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Improved Robust Passivity Criteria for Delayed Neural Networks

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Abstract: This paper investigates the robust passivity problem for neural networks with uncertain system parameters and a time-varying delay. Based on Lyapunov stability theory, ensuring the negative definiteness for the derivatives of the developed Lyapunov-Krasovskii functional (LKF) is necessary in order to derive a passivity criterion. A negative condition on the cubic polynomial over a certain interval is developed in this paper, which introduces some slack matrices to obtain an advanced negative condition. Taking advantage of this condition, an augmented LKF with more system state and delay function information, including several augmented vectors and a single-integral-based term, is constructed. Then some improved passivity criteria for delayed neural networks are derived on top of the proposed LKF and the negative condition. Finally, the effectiveness and superiority of the obtained passivity criteria are validated on two numerical examples.

Keywords: Lyapunov-Krasovskii functional, neural networks, passivity analysis, time-varying delay.

1. INTRODUCTION

Since neural networks have been proposed to describe how biological brains solve problems, extensive research has been conducted on neural networks in many areas [1]. In the implementation of neural networks, time delays are inevitable because of the finite switching speed of amplifiers or internal neuronal communication, leading to unstable neural networks [2]. Consequently, the analysis and synthesis of neural networks with time delays have received considerable attention over the past few decades [3-5].

The concept of passivity initially appeared in circuit theory, which plays a key role in systems analysis [6]. Passivity theory changes the research framework of control theory from traditional signal processing to energy transmission. It provides a powerful framework for analyzing the stability of nonlinear dynamical systems because the passivity properties can ensure that the system is internally stable. Research on the passivity analysis of delayed neural networks has made considerable efforts, and some meaningful works were reported in [7-15].

Lyapunov-Krasovskii functional (LKF) approach is a well-recognized tool for analyzing the passivity of delayed neural networks. A suitable LKF is essential to derive less conservative passivity criteria [16-18]. Various types of LKF have been introduced, such as augmented LKF, integral terms-based LKF, and delay product type (DPT) LKF. A review paper related to constructing LKF can be found in [19]. A new type of augmented single integral, including vector-dependent integral, was presented to study the stability of delayed neural networks in [20]. In [21], some improved time-delay product auxiliary polynomial functions were introduced into the LKF to study the stability analysis of delayed neural networks. A DPT LKF for stability analysis of the delayed neural network was proposed in [22], which considers the delay change rate information and achieves less conservative results. The passivity criteria on the delayed neural networks based on the coupled LKF were developed in [23], indicating that the augmented LKF with delay-dependent matrices can help to derive a good passivity criterion.

It is noted that some of the constructed LKF's derivatives are not linear with respect to the delay function, such as the LKFs in [4,12,14,23]. It is a crucial issue to find negativity conditions of the function with different degrees of the delay function to obtain tractable passivity criteria. Some results on the negative condition of a quadratic polynomial have been reported up to now. The seminal negative condition on the quadratic polynomial was deduced in [24], which has been frequently used in the previous literature. Based on [24], some advanced sufficient conditions [25-30] and sufficient and necessary conditions [4,12,14,31-34] were proposed. Furthermore, as augmented LKF is employed in recent years, the expression after the derivative of the augmented LKF may be related to the cubic polynomial with respect to the delay function. Thus, it is necessary to study the negative condi-

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tion of cubic polynomial. Very recently, a negative condition on cubic polynomial was put forward by utilizing the Taylors formula in [35]. However, how to develop an advanced negative condition on cubic polynomial and apply it to obtain less conservative passivity criteria for delayed neural networks needs further research.

In this paper, the passivity analysis of delayed neural networks is further investigated. Firstly, compared with the negative condition on cubic polynomial in [35], a free-matrix-based equality and more slack matrices are introduced, and a new negative condition of the cubic polynomial is developed. Secondly, a novel LKF with more system state and delay function information is developed, in which single-integral-based integral inequality and augmented LKF philosophy are adopted. Finally, the constraint on the delay function is weakened. The condition $\mu_1 \le \dot{h}(t) \le \mu_2$ is replaced by $\dot{h}(t) \le \mu$, where μ_1, μ_2 and μ are real constants. Several improved passivity criteria are derived using the proposed LKF and the negative condition, and the superiority of the proposed passivity criteria is validated through two numerical examples.

Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are the n-dimensional vector space and the set of all $n\times m$ real matrices, respectively. $\mathbb{N}$ is the set of non-negative integers. I and 0 refer to the identity matrix and zero matrix, respectively. $\mathfrak{L} > 0$ represents the matrix $\mathfrak{L}$ is the symmetric and positive definition. Notations $diag\{\cdots\}$, $*$, and $Sym\{\mathfrak{L}\}$ represent block-diagonal matrix, the symmetric term in symmetric block matrix, and $\mathfrak{L}^T + \mathfrak{L}$, respectively.

2. PRELIMINARIES

Consider neural networks with uncertain system parameters and a time-varying delay, as described in [10]

$$\begin{cases} \dot{x}(t) = -(\mathcal{A}+\Delta\mathcal{A}(t))x(t)+(\mathcal{W}_1+\Delta\mathcal{W}_1(t)) \\ \qquad\qquad \times g(x(t))+(\mathcal{W}_2+\Delta\mathcal{W}_2(t)) \\ \qquad\qquad \times g(x(t-h(t)))+u(t), \\ y(t) = \mathcal{C}_1 g(x(t))+\mathcal{C}_2 g(x(t-h(t))), \\ x(\delta) = \phi(\delta),\ -h \le \delta \le 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the neuron state. $y(t) \in \mathbb{R}^p$ is the output vector. $u(t) \in \mathbb{R}^n$ is the input vector. $g(\cdot) \in \mathbb{R}^q$ is the neuron activation function vector. $\phi(\delta)$ is the initial condition. $h(t)$ is the delay function. $\mathcal{A} = diag\{a_1, a_2, \ldots, a_n\} > 0$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{C}_1$, and $\mathcal{C}_2$ are known real matrices. $\Delta\mathcal{A}(t)$, $\Delta\mathcal{W}_1(t)$, and $\Delta\mathcal{W}_2(t)$ are unknown matrices, and assumed as

$$\left[\Delta\mathcal{A}(t),\ \Delta\mathcal{W}_1(t),\ \Delta\mathcal{W}_2(t)\right] = \mathcal{H}\mathcal{F}(t)\left[\mathscr{E}_1,\ \mathscr{E}_2,\ \mathscr{E}_3\right], \tag{2}$$

where $\mathcal{H}$, $\mathscr{E}_1$, $\mathscr{E}_2$, and $\mathscr{E}_3$ are known matrices. Additionally, the uncertainty parameter $\mathcal{F}(t)$ is assumed to satisfy $\mathcal{F}^T(t)\mathcal{F}(t) \le I$.

The following assumptions are made for both the delay function and the neuron activation function.

Assumption 1: The time-varying delay function $h(t)$ satisfies the following condition

$$0 \le h(t) \le h,\ \dot{h}(t) \le \mu, \tag{3}$$

where h and μ are real constants.

