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Observer-based Sliding Mode Control for Fractional Order Singular Fuzzy Systems

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Abstract: In the paper, observer-based sliding mode control (SMC) for fractional order singular fuzzy (FOSF) systems with order $0 < \alpha < 1$ is studied. The non-fragile FOSF observer is designed to reconstruct the unmeasured states, and a novel fractional order integral sliding function is formulated. Then, the admissibility condition of the FOSF error system is derived, based on the linear matrix inequality (LMI) approach. By using the singular value decomposition approach, the strict LMI-based admissibility condition is improved. Based on the fractional order Lyapunov function and sliding surface, the fractional order SMC is constructed to ensure the reachability of the sliding surface. Two examples are given to illustrate the effectiveness of the methods proposed in the paper.

Keywords: Admissibility, fractional order singular fuzzy (FOSF), linear matrix inequality (LMI), observer-based, sliding mode control (SMC).

1. INTRODUCTION

Recently, fractional order systems have made great progress in theoretical research and engineering application [1,2], since fractional order systems can describe the model of the real world phenomena with memory more concisely and precisely. The linear matrix inequality (LMI) approach is widely used as an effective tool in fractional order systems. Many LMI-based conditions of stability analysis for fractional order systems have been discussed in [3,4]. Furthermore, event-triggered control, sliding mode control, adaptive robust tracking control, and output tracking control have been studied for fractional order systems in [5-9]. Observer-based control has been wildly studied for fractional order systems in [10-13]. The observer-based control for uncertain systems has been developed in [11,12]. Wang *et al*. [10] studied the observer-based control for nonlinear fractional order systems. Moreover, Geng *et al*. [13] give the conditions of the observer-based control for input delay systems.

Singular systems have been widely applied in economics and electricity [14]. The key problem of singular systems is the admissibility analysis, including stability, regularity, and impulse-free. Admissibility conditions for fractional order singular systems have been studied in [15-17]. Robust admissibility conditions for order $1 < \alpha < 2$ and $0 < \alpha < 1$ have been proposed in [15,16]. Wang *et al*. [17] studied the observer-based control for fractional order singular switched systems.

The systems in the real world are mostly nonlinear. The T-S fuzzy model is an effective way to approximate the nonlinear system [18-20]. Since fractional order systems, singular systems and T-S fuzzy systems play significant roles in control theory and application, Therefore, the study of fractional order fuzzy systems or fractional order singular fuzzy (FOSF) systems has attracted more and more researchers. The output feedback control for fuzzy fractional order systems has been studied in [21,22]. Moreover, fault-tolerant control, observer-based control, H_∞ control, and adaptive fuzzy control for fractional order fuzzy systems have been developed in [23-28]. Stability analysis for fractional order fuzzy delayed systems has been studied in [29]. Compared with fractional order fuzzy systems, FOSF systems are more complex because not only the stability analysis, but also regularity and impulsiveness-free need to be considered. Authors of [30,31] studied the output feedback and observer-based control for FOSF systems. Furthermore, the adaptive sliding mode control (SMC) has been developed for FOSF systems in [32].

SMC as a strongly robust strategy has been widely used in industrial [33,34] and electrical equipment [35], since

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its fast response, is particularly robust and insensitive to model errors, matching uncertainties, and external perturbations. Many results have been proposed for constructing the sliding function and controller. In [36,37], the sliding mode controller design and guaranteed cost SMC are considered. Observer-based control is a frequently applied technique to overcome the shortcoming of the unmeasured states. The observer-based SMC scheme has been wildly studied, see [38-42]. In [43], vector integral SMC for fuzzy singular systems has been studied. Many results of SMC have been reported in the field of fractional order control. Mani *et al*. [44] developed the adaptive fractional fuzzy integral SMC for the permanent magnet synchronous motor model. Moreover, adaptive backstepping SMC, neural network SMC, and H_∞ adaptive output feedback for fractional order fuzzy systems have been studied in [45-47]. Some results have been published in engineering, for example, admittance-based telerobotic systems [48], and LCL-type grid-connected converters [49]. SMC of FOSF systems as more general systems have rarely been studied. [32] proposed the conditions of SMC for FOSF systems. However, up to now, observer-based SMC for FOSF systems has been studied.

Observer-based SMC strategy is designed and the admissibility for FOSF systems is studied in the paper. The main contributions of this paper are summarized as follows:

1) The new fractional order integral sliding function is formulated for FOSF systems with order $0 < \alpha < 1$. Compared with the sliding function in [29,44], the constrained conditions of observer-based SMC are relaxed, which can allow us to build LMI-based conditions that ensure the admissibility of FOSF systems.
2) A new non-fragile fractional order fuzzy singular observer is represented. Based on the observer, system state and output can be well estimated. Moreover, the sliding motion of FOSF systems can be well displayed.
3) The condition of FOSF systems is derived to guarantee admissibility for FOSF systems. Furthermore, by using the singular value decomposition approach, the strict LMI-based admissibility condition is improved. Compared with the results in [31], it avoids solving complex matrices.
4) Based on the fractional order Lyapunov function, the fractional order SMC law is proved such that the reachability of sliding function is guaranteed.

This paper is organized as follows: Problem formulation and preliminaries are introduced in Section 2. In Section 3, the non-fragile fuzzy observer and the fractional order sliding surface are formulated. The LMI-based conditions of admissibility are proved for FOSF systems. Then, the fractional order SMC law is designed. In Section 4, two examples are shown to verify the effectiveness of the proposed methods. Section 5 concludes the paper.

Notations: In the paper, $P > 0$ represents the positive definite matrix. Let $\mathbf{R}^{m\times n}$ and $\mathbf{R}^n$ be m by n matrices and n dimensional vectors. I is the identity matrix. $\mathrm{rank}(P)$ denotes the rank of matrix P. $\|P\|$ denotes the norm of P, $\mathrm{sym}(P) = P + P^T$, and $\begin{bmatrix} P & W \\ W^T & P \end{bmatrix} = \begin{bmatrix} P & W \\ * & P \end{bmatrix}$.

2. PROBLEM FORMULATION AND PRELIMINARIES

2.1. Problem formulation

In this paper, the T-S model FOSF system is represented as follows:

Plant Rule i: IF z_1 is H_1^i, $\cdots$, z_2 is H_2^i, $\cdots$, and z_s is H_s^i, THEN

$$ED^\alpha x(t) = (A_i + \Delta A(t))x(t) + B_i(u(t) + f(t,x(t))),$$
$$y(t) = C_i x(t), \tag{1}$$

where $0 < \alpha < 1$, $z_1, z_2, \ldots, z_s$ are premise variables, and H_j^i, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$ are the fuzzy sets. $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^p$ are the system state, the control input and the measurement output. $f(t,x(t))$ is a nonlinear function satisfying $\|f(t,x(t))\| \le \mu_1 + \mu_2\|y(t)\|$, μ_1 and μ_2 are unknown constants. $E \in \mathbf{R}^{n\times n}$ is the singular matrix where $\mathrm{rank}(E) = q < n$. A_i, B_i and C_i are known matrices. $\Delta A(t)$ is the uncertain term satisfying

$$\Delta A(t) = M\Delta_1(t)N_1, \Delta_1^T(t)\Delta_1(t) \le I, \tag{2}$$

where M and N_1 are real matrices, $\Delta_1(t)$ denotes the unknown matrix.

