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Finite-time Incremental Stability Analysis for Nonlinear Switched Systems With Unstable Subsystems

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Abstract: In this paper, finite-time incremental stability (FTIS) and finite-time incremental boundedness (FTIB) are investigated for nonlinear switched systems (NSS) with unstable systems. Firstly, based on the reverse mode-dependent average dwell time (MDADT) method, we propose some sufficient conditions for FTIS of NSS. Secondly, the sufficient conditions for NSS to be FTIB with an incremental performance index are given by the multiple incremental Lyapunov functions. Finally, a distinctive advantage of reverse MDADT method is that each subsystems satisfies FTIB in the activation interval. A example has been provided to show the effectiveness of the theoretic results developed.

Keywords: Finite-time incremental boundedness, finite-time incremental stability, mode-dependent average dwell time.

1. INTRODUCTION

Nowadays, finite-time stability (FTS) and H_∞ control problem have been extensively explored for NSS. Several aspects of FTS have been researched in [1-3], Lyapunov and converse Lyapunov results, regularity properties of the settling-time function have been considered. However, many literatures [4-6] focus on the case where all subsystems are FTS. Moreover, in actual control systems, there are many NSS with non-FTS subsystems, thus the works of this paper have strong theoretical and practical significance to study the FTS and non-FTS systems coexistence. Over the years, many modeling frameworks of FTS subsystems have been developed that describe and quantify the behavior of physical systems. A large variety of FTS and H_∞ control analysis tools are available for NSS, such as Lyapunov's stability theory and passivity theory. Moreover, techniques such as dwell time, average dwell time (ADT) or MDADT [7] have been developed to stabilize the behavior and to achieve performance index for NSS. There are good properties of robustness and disturbance rejections in [8] and from the fact that such control laws exhibit [9].

A large body of research in control literature concerns itself with the topic of FTS and H_∞ control in NSS. The finite-time H_∞ control problem of discrete-time NSS has been explored in [10,11] by using ADT method. The finite-time H_∞ control problem of continuous-time NSS with time-varying delays is discussed in [12] by using ADT method and delta operator approach. In applications, several generalized problems have been considered, such as H_∞ control problem [13], adaptive feedback control [14], FTS [15-17]. By using the back-stepping technique, FTS for triangular control systems described by retarded functional differential equations has been researched in [18]. However, there is unified ADT τ_a for each subsystem, hence, this leads to conservatism, and a systematic performance characterization for each subsystem is largely missing. From the perspective of energy attenuation, if the energy shows the attenuation trend, the whole system inclines to stay at the current subsystem, which may be described as the slow switching. Otherwise, the system will fleetly keep switching to other subsystems until the current subsystem exhibits the stable tendency in a finite-time interval, which may be described as the fast switching. Also, Wang *et al*. [19] investigated the non-fragile H_∞ synchronization issue for a class of discrete-time T-S fuzzy Markov jumps systems by persistent dwell-time (PDT) switching rule. As the PDT switching has been verified to be more flexible than DT and ADT switching mechanism, exploring an appropriate disposing method to deal with nonlinear singularly perturbed systems [20,21] subject to PDT switching in the discrete-time system is of great significance.

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The MDADT approach has been developed to extend the ADT method of NSS. MDADT approach showed that each subsystem has its own DT and does not need to satisfy a common DT. In addition, each subsystem operates independently on each other and will not be affected each other. For example, finite-time H_∞ control problem of NSS can be analyzed using MDADT approach [22-24]. The use of reverse MDADT approach from the NSS with unstable subsystems is introduced in [25], which showed significant improvements on the incremental stability and incremental H_∞ performance characterization. In addition, the reverse MDADT switching rule, as a kind of time-dependent switching rules, is used to manage the switching among nonlinear switched subsystems, and this rule is more general than DT and ADT switching rules. It could be concluded that the (reverse) MDADT switching has the flexibility advantage for FTIS systems, where the switching can or must be designed. To the best of the authors' knowledge, there exists few available literatures on NSS with reverse MDADT switched, which can be considered as the essential motivation of this work. Hence, inspired by on reverse MDADT approach, we analyze FTIS and incremental H_∞ performance index for NSS with finite-time unstable subsystems.

Motivated by the above observations, we study the finite-time incremental stability and incremental H_∞ control problems for NSS with unstable subsystems which is absent in the previous literature. The main contributions to our work may be expressed as follows: First, inspired by the idea of [24], the concept of finite-time incremental stability is proposed to NSS. Second, by introducing incremental Lyapunov function, some sufficient conditions are obtained for FTIS systems with finite-time unstable subsystems. The unstable subsystem involved, makes the finite-time incremental stability analysis much more difficult. Finally, based on reverse MDADT, it is concluded that the resulting nonlinear system is FTIB with a prescribed incremental H_∞ performance in the presence of actuator gain variations.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the NSS of the following form

$$\dot{x}(t) = f_{\sigma(t)}(x(t)),\ x(0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is the system states, $\sigma(t) : [0, \infty) \to \Gamma = \{1, 2, \ldots, m\}$ is the switching signal. For every $i \in \Gamma$, $f_i(\cdot)$ is continuous and locally Lipschitz. Throughout this paper, it is also assumed that, the system trajectory $x(t)$ does not jump at the switching instants. For the switching signal $\sigma(t)$, the switching sequence can be designed as follows: $\{x_i;\ (i_0,t_0),\ \cdots,\ (i_k,t_k),\ \cdots \mid i_k \in \Gamma,\ k = 1,\ 2,\ \cdots\}$, which implies that the subsystems i_k is activated on $t \in [t_k,\ t_{k+1})$.

Definition 1: Given a switching signal $\sigma(t) \in \Gamma$, constants $c_1 \geq 0$, $c_2 \geq 0$ satisfying $c_1 < c_2$, a positive definite matrix R and $T > 0$, for any two initial conditions x_0 and $\hat{x}_0$ corresponding to two solutions $x(t)$ and $\hat{x}(t)$ of system (1), the NSS in (1) is said to be FTIS with respect to (c_1, c_2, T, R, σ) if

$$\begin{aligned} &(x_0 - \hat{x}_0)^T R (x_0 - \hat{x}_0) \leq c_1 \\ &\Longrightarrow (x(t) - \hat{x}(t))^T R (x(t) - \hat{x}(t)) \leq c_2,\ \forall t \in (0,\ T]. \end{aligned}$$

Next, we review a definition [12] that will be used in the following.

