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Fuzzy Controller Design for Discrete-time T-S Fuzzy Systems with Partially Unknown Membership Functions

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Abstract: This paper is concerned with the controller design problem for discrete-time T-S fuzzy systems with partially unknown membership functions. If the membership functions are partially unknown, then the existing stabilization conditions which are based on the parallel distributed compensator (PDC) strategy cannot be applied. To tackle this problem, a new type of fuzzy controller is proposed to close the feedback loop. Based on this new type fuzzy controller, some sufficient stabilization conditions, including membership-function-dependent and independent conditions, are given in the form of LMIs. Finally, two examples are given to illustrate the the effectiveness of the proposed fuzzy controller design approaches.

Keywords: Fuzzy controller, linear matrix inequality, T-S fuzzy systems, unknown membership functions.

1. INTRODUCTION

Takagi-Sugeno (T-S) fuzzy model provides an effective way in representing a complex nonlinear system [1]. It has been applied to various industrial fields. There are many approaches and classical results in the area of stability and performance conditions of fuzzy control systems, and most of the significant results are obtained by using the linear matrix inequality (LMI) approach. For example, the stability analysis of fuzzy control systems has been discussed in [2-4]; the H_∞ controller design problems have been addressed in [3,5-7].

For study on T-S fuzzy control systems, the stability analysis can be seen as a primary work. Many methods have been proposed to reduce the conservatism in stability analysis. As author knows, these proposed methods include but not limited to quadratic Lyapunov function methods [2-4], piecewise Lyapunov function methods [5], fuzzy Lyapunov function approaches [6,7]. Most of these existing results are membership-function-independent.

An effective alternative approach on this topic is the membership-function-dependent or with the knowledge of membership functions. Membership-function-dependent means the control-condition is relative to the membership functions of the fuzzy systems. It has been shown the information of membership functions can be used to reduce the conservativeness in stability analysis [11-13]. First, the boundary information of member-ship functions is cleverly introduced into stability analysis in [14]. By dividing the premise variable operating domain into connected sub-domain, the local boundary information of the membership functions was employed in [15]. The idea of these literatures is to characterize the membership functions by their upper-(or lower-)bounds information, and combine the bounds information into the analysis derivation. Moreover, a parameterized Lyapunov function-based systematic method with a slack matrix technique can be found in [16]. Another idea is to approximate the complex membership functions by some specific simple functions. In this way, the original complex membership functions can be symbolized by the specific parameters of those simple functions, making them easier to be applied. Some results have focused on the Membership-Function-Approximation approach, which can more effectively introduce membership functions information into stability analysis [17-29].

In practice, it is not the case that membership functions are always known, for example, they may contain immeasurable premise variables, or unknown perturbations, or uncertain parameters. In this case, i.e., there are unknown or partially unknown elements in membership functions, the method mentioned earlier are no longer available or restrict the conservatism-reduction effects. Literatures [8-10] consider the fuzzy control issue with unknown or partially unknown membership functions. A switching mechanism is introduced and further modified to construct control gains. The gains are switched depending on the upper and lower bounds of membership functions.

Almost all the aforementioned results about stabil-

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ity analysis or performance controller design problem do not consider the case that the membership functions are unknown or partially unknown. To the best of our knowledge, few research efforts have been devoted to the membership-function-dependent stability analysis for discrete-time T-S fuzzy systems (DFS) with unknown or partially membership functions.

Motivated by the above discussion, this paper is concerned with the fuzzy controller design problem for DFS with partially unknown membership functions. We assume the membership functions are partially unknown, which refer to the ones with unknown parameters or unknown variables. The traditional parallel distribution compensation (PDC) methods are very popular and useful in practical research, thus, not as the switching design method proposed in [8-10], in this paper, the state feedback controller is investigated based on a new type construction similar to the parallel distributed compensation strategy to stabilize the discrete-time T-S fuzzy control systems.

As the main contribution of this paper, a new type of fuzzy controller is given based on the construction of membership functions. The upper and lower bounds of partially unknown membership functions are used to construct the membership functions of the fuzzy controller, to be specific, the membership functions of the fuzzy controller are the convex combination of the known elements and the bounds of partially unknown membership functions of fuzzy system. The convex combination coefficient can be obtained by the solution of linear equations. And the stabilization conditions based on the the new tape fuzzy controller are derived. The main result in this paper is dependent on the upper and lower bounds of unknown membership functions, and it is shown that all the conditions are formulated in terms of LMIs which guarantee the closed-loop fuzzy systems to be quadratic stable. Furthermore, we can use the stabilization conditions obtained in this paper to complete the design problems of H_∞ controllers and filters, and the method in this paper is also applicable to the related problems of time-delay systems. And illustrative examples provided in Section 4 demonstrate the correctness and effectiveness of the proposed methods.