Remark 1: Electric circuit in Fig. 1 can be better described by the FOSF system, see [16,30]. R can be a nonlinear resistance, the relation function between voltage and current is $V_R(t) = 3i - i\sin(i)$. $f(t)$ denotes external environmental interference of source voltages e_1 and e_2. Based on Kirchhoff's laws in [51], the system can be represented as

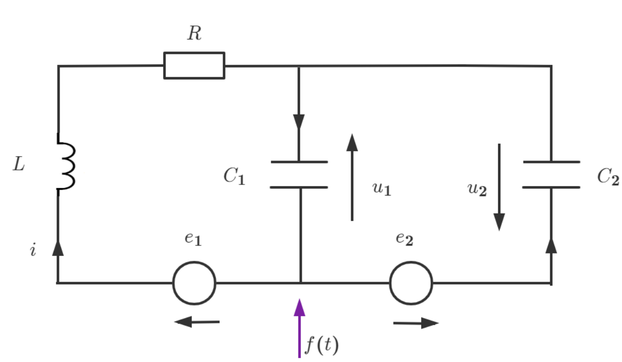

Fig. 1. Electrical circuit illustration.

$$\begin{bmatrix} L & 0 & 0 \\ 0 & -C_2 & C_1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D^\alpha i \\ D^\alpha u_2 \\ D^\alpha u_1 \end{bmatrix} = \begin{bmatrix} 3-\sin(i) & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} i \\ u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 + f(t) \\ e_2 + f(t) \end{bmatrix}.$$

The fuzzy basic function is shown as

$$h_i(z(t)) = \frac{\omega_i z(t)}{\sum_{i=1}^r \omega_i z(t)}, \tag{3}$$

where $\omega_i z(t) = \Pi_{j=1}^s H_j^i(z_j(t))$, $H_j^i(z_j(t))$ are the grade of membership of $z_j(t)$ in set H_j^i. Hence, $h_i(z(t)) \geq 0$ and $\sum_{i=1}^r h_i(z(t)) = 1$, for all $t > 0$.

Then, the FOSF system (1) can be derived as

$$\begin{aligned} &ED^\alpha x(t) = (A_h + \Delta A(t))x(t) + B_h(u(t) + f(t, x(t))), \\ &y(t) = C_h x(t), \end{aligned} \tag{4}$$

where

$$\begin{aligned} &A_h = \sum_{i=1}^r h_i(z(t))A_i, B_h = \sum_{i=1}^r h_i(z(t))B_i, \\ &C_h = \sum_{i=1}^r h_i(z(t))C_i. \end{aligned}$$

2.2. Preliminaries

In the part, to obtain the desired results in this paper, some preliminaries are given in the following. Firstly, the following singular fractional order system is given

$$ED^\alpha x(t) = Ax(t), \tag{5}$$

where $0 < \alpha < 1$, and $\text{rank}(E) = q < n$.

Then, one definition and some necessary lemmas are introduced.

Definition 1 [32]**:** 1) The system (5) is regular if $\det(s^\alpha E - A) \not\equiv 0$.

2) The system (5) is impulse-free if $\deg(\det(sE - A)) = \text{rank}(E)$.
3) The system (5) is asymptotically stable if all the roots of $\det(s^\alpha E - A)$ satisfy $|\arg(\lambda(E, A))| > \frac{\alpha\pi}{2}$.
4) The system (5) is admissible if it is regular, impulse-free and stable.

Lemma 1 [16]**:** System (5) is admissible if and only if there exist matrices $X_1, X_2 \in \mathbf{R}^{q\times q}$, $X_3 \in \mathbf{R}^{(n-q)\times q}$ and $X_4 \in \mathbf{R}^{(n-q)\times(n-q)}$ such that

$$\begin{aligned} &\begin{bmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{bmatrix} > 0, \\ &\text{sym}(aUAVX - bUAVY) < 0, \end{aligned}$$

where U and V are nonsingular matrices satisfying $UEV = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$, $a = \sin(\frac{\pi\alpha}{2})$, $b = \cos(\frac{\pi\alpha}{2})$, and

$$X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}, Y = \begin{bmatrix} X_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Lemma 2 [50]**:** For given a scalar $\epsilon > 0$, $F^T(t)F(t) \leq I$ and matrices P, Q, have

$$PF(t)Q + Q^T F^T(t)P^T \leq \epsilon PP^T + \epsilon^{-1} Q^T Q.$$

Lemma 3 [18]**:** The following inequalities hold

$$\Phi < 0,\ \Phi + HF^{-1}H^T < 0,$$

if and only if

$$\begin{bmatrix} \Phi & H \\ H^T & -F \end{bmatrix} < 0.$$

Lemma 4 [44]**:** Let $x(t) \in \mathbf{R}^n$ be a smooth function of t for $t \geq t_0$. Then there holds

$$\frac{1}{2} D^\alpha (x^T(t)x(t)) \leq x^T(t) D^\alpha x(t),\ \forall 0 < \alpha < 1.$$

3. MAIN RESULTS

In this section, a non-fragile observer is developed. Based on the observer, the system state can be estimated, and the fractional order sliding mode controller can be designed.

3.1. Non-fragile fuzzy observer design

To measure unavailable state of the system (4), the non-fragile observer is developed as follows:

$$\begin{aligned} &ED^\alpha \hat{x}(t) = A_h \hat{x}(t) + (L_h + \Delta L(t))(y(t) - \hat{y}(t)), \\ &\hat{y}(t) = C_h \hat{x}(t), \end{aligned} \tag{6}$$

where $\hat{x}(t)$ and $\hat{y}(t)$ are the estimations of $x(t)$ and $y(t)$. L_h are the gain matrices, and $\Delta L(t)$ satisfies

$$\Delta L(t) = M\Delta_2(t)N_2, \Delta_2^T(t)\Delta_2(t) \leq I. \tag{7}$$

Let $e(t) = x(t) - \hat{x}(t)$. From (4) and (6), obtain

$$\begin{aligned} ED^\alpha e(t) = &\Delta A(t)x(t) + (A_h - L_h C_h - \Delta L(t) C_h) e(t) \\ &+ B_h(u(t) + f(t, x(t))). \end{aligned} \tag{8}$$

Remark 2: In the paper, the non-fragile observer (6) is designed for the FOSF system (4). In practice, the non-fragile observer is often used [35]. If the error system (8) is admissible, then the admissibility of the observer can be ensured. To design the observer, the gain L_h should be determined.

Then, the following sliding surface function is considered

$$s(t) = G_h E\hat{x}(t) + G_h Ee(t) + I^{\alpha}[-G_h(A_h + B_h K_h)\hat{x}(t)], \tag{9}$$

where I represents integration operator, K_h is the gain matrix, G_h is the constant matrix such that $G_h B_h \neq 0$.