Definition 2: For any $t \in (0,\ T_f)$, the switching number $N_\sigma(0,t)$ satisfied

$$\frac{T_i(0,t)}{\tau_{ai}} - N_{0i} \leq N_i(0,t) \leq N_{0i} + \frac{T_i(0,t)}{\tau_{ai}},\ \sigma(t) = i,$$

where $\tau_{ai} > 0$ and an integer $N_{0i} \geq 1$, then τ_{ai} is called reverse MDADT.

3. FINITE-TIME INCREMENTAL STABILITY ANALYSIS

Initially, we investigate the FTIS of system (1) with non-FTIS subsystems and give the following result.

Let $i \in \Gamma_u$ denote the i-th unstable subsystem. If $i \in \Gamma_s$, then the i-th subsystem is FTIS. Let $T^+[0,t)$ and $T^-[0,t)$ denote separatively the total activation time of non-FTIS subsystems and FTIS subsystems in $[0,\ t)$. If there exist Lyapunov functions $V_i(x,\hat{x})$, $\gamma_{i_1} \geq 0$, $\gamma_{i_2} \geq 0$, such that

$$\gamma_{i_1} \|x - \hat{x}\|^2 \leq V_i(x,\hat{x}) \leq \gamma_{i_2} \|x - \hat{x}\|^2, \tag{2}$$

$$\begin{aligned} \dot{V}_i(x,\hat{x}) &= \frac{\partial V_i(x,\hat{x})}{\partial x} f_i(x) + \frac{\partial V_i(x,\hat{x})}{\partial \hat{x}} f_i(\hat{x}) \\ &\leq \alpha_i V_i(x,\hat{x}),\ \alpha_i > 1,\ i \in \Gamma_u, \end{aligned} \tag{3}$$

$$\dot{V}_i(x,\hat{x}) \leq \beta_i V_i(x,\hat{x}),\ 0 < \beta_i < 1,\ i \in \Gamma_s. \tag{4}$$

The following theorem presents the sufficient condition of FTIS of system (1) with non-FTIS subsystems.

Theorem 1: Consider system (1), if exist multiple incremental Lyapunov functions $V_i(x,\hat{x})$ and scalar $\mu_i > 0$ such that

$$V_i(x,\hat{x}) \leq \mu_i V_j(x,\hat{x}),\ i \neq j, \tag{5}$$

$$T^+[0,t) \leq a_0,\ a_0 > 0, \tag{6}$$

and (2)-(4) hold, then system (1) is FTIS with MDADT τ_{ai} If $\mu_i \geq 1$, then

$$\begin{aligned} \tau_{ai} \geq T \ln \mu_i / \{&\ln[\gamma_1 c_2 \lambda_{min}(R)] - \ln[\gamma_2 c_1 \lambda_{max}(R)] \\ &- \beta T - (\alpha - \beta) a_0 - N_{0i} \ln \mu_i\}, \end{aligned} \tag{7}$$

if $0 < \mu_i < 1$, then

$$0 < \tau_{ai} < T \ln \mu_i / \{\ln[\gamma_1 c_2 \lambda_{min}(R)] - \ln[\gamma_2 c_1 \lambda_{max}(R)]$$

$$-\beta T-(\alpha-\beta)a_0-N_{0i}\ln\mu_i\}, \tag{8}$$

where $\alpha=\max_{\forall i\in\Gamma}(\alpha_i)$, $\beta=\max_{\forall i\in\Gamma}(\beta_i)$, $\gamma_1=\min_{\forall i\in\Gamma}(\gamma_{i_1})$, $\gamma_2=\min_{\forall i\in\Gamma}(\gamma_{i_2})$.

Proof: Choose the multiple incremental Lyapunov functions

$$V_i(x,\hat{x})=V_{\sigma(t)}(x(t_k),\hat{x}(t_k)),\ \sigma(t)=i.$$

When $t\in[t_k,t_{t+1})$, from (3) and (4), one has

$$\begin{aligned}&V_{\sigma(t)}(x(t),\hat{x}(t))\\&<e^{\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}\left(x(t_k),\hat{x}(t_k)\right)\sigma(t_k)\in\Gamma_u,\end{aligned} \tag{9}$$

$$\begin{aligned}&V_{\sigma(t)}(x(t),\hat{x}(t))\\&<e^{\beta_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}\left(x(t_k),\hat{x}(t_k)\right)\sigma(t_k)\in\Gamma_s.\end{aligned} \tag{10}$$

At the switching instant t_k, we have $\sigma(t_k)=i$, $\sigma(t_k^-)=j$, $i\neq j$.

Case 1: When $\mu_i\geq 1$, $t\in[t_k,t_{k+1})$, one has

$$V_{\sigma(t)}(x(t),\hat{x}(t))\leq\mu_iV_{\sigma(t_k^-)}(x(t_k^-),\hat{x}(t_k^-)). \tag{11}$$

It follows from (9) and (11) that

$$\begin{aligned}&V_{\sigma(t)}(x(t),\hat{x}(t))\\&\leq e^{\alpha(t-t_0)}\mu_{\sigma(t_k)}\mu_{\sigma(t_{k-1})}\cdots\mu_{\sigma(t_0)}V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0))\\&\leq e^{\alpha(t-t_0)}\prod_{i=1}^{m}\mu_i^{N_{0i}+\frac{T^+(0,t)}{\tau_{ai}}}V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0)),\end{aligned} \tag{12}$$

where $\sigma(t_k)\in\Gamma_u$.
From (10) and (11), we obtain

$$\begin{aligned}&V_{\sigma(t)}(x(t),\hat{x}(t))\\&\leq e^{\beta(t-t_0)}\prod_{i=1}^{m}\mu_i^{N_{0i}+\frac{T^-(0,t)}{\tau_{ai}}}V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0)),\end{aligned} \tag{13}$$

where $\sigma(t_k)\in\Gamma_s$.
For any $t\in(0,T)$, $T(0,t)=T^+(0,t)+T^-(0,t)$, using the iterative method and Definition 2, it is easy to see based on (12) and (13) that

$$\begin{aligned}&V_i(x,\hat{x})\\&\leq e^{\alpha T^+(0,t)+\beta(T-T^+(0,t))}\prod_{i=1}^{m}\mu_i^{N_{0i}+\frac{T(0,t)}{\tau_{ai}}}\\&\quad\times V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)\\&\leq e^{\alpha T^+(0,t)+\beta(T-T^+(0,t))}e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}\\&\quad\times V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right).\end{aligned} \tag{14}$$