This paper is organized as follows: The model description and problem statement are described in Section 2. The results of fuzzy controller design for the discrete-time T-S fuzzy system with partially unknown membership functions are obtained in Section 3. In Section 4, simulation examples are provided to show the effectiveness of the results in this paper. Finally, we conclude this paper in Section 5.

Notation: The superscript T stands for matrix transpositions; R^n denotes the n-dimensional Euclidean space. In block symmetric matrices expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry. A block diagonal matrix with matrices $X_1, X_2, \cdots, X_n$ on its main diagonal is denoted as $diag\{X_1, X_2, \cdots, X_n\}$.

2. PRELIMINARIES

Discrete-time T-S fuzzy system with partially unknown membership functions is considered and a new type fuzzy controller based on the strategy similar to the PDC is proposed to close the feedback loop to form the fuzzy control system.

2.1. System model

T-S fuzzy model is described by fuzzy IF-THEN rules, whose collection represents the approximation of a nonlinear system. The ith rule of a discrete-time fuzzy model is of the following form.

Plant Rule i: IF $\xi_1(t)$ is $\mu_{i1}, \cdots$, and $\xi_s(t)$ is μ_{is}, THEN

$$x(t+1) = A_i x(t) + B_i u(t), \tag{1}$$

where $\xi_i(t)$ $(i = 1, 2, \cdots, s)$ is the premise variable, μ_{ij} $(i = 1, 2, \cdots, r,\ j = 1, 2, \cdots, s)$ is the fuzzy set, r is the number of IF-THEN rules, and s is the number of the premise variables. $x(t) \in R^n$ is the sate-space vector, $u(t) \in R^m$ is the input vector. The matrices A_i, B_i are known constant matrices of appropriate dimensions. Given a pair of $(x(t), u(t))$, the final output of the fuzzy system is inferred as follows:

$$x(t+1) = \sum_{i=1}^{r} \lambda_i(\xi(t))[A_i x(t) + B_i u(t)], \tag{2}$$

where $\lambda_i(\xi(t)) = \frac{\omega_i(\xi(t))}{\sum_{i=1}^{r}\omega_i(\xi(t))}$, $\omega_i(\xi(t)) = \Pi_{j=1}^{s}\mu_{ij}(\xi_j(t))$, $\mu_{ij}(\xi_j(t))$ is the grade of membership of $\xi_j(t)$ in μ_{ij}, and $\omega_i(\xi(t))$ represents the weight of the ith rule. It is easy to check that

$$\lambda_i(\xi(t)) \geq 0,\ i = 1, 2, \cdots, r,$$
$$\sum_{i=1}^{r} \lambda_i(\xi(t)) = 1. \tag{3}$$

In this paper, we always assume

$$\lambda_i(\xi(t)) = l_i(\xi(t)) + h_i(\xi(t)), \tag{4}$$

where $l_i(\xi(t))$ and $h_i(\xi(t))$ are used to represent the known and unknown elements of the membership functions. In addition, suppose that $\overline{h}_i \geq h_i(\xi(t)) \geq \underline{h}_i$, where $\overline{h}_i$ and $\underline{h}_i$ are known upper and lower bounds of $h_i(\xi(t))$.

Hereafter, for convenience, we let $\lambda_i(t) := \lambda_i(\xi(t))$, $h_i(t) := h_i(\xi(t))$, $l_i(t) := l_i(\xi(t))$.

Remark 1: By (4), if $h_i(t) = 0$, $l_i(t) \neq 0$, then the membership functions of the fuzzy system (2) are known; if $h_i(t) \neq 0$, $l_i(t) \neq 0$, then the membership functions of the fuzzy system (2) are partially unknown; if $h_i(t) \neq 0$, $l_i(t) = 0$, then the membership functions of the fuzzy system (2) are completely unknown. In this paper, we always assume $l_i(t) \neq 0$.

2.2. The construction of fuzzy controller

In this paper, the membership functions include unknown parameters, thus, the existing controller and H_∞ performance design approaches in [2-4] which focus on the T-S fuzzy systems with known membership functions can not be applied as the explicit expression of $h_i(t)$ with unknown parameters is unavailable [8]. This fact motivates us to design fuzzy or nonlinear controllers.

Because we assume the unknown parts in membership functions are bounded, i.e., $\underline{h}_i \le h_i(t) \le \bar{h}_i$, we can utilize the convex combination of upper and lower bounds of the partial unknown membership functions to design fuzzy controller.