Assumption 2: The neuron activation function $g(\cdot)$ meets the following condition

$$\varsigma_i^- \le \frac{g_i(\tau_1)-g_i(\tau_2)}{\tau_1-\tau_2} \le \varsigma_i^+,\ \tau_1 \ne \tau_2, \tag{4}$$

and $g_i(0) = 0$, where ς_i^- and ς_i^+ are real constants.

The passivity analysis for the delayed neural networks (1) subject to (2)-(4) will be given in the next section. The following definition and some useful lemmas are necessary for deriving passivity criteria.

Definition 1 [6]**:** For all $t_d \ge 0$, the delayed neural network (1) with $\phi(0) = 0$ is called passive if there exists a scalar $\gamma \ge 0$ such that

$$-\gamma\int_0^{t_d} u^T(t)u(t)\,dt \le 2\int_0^{t_d} u^T(t)y(t)\,dt. \tag{5}$$

Lemma 1 [29]**:** For given a scalar $N \in \mathbb{N}$, any scalars $a < b$, an any vector η, a positive matrix $\mathscr{R} \in \mathbb{R}^{m\times m}$, an any matrix $\mathcal{H} \in \mathbb{R}^{(N+1)n\times k}$, and a differentiable function x in $[a,b] \to \mathbb{R}^n$, the following inequality holds

$$-\int_a^b \dot{x}^T(s)\mathscr{R}\dot{x}(s)\,ds \le 2(\widetilde{\Phi}_N\Psi_N\Omega_N)^T\mathcal{H}\eta + (b-a)\eta^T\mathcal{H}^T\bar{\mathcal{R}}_N^{-1}\mathcal{H}\eta, \tag{6}$$

where

$$\begin{aligned} &\widetilde{\Phi}_N = [\bar{\Phi}_0^T,\ \bar{\Phi}_1^T, \ldots,\ \bar{\Phi}_N^T]^T,\ N \in \mathbb{N}, \\ &\bar{\Phi}_N = [\Phi_N,\ 0,\ \ldots,\ 0],\ N \in \mathbb{N}, \\ &\Phi_N = \begin{cases} [I,\ \ I], & N = 0, \\ \left[\wp_0^N I,\ -\sum_{\iota=0}^N \wp_\iota^N I,\ \wp_\iota^1 I, \ldots,\ \wp_\iota^N I\right], & N \ge 1, \end{cases} \\ &\wp_\iota^k = (-1)^\iota\binom{k}{\iota}\binom{k+1}{\iota},\ \iota,\ k \in \mathbb{N}, \\ &\Psi_N = \begin{cases} diag\{I,\ \ I\}, & N = 0, \\ diag\{\Psi_0,\ I,\ 2I, \ldots,\ NI\}, & N \ge 1, \end{cases} \\ &\Omega_N = \begin{cases} [x^T(b),\ \ x^T(a)]^T, & N = 0, \\ [\Omega_0^T,\ \tfrac{1}{b-a}\ell_1^T,\ \ldots,\ \tfrac{1}{b-a}\ell_N^T]^T, & N \ge 1, \end{cases} \\ &\ell_N^{(a,b)} = \int_a^b \left(\frac{b-s}{b-a}\right)^{N-1} x(s)ds,\ N \in \mathbb{N}^+, \\ &\bar{\mathcal{R}}_N = diag\{\mathscr{R},\ 3\mathscr{R},\ \ldots,\ (2N+1)\mathscr{R}\},\ N \in \mathbb{N}. \end{aligned}$$

Remark 1: Compared with the Bessel-Legendre integral inequality in [16], estimating the integral inequality that appears after the derivation of the LKF does not involve an additional reciprocally convex approach. It means that Lemma 1 is more suitable for the stability analysis of a class of time-varying delay systems, and a similar integral inequality can be found in [5].

Lemma 2 [35]**:** For given a constant $h>0$ and $\Gamma_i \in \mathbb{S}^k$ $(i=0,\,1,\,2,\,3)$, the cubic polynomial function $Z(s)=\Gamma_3 s^3+\Gamma_2 s^2+\Gamma_1 s+\Gamma_0<0$ holds for $s\in[0,\,h]$ such that

$$Z(0)<0,\ Z(h)<0, \tag{7}$$

$$h^3\Gamma_3+Z(0)<0, \tag{8}$$

$$-h^2(3h\Gamma_3+\Gamma_2)+Z(0)<0, \tag{9}$$

$$h^3\Gamma_3-h^2(3h\Gamma_3+\Gamma_2)+Z(0)<0. \tag{10}$$

Lemma 3: For given a constant $h>0$, an integer $k>0$, and matrices $\Gamma_i\in\mathbb{S}^k$, the cubic polynomial $Z(s)=\Gamma_3 s^3+\Gamma_2 s^2+\Gamma_1 s+\Gamma_0<0$ holds for $s\in[0,\,h]$ if there exist constant matrices $M\in\mathbb{R}^{k\times k}$, $N\in\mathbb{R}^{k\times k}$, $H_j\in\mathbb{R}^{k\times k}$ $(j=1,\,2)$ such that

$$\begin{bmatrix}\Gamma_0 & -H_1\\ * & -H_2-H_2^T\end{bmatrix}<0, \tag{11}$$

$$\begin{bmatrix}\widetilde{\Gamma}_{11} & \frac{h}{2}\Gamma_2-H_1+h(M-hN+H_2^T)\\ * & h(\Gamma_3+N+N^T)-H_2-H_2^T\end{bmatrix}<0, \tag{12}$$

where $\widetilde{\Gamma}_{11}=h\left(\Gamma_1+H_1+H_1^T-h(M+M^T)\right)+\Gamma_0$.

Proof: According to the truth that

$$0\le s\le h\Rightarrow s(s-h)\le 0\Rightarrow s^2(s-h)\le 0,$$

and define $\varsigma(s)=[I,\,sI]^T$ for matrices $M\in\mathbb{R}^{k\times k}$, $N\in\mathbb{R}^{k\times k}$ with $M+M^T>0$ and $N+N^T>0$, one has

$$\begin{aligned}&s(s-h)\le 0\\ &\Rightarrow \mathcal{M}(s)=\varsigma^T(s)\begin{bmatrix}-sh(M+M^T) & sM\\ * & 0\end{bmatrix}\varsigma(s)\le 0,\\ &s^2(s-h)\le 0\\ &\Rightarrow \mathcal{N}(s)=\varsigma^T(s)\begin{bmatrix}0 & -shN\\ * & s(N+N^T)\end{bmatrix}\varsigma(s)\le 0.\end{aligned}$$

Then

$$-\mathcal{M}(s)-\mathcal{N}(s)+Z(s)\le 0\Rightarrow Z(s)\le 0,$$

where $Z(s)=\varsigma^T(s)\begin{bmatrix}\Gamma_0+s\Gamma_1 & \frac{s}{2}\Gamma_2\\ * & s\Gamma_3\end{bmatrix}\varsigma(s)$.