From (2), we have

$$\frac{V_i(x,\hat{x})}{\gamma_{i_2}}\leq\|x-\hat{x}\|^2\leq\frac{V_i(x,\hat{x})}{\gamma_{i_1}}. \tag{15}$$

In terms of the inequalities (6), (7) and (12)-(15), one has

$$\begin{aligned}&(x-\hat{x})^TR(x-\hat{x})\\&\leq\lambda_{max}(R)\parallel x-\hat{x}\parallel^2\\&\leq\lambda_{max}(R)\frac{\gamma_2}{\gamma_1}e^{\alpha T^+(0,t)+\beta(T-T^+(0,t))}e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}\\&\quad\times\parallel x_0-\hat{x}_0\parallel^2\\&\leq\frac{\lambda_{max}(R)\gamma_2}{\lambda_{min}(R)\gamma_1}e^{\alpha T^+(0,t)+\beta(T-T^+(0,t))}e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}\\&\quad\times(x_0-\hat{x}_0)^TR(x_0-\hat{x}_0).\end{aligned} \tag{16}$$

Given positive constant c_1, we have

$$(x_0-\hat{x}_0)^TR(x_0-\hat{x}_0)\leq\frac{c_1}{m}\leq c_1. \tag{17}$$

From (16), we have

$$\begin{aligned}&(x-\hat{x})^TR(x-\hat{x})\\&\leq\frac{\lambda_{max}(R)\gamma_2}{\lambda_{min}(R)\gamma_1}e^{(\alpha-\beta)a_0+\beta T}e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}\frac{c_1}{m}.\end{aligned}$$

Due to $\alpha>0$, $0<\beta<1$, one has

$$e^{(m-1)(-1)[(\alpha-\beta)a_0+\beta T]}<1, \tag{18}$$

which implies

$$\begin{aligned}(x-\hat{x})^TR(x-\hat{x})&\leq\frac{mc_2}{c_1}e^{(m-1)(-1)[(\alpha-\beta)a_0+\beta T]}\frac{c_1}{m}\\&\leq c_2.\end{aligned}$$

Case 2: When $0<\mu_i<1$, we have

$$\begin{aligned}\frac{T\ln\mu_i}{\tau_{ai}}&<\ln[\gamma_1c_2\lambda_{min}(R)]-\ln[\gamma_2c_1\lambda_{max}(R)]\\&\quad-\beta T-(\alpha-\beta)a_0-N_{0i}\ln\mu_i.\end{aligned}$$

Furthermore, we can get

$$\begin{aligned}(x-\hat{x})^TR(x-\hat{x})&\leq\frac{mc_2}{c_1}e^{(m-1)(-1)[(\alpha-\beta)a_0+\beta T]}\frac{c_1}{m}\\&\leq c_2.\end{aligned}$$

Then, system (1) is FTIS. □

Remark 1: In the proof of Theorem 1, condition (6) is mainly used to obtain the FTIS of the whole system. Compared with literature [4], our theorems can more truly reflect the characteristics of the system itself by using to reverse MDADT. Each subsystem has its own DT and does not need to satisfy a common dwell time. In addition,each subsystem operates independently on each other and will not be affected each other. The reverse MDADT method is used to obtain FTIS of the whole system. This is introduced mainly to make sure that the whole system stays longer in the stable subsystem than in the unstable subsystem.

4. FINITE-TIME INCREMENTAL H_∞ CONTROL ANALYSIS

In this subsection, we study the following class of NSS:

$$\begin{cases}\dot{x}(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t))\omega(t),\ x(0) = x_0,\\ y(t) = h_{\sigma(t)}(x(t)),\end{cases} \tag{19}$$

where $\omega(t) \in R^m$ is the external disturbance input, $f_i(\cdot)$, $g_i(\cdot)$ and $h_i(\cdot)$ are continuous and locally Lipschitz. It is also assumed that, for any bounded $\omega(t)$ satisfied

$$\int_0^T \omega^T(t)\omega(t)dt \le d,\ d \ge 0,\ T \ge 0.$$

In addition, we further write $x(t)$ and $\hat{x}(t)$ to denote the two system trajectories under the external disturbance input $\omega(t)$ and $\hat{\omega}(t)$, respectively, where $\triangle\omega(t) = \omega(t) - \hat{\omega}(t)$, $\triangle x(t) = x(t) - \hat{x}(t)$, $\triangle y(t) = y(t) - \hat{y}(t)$.

Definition 3: Given a switching signal $\sigma(t) \in \Gamma$, constants $c_1 \ge 0$, $c_2 \ge 0$ satisfying $c_1 < c_2$, a positive definite matrix R and $T > 0$, for any two initial conditions x_0 and $\hat{x}_0$ corresponding to two solutions $x(t)$ and $\hat{x}(t)$ of system (1), the NSS in (1) is said to be FTIB with respect to $(c_1, c_2, T, R, \sigma, d)$ if

$$\begin{aligned}&(x_0 - \hat{x}_0)^T R(x_0 - \hat{x}_0) \le c_1\\ &\Longrightarrow (x(t) - \hat{x}(t))^T R(x(t) - \hat{x}(t)) \le c_2,\ \forall t \in (0,\ T],\\ &\qquad \forall \triangle\omega(t) : \int_0^T \triangle\omega^T(t)\triangle\omega(t)dt \le d.\end{aligned}$$

In what follows, finite-time incremental H_∞ control problem can be described as follows:

Given a constant $\gamma_0 > 0$, if the following conditions can be satisfied

1) System (19) is FTIB;
2) When $x_0 \neq \hat{x}_0$, incremental input $\triangle\omega(t)$ and incremental output $\triangle y(t)$ can be satisfied

$$\begin{aligned}&\int_0^T (e^{-s\xi} - e^{-T\xi}) \triangle y^T(s) \triangle y(s)ds\\ &\le \gamma_0^2 \int_0^T \triangle\omega^T(s) \triangle \omega(s)ds + c_0\|x_0 - \hat{x}_0\|^2,\end{aligned}$$

where ξ and c_0 constants.