Define

$$\hat{\lambda}_i(t) = l_i(t) + \hat{h}_i, \tag{5}$$

where

$$\hat{h}_i = \alpha \underline{h}_i + \beta \bar{h}_i, \tag{6}$$

and

$$\alpha + \beta = 1,\ \alpha \ge 0,\ \beta \ge 0. \tag{7}$$

Functions $\hat{\lambda}_i(t)$ $(i = 1, 2, \cdots, r)$ will be used as the membership function of fuzzy controller, then, the functions $\hat{\lambda}_i(t)$ $(i = 1, 2, \cdots, r)$ must meet the following:

$$\hat{\lambda}_i(t) \ge 0\ (i = 1,\ 2,\ \cdots,\ r),\ \sum_{i=1}^{r} \hat{\lambda}_i(t) = 1.$$

By the express of $\hat{\lambda}_i(t)$ and the constraint (7) on parameters α and β, we have $\hat{\lambda}_i(t) \ge 0$. Thus, we have to choose the right parameters α and β so that

$$\sum_{i=1}^{r} \hat{\lambda}_i(t) = \sum_{i=1}^{r} (l_i(t) + \hat{h}_i) = 1 \tag{8}$$

is satisfied.

Due to the all the unknown parameters or variables only exist in the functions $h_i(t)$ and the membership functions of fuzzy system satisfy $\sum_{i=1}^{r}(l_i(t) + h_i(t)) = 1$, we have

$$\sum_{i=1}^{r} h_i(t) = 0,\ \sum_{i=1}^{r} l_i(t) = 1.$$

Thus, by (6), (8), and with the above two equations, we have

$$\sum_{i=1}^{r} \hat{h}_i = \sum_{i=1}^{r} (\alpha \underline{h}_i + \beta \bar{h}_i) = 0. \tag{9}$$

From (6), (7) and (9), we have the following equation that the non-negative parameters α, β must be satisfied

$$\begin{bmatrix} \sum_{i=1}^{r} \underline{h}_i & \sum_{i=1}^{r} \bar{h}_i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{10}$$

Obviously, the matrix $\begin{bmatrix} \sum_{i=1}^{r} \underline{h}_i & \sum_{i=1}^{r} \bar{h}_i \\ 1 & 1 \end{bmatrix}$ is invertible matrix, therefore, we obtain the unique solution of parameters α, β

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sum_{i=1}^{r} \bar{h}_i - \sum_{i=1}^{r} \underline{h}_i} \begin{bmatrix} \sum_{i=1}^{r} \bar{h}_i \\ -\sum_{i=1}^{r} \underline{h}_i \end{bmatrix}. \tag{11}$$

By (9), we know that

$$\sum_{i=1}^{r} \bar{h}_i \ge 0,\ \sum_{i=1}^{r} \underline{h}_i \le 0. \tag{12}$$

The inequalities (12) ensures $\alpha \ge 0$, $\beta \ge 0$.

Therefore, the solution in the form of can be used as the convex combination parameters. Moreover, the most important of all is that the convex combination parameters α, β are existed and unique. Thus, we can use the bounded of unknown part of membership functions to construct fuzzy controller.

In summary, we have succeed in constructing the functions $\hat{\lambda}_i(t)$ which are dependent on the upper and lower bounds of partially unknown membership functions and satisfy the same properties as the membership functions of the system

$$\hat{\lambda}_i(t) \ge 0,\ \sum_{i=1}^{r} \hat{\lambda}_i(t) = 1. \tag{13}$$

Moreover, functions $\hat{\lambda}_i(t)$ include the known part of membership functions $\lambda_i(t)$ and the information of unknown part in membership functions. The above analysis result is summarized as the following lemma.

Lemma 1: For the unknown elements $h_i(t)$ in membership functions which satisfy $\underline{h}_i \le h_i(t) \le \bar{h}_i$, there exist uniquely determined parameters α and β such that

$$\sum_{i=1}^{r} (l_i(t) + \hat{h}_i(t)) = 1, \tag{14}$$

where

$$\hat{h}_i = \alpha \underline{h}_i + \beta \bar{h}_i,\ \alpha + \beta = 1,\ \alpha \ge 0,\ \beta \ge 0.$$

Based on Lemma 1, We can design the following fuzzy state feed-back controller:

$$u(t) = \sum_{i=1}^{r} \hat{\lambda}_i(t) F_i x(t) = \sum_{i=1}^{r} (l_i(t) + \hat{h}_i(t)) F_i x(t), \tag{15}$$

where $\hat{h}_i(t)$ $(i = 1, 2, \cdots, r)$ are in the form of (6), F_i are controller-gain to be designed.

Combining (2) and (15), the closed-loop system can be described by

$$x(t+1) = \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i(t) \hat{\lambda}_j(t) (A_i + B_i F_j) x(t). \tag{16}$$

Definition 1 [2]**:** The fuzzy system (2) is said to be quadratically stabilizable if there exists a controller as in (15) such that the closed-loop system (16) is quadratically stable.