Furthermore, for given two constant matrices $H_1\in\mathbb{R}^{k\times k}$ and $H_2\in\mathbb{R}^{k\times k}$, we obtain

$$\begin{aligned}\mathcal{H}(s)&=2\begin{bmatrix}H_1\\ H_2\end{bmatrix}[sI\ \ -I]\varsigma(s)\\ &=\begin{bmatrix}s(H_1+H_1^T) & -H_1+sH_2^T\\ * & -H_2-H_2^T\end{bmatrix}\varsigma(s)=0.\end{aligned}$$

Therefore, one has

$$\begin{aligned}\mathcal{Z}(s)&=-\mathcal{M}(s)-\mathcal{N}(s)+\mathcal{H}(s)+Z(s)\le 0\\ &\Rightarrow Z(s)\le 0,\end{aligned}$$

then, $\mathcal{Z}(s)$ can be rewritten as follows:

$$\mathcal{Z}(s)=\varsigma^T(s)\bar{\Gamma}(s)\varsigma(s),$$

where

$$\begin{aligned}\bar{\Gamma}(s)&=\begin{bmatrix}\bar{\Gamma}_{11} & \frac{s}{2}\Gamma_2-H_1+s(M-hN+H_2^T)\\ * & s(\Gamma_3+N+N^T)-H_2-H_2^T\end{bmatrix},\\ \bar{\Gamma}_{11}&=s\left(\Gamma_1+H_1+H_1^T-h(M+M^T)\right)+\Gamma_0.\end{aligned}$$

If conditions (11) and (12) are satisfied. We obtain $\mathcal{Z}(s)<0$ for $\forall s\in[0,\,h]$, which completes the proof. □

Remark 2: Inspired by [4,30,33], a new negative condition for the cubic polynomial can be obtained in Lemma 3. Compared with the sufficient condition in Lemma 2, a free-matrix-based equality $\mathcal{H}(s)$ and slack matrices M and N can be introduced, which contributes to derive an advanced negative condition.

Lemma 4 [36]**:** For given appropriate dimensions matrices $\mathcal{H}$, $\mathscr{E}$ and $\mathcal{F}(t)$, and $\mathcal{F}(t)$ satisfying $\mathcal{F}^T(t)\mathcal{F}(t)\le I$. If there exists any positive scalar κ such that

$$\mathcal{H}\mathcal{F}(t)\mathscr{E}+(\mathcal{H}\mathcal{F}(t)\mathscr{E})^T\le\kappa^{-1}\mathcal{H}\mathcal{H}^T+\kappa\mathscr{E}^T\mathscr{E}. \tag{13}$$

3. MAIN RESULTS

For simplicity, the notations are defined following:

$$\begin{aligned}
&\ell_N^{(a,b)}=\int_a^b\left(\frac{b-s}{b-a}\right)^N x(s)ds,\ \ N\in\mathbb{N},\\
&\kappa_N(t)=\ell_N^{(t-h,t)}(t),\ \upsilon_N(t)=\ell_N^{(t-h(t),t)}(t),\\
&\vartheta_N(t)=\ell_N^{(t-h,t-h(t))}(t),\\
&\eta_1(t)=\left[x^T(t),x^T(t-h),\kappa_0^T(t),\kappa_1^T(t)\right]^T,\\
&\eta_2(t)=\left[\eta_1^T(t),\kappa_1^T(t)\right]^T,\\
&\eta_3(s)=\left[x^T(s),\dot{x}^T(s)\right]^T,\\
&\eta_4(s)=\left[x^T(s),g^T(x(s))\right]^T,\\
&\eta_5(s)=\left[\dot{x}^T(s),\eta_4^T(s)\right]^T,\\
&\eta_6(t,s)=\left[x^T(t),\ x^T(s),\ \textstyle\int_s^t x^T(\theta)d\theta\right]^T,\\
&\xi(t)=[x^T(t),g^T(x(t)),x^T(t-h(t)),g^T(x(t-h(t))),\\
&\qquad x^T(t-h),\ g^T(x(t-h)),\frac{\upsilon_0^T(t)}{h(t)},\frac{\upsilon_1^T(t)}{h(t)},\\
&\qquad \frac{\vartheta_0^T(t)}{h-h(t)},\frac{\vartheta_1^T(t)}{h-h(t)},\dot{x}^T(t),\dot{x}^T(t-h),u^T(t)]^T,\\
&\mathcal{L}_1=diag\left\{\varsigma_1^+,\varsigma_2^+,\ldots,\varsigma_n^+\right\},\\
&\mathcal{L}_2=diag\left\{\varsigma_1^-,\varsigma_2^-,\ldots,\varsigma_n^-\right\},\\
&e_i=\left[0_{n\times(i-1)n},\ I_n,\ 0_{n\times(13-i)n}\right],\ i=1,2,\ldots,13.
\end{aligned}$$

In the following, we establish a robust passivity criterion for the delayed neural networks (1).

Theorem 1: For given positive scalars h, μ, κ, and γ, the delayed neural networks (1) with (2)-(4) is robust passive if there exist symmetric matrices $P_k>0$, $Q_i>0$, $R>0$, diagonal matrices $\Omega_j>0$, $\Gamma_i>0$, $\Gamma_{kl}>0$, and any matrices M_k, H_k, G_i, $D+D^T>0$, $F+F^T>0$ ($k=1,2$; $l=2,3$ ($l>k$); $i=k,3$; $j=i,4$), such that the following linear matrix inequalities (LMIs) hold:

$$\begin{bmatrix} \Upsilon_0+\kappa\Theta_2^T\Theta_2 & -H_1 & \sqrt{h}M_2 & \Theta_1 \\ * & -H_2-H_2^T & 0 & 0 \\ * & * & -\widetilde{R} & 0 \\ * & * & * & -\kappa I \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} \Im_{11}+\kappa\Theta_2^T\Theta_2 & \Im_{12} & \sqrt{h}M_1 & \Theta_1 \\ * & \Im_{22} & 0 & 0 \\ * & * & -\widetilde{R} & 0 \\ * & * & * & -\kappa I \end{bmatrix} < 0, \quad (15)$$

where

$$\Im_{11}=h(\Upsilon_1+H_1+H_1^T-h(D+D^T))+\Upsilon_0,$$

$$\Im_{12}=\frac{h}{2}\Upsilon_2-H_1+h(D-hF+H_2^T),$$

$$\Im_{22}=h(\Upsilon_3+F+F^T)-H_2-H_2^T,\ \Upsilon_0=\Upsilon(0),$$

$$\Upsilon_1=\frac{1}{6}[-2\Upsilon(-1)-3\Upsilon(0)+6\Upsilon(1)-\Upsilon(2)],$$