Assumption 1: $G_h E = HC_h$ holds if the following function holds

$$\text{rank}\begin{bmatrix} G_h E \\ C_h \end{bmatrix} = \text{rank}(C_h). \tag{10}$$

Remark 3: Since system state $x(t)$ and estimation error $e(t)$ are unknown, the sliding function (9) is not well defined. However, $e(t) = y(t) - \hat{y}(t)$ is available. If $G_h E = HC_h$ holds, $s(t) = Hy(t) + I^{\alpha}[-G_h(A_h + B_h K_h)\hat{x}(t)]$ can be designed. When G_h is found such that $G_h B_h \neq 0$. Then, a set $\bar{G}_h$ can be found such that $\bar{G}_i B_i = \bar{G}_j B_j (i \neq j)$, where $\bar{G}_1 = G_1$, and $\bar{G}_i = G_1 B_1 (G_i B_i)^{-1} G_i$. In the paper, G_h can be chosen as $G_h = \bar{G}_h$.

Based on the fractional order SMC law in [6,44], get

$$D^{\alpha}s(t) = G_h(A_h e(t) + \Delta A(t)x(t)) + G_h B_h(u(t) + f(t,x(t)) - K_h\hat{x}(t)). \tag{11}$$

Let $u_e(t)$ be the equivalent control law, get

$$u_e(t) = K_h\hat{x}(t) - f(t,x(t)) - \hat{G}(A_h e(t) + \Delta A(t)x(t)), \tag{12}$$

where $\hat{G}_h = (G_h B_h)^{-1} G_h$.

Substituting (12) into (4) and (8), assume $\bar{G}_h = I - B_h\hat{G}_h$, we have

$$ED^{\alpha}x(t) = (A_h + \bar{G}_h\Delta A(t) + B_h K_h)x(t) - (B_h\hat{G}_h A_h + B_h K_h)e(t), \tag{13}$$

$$ED^{\alpha}e(t) = (\bar{G}_h\Delta A(t) + B_h K_h)x(t) + (A_h - L_h C_h - \Delta L(t)C_h - B_h K_h B_h \hat{G}_h A_h)e(t). \tag{14}$$

According to (13) and (14), obtain

$$\bar{E}D^{\alpha}\bar{x}(t) = \bar{A}_h(t)\bar{x}(t), \tag{15}$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \ \bar{A}_h(t) = \bar{A}_h + \bar{M}\bar{\Delta}(t)\bar{N},$$

$$\bar{\Delta}(t) = \begin{bmatrix} \Delta_1(t) & 0 \\ 0 & \Delta_2(t) \end{bmatrix}, \ \bar{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix},$$

$$\bar{A}_h = \begin{bmatrix} A_h + B_h K_h & -B_h\hat{G}_h A_h - B_h K_h \\ B_h K_h & A_h - L_h C_h - B_h K_h - B_h\hat{G}_h A_h \end{bmatrix},$$

$$\bar{M} = \begin{bmatrix} \bar{G}_h M & 0 \\ \bar{G}_h M & -M \end{bmatrix}, \ \bar{N} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 C_h \end{bmatrix}.$$

3.2. Admissibility analysis

In the subsection, the LMI-based conditions are developed to ensure (15) robustly admissible.

Theorem 1: The FOSF system (15) is robustly admissible with gain matrices K_h and L_h if there exist matrices $X_{11i}, X_{12i}, X_{13i}, X_{14i}, X_{21i}, X_{22i}, X_{23i}, X_{24i}$, and the scalars ϵ_i, $i = \{1, 2, \ldots, r\}$ such that

$$\begin{bmatrix} X_{11i} & X_{12i} \\ -X_{12i} & X_{11i} \end{bmatrix} > 0, \tag{16}$$

$$\begin{bmatrix} X_{21i} & X_{22i} \\ -X_{22i} & X_{21i} \end{bmatrix} > 0, \tag{17}$$

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & N_1 V_1 P_{1i} & 0 \\ * & \Phi_{22} & 0 & N_2 C_h V_1 P_{2i} \\ * & * & -\epsilon_i I & 0 \\ * & * & * & -\epsilon_i I \end{bmatrix} < 0, \tag{18}$$

where a and b are given in Lemma 2, U_1 and V_1 are chosen nonsingular matrices satisfying

$$U_1 E V_1 = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \ U = \begin{bmatrix} U_1 & 0 \\ 0 & U_1 \end{bmatrix}, \ V = \begin{bmatrix} V_1 & 0 \\ 0 & V_1 \end{bmatrix},$$

and

$$\Phi_{11} = \text{sym}(U_1 A_h V_1 P_{1i} + U_1 B_h K_h V_1 P_{1i}) + \epsilon_i U_1 \bar{G}_h M M^T \bar{G}_h^T U_1^T,$$

$$\Phi_{12} = -U_1 B_h \hat{G}_h A_h V_1 P_{2i} - U_1 B_h K_h V_1 P_{2i} + P_{1i}^T V_1^T K_h^T B_h^T U_1^T + U_1 \bar{G}_h M M^T \bar{G}_h^T U_1^T,$$

$$\Phi_{22} = \text{sym}(U_1 A_h V_1 P_{2i} - U_1 L_h C_h V_1 P_{2i} - U_1 B_h K_h V_1 P_{2i} - U_1 B_h \hat{G}_h A_h V_1 P_{2i}) + \epsilon_i U_1 M M^T U_1^T,$$

$$X_{1i} = \begin{bmatrix} X_{11i} & 0 \\ X_{13i} & X_{14i} \end{bmatrix}, \ X_{2i} = \begin{bmatrix} X_{21i} & X_{23i} \\ 0 & X_{24i} \end{bmatrix},$$

$$Y_{1i} = \begin{bmatrix} X_{12i} & 0 \\ 0 & 0 \end{bmatrix}, \ Y_{2i} = \begin{bmatrix} X_{22i} & 0 \\ 0 & 0 \end{bmatrix},$$

$$X_i = \begin{bmatrix} X_{1i} & 0 \\ 0 & X_{2i} \end{bmatrix}, \ Y_i = \begin{bmatrix} Y_{1i} & 0 \\ 0 & Y_{2i} \end{bmatrix},$$

$$P_{1i} = aX_{1i} - bY_{1i}, \ P_{2i} = aX_{2i} - bY_{2i}.$$

Proof: From the system (15), and Lemma 1, get that

$$\text{sym}(aU\bar{A}_h(t)VX_i - bU\bar{A}_h(t)VY_i) < 0. \tag{19}$$

Then, have

$$\text{sym}(aU\bar{A}_h VX_i - bU\bar{A}_h VY_i) + \text{sym}(aU\bar{M}\bar{\Delta}(t)\bar{N}VX_i - bU\bar{M}\bar{\Delta}(t)\bar{N}VY_i) < 0. \tag{20}$$