3. MAIN RESULTS

In this section, two kinds of stabilization conditions are presented. One is membership-function-independent stabilization conditions. The other one is membership-function-dependent condition. The latter one is more relaxed than the prior in some cases, although it is difficult to conclude that the membership-function-dependent condition can get less conservative the independent ones from the point of theory of view. In the subsequent discussion, the origin $x = 0$ is assumed to be the only equilibrium point of the fuzzy control system.

3.1. Membership-function-independent stabilization conditions

Theorem 1: The equilibrium of the DFS of (2) is quadratically stabilizable via the fuzzy controller (15) if there exist matrices $Q > 0$, M_i $(i = 1, 2, \cdots, r)$ such that the following linear matrices inequalities hold

$$\begin{bmatrix} -Q & QA_i^T + M_j^T B_i^T F_j \\ A_iQ + B_iM_j & -Q \end{bmatrix} < 0. \tag{17}$$

Moreover, the fuzzy feedback gains are $F_j = M_jQ^{-1}$ $(j = 1, 2, \cdots, r)$.

Proof: Consider a candidate of quadratic function $V(x(t)) = x^T(t)Px(t)$, where $P > 0$. The equilibrium of DFS of (16) is quadratically stable if

$$\begin{aligned} &V(x(t+1)) - V(x(t)) \\ &= x^T(t)\Big\{\sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_i(t)\hat{\lambda}_j(t)(A_i + B_iF_j)^T \\ &\quad \times P(A_i + B_iF_j) - P\Big\}x(t) < 0 \ (\forall x(t) \neq 0), \end{aligned} \tag{18}$$

or if

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_i(t)\hat{\lambda}_j(t)(A_i + B_iF_j)^T P(A_i + B_iF_j) - P < 0, \tag{19}$$

which is equivalent to

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_i(t)\hat{\lambda}_j(t)\begin{bmatrix} -P & (A_i + B_iF_j)^T P \\ P(A_i + B_iF_j) & -P \end{bmatrix} < 0. \tag{20}$$

Let $Q = P^{-1}$, $M_j = F_jQ$. Premultiply and postmultiply the left-hand side of (20) by $diag\{Q, Q\}$, we have

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_i(t)\hat{\lambda}_j(t)\begin{bmatrix} -Q & QA_i^T + M_j^T B_i^T F_j \\ A_iQ + B_iM_j & -Q \end{bmatrix} < 0. \tag{21}$$

Thus, we get

$$\begin{bmatrix} -Q & QA_i^T + M_j^T B_i^T F_j \\ A_iQ + B_iM_j & -Q \end{bmatrix} < 0. \tag{22}$$

□

Remark 2: Based on the new type fuzzy controller (15), we obtained the stabilization condition described as Theorem 1. From the proof of Theorem 1, it easily can be seen that although the membership functions include unknown elements, the stabilization condition is as the same as the one when the membership functions are complete known, because of the information of membership functions not involved in the stabilization conditions. Therefore, we can conclude that all the existing membership-function-independent stabilization conditions within the membership functions completely known are all available for the case that the system is with partially unknown membership functions via fuzzy controller (15) proposed in this paper. Therefore, we can readily obtain the following conclusions.

Theorem 2: The equilibrium of the DFS of (2) is quadratically stabilizable via the fuzzy controller (15) if there exist matrices $Q > 0$, M_i $(i = 1, 2, \cdots)$ and symmetric matrices Z_{ij} $(i, j = 1, 2, \cdots, r)$ such that the following linear matrices inequalities hold

$$\begin{bmatrix} -Q - Z_{ii} & Q_{ij}^T \\ Q_{ij} & -Q \end{bmatrix} < 0 \ (i = 1, 2, \cdots, r), \tag{23}$$

$$\begin{bmatrix} -Q - Z_{ij} & Q_{ij}^T \\ Q_{ij} & -Q \end{bmatrix} < 0 \ (i < j), \tag{24}$$

$$\begin{bmatrix} -Q - Z_{ij} & Q_{ij}^T \\ Q_{ij} & -Q \end{bmatrix} < 0 \ (i > j), \tag{25}$$

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1r} \\ Z_{12} & Z_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Z_{1r} & Z_{2r} & \cdots & Z_{rr} \end{bmatrix} < 0, \tag{26}$$

where $Q_{ij} = A_iQ + B_iM_j$.