$$\Upsilon_2=\frac{1}{2}[\Upsilon(-1)-2\Upsilon(0)+\Upsilon(1)],$$

$$\Upsilon_3=\frac{1}{6}[-\Upsilon(-1)+3\Upsilon(0)-3\Upsilon(1)+\Upsilon(2)],$$

$$\Upsilon(h(t))=\Xi_1(h(t))+\Xi_2,$$

$$\begin{aligned}\Xi_1(h(t))=&\,Sym\{\aleph_1^Th(t)P_1\aleph_2+\aleph_3^TP_2\aleph_4+\aleph_7^TS\aleph_8\\ &+\aleph_{15}^TQ_3\aleph_{16}+((e_2^T-e_1^T\mathcal{L}_2)\Omega_1\\ &+(e_1^T\mathcal{L}_1-e_2^T)\Omega_2)e_{11}+((e_6^T-e_5^T\mathcal{L}_2)\Omega_3\\ &+(e_5^T\mathcal{L}_1-e_6^T)\Omega_4)e_{12}+\mathcal{G}_1^TM_1+\mathcal{G}_2^TM_2\\ &+\aleph_{17}^T\aleph_{18}-(\mathcal{C}_1e_2+\mathcal{C}_2e_4)^Te_{13}\}\\ &+\mu\aleph_1^TP_1\aleph_1+h\aleph_5^TS\aleph_5-h\aleph_6^TS\aleph_6\\ &+\aleph_9^TQ_1\aleph_9-(1-\mu)\aleph_{10}^TQ_1\aleph_{10}\\ &+\aleph_{11}^TQ_2\aleph_{11}-\aleph_{12}^TQ_2\aleph_{12}+\aleph_{13}^TQ_3\aleph_{13}\\ &-\aleph_{14}^TQ_3\aleph_{14}+he_{11}^TRe_{11}-e_{13}^T\gamma Ie_{13},\end{aligned}$$

$$\begin{aligned}\Xi_2=&\,Sym\{(e_2-\mathcal{L}_2e_1)^T\Gamma_1(\mathcal{L}_1e_1-e_2)\\ &+(e_4-\mathcal{L}_2e_3)^T\Gamma_2(\mathcal{L}_1e_3-e_4)\\ &+(e_6-\mathcal{L}_2e_5)^T\Gamma_3(\mathcal{L}_1e_5-e_6)\\ &+((e_2-e_4)-\mathcal{L}_2(e_1-e_3))^T\Gamma_{12}(\mathcal{L}_1(e_1-e_3)\\ &-(e_2-e_4)+((e_2-e_6)\\ &-\mathcal{L}_2(e_1-e_5))^T\Gamma_{13}(\mathcal{L}_1(e_1-e_5)\\ &-(e_2-e_6)+((e_4-e_6)-\mathcal{L}_2(e_3-e_5))^T\\ &\times\Gamma_{23}(\mathcal{L}_1(e_3-e_5)-(e_4-e_6)\},\end{aligned}$$

$$\Theta_1=\left[e_1^TG_1\mathcal{H}+e_{11}^TG_2\mathcal{H}+e_{12}^TG_3\mathcal{H}\right]^T,$$

$$\Theta_2=\left[e_1^T\mathscr{E}_1^T+e_2^T\mathscr{E}_2^T+e_4^T\mathscr{E}_3^T\right]^T,$$

$$\aleph_1=\left[e_1^T,\ e_5^T,\ h(t)e_7^T+(h-h(t))e_9^T\right]^T,$$

$$\aleph_2=\left[e_{11}^T,\ e_{12}^T,\ e_1^T-e_5^T\right]^T,$$

$$\aleph_3=\left[\aleph_1^T,\ -e_5^T+h(t)e_7^T+(h-h(t))e_9^T\right]^T,$$

$$\begin{aligned}\aleph_4=&\left[\aleph_2^T,\ h^2(t)e_8^T+h(t)(h-h(t))e_9^T\right.\\ &\left.+(h-h(t))^2e_{10}^T\right]^T,\end{aligned}$$

$$\aleph_5=\left[e_1^T,\ e_{11}^T\right]^T,\quad \aleph_6=\left[e_5^T,\ e_{12}^T\right]^T,$$

$$\aleph_7=\left[e_8^T+e_{10}^T,\ e_1^T-e_5^T\right]^T,$$

$$\aleph_8=\left[e_1^T-e_5^T,\ e_{11}^T-e_{12}^T\right]^T,\ \aleph_9=\left[e_1^T,\ e_2^T\right]^T,$$

$$\aleph_{10}=\left[e_3^T,\ e_4^T\right]^T,\ \aleph_{11}=\left[e_{11}^T,\ e_1^T,\ e_2^T\right]^T,$$

$$\aleph_{12}=\left[e_{12}^T,\ e_5^T,\ e_6^T\right]^T,\ \aleph_{13}=\left[e_1^T,\ e_1^T,\ 0\right]^T,$$

$$\aleph_{14}=\left[e_1^T,\ e_5^T,\ h(t)e_7^T+(h-h(t))e_9^T\right]^T,$$

$$\begin{aligned}\aleph_{15}=&\left[he_1^T,\ h(t)e_7^T+(h-h(t))e_9^T,\right.\\ &h(h(t)e_7^T+(h-h(t))e_9^T)-h^2(t)e_8,\\ &\left.h(t)(h-h(t))e_9^T-(h-h(t))^2e_{10}^T\right]^T,\end{aligned}$$

$$\aleph_{16}=\left[e_{11}^T,\ 0,\ e_1^T\right]^T,\ \aleph_{17}=G_1e_1+G_2e_{11}+G_3e_{12},$$

$$\aleph_{18}=-e_{11}-\mathcal{A}e_1+\mathcal{W}_1e_2+\mathcal{W}_2e_4+e_{13},$$

$$\widetilde{R}=diag\{R,\ 3R,\ 5R\},\ \mathcal{G}_1=\bar{\Phi}_2\Psi_2\Omega_2^a,$$

$$\mathcal{G}_2=\bar{\Phi}_2\Psi_2\Omega_2^b,$$

$$\bar{\Phi}_2=\begin{bmatrix} I & -I & 0 & 0 \\ I & I & -2I & 0 \\ I & -I & -6I & 6I \end{bmatrix},\ \Psi_2=diag\{I,\ I,\ I,\ 2I\},$$

$$\Omega_2^a=\left[e_1^T,\ e_3^T,\ e_7^T,\ e_8^T\right]^T,\ \Omega_2^b=\left[e_3^T,\ e_5^T,\ e_9^T,\ e_{10}^T\right]^T.$$

Proof: The LKF constructed as follows:

$$V(t)=\sum_{i=1}^{5}V_i(t), \quad (16)$$

where

$$V_1(t)=\eta_1^T(t)h(t)P_1\eta_1(t)+\eta_2^T(t)P_2\eta_2(t),$$

$$\begin{aligned}V_2(t)=&\,h\int_{t-h}^{t}\eta_3^T(s)S\eta_3(s)ds\\ &-(\int_{t-h}^{t}\eta_3(s)ds)^TS(\int_{t-h}^{t}\eta_3(s)ds),\end{aligned}$$

$$\begin{aligned}V_3(t)=&\int_{t-h(t)}^{t}\eta_4^T(s)Q_1\eta_4(s)ds\\ &+\int_{t-h}^{t}\eta_5^T(s)Q_2\eta_5(s)ds\\ &+\int_{t-h}^{t}\eta_6^T(s)Q_3\eta_6(s)ds,\end{aligned}$$

$$\begin{aligned}V_4(t)=&\,2\sum_{i=1}^{n}\int_0^{x(t)}[\sigma_{1i}f_{i1}(s)+\sigma_{2i}f_{i2}(s)]ds\\ &+2\sum_{i=1}^{n}\int_0^{x(t-h)}[\sigma_{3i}f_{i1}(s)+\sigma_{4i}f_{i2}(s)]ds,\end{aligned}$$