If the functions $V_i(x,\hat{x})$ satisfied the following conditions

$$\gamma_{i_1}\|x - \hat{x}\|^2 \le V_i(x,\hat{x}) \le \gamma_{i_2}\|x - \hat{x}\|^2, \tag{20}$$
$$\dot{V}_i(x,\hat{x}) \le \alpha_i V_i(x,\hat{x}) - \phi(s), \alpha_i > 1,\ i \in \Gamma_u, \tag{21}$$
$$\dot{V}_i(x,\hat{x}) \le \beta_i V_i(x,\hat{x}) - \phi(s), 0 < \beta_i < 1, i \in \Gamma_s, \tag{22}$$

where

$$\phi(s) = \triangle y^T(s) \triangle y(s) - \gamma^2 \triangle \omega^T(s) \triangle \omega(s),\ \gamma > 0.$$

Remark 2: Condition (20) implies that $V_i(x,\hat{x})$ are nonnegative continuous radially unbounded functions. When both FTIS and non-FTIS coexist, inequalities (21) and (22) can be used to deal with the FTIS of the whole system. By resorting to the inequalities (21) and (22), the weight incremental disturbance attenuation property can be ensured for system (19).

Next, we will give the sufficient condition of FTIB and the performance index of incremental gain for system (1).

Theorem 2: Consider system (19), if exist multiple incremental Lyapunov functions $V_i(x,\hat{x})$ and $\mu_i > 0$ such that

$$V_i(x,\hat{x}) \le \mu_i V_j(x,\hat{x}), i \neq j, \tag{23}$$
$$T^+[0,t) \le a_0, a_0 > 0, \rho > 0, \tag{24}$$

(20)-(22) hold, and MDADT τ_{ai} satisfies

1) When $\mu_i \ge 1$,

$$\begin{aligned}\tau_{ai} \ge T\ln\mu_i/\{\{&\ln[\gamma_1 c_2 \lambda_{min}(R)]\\ &- \ln[\gamma_2 c_1 \lambda_{max}(R) e^{(\alpha-\beta)a_0+\beta T}\\ &+ \lambda_{max}(R)\lambda_{min}(R)\gamma^2 d e^{\alpha T}\gamma_1]\}/m - N_{0i}\ln\mu_i\}.\end{aligned} \tag{25}$$

2) When $0 < \mu_i < 1$,

$$\begin{aligned}0 &< \tau_{ai}\\ &< T\ln\mu_i/\{\{\ln[\gamma_1 c_2 \lambda_{min}(R)\\ &\quad - \lambda_{max}(R)\lambda_{min}(R)\gamma^2 d e^{\alpha T}]\\ &\quad - \ln[\gamma_2 c_1 \lambda_{max}(R) e^{(\alpha-\beta)a_0+\beta T}]\}/m - N_{0i}\ln\mu_i\},\end{aligned} \tag{26}$$

where $\alpha = \max_{\forall i\in\Gamma}(\alpha_i)$, $\beta = \max_{\forall i\in\Gamma}(\beta_i)$, $\gamma_1 = \min_{\forall i\in\Gamma}(\gamma_{i_1})$, $\gamma_2 = \min_{\forall i\in\Gamma}(\gamma_{i_2})$, then system (19) is FTIB and has an incremental H_∞ performance index $\gamma_0 > 0$.

Proof: When $t \in [t_k,\ t_{k+1})$, from (21) and (22), we can get

$$\begin{aligned}&\dot{V}_i(x,\hat{x}) \le \alpha_i V_i(x,\hat{x}) + \gamma^2 \triangle \omega^T(t) \triangle \omega(t),\\ &\qquad \alpha_i > 1,\ i \in \Gamma_u,\\ &\dot{V}_i(x,\hat{x}) \le \beta_i V_i(x,\hat{x}) + \gamma^2 \triangle \omega^T(t) \triangle \omega(t),\\ &\qquad 0 < \beta_i < 1,\ i \in \Gamma_s.\end{aligned}$$

This leads to

$$\begin{aligned}&V_{\sigma(t)}(x(t),\hat{x}(t))\\ &\le e^{\pi_{\sigma(t_k)}}(t - t_k)V_{\sigma(t_k)}(x(t_k),\hat{x}(t_k))\\ &\quad + \int_{t_s}^t e^{\pi_{\sigma(t_k)}(t-s)}\gamma^2 \triangle \omega^T(s) \triangle \omega(s)ds,\end{aligned} \tag{27}$$

where if $\sigma(t) \in \Gamma_u$, we have $\pi_{\sigma(t)} = \alpha_i$; if $\sigma(t) \in \Gamma_s$, we obtain $\pi_{\sigma(t)} = \beta_i$.

Case 1: When $\mu_i \geq 1$, $t \in [t_k, t_{k+1})$, from (23), one has

$$V_{\sigma(t)}\left(x(t),\hat{x}(t)\right) \leq \mu_i V_{\sigma(t_k^-)}\left(x(t_k^-),\hat{x}(t_k^-)\right). \tag{28}$$

This gives us

$$\begin{aligned}
&V_{\sigma(t)}\left(x(t),\hat{x}(t)\right)\\
&\leq e^{\pi_{\sigma(t_k)}(t-t_k)}\mu_i V_{\sigma(t_k^-)}\left(x(t_k^-),\hat{x}(t_k^-)\right)\\
&\quad+\int_{t_k}^{t} e^{\pi_{\sigma(t_k)}}(t-s)\gamma^2 \triangle\omega^T(s)\triangle\omega(s)ds.
\end{aligned} \tag{29}$$