Proof: The proving process could refer to [2]. □

Theorem 3: The equilibrium of the DFS of (2) is quadratically stabilizable via the fuzzy controller (15) if there exist matrices $Q > 0$, M_i, Z_{ii} $(i = 1, 2, \cdots)$ and $Z_{ij} = Z_{ji}^T$ $(i \neq j)$ such that the following linear matrices inequalities hold

$$\begin{bmatrix} -Q - Z_{ii} & Q_{ii}^T \\ Q_{ii} & -Q \end{bmatrix} < 0 \ (i = 1, 2, \cdots, r), \tag{27}$$

$$\begin{bmatrix} -Q - Z_{ij} & Q_{ij}^T \\ Q_{ij} & -Q \end{bmatrix} < 0 \ (i < j), \tag{28}$$

$$\begin{bmatrix} -Q - Z_{ij} & Q_{ij}^T \\ Q_{ij} & -Q \end{bmatrix} < 0 \ (i > j), \tag{29}$$

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1r} \\ Z_{21} & Z_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1} & Z_{r2} & \cdots & Z_{rr} \end{bmatrix} < 0, \tag{30}$$

where $Q_{ij} = A_iQ + B_iM_j$.

Proof: The proving process could refer to [3]. □

Remark 3: Based on Lemma 1, a new type of fuzzy controller in form of (15) is proposed in this paper. And, some membership-function-independent stabilization conditions are derived via Lyapunov- Function method. As the statement in [3], the stabilization condition in Theorem 3 is less conservative than the conditions in Theorems 1 and 2.

3.2. Membership-function-dependent condition

In this section, membership-function-dependent stabilization condition is derived via the bounds of partially unknown membership functions.

Theorem 4: The equilibrium of the DFS of (2) is quadratically stabilizable via the fuzzy controller (15) if there exist matrices $Q > 0$, M_i, R_{ij}, Z_{ii} $(i = 1, 2, \cdots)$ and $Z_{ij} = Z_{ji}^T (i \neq j)$ such that the following linear matrices inequalities hold

$$\begin{bmatrix} -Q-R_{ii}+\Pi_{ii}-Z_{ii} & Q_{ii}^T \\ Q_{ii} & -Q \end{bmatrix} < 0 \ (i = 1,\ 2,\ \cdots,\ r), \tag{31}$$

$$\begin{bmatrix} -Q-R_{ij}+\Pi_{ij}-Z_{ij} & Q_{ij}^T \\ Q_{ij} & -Q \end{bmatrix} < 0 \ (i < j), \tag{32}$$

$$\begin{bmatrix} -Q-R_{ij}+\Pi_{ij}-Z_{ij} & Q_{ji}^T \\ Q_{ij} & -Q \end{bmatrix} < 0 \ (i > j), \tag{33}$$

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1r} \\ Z_{21} & Z_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1} & Z_{r2} & \cdots & Z_{rr} \end{bmatrix} < 0, \tag{34}$$

where $Q_{ij} = A_iQ + B_iM_j$, $\Pi_{ij} = -R_{ij} + \sum_{l=1}^r \bar{\bar{\lambda}}_l R_{il} - \sum_{k=1}^r \Delta_k R_{kj} + \sum_{k=1}^r \sum_{l=1}^r \Delta_k \bar{\bar{\lambda}}_l R_{kl}$, $\Delta_k = \frac{\bar{h}_k - \underline{h}_k}{2}$.

Proof: Recalling the properties that $\lambda_i(t) \geq 0$, $\underline{h}_i \leq h_i(t) \leq \bar{h}_i$. And we assume $\hat{\lambda}_i(t) \leq \bar{\bar{\lambda}}_i (\leq 1)$. Thus, for any matrix $E_{ij} \geq 0$, we have the following inequalities

$$(\lambda_i(t) + \Delta_i)(-\hat{\lambda}_j + \bar{\bar{\lambda}}_j)E_{ij} \geq 0,\ \forall\ i,\ j = 1,\ 2,\ \cdots,\ r,$$

where $\Delta_i = \frac{\bar{h}_i - \underline{h}_i}{2}$.

Therefore,

$$\sum_{i=1}^r \sum_{j=1}^r (\lambda_i(t) + \Delta_i)(-\hat{\lambda}_j + \bar{\bar{\lambda}}_j)E_{ij} \geq 0.$$

Expanding the left-hand of above inequality, by $\sum_{i=1}^r \lambda_i(t) = \sum_{j=1}^r \hat{\lambda}_j(t) = 1$, we obtain

$$\sum_{i=1}^r \sum_{j=1}^r \lambda_i(t)\hat{\lambda}_j(t)\Big(- E_{ij} + \sum_{l=1}^r \bar{\bar{\lambda}}_l E_{il} - \sum_{k=1}^r \Delta_k E_{kj} + \sum_{k=1}^r \sum_{l=1}^r \Delta_k \bar{\bar{\lambda}}_l E_{kl}\Big) \geq 0. \tag{35}$$