Hence,

$$\text{sym}\left(\begin{bmatrix} U_1 & 0 \\ 0 & U_1 \end{bmatrix}\right.$$

$$\times\begin{bmatrix}A_h+B_hK_h & -B_h\hat{G}_hA_h-B_hK_h\\ B_hK_h & A_h-L_hC_h-B_hK_h-B_h\hat{G}_hA_h\end{bmatrix}$$
$$\times\begin{bmatrix}V_1&0\\0&V_1\end{bmatrix}\left(a\begin{bmatrix}X_{1i}&0\\0&X_{2i}\end{bmatrix}-b\begin{bmatrix}Y_{1i}&0\\0&Y_{2i}\end{bmatrix}\right)\Bigg)$$
$$+\operatorname{sym}\left(\begin{bmatrix}U_1&0\\0&U_1\end{bmatrix}\begin{bmatrix}\bar{G}_hM&0\\\bar{G}_hM&-M\end{bmatrix}\begin{bmatrix}\Delta_1(t)&0\\0&\Delta_2(t)\end{bmatrix}\right.$$
$$\times\begin{bmatrix}N_1&0\\0&N_2C_h\end{bmatrix}\begin{bmatrix}V_1&0\\0&V_1\end{bmatrix}\left(a\begin{bmatrix}X_{1i}&0\\0&X_{2i}\end{bmatrix}\right.$$
$$\left.\left.-b\begin{bmatrix}Y_{1i}&0\\0&Y_{2i}\end{bmatrix}\right)\right)<0, \tag{21}$$

where

$$\bar{\Delta}^T(t)\bar{\Delta}(t)=\begin{bmatrix}\Delta_1(t)&0\\0&\Delta_2(t)\end{bmatrix}^T\begin{bmatrix}\Delta_1(t)&0\\0&\Delta_2(t)\end{bmatrix}\leq\begin{bmatrix}I&0\\0&I\end{bmatrix}.$$

Let $P_{1i}=aX_{1i}-bY_{1i}$, and $P_{2i}=aX_{2i}-bY_{2i}$, obtain

$$\operatorname{sym}\left(\begin{bmatrix}U_1&0\\0&U_1\end{bmatrix}\right.$$
$$\times\begin{bmatrix}A_h+B_hK_h & -B_h\hat{G}_hA_h-B_hK_h\\ B_hK_h & A_h-L_hC_h-B_hK_h-B_h\hat{G}_hA_h\end{bmatrix}$$
$$\left.\times\begin{bmatrix}V_1&0\\0&V_1\end{bmatrix}\begin{bmatrix}P_{1i}&0\\0&P_{2i}\end{bmatrix}\right)$$
$$+\operatorname{sym}\left(\begin{bmatrix}U_1&0\\0&U_1\end{bmatrix}\begin{bmatrix}\bar{G}_hM&0\\\bar{G}_hM&-M\end{bmatrix}\begin{bmatrix}\Delta_1(t)&0\\0&\Delta_2(t)\end{bmatrix}\right.$$
$$\left.\times\begin{bmatrix}N_1&0\\0&N_2C_h\end{bmatrix}\begin{bmatrix}V_1&0\\0&V_1\end{bmatrix}\begin{bmatrix}P_{1i}&0\\0&P_{2i}\end{bmatrix}\right)<0. \tag{22}$$

According to Lemma 2, get that

$$\operatorname{sym}\left(\begin{bmatrix}U_1(A_h+B_hK_h)V_1P_{1i} & U_1(-B_h\hat{G}_hA_h-B_hK_h)V_1P_{2i}\\ U_1B_hK_hV_1P_{1i} & U_1(A_h-L_hC_h-B_hK_h-B_h\hat{G}_hA_h)V_1P_{2i}\end{bmatrix}\right)$$
$$+\epsilon_h\begin{bmatrix}U_1&0\\0&U_1\end{bmatrix}\begin{bmatrix}\bar{G}_hM&0\\\bar{G}_hM&-M\end{bmatrix}\left(\begin{bmatrix}U_1&0\\0&U_1\end{bmatrix}\begin{bmatrix}\bar{G}_hM&0\\\bar{G}_hM&-M\end{bmatrix}\right)^T$$
$$+\epsilon_h^{-1}\left(\begin{bmatrix}N_1&0\\0&N_2C_h\end{bmatrix}\begin{bmatrix}V_1&0\\0&V_1\end{bmatrix}\begin{bmatrix}P_{1i}&0\\0&P_{2i}\end{bmatrix}\right)^T$$
$$\times\begin{bmatrix}N_1&0\\0&N_2C_h\end{bmatrix}\begin{bmatrix}V_1&0\\0&V_1\end{bmatrix}\begin{bmatrix}P_{1i}&0\\0&P_{2i}\end{bmatrix}<0. \tag{23}$$

From (23), obtain

$$\operatorname{sym}\left(\begin{bmatrix}U_1(A_h+B_hK_h)V_1P_{1i} & U_1(-B_h\hat{G}_hA_h-B_hK_h)V_1P_{2i}\\ U_1B_hK_hV_1P_{1i} & U_1(A_h-L_hC_h-B_hK_h-B_h\hat{G}_hA_h)V_1P_{2i}\end{bmatrix}\right)$$
$$+\epsilon_h\begin{bmatrix}U_1\bar{G}_hM&0\\U_1\bar{G}_hM&-U_1M\end{bmatrix}\begin{bmatrix}U_1\bar{G}_hM&0\\U_1\bar{G}_hM&-U_1M\end{bmatrix}^T$$
$$+\epsilon_h^{-1}\begin{bmatrix}N_1V_1P_{1i}&0\\0&N_2C_hV_1P_{2i}\end{bmatrix}^T\begin{bmatrix}N_1V_1P_{1i}&0\\0&N_2C_hV_1P_{2i}\end{bmatrix}$$
$$<0. \tag{24}$$

Then, according to Lemma 3, we have

$$\begin{bmatrix}\Phi_{11}&\Phi_{12}&N_1V_1P_{1i}&0\\ *&\Phi_{22}&0&N_2C_hV_1P_{2i}\\ *&*&-\epsilon_iI&0\\ *&*&*&-\epsilon_iI\end{bmatrix}<0, \tag{25}$$

where

$$\Phi_{11}=\operatorname{sym}(U_1A_hV_1P_{1i}+U_1B_hK_hV_1P_{1i})+\epsilon_iU_1\bar{G}_hMM^T\bar{G}_h^TU_1^T,$$
$$\Phi_{12}=-U_1B_h\hat{G}_hA_hV_1P_{2i}-U_1B_hK_hV_1P_{2i}+P_{1i}^TV_1^TK_h^TB_h^TU_1^T+U_1\bar{G}_hMM^T\bar{G}_h^TU_1^T,$$
$$\Phi_{22}=\operatorname{sym}(U_1A_hV_1P_{2i}-U_1L_hC_hV_1P_{2i}-U_1B_hK_hV_1P_{2i}-U_1B_h\hat{G}_hA_hV_1P_{2i})+\epsilon_iU_1MM^TU_1^T.$$

From (25), (15) can be obtained. Therefore, the FOSF system (15) is robustly admissible. □

Remark 4: It is easy to see that there are some nonlinear terms in Theorem 1, such as $U_1B_hK_hV_1P_{1i}$ and $U_1L_hC_hV_1P_{2i}$. Although some terms can be solved by defining $Z_i=K_hV_1P_{1i}$. The terms are still difficult to solve, such as $U_1L_hC_hV_1P_{2i}$. Hence, the singular value decomposition approach will be used to solve the nonlinear terms in [11,17].