$$V_5(t)=\int_{t-h}^{t}\int_{\theta}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta,$$

with

$$f_{i1}(s) = \varsigma_i^+ s - g_i(s),\ f_{i2}(s) = g_i(s) - \varsigma_i^- s.$$

Taking the time-derivative of the LKF $V(t)$ yields

$$\dot{V}_1(t) \le \xi^T(t)[Sym\{\aleph_1^T h(t) P_1 \aleph_2 + \aleph_3^T P_2 \aleph_4\} + \mu \aleph_1^T P_1 \aleph_1]\xi(t), \quad (17)$$

$$\dot{V}_2(t) = \xi^T(t)[h\aleph_5^T S \aleph_5 - h\aleph_6^T S \aleph_6 - Sym\{\aleph_7^T S \aleph_8\}]\xi(t), \quad (18)$$

$$\dot{V}_3(t) \le \xi^T(t)[\aleph_9^T Q_1 \aleph_9 - (1-\mu)\aleph_{10}^T Q_1 \aleph_{10} + \aleph_{11}^T Q_2 \aleph_{11} - \aleph_{12}^T Q_2 \aleph_{12} + \aleph_{13}^T Q_3 \aleph_{13} - \aleph_{14}^T Q_3 \aleph_{14} + Sym\{\aleph_{15}^T Q_3 \aleph_{16}\}]\xi(t), \quad (19)$$

$$\dot{V}_4(t) = 2\xi^T(t)[((e_2^T - e_1^T\mathcal{L}_2)\Omega_1 + (e_1^T\mathcal{L}_1 - e_2^T)\Omega_2)e_{11} + ((e_6^T - e_5^T \times \mathcal{L}_2)\Omega_3 + (e_5^T\mathcal{L}_1 - e_6^T)\Omega_4)e_{12}]\xi(t), \quad (20)$$

$$\dot{V}_5(t) = h\xi^T(t)e_{11}^T R e_{11}\xi(t) - \int_{t-h(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s)R\dot{x}(s)ds, \quad (21)$$

where $\Omega_1 = diag\{\sigma_{11}, \dots, \sigma_{1n}\} > 0$, $\Omega_2 = diag\{\sigma_{21}, \dots, \sigma_{2n}\} > 0$, $\Omega_3 = diag\{\sigma_{31}, \dots, \sigma_{3n}\} > 0$, and $\Omega_4 = diag\{\sigma_{41}, \dots, \sigma_{4n}\} > 0$.

Using Lemma 1 with $N = 2$ to estimate the above integral terms, we obtain

$$-\int_{t-h(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds \le \xi^T(t)[2\mathcal{G}_1^T M_1 + h(t)M_1^T\widetilde{R}^{-1}M_1]\xi(t), \quad (22)$$

$$-\int_{t-h}^{t-h(t)} \dot{x}^T(s)R\dot{x}(s)ds \le \xi^T(t)[2\mathcal{G}_2^T M_2 + (h-h(t))M_2^T\widetilde{R}^{-1}M_2]\xi(t). \quad (23)$$

Considering inequality (4), there exist positive diagonal matrices $\Gamma_i > 0$ and $\Gamma_{kl} > 0$ ($i = 1, 2, 3$; $k = 1, 2$; $l = 2, 3$ $(l > k)$), we have

$$2\left[\mathcal{L}_1\rho_i - g(\rho_i)\right]^T \Gamma_i \left[g(\rho_i) - \mathcal{L}_2\rho_i\right] \ge 0, \quad (24)$$

$$2\left[\mathcal{L}_1(\rho_k - \rho_l) - (g(\rho_k) - g(\rho_l))\right]^T \Gamma_{kl} \times \left[(g(\rho_k) - g(\rho_l)) - \mathcal{L}_2(\rho_k - \rho_l)\right] \ge 0, \quad (25)$$

where $\rho_1 = x(t)$, $\rho_2 = x(t-h(t))$, and $\rho_3 = x(t-h)$. Then, we obtain

$$2\xi^T(t)\Xi_2\xi(t) \ge 0. \quad (26)$$

Furthermore, for given any matrices G_j ($j = 1, 2, 3$), we can obtain the following zero equation

$$0 = 2[x^T(t)G_1 + \dot{x}^T(t)G_2 + \dot{x}^T(t-h)G_3][-\dot{x}(t) - (\mathcal{A} + \Delta\mathcal{A}(t))x(t) + (\mathcal{W}_1 + \Delta\mathcal{W}_1(t))g(x(t)) + (\mathcal{W}_2 + \Delta\mathcal{W}_2(t))g(x(t-h(t))) + u(t)] = 2\xi^T(t)(\aleph_{17}^T(\aleph_{18} + \mathcal{D}))\xi(t), \quad (27)$$

where $\mathcal{D} = \mathcal{H}\mathcal{F}(t)\left[\mathscr{E}_1 e_1,\ \mathscr{E}_2 e_2,\ \mathscr{E}_3 e_4\right]$.

By Lemma 4, there exists a scalar $\kappa > 0$, we obtain

$$2\xi^T(t)\aleph_{17}^T\mathcal{D}\xi(t) \le \xi^T(t)(\kappa^{-1}\Theta_1\Theta_1^T + \kappa\Theta_2^T\Theta_2)\xi(t). \quad (28)$$

To sum up, one has

$$\dot{V}(t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \le \xi^T(t)\Omega\xi(t), \quad (29)$$

where

$$\Omega = \Upsilon(h(t)) + \kappa^{-1}\Theta_1\Theta_1^T + h(t)M_1^T\widetilde{R}^{-1}M_1 + (h-h(t))M_2^T\widetilde{R}^{-1}M_2.$$

Finally, it can be seen that $\Upsilon(h(t))$ is a cubic polynomial concerning the delay function $h(t)$, Using Lemma 3 and Schur complement, $\xi^T(t)\Omega\xi(t) < 0$ holds if LMIs (14)-(15) are satisfied, which means delayed neural networks (1) is robust passive. This completes the proof. □

Remark 3: The Wirtinger-based inequality and the reciprocally convex approach in [7,8], the free-matrix-based inequality in [10], the extended free-weighting matrices inequality in [11], and the DPT LKF in [13] were used to investigate the robust passivity problem for delayed neural networks (1). However, these methods remain conservative. To obtain a less conservative passivity criterion, the augmented LKF (16) and the generalized inequality in Lemma 1 are used to derive the passivity condition. Then, the cubic polynomial inequality of the criterion is addressed by Lemma 3. Finally, an improved robust passivity condition for delayed neural networks (1) is presented in Theorem 1.

Remark 4: If applying the high polynomial inequalities in [4,33] to handle the cubic polynomial, it will place the nonlinear terms $h(t)M_1^T\widetilde{R}^{-1}M_1$ and $(h-h(t))M_2^T\widetilde{R}^{-1}M_2$ on the non-diagonal of the block matrix. It is not easy to address the two nonlinear terms. However, Lemma 3 shows the ability to subtly transfer two nonlinear terms to the diagonal of the matrix.