Consider a candidate of quadratic Lyapunov function $V(x(t)) = x^T(t)Px(t)$, where $P > 0$. Following the similar line of the proof of Theorem 1, we have

$$\begin{aligned} &V(x(t+1)) - V(x(t)) \\ &= x^T(t)\Big\{\sum_{i=1}^r \sum_{j=1}^r \lambda_i(t)\hat{\lambda}_j(t)[(A_i + B_iF_j)^T \\ &\quad \times P(A_i + B_iF_j) - P]\Big\}x(t) \\ &\leq x^T(t)\Big\{\sum_{i=1}^r \sum_{j=1}^r \lambda_i(t)\hat{\lambda}_j(t)[W_{ij}^T PW_{ij} - E_{ij} \\ &\quad + \Lambda_{ij} - P]\Big\}x(t) \\ &= x^T(t)\sum_{i=1}^r \lambda_i(t)\hat{\lambda}_i(t)\big[W_{ii}^T PW_{ii} - E_{ii} + \Lambda_{ii}c - P\big]x(t) \\ &\quad + x^T(t)\sum_{i=1}^{r-1} \sum_{i<j}^{r} \lambda_i(t)\hat{\lambda}_i(t)\big[W_{ij}^T PW_{ij} - E_{ij} + \Lambda_{ij} \\ &\quad - P\big]x(t) \\ &\quad + x^T(t)\sum_{i=1}^{r} \sum_{i>j}^{r-1} \lambda_i(t)\hat{\lambda}_i(t)\big[W_{ij}^T PW_{ij} - E_{ij} + \Lambda_{ij} \\ &\quad - P\big]x(t), \end{aligned}$$

where $W_{ij} = A_i + B_iF_j$, $\Lambda_{ij} = \sum_{l=1}^r \bar{\bar{\lambda}}_l E_{il} - \sum_{k=1}^r \Delta_k E_{kj} + \sum_{k=1}^r \sum_{l=1}^r \Delta_k \bar{\bar{\lambda}}_l E_{kl}$.

If the following

$$W_{ii}^T PW_{ii} - E_{ii} + \Lambda_{ii} - P < X_{ii}\ (i = 1,\ 2,\ \cdots,\ r), \tag{36}$$

$$W_{ij}^T PW_{ij} - E_{ij} + \Lambda_{ij} - P < X_{ij}\ (i < j), \tag{37}$$

$$W_{ij}^T PW_{ij} - E_{ij} + \Lambda_{ij} - P < X_{ij}^T\ (i > j), \tag{38}$$

$$\begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{1r} & \cdots & X_{rr} \end{bmatrix} < 0 \tag{39}$$

hold, we have

$$\begin{aligned} &V(x(t+1)) - V(x(t)) \\ &< \begin{bmatrix} \lambda_1(t)x(t) \\ \vdots \\ \lambda_r(t)x(t) \end{bmatrix}^T \begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{1r} & \cdots & X_{rr} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \lambda_1(t)x(t) \\ \vdots \\ \lambda_r(t)x(t) \end{bmatrix} < 0. \end{aligned} \tag{40}$$

Thus, the equilibrium of DFS of (16) is quadratically stable.

Let $Q=P^{-1}$, $Q_{ij}=QW_{ij}Q$, $R_{ij}=QE_{ij}Q$, $\Pi_{ij}=Q\Lambda_{ij}Q$. Premultiply and postmultiply (37), (38) and (39) by Q, and according to Schur Complement, the inequalities (31), (32) and (33) are obtained. Premultiply and postmultiply (40) with $diag\{Q,Q,\cdots,Q\}$, we get the inequality (34).□

Remark 4: The membership-function-dependent stabilization condition is derived via the bounds of membership functions. It is difficult to prove that the membership-function-dependent condition is more relaxed than the independent ones from the theory point of view. However, the proposed examples in Section 4 show that the membership-function-dependent condition is less conservative than the independent ones in some cases.

3.3. Algorithms

Based on Theorems 3 and 4, the following algorithms are provided to stabilize the fuzzy system with partially unknown membership functions.

Algorithm: For the given T-S fuzzy system (2) with partially unknown membership functions, a feasible solution to the fuzzy controller (15) is obtained by solving the following problem:

Step 1: According to the given T-S fuzzy system with partially unknown membership functions, calculate the upper and lower bounds of unknown membership functions $h_i(t)$, and obtain the value of Δ_i.

Step 2: The convex combination parameters α, β can be obtained by solving (10). And the upper bounds of $\hat{\lambda}_i(t)$ can also be obtained.

Step 3: A feasible solution to the fuzzy controller (15) is obtained by solving the LMIs (31), (32), (33), (34) in Theorem 4 (or LMIs (27), (28), (29), (30) in Theorem 3).

Then, the controller gains can be constructed as $F_i=M_iQ^{-1}$.