Theorem 2: The FOSF system (15) is robustly admissible with gain matrices K_h and L_h if there exist matrices X_{11i}, X_{12i}, X_{13i}, X_{14i}, X_{21i}, X_{22i}, X_{23i}, X_{24i}, W_{1i}, W_{2i}, Z_i and the scalars ϵ_i, $i=\{1, 2, \dots, r\}$ such that

$$\begin{bmatrix}X_{11i}&X_{12i}\\-X_{12i}&X_{11i}\end{bmatrix}>0, \tag{26}$$
$$\begin{bmatrix}X_{21i}&X_{22i}\\-X_{22i}&X_{21i}\end{bmatrix}>0, \tag{27}$$
$$\begin{bmatrix}\Phi_{11}&\Phi_{12}&N_1V_1P_{1i}&0\\ *&\Phi_{22}&0&N_2C_hV_1P_{2i}\\ *&*&-\epsilon_iI&0\\ *&*&*&-\epsilon_iI\end{bmatrix}<0, \tag{28}$$

where a and b are given in Lemma 2, U_1, V_1, U_2, S_h and V_2 are chosen nonsingular matrices satisfying

$$U_1EV_1=\begin{bmatrix}I_q&0\\0&0\end{bmatrix},\ U=\begin{bmatrix}U_1&0\\0&U_1\end{bmatrix},\ V=\begin{bmatrix}V_1&0\\0&V_1\end{bmatrix},$$
$$C_hV_1=U_2\begin{bmatrix}S_h&0\end{bmatrix}V_2^T,$$

and

$$\Phi_{11}=\operatorname{sym}(U_1A_hV_1P_{1i}+U_1B_hW_{1i})$$

$$
\begin{aligned}
&+\epsilon_i U_1 \bar{G}_h MM^T \bar{G}_h^T U_1^T,\\
\Phi_{12} &= -U_1 B_h \hat{G}_h A_h V_1 P_{2i} - U_1 B_h W_{2i} + W_{1i}^T B_h^T U_1^T\\
&\quad + U_1 \bar{G}_h MM^T \bar{G}_h^T U_1^T,\\
\Phi_{22} &= \mathrm{sym}(U_1 A_h V_1 P_{2i} - U_1 Z_i C_h V_1 - U_1 B_h W_{2i}\\
&\quad - U_1 B_h \hat{G}_h A_h V_1 P_{2i}) + \epsilon_h U_1 MM^T U_1^T,\\
X_{1i} &= \begin{bmatrix} X_{11i} & 0 \\ X_{13i} & X_{14i} \end{bmatrix},\ X_2 = \begin{bmatrix} X_{21i} & X_{23i} \\ 0 & X_{24i} \end{bmatrix},\\
Y_{1i} &= \begin{bmatrix} X_{12i} & 0 \\ 0 & 0 \end{bmatrix},\ Y_{2i} = \begin{bmatrix} X_{22i} & 0 \\ 0 & 0 \end{bmatrix},\\
X_i &= \begin{bmatrix} X_{1i} & 0 \\ 0 & X_{2i} \end{bmatrix},\ Y_i = \begin{bmatrix} Y_{1i} & 0 \\ 0 & Y_{2i} \end{bmatrix},\\
P_{1i} &= aX_{1i} - bY_{1i},\ P_{2i} = aX_{2i} - bY_{2i},\\
P_{2i} &= V_2 \begin{bmatrix} P_{21i} & 0 \\ P_{22i} & P_{23i} \end{bmatrix} V_2^T,\ \bar{P}_i = U_2 S_h P_{21i} S_h^{-1} U_2^{-1}.
\end{aligned}
$$

The gain matrices are

$$
\begin{aligned}
K_h &= W_{1i}(V_1 P_{1i})^{-1},\\
L_h &= Z_i U_2 S_h P_{21i}^{-1} S_h^{-1} U_2^{-1}.
\end{aligned} \tag{29}
$$

Proof: According to

$$
C_h V_1 = U_2 \begin{bmatrix} S_h & 0 \end{bmatrix} V_2,\ P_{2i} = V_2 \begin{bmatrix} P_{21i} & 0 \\ P_{22i} & P_{23i} \end{bmatrix} V_2^T,
$$

we can get that the matrix $\bar{P}_i = U_2 S_h P_{21i} S_h^{-1} U_2^{-1}$ satisfies $C_h V_1 P_{2i} = \bar{P}_i C_h V_1$.

Let $K_h V_1 P_{1i} = W_{1i}$, $K_h V_1 P_{2i} = W_{2i}$, and $L_h \bar{P}_i = Z_i$, Theorem 2 can be directly derived by Theorem 1. □

Remark 5: By using singular value decomposition approach (see Lemma 6 in [17]), Theorem 2 based on the strict LMI can be proved, and it can be easily solved. It is worth mentioning that the above conditions are equivalent to these of integer order systems, when $\alpha = 1$.

3.3. Sliding mode control

In this subsection, the SMC control law is developed based on fractional order Lyapunov function, which can ensure the system trajectories can be kept on the sliding surface.

For the FOSF system (15), there exist the unknown scalars λ_1, λ_2, ξ_1 and ξ_2 satisfying

$$
\begin{aligned}
&\|e(t)\| \le \lambda_1 \|y(t)\| + \lambda_2 \|\hat{y}(t)\|,\\
&\|T\|\|e(t)\| \le \xi_1 \|y(t)\| + \xi_2 \|\hat{y}(t)\|,
\end{aligned}
$$

where $T = G_h B_h$.

Then, assume that $\bar{\xi}_1$, $\bar{\xi}_2$, $\bar{\mu}_1$, and $\bar{\mu}_2$ are the estimates of ξ_1, ξ_2, μ_1, and μ_2. The errors are $\hat{\xi}_1 = \bar{\xi}_1 - \xi_1$, $\hat{\xi}_2 = \bar{\xi}_2 - \xi_2$, $\hat{\mu}_1 = \bar{\mu}_1 - \mu_1$ and $\hat{\mu}_2 = \bar{\mu}_2 - \mu_2$. The SMC is

$$
\begin{aligned}
u(t) &= K_h \hat{x}(t) - T^{-1} G_h \Delta A(t) x(t) - [T^{-1}(\bar{\xi}_1 \|y(t)\|\\
&\quad + \bar{\xi}_2 \|\hat{y}(t)\| + \rho) + \bar{\mu}_1 + \bar{\mu}_2 \|y(t)\|]\mathrm{sgn}(s(t)),
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
&D^\alpha \bar{\mu}_1 = c_{\mu_1} \|T\|\|s(t)\|,\ D^\alpha \bar{\mu}_2 = c_{\mu_2} \|y(t)\|\|T\|\|s(t)\|,\\
&D^\alpha \bar{\xi}_1 = c_{\xi_1} \|y(t)\|\|s(t)\|,\ D^\alpha \bar{\xi}_2 = c_{\xi_2} \|\hat{y}(t)\|\|s(t)\|,
\end{aligned} \tag{31}
$$

and c_{μ_1}, c_{μ_2}, c_{ξ_1}, c_{ξ_2}, and ρ are positive constants.

Theorem 3: From the sliding surface (9), the system state can reach the sliding surface in a finite time based on the SMC (30) with the gain matrices K_h and L_h solved by Theorem 2.