When $t \in (0, T)$, it can be shown that

$$\begin{aligned}
&V_{\sigma(t)}\left(x(t),\hat{x}(t)\right)\\
&\leq e^{\pi_{\sigma(t_k)}(t-t_k)}\mu_{\sigma(t_k)}(e^{\pi_{\sigma(t_{k-1})}(t_k-t_{k-1})}\\
&\quad\times V_{\sigma(t_{k-1})}\left(x(t_{k-1}),\hat{x}(t_{k-1})\right)\\
&\quad+\int_{t_{k-1}}^{t_k} e^{\pi_{\sigma(t_{k-1})(t_k-s)}}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds)\\
&\quad+\int_{t_k}^{t} e^{\pi_{\sigma(t_k)(t-s)}}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\
&\leq \mu_{\sigma(t_k)}\mu_{\sigma(t_{k-1})}\mu_{\sigma(t_{k-2})}\cdots\mu_{\sigma(t_1)}\mu_{\sigma(t_0)}e^{\pi_{\sigma(t_k)}(t-t_k)}\\
&\quad\times e^{\pi_{\sigma(t_{k-1})}(t_k-t_{k-1})}\cdots e^{\pi_{\sigma(t_0)}(t_1-t_0)}V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)\\
&\quad+\mu_{\sigma(t_k)}\mu_{\sigma(t_{k-1})}\mu_{\sigma(t_{k-2})}\cdots\mu_{\sigma(t_1)}\mu_{\sigma(t_0)}\\
&\quad\times e^{\pi_{\sigma(t_k)}(t-t_k)}e^{\pi_{\sigma(t_{k-1})}(t_k-t_{k-1})}\cdots e^{\pi_{\sigma(t_1)}(t_2-t_1)}\\
&\quad\times\int_{t_0}^{t_1} e^{\pi_{\sigma(t_0)}(t_1-s)}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\
&\quad+\cdots+\mu_{\sigma(t_k)}e^{\pi_{\sigma(t_k)}(t-t_k)}\\
&\quad\times\int_{t_{k-1}}^{t_k} e^{\pi_{\sigma(t_{k-1})}(t_k-s)}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\
&\quad+\int_{t_k}^{t} e^{\pi_{\sigma(t_k)}(t-s)}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\
&< e^{\sum_{i=1}^{m}N_i(0,t)\ln\mu_i}e^{\alpha T^+[0,t)+\beta T^-[0,t)}V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)\\
&\quad+\int_{t_0}^{T} e^{\pi_{\sigma(t_k)}(t-s)}e^{\sum_{i=1}^{m}N_i(s,t)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds.
\end{aligned} \tag{30}$$

From $\forall\triangle\omega(t): \int_0^T\triangle\omega^T(s)\triangle\omega(s)ds \leq d,\ \alpha_i > 1,\ 1 > \beta_i > 0$, we have

$$\begin{aligned}
&\int_{t_0}^{T} e^{\pi_{\sigma(t_k)}(t-s)}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\
&=-\frac{1}{\pi_{\sigma(t_k)}}\left[e^{\pi_{\sigma(t_k)}(t-T)}-e^{\pi_{\sigma(t_k)}(t-t_0)}\right]d\gamma^2\\
&\leq\frac{1}{\pi_{\sigma(t_k)}}e^{\pi_{\sigma(t_k)}T}d\gamma^2\\
&\leq\frac{1}{\alpha}e^{\alpha T}d\gamma^2\leq e^{\alpha T}d\gamma^2.
\end{aligned}$$

From (30), one has

$$\begin{aligned}
&V_{\sigma(t)}\left(x(t),\hat{x}(t)\right)\\
&< e^{\sum_{i=1}^{m}N_i(0,t)\ln\mu_i}e^{\alpha T^+[0,t)+\beta T^-[0,t)}V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)\\
&\quad+\int_{t_0}^{T} e^{\pi_{\sigma(t_k)}(t-s)}e^{\sum_{i=1}^{m}N_i(s,t)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\
&\leq e^{(\alpha-\beta)a_0+\beta T}e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)\\
&\quad+e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}e^{\alpha T}d\gamma^2.
\end{aligned} \tag{31}$$

From (20), we can get

$$\frac{V_i(x,\hat{x})}{\gamma_{i_2}} \leq \| x-\hat{x} \|^2 \leq \frac{V_i(x,\hat{x})}{\gamma_{i_1}}. \tag{32}$$

Given constant c_1, we have $(x_0-\hat{x}_0)^T R(x_0-\hat{x}_0) \leq c_1$. Lastly, it follows from (28)-(32) that

$$\begin{aligned}
&(x-\hat{x})^T R(x-\hat{x})\\
&\leq\lambda_{max}(R)\parallel x-\hat{x}\parallel^2\\
&\leq\lambda_{max}(R)[e^{(\alpha-\beta)a_0+\beta T}e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}\\
&\quad\times V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)+e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}e^{\alpha T}d\gamma^2]/\gamma_1\\
&\leq\frac{\lambda_{max}(R)\gamma_2}{\lambda_{min}(R)\gamma_1}e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}e^{(\alpha-\beta)a_0+\beta T}c_1\\
&\quad+\lambda_{max}(R)e^{\sum_{i=1}^{m}\left(N_{0i}+\frac{T(0,t)}{\tau_{ai}}\right)\ln\mu_i}e^{\alpha T}d\gamma^2]/\gamma_1.
\end{aligned} \tag{33}$$

Substituting (25) into (33) yields $(x-\hat{x})^T R(x-\hat{x}) \leq c_2$.

Case 2: When $0 < \mu_i < 1$, we can get

$$\begin{aligned}
&V_{\sigma(t)}\left(x(t),\hat{x}(t)\right)\\
&< e^{\sum_{i=1}^{m}N_i(0,t)\ln\mu_i}e^{\alpha T^+[0,t)+\beta T^-[0,t)}V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)\\
&\quad+\int_{t_0}^{T} e^{\pi_{\sigma(t_k)}(t-s)}e^{\sum_{i=1}^{m}N_i(s,t)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\
&\leq e^{(\alpha-\beta)a_0+\beta T}e^{\sum_{i=1}^{m}\left(\frac{T(0,t)}{\tau_{ai}}-N_{0i}\right)\ln\mu_i}V_{\sigma(t_0)}\left(x(t_0),\hat{x}(t_0)\right)\\
&\quad+e^{\alpha T}d\gamma^2.
\end{aligned}$$

Thus, it has

$$\begin{aligned}
&(x-\hat{x})^T R(x-\hat{x})\\
&\leq\lambda_{max}(R)\parallel x-\hat{x}\parallel^2\\
&\leq\frac{\lambda_{max}(R)\gamma_2}{\lambda_{min}(R)\gamma_1}e^{\sum_{i=1}^{m}\left(\frac{T(0,t)}{\tau_{ai}}-N_{0i}\right)\ln\mu_i}e^{(\alpha-\beta)a_0+\beta T}c_1\\
&\quad+\lambda_{max}(R)e^{\alpha T}d\gamma^2/\gamma_1.
\end{aligned} \tag{34}$$

According to (26) and (34), one has $(x-\hat{x})^T R(x-\hat{x}) \leq c_2$.