4. NUMERICAL EXAMPLES

4.1. Membership functions with unknown variables

Consider the follow discrete-time fuzzy system

$$x(t+1)=\sum_{i=1}^{2}\lambda_i[A_ix(t)+B_iu(t)], \tag{41}$$

where

$$A_1=\begin{bmatrix}0.1+2\sigma^2 & 0.6\\ 1 & 1\end{bmatrix},\ A_2=\begin{bmatrix}0.1 & 0.6\\ 1 & 1\end{bmatrix},$$
$$B_1=\begin{bmatrix}1\\1\end{bmatrix},\ B_2=\begin{bmatrix}1\\1\end{bmatrix}.$$

And

$$\lambda_1(x(t))=1-\frac{1}{2}\frac{x_1^2(t)}{\sigma^2}-\frac{1}{2}\frac{x_2^2(t)}{\sigma^2},$$
$$\lambda_2(x(t))=\frac{1}{2}\frac{x_1^2(t)}{\sigma^2}+\frac{1}{2}\frac{x_2^2(t)}{\sigma^2}, \tag{42}$$

with $x_1(t), x_2(t)\in[0,\sigma]$.

From (42), we have

$$l_1(t)=1-\frac{1}{2}\frac{x_2^2(t)}{\sigma^2},\ h_1(t)=\frac{1}{2}\frac{x_1^2(t)}{\sigma^2},$$
$$l_2(t)=\frac{1}{2}\frac{x_2^2(t)}{\sigma^2},\ h_2(t)=\frac{1}{2}\frac{x_1^2(t)}{\sigma^2}. \tag{43}$$

$h_1(t)$ and $h_2(t)$ are the unknown elements of membership functions $\lambda_i(x(t))$, which satisfy $-0.5\le h_1(t)\le 0$, $0\le h_2(t)\le 0.5$.

By Lemma 1, we can get $\alpha=\beta=0.5$, thus

$$\hat{\lambda}_1(t)=l_1(t)+\hat{h}_1=1-\frac{1}{2}\frac{x_2^2(t)}{\sigma^2}-\frac{1}{4},$$
$$\hat{\lambda}_2(t)=l_2(t)+\hat{h}_2=\frac{1}{2}\frac{x_2^2(t)}{\sigma^2}+\frac{1}{4}, \tag{44}$$

and

$$\Delta_1=\frac{1}{4},\ \Delta_2=\frac{1}{4},\ \bar{\hat{\lambda}}_1(t)=\frac{3}{4},\ \bar{\hat{\lambda}}_2(t)=\frac{3}{4}. \tag{45}$$

In addition, suppose $\sigma=1$. After implementing the LMIs in Theorems 3 and 4, we find there exists feasible solution. The results of Theorem 4 are as the following

$$tmin=-0.0713,$$
$$Q=10^3*\begin{bmatrix}2.4865 & 0.1348\\ 0.13487 & 0.02058\end{bmatrix},$$
$$M_1=10^3*\begin{bmatrix}-3.7967\\ -0.3464\end{bmatrix}^T,\ M_2=10^3*\begin{bmatrix}-3.8671\\ -0.3506\end{bmatrix}^T,$$
$$F_1=\begin{bmatrix}-0.9483\\ -10.6749\end{bmatrix}^T,\ F_2=\begin{bmatrix}-0.9751\\ -10.7021\end{bmatrix}^T.$$

4.2. Membership functions with unknown parameters

Consider a DFS of (2) with

$$A_1=\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix},\ A_2=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix},$$
$$B_1=\begin{bmatrix}1\\2\end{bmatrix},\ B_2=\begin{bmatrix}1\\1\end{bmatrix},$$

and the membership functions are assumed to be in the form

$$\lambda_1(t)=l_1(t)+h_1(t)=\frac{1}{4}x_1^2+\frac{1}{4}\delta^2(x_1(t)),$$
$$\lambda_2(t)=l_2(t)+h_2(t)=1-\frac{1}{4}x_1^2-\frac{1}{4}\delta^2(x_1(t)), \tag{46}$$

where $\delta(x_1(t))$ is an unknown parameter which satisfies $0\le\delta(x_1(t))\le 1$ with $x_1(t)\in[0,1]$.

