}]=\frac{7\epsilon}{\bar{\mu}}.\end{aligned}$$

For all $i\in\mathcal{I}_{m+n}-\mathcal{I}_m$ and $l_0<\hat{k}_t<\bar{k}-1-(2m+n-1)B$, we have

$$\begin{aligned}&E[\|\rho_i(\hat{k}_t)\|^2]\\ &\leq E\big[[\chi_{ix}(\hat{k}_t)\|x_i(\hat{k}_t)-P_Y[x_i(\hat{k}_t)]\|+\|\gamma(\hat{k}_t)\chi_{i\xi}(\hat{k}_t)\|]^2\\ &\quad+\sum_{j=1}^{m+n}\chi_{ij}(\hat{k}_t)\|x_j(\hat{k}_t)-P_Y[x_j(\hat{k}_t)]\|\big]\\ &\quad-E\big[\|x_i(\hat{k}_t+1)-P_Y[x_i(\hat{k}_t+1)]\|^2\big]\end{aligned}$$

$$\leq (M+\epsilon)+\epsilon-[(M+\epsilon)-\frac{7\epsilon}{\bar{\mu}}] \leq \epsilon+\frac{7\epsilon}{\bar{\mu}}.$$

Since ϵ is arbitrarily small, for any $\varpi > 0$, we can take large enough N such that $E[\|\rho_i(k)\|^2] < \varpi$. By Lemma 5, there exist $\theta_j(k) > 0$ and $0 < \beta_1 < 1/(2m+n)$, such that $\|[\Psi(k,s)]_{ij} - \theta_j(k)\| \leq 2C_\mu(1-2\beta_1)^{\frac{k-s}{B_0}}$, where $B_0 = (2m+n-1)B$, $C_\mu = 1/(1-2\beta_1)$ and $k \geq s > \underline{k}_2$. From (4), we have $Z_i(k+1) = \sum_{j=1}^{2m+n}[\Phi(k)]_{ij}Z_j(k) + \gamma(k)\sum_{j=1}^{2m+n}[\Xi(k)]_{ij}\bar{\xi}_j^T(k) + \bar{\rho}_i(k) = \Delta_{1i}(k) + \Delta_{2i}(k) + \Delta_{3i}(k) + \Delta_{4i}(k) + \Delta_{5i}(k)$, where $\Delta_{1i}(k) = \sum_{j=1}^{2m+n}[\Psi(k,s)]_{ij}Z_j(s)$, $\Delta_{2i}(k) = \sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}[\Psi(k,r)]_{ij}\bar{\rho}_j(r-1)$, $\Delta_{3i}(k) = \bar{\rho}_i(k)$, $\Delta_{4i}(k) = \sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}[\Psi(k,r)]_{ij}\gamma(r-1)\sum_{u=1}^{2m+n}[\Xi(r-1)]_{ju}\bar{\xi}_u^T(r-1)$, $\Delta_{5i}(k) = \gamma(k)\sum_{j=1}^{2m+n}[\Xi(k)]_{ij}\bar{\xi}_j^T(k)$. Let $\bar{Z}(k+1) = \frac{1}{2m+n}\sum_{i=1}^{2m+n}Z_i(k+1)$. Hence,

$$\begin{aligned}&\|Z_i(k+1)-Z_j(k+1)\|^2\\ &\leq 4\max_i\|Z_i(k+1)-\bar{Z}(k+1)\|^2. \qquad \text{(C.1)}\end{aligned}$$

Take the expectation for each term of the expansion on the right-hand side of the inequality,

$$\begin{aligned}&E\left[\|\Delta_{1i}(k)-\frac{1}{2m+n}\sum_{i=1}^{2m+n}\Delta_{1i}(k)\|^2\right]\\ &\leq E\left[\left\|\sum_{j=1}^{2m+n}\left[([\Psi(k,s)]_{ij}-\theta_j(s))\right.\right.\right.\\ &\quad\left.\left.\left.-\frac{1}{2m+n}\sum_{i=1}^{2m+n}([\Psi(k,s)]_{ij}-\theta_j(s))\right]\right\|^2\sup_j\{\|Z_j(s)\|^2\}\right]\\ &\leq\left[\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k-s}{B_0}}\right]^2\max_j\{\|Z_j(s)\|^2\}. \qquad \text{(C.2)}\end{aligned}$$