Proof: The following Lyapunov function is chosen

$$
\begin{aligned}
V(t) &= V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t)\\
&= \frac{1}{2} s^T(t) s(t) + \frac{1}{2c_{\mu_1}} \hat{\mu}_1^2 + \frac{1}{2c_{\mu_2}} \hat{\mu}_2^2 + \frac{1}{2c_{\xi_1}} \hat{\xi}_1^2\\
&\quad + \frac{1}{2c_{\xi_2}} \hat{\xi}_2^2.
\end{aligned} \tag{32}
$$

Based on Lemma 4, have

$$
\begin{aligned}
D^\alpha V_1(t) &\le s^T(t) D^\alpha s(t) = s^T(t) \times [G_h(A_h e(t)\\
&\quad + \Delta A(t) x(t)) + G_h B_h(u(t) + f(t, x(t))\\
&\quad - K_h \hat{x}(t))].
\end{aligned} \tag{33}
$$

From (30), the above inequality can be rewritten as

$$
\begin{aligned}
D^\alpha V_1(t) &\le s^T(t)[G_h(A_h e(t) + B_h(f(t, x(t))\\
&\quad - K_h \hat{x}(t))) - T(T^{-1}(\bar{\xi}_1 \|y(t)\| + \bar{\xi}_2 \|\hat{y}(t)\|\\
&\quad + \rho) + \bar{\mu}_1 + \bar{\mu}_2 \|y(t)\|)\mathrm{sgn}(s(t))\\
&\quad - K_h \hat{x}(t)]\\
&\le s^T(t)[G_h A_h e(t) + T(f(t, x(t)) - K_h \hat{x}(t))\\
&\quad - T(T^{-1}(\bar{\xi}_1 \|y(t)\| + \bar{\xi}_2 \|\hat{y}(t)\| + \rho)\\
&\quad + \bar{\mu}_1 + \bar{\mu}_2 \|y(t)\|)\mathrm{sgn}(s(t)) - K_h \hat{x}(t)].
\end{aligned} \tag{34}
$$

Then, we have

$$
\begin{aligned}
D^\alpha V(t) &\le s^T(t)[G_h(A_h e(t) + B_h(f(t, x(t))\\
&\quad - K_h \hat{x}(t))) - T(T^{-1}(\bar{\xi}_1 \|y(t)\| + \bar{\xi}_2 \|\hat{y}(t)\|\\
&\quad + \rho) + \bar{\mu}_1 + \bar{\mu}_2 \|y(t)\|)\mathrm{sgn}(s(t))\\
&\quad - K_h \hat{x}(t)] + \frac{\hat{\mu}_1}{c_{\mu_1}} D^\alpha \hat{\mu}_1 + \frac{\hat{\mu}_2}{c_{\mu_2}} D^\alpha \hat{\mu}_2\\
&\quad + \frac{\hat{\xi}_1}{c_{\xi_1}} D^\alpha \hat{\xi}_1 + \frac{\hat{\xi}_2}{c_{\xi_2}} D^\alpha \hat{\xi}_2\\
&\le \|s(t)\|(\xi_1 \|y(t)\| + \xi_2 \|\hat{y}(t)\|)\\
&\quad + \|s(t)\|\|T\|(\mu_1 + \mu_2 \|y(t)\|)\\
&\quad - \|s(t)\|(\bar{\xi}_1 \|y(t)\| + \bar{\xi}_2 \|\hat{y}(t)\|\\
&\quad + \rho) - \|T\|\|s(t)\|(\bar{\mu}_1 + \bar{\mu}_2 \|y(t)\|)\\
&\quad + \hat{\mu}_1 \|T\|\|s(t)\| + \hat{\mu}_2 \|y(t)\|\|T\|\|s(t)\|\\
&\quad + \hat{\xi}_1 \|y(t)\|\|s(t)\| + \hat{\xi}_2 \|\hat{y}(t)\|\|s(t)\|
\end{aligned}
$$

$$= -\rho\|s(t)\|. \tag{35}$$

According to Lemma 3 in [7] and (35), have

$$D^{\alpha}V(t) \leq -\rho\|s(t)\| \leq -\sqrt{2}\rho V^{\frac{1}{2}}(t). \tag{36}$$

Then, according to Lemma 8 in [2] and (36), we can get $t^* \leq (\frac{\alpha}{\sqrt{2}\rho}V^{\alpha-\frac{1}{2}}(0)\frac{\Gamma(\alpha)\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})})^{\frac{1}{\alpha}}$ such that the system states can reach the sliding surface (9) in finite time. This completes the proof. □

Remark 6: If the system (4) has the same matrices, i.e., $B = B_1 = \ldots = B_n$, and $C = C_1 = \ldots = C_n$. The results proposed in the paper are still applicable. Hence, the results are less conservative.

After the SMC design, the detailed process is outlined below:

1) Choose appropriate matrices G_h satisfying $G_hB_h \neq 0$, and choose U_1, U_2, V_1, and V_2 satisfying

$$U_1EV_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, C_hV_1 = U_2\begin{bmatrix} S_h & 0 \end{bmatrix}V_2.$$

2) Determine the gain matrices K_h and L_h by solving the (26)-(29) in Theorem 2.
3) Design the sliding surface (9), and choose the parameters ρ, c_{μ_1}, c_{μ_2}, c_{ξ_1}, and c_{ξ_2} in (30), (31).
4) Apply the designed control law (30) to the plant.

4. ILLUSTRATIVE EXAMPLES

In the section, two examples to show the effectiveness of our results.

4.1. Example 1

Consider the following FOSF system

$$\begin{aligned} ED^{\frac{1}{3}}x(t) &= \sum_{i=1}^{3} h_i(z(t))[(A_i + \Delta A(t))x(t) \\ &\quad + B_i(u(t) + f(t,x(t)))], \\ y(t) &= \sum_{i=1}^{3} h_i(z(t))C_ix(t), \end{aligned} \tag{37}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.1 & 0.3 & 1 \\ 0.2 & 0.1 & 0.5 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}, N_1 = I_3, N_2 = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.1 \\ 0.2 & 0.3 \end{bmatrix},$$

$$\Delta_1(t) = \Delta_2(t) = \begin{bmatrix} \frac{\sin(t)}{4} & 0 & 0 \\ 0 & \frac{\cos(t)}{4} & 0 \\ 0 & 0 & \frac{\sin(t)}{4} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1.5 \\ -0.5 \\ 2 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},$$

$$f(t,x(t)) = (\sin^2(t) - \cos(t) - 0.1)x_1(t),$$

$$C_1 = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 2 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$h_1(z(t)) = \frac{\sin(x_1^2)}{6}, \; h_2(z(t)) = \frac{1+\cos(x_1)}{6},$$

$$h_3(z(t)) = 1 - h_1(z(t)) - h_2(z(t)).$$

It is worth mentioning that $B_1 \neq B_2 \neq B_3$ and $C_1 \neq C_2 \neq C_3$. We can choose

$$G_1 = G_2 = G_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

such that $G_1B_1 = G_2B_2 = G_3B_3 = 3$ is nonsingular, and

$$\hat{G}_1 = \hat{G}_2 = \hat{G}_3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \bar{G}_1 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

$$\bar{G}_2 = \begin{bmatrix} 0.5 & -0.5 & -0.5 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \bar{G}_3 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Choose

$$U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

such that

$$U_1EV_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, S_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, S_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By solving (26)-(29) in Theorem 2, the gain matrices K_h, L_h and ϵ_i can be obtained

$$K_1 = \begin{bmatrix} 0.1054 & -52.9027 & 2.1942 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.0883 & 13.2260 \\ -8.7087 & 0.1871 \\ -8.7510 & 13.1509 \end{bmatrix}, \epsilon_1 = 2.9694,$$

$$K_2 = \begin{bmatrix} 0.5217 & 2.2231 & -1.0883 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0.2042 & -5.6773 \\ 2.7990 & 0.0357 \\ 2.9439 & -5.5744 \end{bmatrix}, \epsilon_2 = 2.9342,$$