From (46), we have

$$\alpha=\beta=\frac{1}{2},\ \hat{h}_1=\frac{1}{8},\ \hat{h}_2=-\frac{1}{8},$$

$$\hat{\lambda}_1(t) = \frac{1}{4}x_1^2 + \frac{1}{8},$$
$$\hat{\lambda}_2(t) = 1 - \frac{1}{4}x_1^2 - \frac{1}{8}, \tag{47}$$

thus we can get

$$\Delta_1 = \Delta_2 = \frac{1}{8},$$
$$\bar{\bar{\lambda}}_1(t) = \frac{3}{8},\ \bar{\bar{\lambda}}_2(t) = \frac{7}{8}. \tag{48}$$

In this case, we find there is not infeasible solution to the LMIs in Theorem 3. After implementing the LMIs in Theorem 4, the results are

$$tmin = -0.0093,$$
$$Q = \begin{bmatrix} 66.3569 & -19.8357 \\ -19.8357 & 12.4737 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} -55.0399 \\ 19.2910 \end{bmatrix}^T,\ M_2 = \begin{bmatrix} -62.4035 \\ 22.2404 \end{bmatrix}^T,$$
$$F_1 = \begin{bmatrix} -0.6998 & 0.4336 \end{bmatrix},\ F_2 = \begin{bmatrix} -0.7766 & 0.5479 \end{bmatrix}. \tag{49}$$

From Figs. 1 and 2, it can be seen that the fuzzy controller with gains (49) stabilizes the open-loop sysetm.

Remark 5: In fact, another feasible controller design scheme for T-S fuzzy systems with unknown partially unknown membership functions is construct a controller with fixed gains.

$$u(t) = Fx(t). \tag{50}$$

In this case, we also find there is no feasible solution to stabilize the fuzzy system.

From the proposed examples in Section 4, it can be seen that the stabilization conditions presented in this paper are effective. Moreover, the existing controller design approaches which focus on the fuzzy systems cannot be applied as the explicit expression of $\delta(t)$ is unavailable.

It should be point out that the computational burdens are not increase to implement the fuzzy controller. In addition, from the theory point of view, it is difficult to prove that the membership-function-dependent conditions can get less conservative than the independent ones. However, the proposed examples illustrate that the membership-function-dependent condition usually is less conservative than the independent ones in some cases.

5. CONCLUSION

This paper has addressed the fuzzy controller design problem for a class of discrete-time T-S fuzzy systems with partially unknown membership functions. Membership-function-dependent and membership-function-independent stabilization conditions based on fuzzy controller design strategy have been given, respectively. The design methods proposed in this paper, can also be applied in the issues such as the H_∞ controller design and filter design for fuzzy systems with partially membership functions. Finally, two examples have been given to illustrate the effectiveness of the proposed approaches.

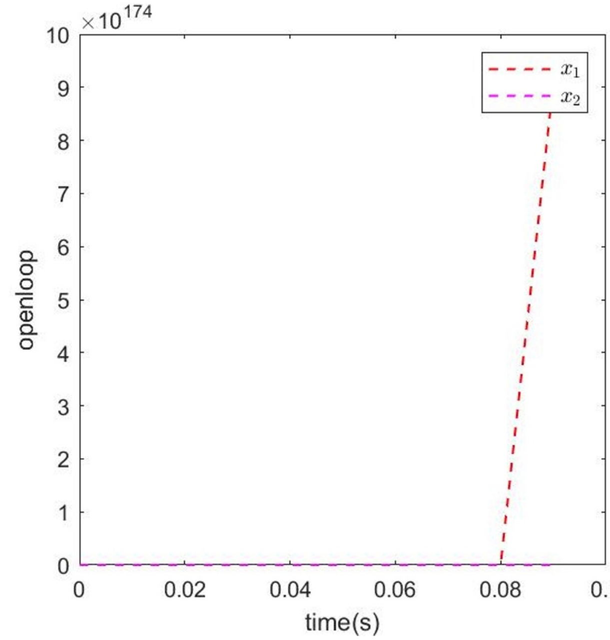

Fig. 1. State response of open-loop system with $0 \le \delta(x_1(t)) \le 1$.

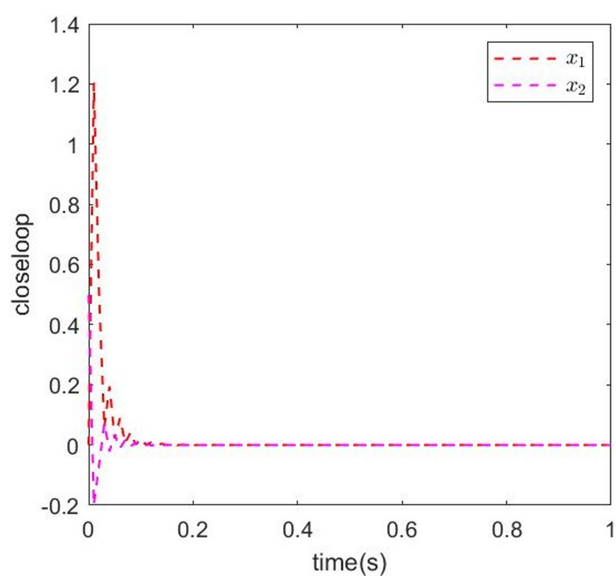

Fig. 2. State response of closed-loop system with $0 \le \delta(x_1(t)) \le 1$.

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