And similarly, we can obtain

$$\begin{aligned}&E\left[\|\Delta_{2i}(k)-\frac{1}{2m+n}\sum_{i=1}^{2m+n}\Delta_{2i}(k)\|^2\right]\\ &\leq E\left[\left\|\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}\left[\big([\Psi(k,r)]_{ij}-\theta_j(r)\big)\right.\right.\right.\\ &\quad\left.\left.-\frac{1}{2m+n}\sum_{i=1}^{2m+n}([\Psi(k,r)]_{ij}-\theta_j(r))\right]\right\|^2\\ &\quad\left.\times\sup_{j,r}\{\|\bar{\rho}_j(r-1)\|^2\}\right]\\ &\leq\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2\varpi, \qquad \text{(C.3)}\end{aligned}$$

$$E\left[\|\Delta_{3i}(k)-\frac{1}{2m+n}\sum_{i=1}^{2m+n}\Delta_{3i}(k)\|^2\right]\leq 2\varpi, \qquad \text{(C.4)}$$

$$\begin{aligned}&E\left[\|\Delta_{4i}(k)-\frac{1}{2m+n}\sum_{i=1}^{2m+n}\Delta_{4i}(k)\|^2\right]\\ &\leq\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}C_\mu(4-8\beta_1)^{\frac{k-r}{B_0}}\right]^2\\ &\quad\times[(2m+n)(1-\mu)T\sigma_\xi]^2, \qquad \text{(C.5)}\end{aligned}$$

$$\begin{aligned}&E\left[\|\Delta_{5i}(k)-\frac{1}{2m+n}\sum_{i=1}^{2m+n}\Delta_{5i}(k)\|^2\right]\\ &\leq 8[\gamma(k)(2m+n)(1-\mu)T\sigma_\xi]^2. \qquad \text{(C.6)}\end{aligned}$$

Using the inequality $2xy \leq x^2+y^2$, and (C.2)-(C.6), we have

$$\begin{aligned}&E\left[2[\Delta_{1i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{1i}(k)}{2m+n}]^T[\Delta_{2i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{2i}(k)}{2m+n}]\right]\\ &\leq\left[\sum_{j=1}^{2m+n}4C_\mu(1-2\beta)^{\frac{k}{B_0}}\right]^2\|Z_j(s)\|^2\\ &\quad+\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2\varpi. \qquad \text{(C.7)}\end{aligned}$$

And similarly, then

$$\begin{aligned}&E\left[2[\Delta_{1i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{1i}(k)}{2m+n}]^T[\Delta_{3i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{3i}(k)}{2m+n}]\right]\\ &\leq\left[\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k}{B_0}}\right]^2\|Z_j(s)\|^2+2\varpi,\\ &E\left[2[\Delta_{2i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{2i}(k)}{2m+n}]^T[\Delta_{3i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{3i}(k)}{2m+n}]\right]\\ &\leq\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2\varpi+2\varpi, \qquad \text{(C.8)}\\ &E\left[2[\Delta_{4i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{4i}(k)}{2m+n}]^T[\Delta_{5i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{5i}(k)}{2m+n}]\right]\\ &\leq\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2\\ &\quad\times[(2m+n)(1-\mu)T\sigma_\xi]^2\\ &\quad+8[\gamma(k)(2m+n)(1-\mu)T\sigma_\xi]^2, \qquad \text{(C.9)}\\ &E\left[\left[\Delta_{1i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{1i}(k)}{2m+n}\right]^T\left[\Delta_{4i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{4i}(k)}{2m+n}\right]\right]\\ &=0,\\ &E\left[\left[\Delta_{1i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{1i}(k)}{2m+n}\right]^T\left[\Delta_{5i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{5i}(k)}{2m+n}\right]\right]\\ &=0,\end{aligned}$$

$$
\begin{aligned}
&E\left[\left[\Delta_{2i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{2i}(k)}{2m+n}\right]^T\left[\Delta_{4i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{4i}(k)}{2m+n}\right]\right]\\
&=0,\\
&E\left[\left[\Delta_{2i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{2i}(k)}{2m+n}\right]^T\left[\Delta_{5i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{5i}(k)}{2m+n}\right]\right]\\
&=0,\\
&E\left[\left[\Delta_{3i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{3i}(k)}{2m+n}\right]^T\left[\Delta_{4i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{4i}(k)}{2m+n}\right]\right]\\
&=0,\\
&E\left[\left[\Delta_{3i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{3i}(k)}{2m+n}\right]^T\left[\Delta_{5i}(k)-\frac{\sum_{i=1}^{2m+n}\Delta_{5i}(k)}{2m+n}\right]\right]\\
&=0.
\end{aligned}
$$