Next, we give incremental H_∞ performance index. From (21), (22) and (31), one has

$$\begin{aligned}
&V_{\sigma(t)}\left(x(t),\hat{x}(t)\right)\\
&\leq e^{(\alpha-\beta)a_0+\beta T}e^{\sum_{i=1}^{m}N_i(0,t)\ln\mu_i}V_{\sigma(t_0)}\left((x(t_0),\hat{x}(t_0)\right)
\end{aligned}$$

$$-\int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m N_i(s,t)\ln\mu_i}\phi(s)ds,$$

where

$$\phi(s) = \triangle y^T(s)\triangle y(s) - \gamma^2\triangle\omega^T(s)\triangle\omega(s),\ \gamma > 0.$$

Due to $V_{\sigma(t)}(x(t),\hat{x}(t)) > 0$, it follows that

$$\begin{aligned}&\int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m N_i(s,t)\ln\mu_i}\triangle y^T(s)\triangle y(s)ds\\ &\le e^{(\alpha-\beta)a_0+\beta T} e^{\sum_{i=1}^m N_i(0,t)\ln\mu_i} V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0))\\ &\quad + \int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m N_i(s,t)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds.\end{aligned} \tag{35}$$

Case 3: When $\mu_i \ge 1$, we have

$$\begin{aligned}\tau_{ai} \ge\ & T\ln\mu_i/\{\{\ln[\gamma_1 c_2\lambda_{min}(R)] - \ln[\gamma_2 c_1\lambda_{max}(R)\\ &\times e^{(\alpha-\beta)a_0+\beta T} + \lambda_{max}(R)\lambda_{min}(R)\gamma^2 d e^{\alpha T}\gamma_1]\}/m\\ &- N_{0i}\ln\mu_i\}.\end{aligned}$$

Using (35), it further has

$$\begin{aligned}(\mathrm{I}) &= \int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m N_i(s,t)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\ &\le \int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m \left(N_{0i}+\frac{T(s,t)}{\tau_{ai}}\right)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\ &\le e^{\sum_{i=1}^m N_{0i}\ln\mu_i}\gamma^2\int_0^T e^{\left[\pi_{\sigma(t)}+\sum_{i=1}^m \frac{\ln\mu_i}{\tau_{ai}}\right](t-s)}\triangle\omega^T(s)\triangle\omega(s)ds.\end{aligned}$$

Let $m_1 = e^{\sum_{i=1}^m N_{0i}\ln\mu_i}$, $m_2 = \pi_{\sigma(t)} + \sum_{i=1}^m \frac{\ln\mu_i}{\tau_{ai}}$, we can obtain

$$\begin{aligned}&m_1\gamma^2\int_0^T\left(\int_s^T e^{m_2(t-s)}\triangle\omega^T(s)\triangle\omega(s)dt\right)ds\\ &= m_1\gamma^2\int_0^T \frac{1}{m_2}\left(e^{m_2(T-s)}-1\right)\triangle\omega^T(s)\triangle\omega(s)ds\\ &\le \frac{m_1\gamma^2}{m_2}(e^{m_2T}-1)\int_0^T\triangle\omega^T(s)\triangle\omega(s)ds.\end{aligned} \tag{36}$$

$$\begin{aligned}(\mathrm{II}) &= \int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m N_i(s,t)\ln\mu_i}\triangle y^T(s)\triangle y(s)ds\\ &\ge \int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m\left(\frac{T_i(s,t)}{\tau_{ai}}-N_{0i}\right)\ln\mu_i}\triangle y^T(s)\triangle y(s)ds\\ &\ge e^{-\sum_{i=1}^m N_{0i}\ln\mu_i}\int_0^T e^{\left[\pi_{\sigma(t)}+\sum_{i=1}^m\frac{\ln\mu_i}{\tau_{ai}}\right](t-s)}\triangle y^T(s)\triangle y(s)ds.\end{aligned}$$

Let $m_3 = e^{-\sum_{i=1}^m N_{0i}\ln\mu_i}$, it has

$$\begin{aligned}&m_3\int_0^T\left(\int_s^T e^{m_2(t-s)}\triangle y^T(s)\triangle y(s)dt\right)ds\\ &= \frac{m_3}{m_2}\int_0^T\left(e^{m_2(T-s)}-1\right)\triangle y^T(s)\triangle y(s)ds.\end{aligned} \tag{37}$$

Let $m_4 = e^{(\alpha-\beta)a_0+\beta T}e^{\sum_{i=1}^m N_i(0,t)\ln\mu_i}$, $m_5 = \sum_{i=1}^m \frac{\ln\mu_i}{\tau_{ai}}$, it has

$$\begin{aligned}&m_4\int_0^T e^{tm_5}V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0))dt\\ &= \frac{m_4}{m_5}\left(e^T-1\right)V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0)).\end{aligned} \tag{38}$$

Thus, we can get

$$\begin{aligned}&\int_0^T\left(e^{-sm_2}-e^{-Tm_2}\right)\triangle y^T(s)\triangle y(s)ds\\ &\le \gamma_0^2\int_0^T\triangle\omega^T(s)\triangle\omega(s)ds + c_0\|x_0-\hat{x}_0\|^2,\end{aligned}$$

where

$$\gamma_0^2 = \frac{m_1\gamma^2(e^{Tm_2}-1)}{m_3e^{Tm_2}}, c_0 = \frac{m_2m_4(e^T-1)}{m_3m_5e^{Tm_2}}.$$

Case 4: When $0 < \mu_i < 1$, it has

$$\begin{aligned}0 &< \tau_{ai}\\ &< T\ln\mu_i/\{\{\ln[\gamma_1 c_2\lambda_{min}(R)\\ &\quad - \lambda_{max}(R)\lambda_{min}(R)\gamma^2 d e^{\alpha T}]\\ &\quad - \ln[\gamma_2 c_1\lambda_{max}(R)e^{(\alpha-\beta)a_0+\beta T}]\}/m - N_{0i}\ln\mu_i\}.\end{aligned}$$

From (35), we have

$$\begin{aligned}&\int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m N_i(s,t)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\ &\le \int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m\left(\frac{T_i(s,t)}{\tau_{ai}}-N_{0i}\right)\ln\mu_i}\gamma^2\triangle\omega^T(s)\triangle\omega(s)ds\\ &\le e^{-\sum_{i=1}^m N_{0i}\ln\mu_i}\\ &\quad\times\gamma^2\int_0^T e^{\left(\pi_{\sigma(t)}+\sum_{i=1}^m\frac{\ln\mu_i}{\tau_{ai}}\right)(t-s)}\triangle\omega^T(s)\triangle\omega(s)ds.\end{aligned}$$