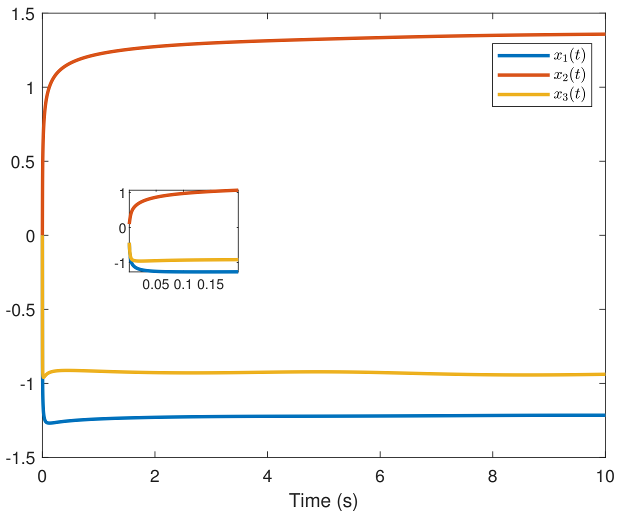

Fig. 2. State response $x(t)$.

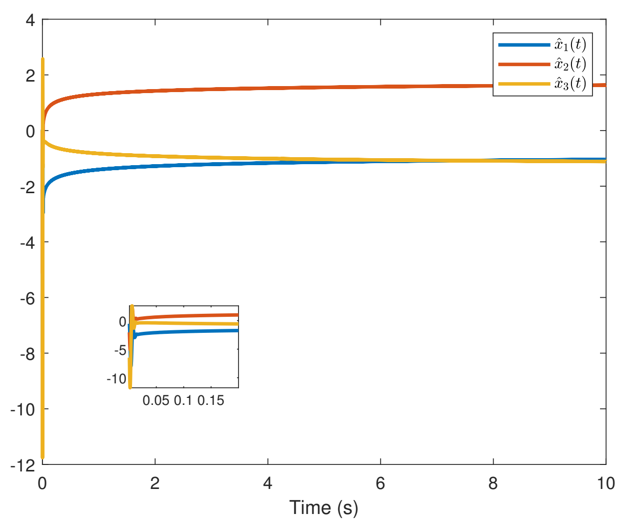

Fig. 3. State estimation $\hat{x}(t)$.

$$K_3 = \begin{bmatrix} -3.4123 & -6.3039 & -1.2016 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} -0.5629 & 2.6542 \\ -1.9725 & 7.8628 \\ 0.6551 & 2.0097 \end{bmatrix}, \ \epsilon_3 = 9.9629.$$

Then, the parameters can be chosen as $\rho = 0.1$, $c_{\mu_1} = 0.15$, $c_{\mu_2} = 0.05$, $c_{\xi_1} = 0.2$, and $c_{\xi_2} = 0.1$.

Figs. 2-4 show the state response, state estimation, and estimation error of the FOSF system. Figs. 5-6 show the sliding mode surface $s(t)$ and the control signal $u(t)$. From Figs. 2-4, we can know that the FOSF system (37) is admissible under the SMC scheme and the the initial condition $x_0 = [-1, 0.2, -0.5]$. Fig. 7 shows that $\bar{\mu}_1$, $\bar{\mu}_2$, $\bar{\xi}_1$, and $\bar{\xi}_2$ are bounded.

Remark 7: One advantage of Theorem 2 is that it is based on strict LMI, which avoids the computation complexity. The other one is that it can be viewed as a gen-

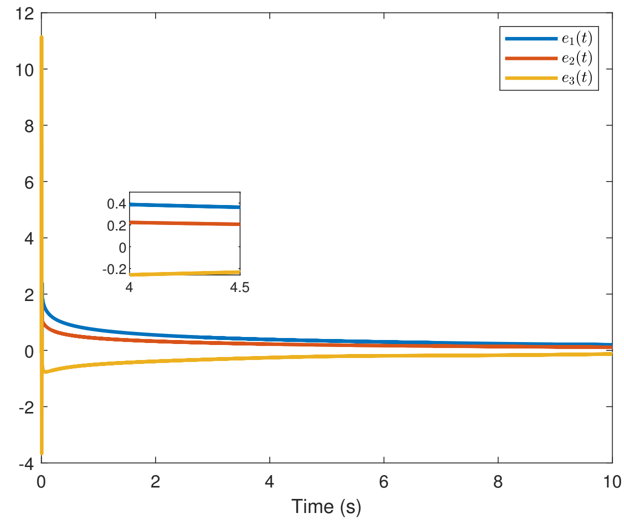

Fig. 4. Estimation error $e(t)$.

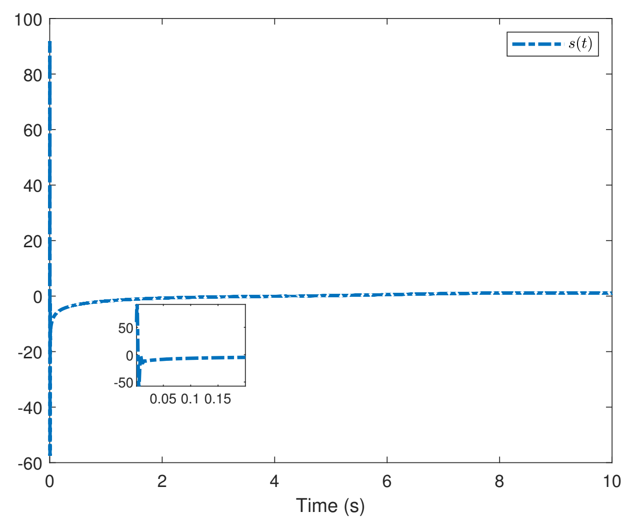

Fig. 5. Sliding mode surface function $s(t)$.

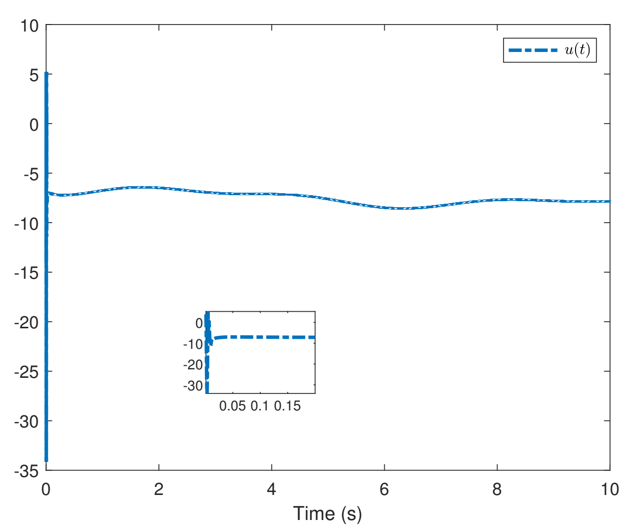

Fig. 6. Control input $u(t)$.