Therefore, taking expectation on the both side of (C.1), by (C.2)-(C.9), we can get

$$
\begin{aligned}
&E\left[\|Z_i(k+1)-Z_j(k+1)\|^2\right]\\
&\leq 12\Big[\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k}{B_0}}\Big]^2\|Z_j(s)\|^2\\
&\quad+12\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2\varpi+24\varpi\\
&\quad+8\left[\sum_{r=s+1}^{k}\sum_{j=1}^{2m+n}4C_\mu(1-2\beta_1)^{\frac{k-r}{B_0}}\right]^2\\
&\quad\times[(2m+n)(1-\mu)T\sigma_\xi]^2\\
&\quad+64[\gamma(k)(2m+n)(1-\mu)T\sigma_\xi]^2.
\end{aligned}
$$

It is easy to obtain that for any $\varpi_1>0$, we can take small enough ϖ and large enough N such that $E[\|Z_i(k)-Z_j(k)\|^2]<\varpi_1$, for any $k>\hat{k}_t>N$, $i\in\mathcal{I}_{m+n}$. Let $\bar{X}_i=\{x\in X_i|(M+\epsilon)-\frac{7\epsilon}{\bar{\mu}}<E[\|x-P_X(x)\|^2]<M+\epsilon]\}$. Since $(M+\epsilon)-\frac{7\epsilon}{\bar{\mu}}>0$, $\bar{X}_i\bigcap X=\varnothing$. Let $\kappa=\min_i\max_j\{E[\|x-P_{X_j}(x)\|^2|x\in\bar{X}_i]\}$. If we choose $\varpi_1<\kappa$, then

$$
\begin{aligned}
&E[\|x_i(k+1)-x_j(k+1)\|^2]\\
&<\varpi_1<\kappa\leq\max_{i,j}\{E[\|x_i(k+1)-P_{X_j}[x_i(k+1)]\|^2\}\\
&\leq\max_{i,j}\{E[\|x_i(k+1)-x_j(k+1)\|^2\},
\end{aligned}
$$

for $i,j\in\mathcal{I}_{m+n}$, which is a contradiction. Therefore, $M=0$, i.e., $\lim_{k\to\infty}E[\|Z_i(k)-P_X[Z_i(k)]\|^2]=0$. □

CONFLICT OF INTEREST

All authors declare that no conflict of interest exists.

REFERENCES

[1] R. Rajasekar and P. Prakasam, "Proposed coordinating multiple sampling tasks in sensor field using geometric progression algorithm for efficient data collection in wireless sensor networks," *Wireless Personal Communications*, vol. 82, no. 3, pp. 1809-1824, 2015.

[2] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations,"*IEEE Transactions on Automatic Control*, vol 49, no. 9, pp. 1465-1476, 2004.

[3] M. Campbel, "Collision monitoring within satellite clusters," *IEEE Transactions on Control Systems Technology*, vol. 13, no. 1, pp. 42-55, 2005.

[4] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520-1533, 2004.

[5] A. Nedic, A. Ozdaglar, and P. A. Parrilo, "Constrained consensus and optimization in multi-agent networks," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 922-938, 2010.

[6] X. Li, Z. Sun, Y. Tang, et al., "Adaptive event-triggered consensus of multiagent systems on directed graphs," *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1670-1685, 2021.

[7] H. Wang, W. Ren, W, Yu, et al., "Fully distributed consensus control for a class of disturbed second-order multi-agent systems with directed networks," *Automatica*, 2021.

[8] P. Lin, W. Ren, and H. Gao, "Distributed velocity-constrained consensus of discrete-time multi-agent systems with nonconvex constraints, switching topologies, and delays," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5788-5794, 2017.

[9] Y. Zhang, X. Li, and L. Wang, "Distributed H_∞ consensus of heterogeneous multi-agent systems with nonconvex constraints', *ISA Transactions*, 2022.

[10] M. Lu, J. Wu, X. Zhan, et al., "Consensus of second-order heterogeneous multi-agent systems with and without input saturation," *ISA Transactions*, vol. 126, 14-20, 2022.

[11] X. Li, C. Li, Y. Yang, et al., "Consensus for heterogeneous multi-agent systems with nonconvex input constraints and nonuniform time delays," *Journal of the Franklin Institute*, vol. 357, no. 6, pp. 3622-3635, 2020.

[12] P. Liu, F. Xiao, B. Wei, et al., "Distributed constrained optimization problem of heterogeneous linear multi-agent systems with communication delays," *Systems & Control Letters*, vol. 155, 2021.

[13] X. Li, C. Li, and Y. Yang, "Heterogeneous linear multi-agent consensus with nonconvex input constraints and switching graphs," *Information Sciences*, vol. 501, pp. 397-405, 2019.

[14] L. Mo, S. Guo, and Y. Yu, "Mean-square consensus of heterogeneous multi-agent systems with nonconvex constraints, Markovian switching topologies and delays," *Neurocomputing*, vol. 291, pp. 167-174, 2018.