Let $G_1 = e^{-\sum_{i=1}^m N_{0i}\ln\mu_i}$, $G_2 = \pi_{\sigma(t)} + \sum_{i=1}^m\frac{\ln\mu_i}{\tau_{ai}}$, it has

$$\begin{aligned}&G_1\gamma^2\int_0^T\left(\int_s^T e^{G_2(t-s)}\triangle\omega^T(s)\triangle\omega(s)dt\right)ds\\ &= G_1\gamma^2\int_0^T\frac{1}{G_2}\left(e^{G_2(T-s)}-1\right)\triangle\omega^T(s)\triangle\omega(s)ds\\ &\le \frac{G_1\gamma^2}{G_2}\left(e^{G_2T}-1\right)\int_0^T\triangle\omega^T(s)\triangle\omega(s)ds.\end{aligned} \tag{39}$$

And we can further get

$$\begin{aligned}&\int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m N_i(s,t)\ln\mu_i}\triangle y^T(s)\triangle y(s)ds\\ &\ge \int_0^T e^{\pi_{\sigma(t)}(t-s)} e^{\sum_{i=1}^m\left(\frac{T(s,t)}{\tau_{ai}}+N_{0i}\right)\ln\mu_i}\triangle y^T(s)\triangle y(s)ds\end{aligned}$$

$$\geq e^{\sum_{i=1}^{m} N_{0i}\ln\mu_i}\int_0^T e^{\left[\pi_{\sigma(t)}+\sum_{i=1}^{m}\frac{\ln\mu_i}{\tau_{ai}}\right](t-s)}\triangle y^T(s)\triangle y(s)ds. \quad (40)$$

Let $G_3 = e^{\sum_{i=1}^{m} N_{0i}\ln\mu_i}$, it has

$$G_3\int_0^T\left(\int_s^T e^{G_2(t-s)}\triangle y^T(s)\triangle y(s)dt\right)ds = \frac{G_3}{G_2}\int_0^T\left(e^{G_2(T-s)}-1\right)\triangle y^T(s)\triangle y(s)ds.$$

Let $G_4 = e^{(\alpha-\beta)a_0+\beta T}e^{-\sum_{i=1}^{m} N_i(0,t)\ln\mu_i}$, $G_5 = \sum_{i=1}^{m}\frac{\ln\mu_i}{\tau_{ai}}$, it has

$$G_4\int_0^T e^{tG_5}V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0))\,dt = \frac{G_4}{G_5}\left(e^T-1\right)V_{\sigma(t_0)}(x(t_0),\hat{x}(t_0)). \quad (41)$$

Thus, we have

$$\int_0^T\left(e^{-sG_2}-e^{-TG_2}\right)\triangle y^T(s)\triangle y(s)ds \leq \gamma_0^2\int_0^T\triangle\omega^T(s)\triangle\omega(s)ds + c_0'\|x_0-\hat{x}_0\|^2,$$

where

$$\gamma_0^2 = \frac{G_1\gamma^2(e^{TG_2}-1)}{G_3e^{TG_2}},\ c_0' = \frac{G_2G_4(e^T-1)}{G_3G_5e^{TG_2}}.$$

Then, one has that

$$\int_0^T\left(e^{-s\xi}-e^{-T\xi}\right)\triangle y^T(s)\triangle y(s)ds \leq \gamma_0^2\int_0^T\triangle\omega^T(s)\triangle\omega(s)ds + c_0\|x_0-\hat{x}_0\|^2. \quad (42)$$

□

Remark 3: Inequality (42) describes the weighted interference attenuation level of from $\triangle\omega(s)$ to $\triangle y(s)$. By using the reverse MDADT method, the $\xi = \pi_{\sigma(t)} + \sum_{i=1}^{m}\frac{\ln\mu_i}{\tau_{ai}}$ can be obtained and the weighted term $e^{-s\xi}-e^{-T\xi}$ cannot be eliminated in (44).

5. EXAMPLE

In this section, the developed theory about FTIS will be demonstrated using a practical example system.

Consider the tunnel-diode circuit shown in Fig. 1, where the tunnel diode is characterized by $i_R = h(\upsilon_R)$. The energy-storing elements in this circuit are the capacitor C and the inductor L. Assuming they are linear and time invariant, we can model them by

$$i_C = C\frac{dv_C}{dt},\ \upsilon_L = L\frac{di_L}{dt},$$

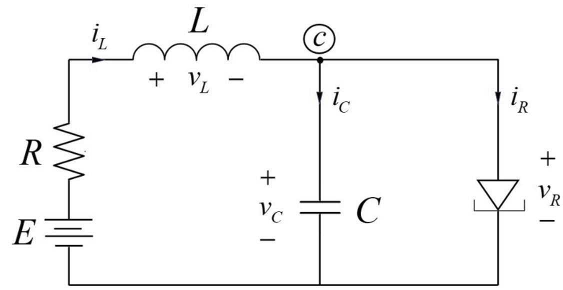

Fig. 1. The tunnel-diode circuit system.

where i and υ are the current through and the voltage across an element, with the subscript specifying the element. To write a state model for the system, let us take $x_1 = \upsilon_C$ and $x_2 = i_L$ as the state variables and $u = E$ as a constant input. To write the state equation for x_1, we need to express i_C as a function of the state variables x_1, x_2 and the input u. Using Kirchhoff's law, we have

$$i_C + i_R - i_L = 0,\ v_C - E + Ri_L + \upsilon_L = 0.$$

Hence, we can now write the state model for the circuit as

$$\dot{x}_1 = \frac{1}{C}[-h(x_1)+x_2],$$
$$\dot{x}_2 = \frac{1}{L}[-x_1-Rx_2+u].$$

By choosing different parameters, we can get the following two subsystems

$$f_1(x) = \begin{pmatrix}-2x_1-x_1^3\\ -3x_2\end{pmatrix}, \quad (43)$$
$$f_2(x) = \begin{pmatrix}x_1+x_2\\ -x_1+x_2\end{pmatrix}. \quad (44)$$

Choose the storage functions as follows:

$$V_1(x,\hat{x}) = \frac{1}{32}(x_1-\hat{x}_1)^2 + \frac{1}{32}(x_2-\hat{x}_2)^2, \quad (45)$$
$$V_2(x,\hat{x}) = 4(x_1-\hat{x}_1)^2 + 4(x_2-\hat{x}_2)^2. \quad (46)$$