Remark 5: Motivated by the first-order Bessel-Legendre inequality in [16], the LKF $V_2(t)$ is proposed. The LKF $V_2(t)$ as one of LKF to analyze passivity for delayed neural networks is the first attempt work, which is actually an extension of single-integral-based LKF.

We apply Lemma 2 to address the cubic polynomial regarding the delay function in the proof of Theorem 1, and the following corollary can be obtained.

Corollary 1: For given positive scalars h, μ, κ, and γ, the delayed neural networks (1) with (2)-(4) is robust passive if there exist symmetric matrices $P_k > 0$, $Q_i > 0$,

$R > 0$, diagonal matrices $\Omega_j > 0$, $\Gamma_i > 0$, $\Gamma_{kl} > 0$, and any matrices M_k, G_i, $(k = 1, 2; l = 2, 3\ (l > k); i = k, 3; j = i, 4)$, such that the following LMIs hold

$$\begin{bmatrix} \Upsilon(0)+\kappa\Theta_2^T\Theta_2 & \sqrt{h}M_2 & \Theta_1 \\ * & -\widetilde{R} & 0 \\ * & * & -\kappa I \end{bmatrix} < 0, \tag{30}$$

$$\begin{bmatrix} \Upsilon(h)+\kappa\Theta_2^T\Theta_2 & \sqrt{h}M_1 & \Theta_1 \\ * & -\widetilde{R} & 0 \\ * & * & -\kappa I \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} h^3\Upsilon_3+\Upsilon(0)+\kappa\Theta_2^T\Theta_2 & \sqrt{h}M_2 & \Theta_1 \\ * & -\widetilde{R} & 0 \\ * & * & -\kappa I \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} \Pi_1+\kappa\Theta_2^T\Theta_2 & \sqrt{h}M_2 & \Theta_1 \\ * & -\widetilde{R} & 0 \\ * & * & -\kappa I \end{bmatrix} < 0, \tag{33}$$

$$\begin{bmatrix} \Pi_2+\kappa\Theta_2^T\Theta_2 & \sqrt{h}M_2 & \Theta_1 \\ * & -\widetilde{R} & 0 \\ * & * & -\kappa I \end{bmatrix} < 0, \tag{34}$$

where

$$\Pi_1 = -h^2(3h\Upsilon_3+\Upsilon_2)+\Upsilon(0),$$
$$\Pi_2 = h^3\Upsilon_3-h^2(3h\Upsilon_3+\Upsilon_2)+\Upsilon(0),$$

and other notations are same as Theorem 1.

When the uncertainties are disappeared, i.e., $\Delta\mathcal{A}(t) = 0$, $\Delta\mathcal{W}_1(t) = 0$, and $\Delta\mathcal{W}_2(t) = 0$ in the system (1). The system model (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = -\mathcal{A}x(t)+\mathcal{W}_1 g(x(t)) \\ \qquad\quad +\mathcal{W}_2 g(x(t-h(t)))+u(t), \\ y(t) = \mathcal{C}_1 g(x(t))+\mathcal{C}_2 g(x(t-h(t))), \\ x(\delta) = \phi(\delta), \ -h \le \delta \le 0. \end{cases} \tag{35}$$

Next, the following passivity criterion for delayed neural networks (35) based on Theorem 1 can be presented.

Theorem 2: For given positive scalars h, μ and γ, the delayed neural networks (35) satisfies conditions (2)-(4) is passive if there exist symmetric matrices $P_k > 0$, $Q_i > 0$, $R_j > 0$, diagonal matrices $\Omega_j > 0$, $\Gamma_i > 0$, $\Gamma_{kl} > 0$, and any matrices M_k, H_k, G_i, $D+D^T > 0$, $F+F^T > 0$ $(k = 1, 2; l = 2, 3\ (l > k); i = k, 3; j = i, 4)$ such that

$$\begin{bmatrix} \Upsilon_0 & -H_1 & \sqrt{h}M_2 \\ * & -H_2-H_2^T & 0 \\ * & * & -\widetilde{R} \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} \Im_{11} & \Im_{12} & \sqrt{h}M_1 \\ * & \Im_{22} & 0 \\ * & * & -\widetilde{R} \end{bmatrix} < 0, \tag{37}$$

where all notations are same as Theorem 1.

Proof: The proof is omitted since this proof is same as Theorem 1. □

Remark 6: Theorem 2 offers a criterion to check the passivity of delayed neural networks (35) with (2)-(4). Compared with the work in [7,9,10,13,15,23], an augmented LKF, including the delay-product term and the single-integral-based term, and a new negative condition of the cubic polynomial with slack matrices are adopted to reduce the conservatism of the resulting criterion.

Applying Lemma 2 instead of Lemma 3 to deal with the cubic polynomial inequality in Theorem 2, the following corollary can be established.

Corollary 2: For given positive scalars h, μ, and γ, the delayed neural networks (35) satisfies conditions (2)-(4) is passive if there exist symmetric matrices $P_k > 0$, $Q_i > 0$, $R_j > 0$, diagonal matrices $\Omega_j > 0$, $\Gamma_i > 0$, $\Gamma_{kl} > 0$, and any matrices M_k, G_i $(k = 1, 2; l = 2, 3\ (l > k); i = k, 3; j = i, 4)$ such that

$$\begin{bmatrix} \Upsilon(0) & \sqrt{h}M_2 \\ * & -\widetilde{R} \end{bmatrix} < 0, \tag{38}$$

$$\begin{bmatrix} \Upsilon(h) & \sqrt{h}M_1 \\ * & -\widetilde{R} \end{bmatrix} < 0, \tag{39}$$

$$\begin{bmatrix} h^3\Upsilon_3+\Upsilon(0) & \sqrt{h}M_2 \\ * & -\widetilde{R} \end{bmatrix} < 0, \tag{40}$$

$$\begin{bmatrix} -h^2(3h\Upsilon_3+\Upsilon_2)+\Upsilon(0) & \sqrt{h}M_2 \\ * & -\widetilde{R} \end{bmatrix} < 0, \tag{41}$$

$$\begin{bmatrix} h^3\Upsilon_3-h^2(3h\Upsilon_3+\Upsilon_2)+\Upsilon(0) & \sqrt{h}M_2 \\ * & -\widetilde{R} \end{bmatrix} < 0, \tag{42}$$

where notations are same as Theorem 1 and Corollary 1.

Remark 7: Since some slack matrices H_k, D and F are added, it is clear to observe that the presented passivity criteria in Theorems 1 and 2 require more decision variables. However, with the rapid development of computer technology, solving the matrix conditions for such levels of decision variables in Theorems 1 and 2 is no longer a problem for high-performance computers.