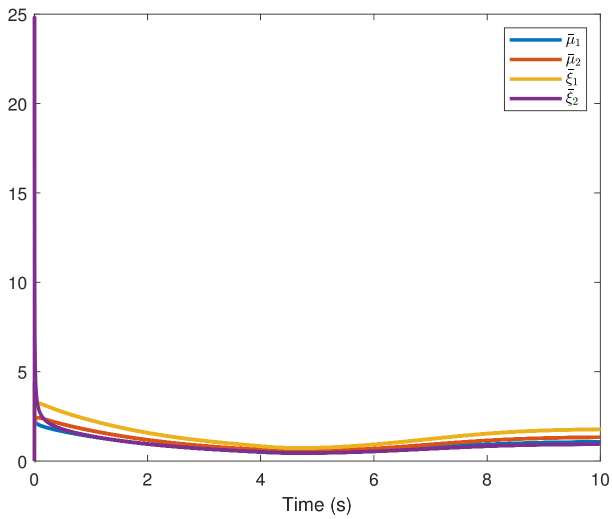

Fig. 7. Parameter estimation.

eralization of the results of integer order systems. When $\alpha = 1$, the FOSF systems reduce to the integer order fuzzy singular systems considered in [38-42]. Compared with the results in [31], Theorem 2 avoids solving complex matrices and reduces the computational cost.

4.2. Example 2

Consider the electrical circuit in Fig. 1. Let $\alpha = 0.6$, inductor $L = 1$, capacitances $C_1 = 1$ $C_2 = 2$, $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T = \begin{bmatrix} i & u_2 & u_1 \end{bmatrix}^T$, and membership functions $h_1(z(t)) = \frac{1-\sin(x_1)}{2}$, $h_2(z(t)) = 1 - h_1(z(t))$.

Then, obtain

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 4 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (38)$$

Consider the system output, internal and external environmental disturbances

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, f(t) = (\sin(t) + 0.1)x_1(t),$$
$$M = \begin{bmatrix} 0.3 & 0.4 & 0.48 \\ 0.4 & 0.9 & 0.5 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}, N_1 = I_3, N_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix},$$
$$\Delta_1(t) = \Delta_2(t) = \begin{bmatrix} \frac{\sin(t)}{3} & 0 & 0 \\ 0 & \frac{\cos(t)}{3} & 0 \\ 0 & 0 & \frac{\sin(t)}{3} \end{bmatrix}. \quad (39)$$

Choose

$$G_1 = G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, the gain matrices K_h, L_h and ϵ_i can be obtained

$$K_1 = \begin{bmatrix} -22.2464 & -36.2670 & -18.7884 \\ -3.9176 & -12.3690 & -20.8654 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 22.7262 \\ -2.4684 \\ -9.2483 \end{bmatrix}, \epsilon_1 = 7.2249,$$
$$K_2 = \begin{bmatrix} 6.9307 & 25.6040 & 79.0750 \\ -10.9817 & -22.9383 & -40.0365 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} 15.9354 \\ -1.1346 \\ -5.7224 \end{bmatrix}, \epsilon_2 = 6.1437.$$

Fig. 8 shows the state response, state estimation, and estimation error in Example 2, which implies the effectiveness of the method about observer-based sliding mode controller design in the paper.

5. CONCLUSION

In the paper, the problem of observer-based SMC for FOSF systems has been studied. Firstly, the non-fragile observer and sliding mode surface is designed for FOSF systems. Then, the sliding motion of the error system has been proposed. Furthermore, the condition with the bilinear term of admissibility has been obtained for FOSF systems. Thus, by using the singular value decomposition approach, the condition based on LMI has been improved. Moreover, by designing the SMC law, the system state can reach the sliding mode surface in a finite time. Two examples have been used to verify the effectiveness of the proposed methods.

In the future, we will focus on H_∞ observer-based SMC for FOSF systems.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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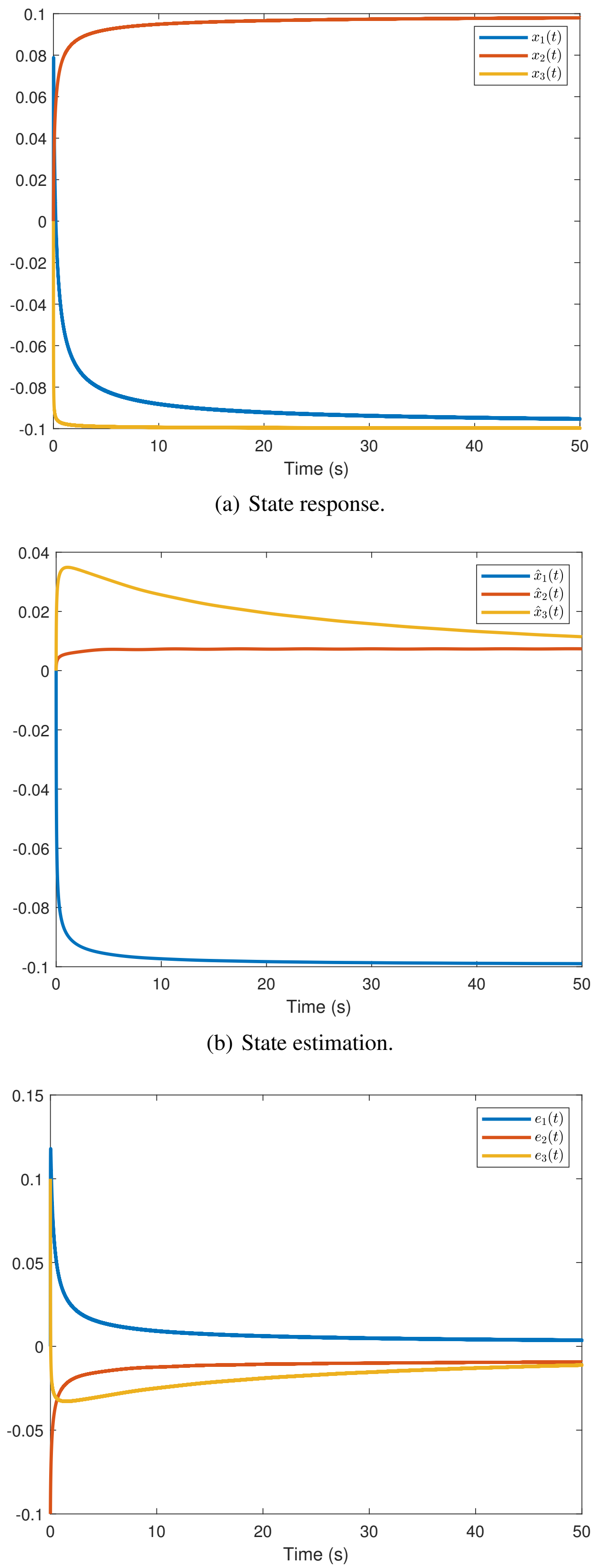

(a) State response.

(b) State estimation.

(c) Estimation error.

Fig. 8. State response, state estimation, and estimation error in Example 2.

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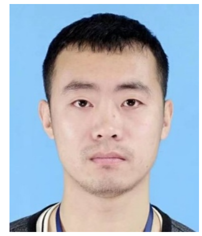

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