Thus, we have

$$\begin{aligned}\dot{V}_1(x,\hat{x}) &\leq -\frac{1}{8}(x_1-\hat{x}_1)^2 - \frac{3}{16}(x_2-\hat{x}_2)^2 \\ &\quad -\frac{1}{16}(x_1-\hat{x}_1)^2\left(x_1^2+x_1\hat{x}_1+\hat{x}_1^2\right) \\ &\leq \frac{1}{2}[\frac{1}{32}(x_1-\hat{x}_1)^2 + \frac{1}{32}(x_2-\hat{x}_2)^2] \\ &\leq \frac{1}{2}V_1(x,\hat{x}), \end{aligned} \quad (47)$$

$$\begin{aligned}\dot{V}_2(x,\hat{x}) &= 8(x_1-\hat{x}_1)^2 + 8(x_2-\hat{x}_2)^2 \\ &\leq 2.1V_2(x,\hat{x}), \end{aligned} \quad (48)$$

where $\alpha = 2.1$, $\beta = 0.5$, $\gamma_1 = \gamma_2 = 2$. Let $c_1 = 1$, $c_2 = 128$, $T = 3$, $a_0 = 0.01$, $R = 1$, $t = 1$, $N_{0i} = 1$, $\mu = 2$,

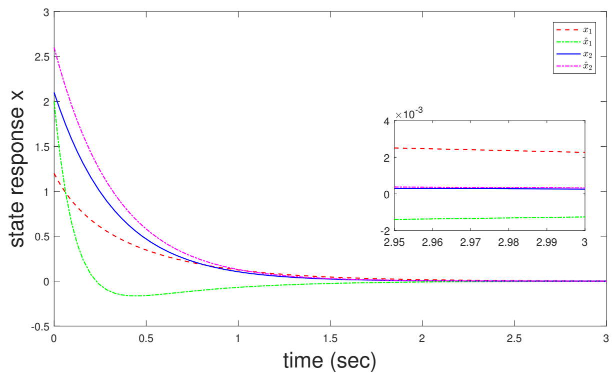

Fig. 2. The state response of subsystem (43).

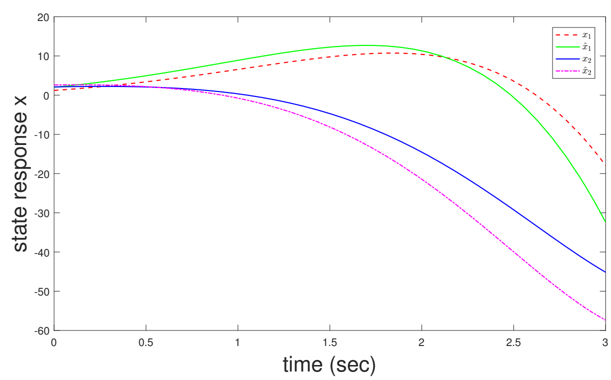

Fig. 3. The state response of subsystem (44).

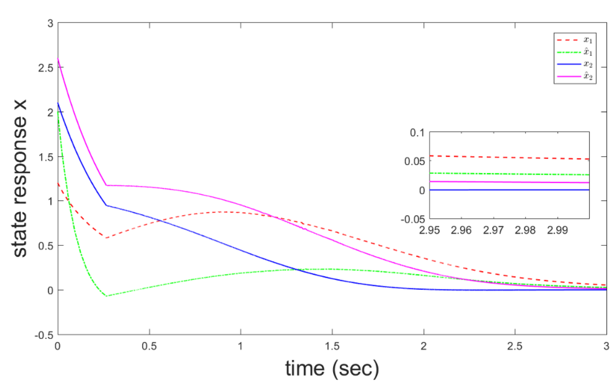

Fig. 4. The state response of subsystem (43) and (44).

$\sigma(t) \in \{1,2\}$, we can obtain the MDADT $\tau_{ai} \geq 0.7868$. Choosing the initial states $x_1(0) = 1.2$, $\hat{x}_1(0) = 2$, $x_2(0) = 2.1$, $\hat{x}_2(0) = 2.6$. Obviously, $(x_0 - \hat{x}_0)^T R(x_0 - \hat{x}_0) = \|x_0 - \hat{x}_0\|^2 = 0.89 < 1$.

The simulation results are depicted in Figs. 2-4. However, Fig. 2 shows the response $(x-\hat{x})^T R(x-\hat{x}) = \|x - \hat{x}\|^2 < 128$ of the subsystem (43), i.e., the subsystem (43) is FTIS. Moreover, Fig. 3 implies the response $(x-\hat{x})^T R(x-\hat{x}) = \|x-\hat{x}\|^2 = \|x_1 - \hat{x}_1\|^2 + \|x_2 - \hat{x}_2\|^2 \simeq 339 > 128$ of the subsystem (44), i.e., the subsystem (44) is FTI unstable. According to Theorem 1, Fig. 4 implies that the whole switched system satisfies $(x-\hat{x})^T R(x-\hat{x}) = \|x-\hat{x}\|^2 < 128$ in $[0, 3]$. In other words, the switched system (43) and (44) is FTIS.

Table 1. The change in MDADT τ_{ai} when $\mu = 2$.

α	1.1	1.1	1.2	1.5
β	0.1	0.6	0.1	0.5
$\tau_{ai} \geq$	0.540271	0.883409	0.540412	0.785026

Table 2. The change in MDADT τ_{ai} when $\mu = \frac{1}{2}$.

α	500	500	800	1000
β	0.4	0.8	0.4	0.99
$0 < \tau_{ai} <$	3.195098	1.125956	0.569582	0.28044

In fact, when $\mu = 2$, Table 1 shows that if α and β are increasing, then the value range of MDADT τ_{ai} is reduced. In addition, when $\mu = \frac{1}{2}$, Table 2 shows that the interval range of MDADT τ_{ai} is also gradually reduced along with α and β. Different parameters α and β lead to different convergence rates of the system. It is not difficult to see that the total MDADT of FTI unstable subsystems are very conservative to some extent by virtue of the method proposed to this note. When the system is running, the MDADT of the whole system in the unstable subsystem is not always less than a_0. However, this article always assumes that the MDADT in the unstable subsystem is less than a_0. Hence, how to reduce the conservatism is our future work.

6. CONCLUSION

This mote solves the FTIS problem of NSS with unstable subsystems. New FTIS results are developed. By using the reverse MDADT approach, a sufficient condition has been obtained for NSS to be FTIB with incremental H_∞ performance index. For nonlinear switched systems with unstable systems, the reverse MDADT is designed to achieve finite-time incremental stability regulation.

There exists several potential future research directions arising from this paper: i) One possible research direction is to extend these results to time-varying vector fields, in order to cover time-varying parameters and time-varying delays. ii) Based on the proposed reverse MDADT, stochastic finite-time incremental stability will be future studied for stochastic switched system.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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