Remark 8: If we take the input vector $u(t) = 0$ and remove $u(t)$ from $\xi(t)$, the passivity problem of delayed neural networks is easily transformed into a stability problem. In contrast to the existing stability results in [4,5,14,18,20,21], this paper's main differences and advantages are that the condition $\dot{h}(t) \le \mu$ is weaker than $\mu_1 \le \dot{h}(t) \le \mu_2$ in [4,5,14,18,20,21], where μ_1 and μ_2 are real constants. Moreover, this paper incorporates a class of uncertain delayed neural networks. Then the considered delayed neural networks in this paper is more general. Furthermore, in the case of $\mu_1 \le \dot{h}(t) \le \mu_2$, the obtained stability criterion will generally be less conservative by considering augmented LKF with more delay information in combination with the proposed negative condition of the cubic polynomial.

Remark 9: It is worth pointing out that the proposed method is generalized. The authors focus on extending the proposed idea to other control topics in the future, such as H_∞ control [2] and state estimation [3,19].

4. NUMERICAL EXAMPLES

Two numerical examples are given to illustrate the improvement and the effectiveness of the proposed criteria in this section.

Example 1: Consider the following uncertain delayed neural networks (1) with parameters

$$\mathcal{A} = \begin{bmatrix} 2.2 & 0.0 \\ 0.0 & 1.5 \end{bmatrix}, \ \mathcal{W}_1 = \begin{bmatrix} 1.0 & 0.6 \\ 0.1 & 1.5 \end{bmatrix},$$
$$\mathcal{W}_2 = \begin{bmatrix} 1.0 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \ \mathcal{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathcal{C}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\mathcal{H} = 0.1I, \ \mathscr{E}_1 = 0.1I, \ \mathscr{E}_2 = 0.2I, \ \mathscr{E}_3 = 0.3I,$$

and the nonlinear function $g_i(x_i) = 0.5(|x_i+1| - |x_i-1|)$ $(i = 1, 2)$, one has

$$\mathcal{L}_1 = diag\{1, \ 1\}, \ \mathcal{L}_2 = 0.$$

For various μ, the obtained maximum upper bounds (MUBs) by Corollary 1, Theorem 1 and some similar methods are listed in Table 1. The results show that the MUBs by Theorem 1 outperform others, illustrating that the proposed robust passivity criterion is less conservatism than existing robust passivity criteria. Moreover, the obtained MUBs by Theorem 1 are larger than those by Corollary 1, which means that the proposed negative condition of the cubic polynomial is more advanced than that in [35].

Example 2: Consider the following delayed neural networks (35) with parameters:

$$\mathcal{A} = \begin{bmatrix} 2.2 & 0.0 \\ 0.0 & 1.8 \end{bmatrix}, \ \mathcal{W}_1 = \begin{bmatrix} 1.2 & 1.0 \\ -0.2 & 0.3 \end{bmatrix},$$
$$\mathcal{W}_2 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix}, \ \mathcal{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathcal{C}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and the nonlinear function $g_i(x_i) = 0.5(|x_i+1| - |x_i-1|)$ $(i = 1, 2)$, one has

$$\mathcal{L}_1 = diag\{1, \ 1\}, \ \mathcal{L}_2 = 0.$$

Table 1. The MUBs h for various μ (Example 1).

μ	0.3	0.5	0.7	1.0
[7]	1.909	1.900	1.895	1.887
[8]	2.135	2.054	1.915	1.907
[10]	2.411	2.311	2.233	2.186
[11]	2.930	2.665	2.499	-
[13]	3.169	2.910	2.767	2.562
Corollary 1	3.541	2.947	2.781	2.697
Theorem 1	4.163	3.330	3.053	2.953

Table 2. The MUBs h for various μ (Example 2).

μ	0.5	0.9	1.0
[7]	3.043	2.842	2.803
[15]	3.104	2.906	2.881
[9]	3.112	2.941	2.905
[10]	3.587	3.220	3.184
[13]	3.656	3.322	3.180
[23]	5.344	3.684	-
Corollary 2	4.863	3.578	3.542
Theorem 2	5.507	3.926	3.898

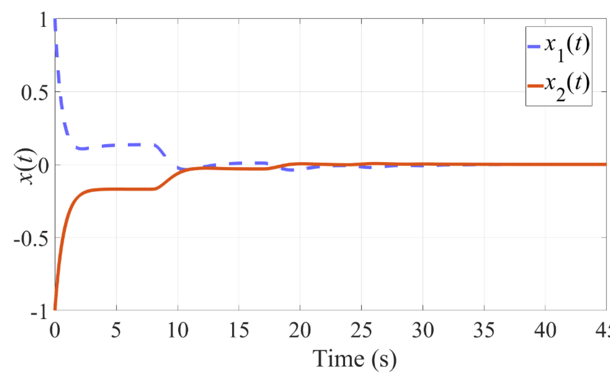

Fig. 1. State response of delayed neural networks (35).

For Example 2, by [7,9,10,13,15,23], Corollary 2, and Theorem 2, the MUBs h for various μ are listed in Table 2. From Table 2, the obtained results by Theorem 2 are extremely superior to other methods in [7,9,10,13,15,23], and Corollary 2. Furthermore, given the initial state $x(0) = [1, -1]^T$ and a time-varying delay function $h(t) = \frac{5.507}{2} + \frac{5.507}{2}\sin\left(\frac{t}{5.507}\right)$. The state responses of delayed neural networks (35) are shown in Fig. 1, and the effectiveness of the proposed method can be checked.

5. CONCLUSION

In this paper, the passivity analysis for delayed neural networks has been studied. A sufficient condition on the cubic polynomial inequality and a new augmented LKF have been developed. Improved passivity criteria for delayed neural networks have been derived. Finally, the superiority of the improved passivity criteria has been verified through numerical examples.

CONFLICT OF INTEREST

The authors declare that there is no competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

REFERENCES

[1] G. P. Liu, *Nonlinear Identification and Control: A Neural Network Approach*, Springer Science & Business Media, 2001.

[2] Y. Lin, G. Zhuang, J. Xia, W. Sun, and J. Zhao, "Asynchronous H_∞ dynamic output feedback control for Markovian jump neural networks with time-varying delays," *International Journal of Control, Automation and Systems*, vol. 20, no. 3, pp. 909-923, March 2022.

[3] H. Shen, Z. Huang, J. Cao, and J. H. Park, "Exponential H_∞ filtering for continuous-time switched neural networks under persistent dwell-time switching regularity," *IEEE Transactions on Cybernetics*, vol. 50, no. 6, pp. 2440-2449, June 2020.

[4] Y. Chen and G. Chen, "Stability analysis of delayed neural networks based on a relaxed delay-product-type Lyapunov functional," *Neurocomputing*, vol. 439, pp. 340-347, June 2021.

[5] X. M. Zhang, Q. L. Han, and Z. Zeng, "Hierarchical type stability criteria for delayed neural networks via canonical Bessel-Legendre inequalities," *IEEE Transactions on Cybernetics*, vol. 48, no. 5, pp. 1660-1671, May 2018.

[6] R. Lozano, B. Brogliato, O. Egeland, and B. Maschke, *Dissipative Systems Analysis and Control: Theory and Applications*, Springer Science & Business Media, 2013.

[7] H. B. Zeng, J. H. Park, and H. Shen, "Robust passivity analysis of neural networks with discrete and distributed delays," *Neurocomputing*, vol. 149, pp. 1092-1097, February 2015.