[15] X. Li, Z. Yu, Z. Li, et al., "Group consensus via pinning control for a class of heterogeneous multi-agent systems with input constraints," *Information Sciences*, vol. 542, pp. 247-262, 2021.

[16] L. Mo, Y. Yu, G. Ren, et al., "Constrained consensus of continuous-time heterogeneous multi-agent networks with nonconvex constraints and delays," *Journal of Systems Science and Complexity*, vol. 35, pp. 105-122, 2022.

[17] J. Huang, L. Chen, X. Xie, et al., "Distributed event-triggered consensus control for heterogeneous multi-agent systems under fixed and switching topologies," *International Journal of Control, Automation, and Systems*, vol. 17, no. 8, pp. 1946-1956, 2019.

[18] J. Zhang, D. Ding, L. Qing, et al., "Event-triggered control for heterogeneous discrete-time multi-agent systems subject to uncertainties and noises," *International Journal of Control, Automation, and Systems*, vol. 18, no. 3, pp. 661-671, 2020.

[19] J. Long, W. Wang, G. Wen, et al., "Output feedback based adaptive consensus tracking for uncertain heterogeneous multi-agent systems with event-triggered communication," *Automatica*, vol. 136, 110049, 2022.

[20] C. Sun, M. Ye, and G. Hu, "Distributed optimization for two types of heterogeneous multiagent systems," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 32, no. 3, pp. 1314-1324, 2021.

[21] Z. Gao, H. Zhang, Y. Cai, et al., "Finite-time event-triggered output consensus of heterogeneous fractional-order multiagent systems with intermittent communication," *IEEE Transactions on Cybernetics*, 2021.

[22] L. He and W. Dong, "Distributed adaptive consensus tracking control for heterogeneous nonlinear multi-agent systems," *ISA Transactions*, 2022.

[23] H. Du, G. Wen, D. Wu, et al., "Distributed fixed-time consensus for nonlinear heterogeneous multi-agent systems," *Automatica*, vol. 103, 108797, 2022.

[24] L. Mo and H. Zhang, "Constrained consensus of heterogeneous networks with nonuniform constraints and nonuniform communication delays," *Control Theory & Applications*, vol. 40, 2023.

[25] T. Li and J. Zhang, "Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions," *Automatica*, vol. 45, no. 8, pp. 1929-1936, 2009.

[26] T. Li and J. Zhang, "consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2043-2057, 2010.

[27] S. S. Ram, A. Nedić, and V. V Veeravalli, "Distributed stochastic subgradient projection algorithms for convex optimization," *Journal of Optimization Theory and Application*, vol. 147, no. 3, pp. 516-545, 2010.

[28] S. Guo, L. Mo, and Y. Yu, "Mean-square consensus of heterogeneous multi-agent systems with communication noises," *Journal of the Franklin Institute*, vol. 355, no. 8, pp. 3717-3736, 2018.

[29] F. Sun, C. Lei, and J. Kurths, "Consensus of heterogeneous discrete-time multi-agent systems with noise over Markov switching topologies," *International Journal of Robust and Nonlinear Control*, vol. 31, no. 5, pp. 1530-1541, 2021.

[30] H. Chen, *Stochastic Approximation and its Application*, Springer-Verlag New York Inc. 2002.

[31] P. Lin, W. Ren, and Y. Song, "Distributed multi-agent optimization subject to nonidentical constraints and communication delays," *Automatica*, vol. 65, pp. 120-131, 2016.

Haokun Hu received his M.S. degree from the School of Mathematics and Statistics, Beijing Technology and Business University, Beijing, China, in 2022. He is currently pursuing a Ph.D. degree from the School of Mathematics and Statistics from Central South University, Changsha, China. His research interests include multi-agent systems, stochastic system, machine learning, and distributed control.

Lipo Mo received his B.S. degree in mathematics and applied mathematics from Shihezi university, Shihezi, China, in 2003, and a Ph.D. degree in control theory from the School of Mathematics and Systems Science, Beihang University, Beijing, China, in 2010. He is currently a Professor with the School of E-Business and Logistics, Beijing Technology and Business University, and the School of Mathematics and Statistics, Beijing Technology and Business University, Beijing. His research interests include systems engineering, distributed control and optimization, big data and applied statistics, and supply chain management.

Fei Long received his M.S. degree from the Department of Mathematics, Guizhou University, China, in 2001, and a Ph.D. degree from the College of Automation from Southeast University, Nanjing, in 2006. Currently, he is a professor at Guizhou Institute of Technology. His research interests include switched systems and multi-agent systems.