[8] Z. Chen, X. Wang, S. Zhong, and J. Yang, "Improved delay-dependent robust passivity criteria for uncertain neural networks with discrete and distributed delays," *Chaos, Solitons & Fractals*, vol. 103, pp. 23-32, October 2017.

[9] M. V. Thuan, H. Trinh, and L. Hien, "New inequality-based approach to passivity analysis of neural networks with interval time-varying delay," *Neurocomputing*, vol. 194, pp. 301-307, June 2016.

[10] S. P. Xiao, H. H. Lian, H. B. Zeng, G. Chen, and W. H. Zheng, "Analysis on robust passivity of uncertain neural networks with time-varying delays via free-matrix-based integral inequality," *International Journal of Control, Automation and Systems*, vol. 15, no. 5, pp. 2385-2394, July 2017.

[11] M. J. Park, O. M. Kwon, and J. H. Ryu, "Passivity and stability analysis of neural networks with time-varying delays via extended free-weighting matrices integral inequality," *Neural Networks*, vol. 106, pp. 67-78, October 2018.

[12] X. M. Zhang, Q. L. Han, X. Ge, and B. L. Zhang, "Delay-variation-dependent criteria on extended dissipativity for discrete-time neural networks with time-varying delay," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 34, no. 3, pp. 1578-1587, March 2023)

[13] B. Yang, J. Wang, M. Hao, and H. Zeng, "Further results on passivity analysis for uncertain neural networks with discrete and distributed delays," *Information Sciences*, vol. 430, pp. 77-86, March 2018.

[14] F. Long, C. K. Zhang, Y. He, Q. G. Wang, Z. M. Gao, and M. Wu, "Hierarchical passivity criterion for delayed neural networks via a general delay-product-type Lyapunov-Krasovskii functional," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 34, no. 1, pp. 421-432, January 2023.

[15] H. J. Yu, Y. He, and M. Wu, "Improved robust passive criteria of neural networks with discrete and distributed delays based on extended reciprocally convex matrix inequality," *International Journal of Control, Automation and Systems*, vol. 16, no. 5, pp. 2480-2488, July 2018.

[16] A. Seuret and F. Gouaisbaut, "Stability of linear systems with time-varying delays using Bessel-Legendre inequalities," *IEEE Transactions on Automatic Control*, vol. 63, no. 1, pp. 225−232, January 2018.

[17] H. B. Zeng, Y. He, and K. L. Teo, "Monotone-delay-interval-based lyapunov functionals for stability analysis of systems with a periodically varying delay," *Automatica*, vol. 138, 110030, April 2022.

[18] Y. Chen and G. Chen, "Improved results on admissibility analysis for singular systems with periodically time-varying delay," *Journal of the Franklin Institute*, vol. 360, no. 8, pp. 5780-5793, May 2023.

[19] X. M. Zhang, Q. L. Han, X. Ge, and D. Ding, "An overview of recent developments in Lyapunov-Krasovskii functionals and stability criteria for recurrent neural networks with time-varying delays," *Neurocomputing*, vol. 313, pp. 392-401, November 2018.

[20] Z. M. Gao, Y. He, and M. Wu, "Improved stability criteria for the neural networks with time-varying delay via new augmented Lyapunov-Krasovskii functional," *Applied Mathematics and Computation*, vol. 349, pp. 258-269, May 2019.

[21] Z. Li, H. Yan, H. Zhang, X. Zhan, and C. Huang, "Stability analysis for delayed neural networks via improved auxiliary polynomial-based functions," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 30, no. 8, pp. 2562-2568, August 2019.

[22] C. K. Zhang, Y. He, L. Jiang, Q. G. Wang, and M. Wu, "Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality," *IEEE Transactions on Cybernetics*, vol. 47, no. 10, pp. 3040-3049, October 2017.

[23] X. M. Zhang, Q. L. Han, X. Ge, and B. L. Zhang, "Passivity analysis of delayed neural networks based on Lyapunov-Krasovskii functionals with delay-dependent matrices," *IEEE Transactions on Cybernetics*, vol. 50, no. 3, pp. 946-956, March 2020.

[24] J. H. Kim, "Further improvement of Jensen inequality and application to stability of time-delayed systems," *Automatica*, vol. 64, pp. 121-125, February 2016.

[25] J. Chen, X. M. Zhang, J. H. Park, and S. Xu, " Improved stability criteria for delayed neural networks using a quadratic function negative-definiteness approach," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 33, no. 3, pp. 1348-1354, March 2022.

[26] C. K. Zhang, F. Long, Y. He, W. Yao, L. Jiang, and M. Wu, “A relaxed quadratic function negative-determination lemma and its application to time-delay systems,” *Automatica*, vol. 113, 108764, March 2020.

[27] T. H. Lee, “Geometry-based conditions for a quadratic function: application to stability of time-varying delay systems,” *IEEE Access*, vol. 8, pp. 92462-92468, May 2020.

[28] H. B. Zeng, H. C. Lin, Y. He, C. K. Zhang, and K. L. Teo, “ Improved negativity condition for a quadratic function and its application to systems with time-varying delay,” *IET Control Theory & Applications*, vol. 14, no. 18, pp. 2989-2993, October 2020.

[29] Y. Chen and Y. Q. Li, “Stability analysis for time-delay systems via a novel negative condition of the quadratic polynomial function,” *International Journal of Control, Automation and Systems*, vol. 19, no. 9, pp. 3159-3167, May 2021.

[30] X. M. Zhang, Q. L. Han, and X. Ge, “Sufficient conditions for a class of matrix-valued polynomial inequalities on closed intervals and application to H_∞ filtering for linear systems with time-varying delays,” *Automatica*, vol. 125, 109390, March 2021.

[31] F. S. S. de Oliveira and F. O. Souza, “Further refinements in stability conditions for time-varying delay systems,” *Applied Mathematics and Computation*, vol. 369, 124866, March 2020.

[32] J. M. Park and P. G. Park, “Finite-interval quadratic polynomial inequalities and their application to time-delay systems,” *Journal of the Franklin Institute*, vol. 357, no. 7, pp. 4316-4327, May 2020.

[33] X. M. Zhang, Q. L. Han, and X. Ge, “Novel stability criteria for linear time-delay systems using Lyapunov-Krasovskii functionals with a cubic polynomial on time-varying delay,” *IEEE/CAA Journal of Automatica Sinica*, vol. 8, no. 1, pp. 77-85, January 2021.

[34] J. H. Kim, “Improvement of stability of time-delayed linear systems via new constrained quadratic matrix inequality,” *Journal of Electrical Engineering & Technology* vol. 16, no. 5, pp. 2777-2782, June 2021.

[35] F. Long, C. K. Zhang, Y. He, Q. G. Wang, and M. Wu, “A sufficient negative-definiteness condition for cubic functions and application to time-delay systems,” *International Journal of Robust and Nonlinear Control*, vol. 31, no. 15, pp. 7361-7371, July 2021.

[36] I. R. Petersen and C. V. Hollot, “A riccati equation approach to the stabilization of uncertain linear systems,” *Automatica*, vol. 22, no. 4, pp. 397-411, July 1986.

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