International Journal of Control, Automation and Systems 20(5) (2022) 1723-1734
http://dx.doi.org/10.1007/s12555-021-0392-6
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Quasi-projective Synchronization for Caputo Type Fractional-order Complex-valued Neural Networks with Mixed Delays

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Abstract: Without decomposing the complex-valued systems into two real-valued subsystems, this paper investigates quasi-projective synchronization (QPS) problem for Caputo type fractional-order complex-valued neural networks (FOCVNNs) with mixed delays by choosing suitable controllers. To realize QPS, the linear feedback controller and adaptive feedback controller are designed, by constructing suitable Lyapunov function, utilizing the fractional Razumikhin theorem and the properties of Mittag-Leffler function and inequality technique, and several sufficient criteria for QPS of FOCVNNs with mixed delays are derived. In addition, the upper bound of the error of QPS is estimated. Finally, two numerical examples are simulated to verify the effectiveness and feasibility of the proposed results.

Keywords: Adaptive feedback control, fractional Razumikhin theorem, fractional-order complex-valued neural networks, linear feedback control, mixed delays, quasi-projection synchronization.

1. INTRODUCTION

In 1696, Leibniz extended the integral calculus in the general sense to obtain fractional calculus. In recent decades, in the wake of developments in science, fractional calculus has been generally applied to many scientific fields, such as viscoelastic materials [1,2], biology [3–5], molecular diffusion theory [6,7], and image processing [8,9]. The main advantages of fractional calculus are infinite memory and more freedom. Therefore, it is very meaningful to study fractional order systems.

The occurrence of synchronization is a collective behavior, due to its applications in signal processing [10], secure communication [11], image encryption [12] and so on. So far, many types of synchronization have been studied, such as complete synchronization [13,14], phase synchronization [15,16], exponential synchronization [17, 18]. As we all know, most networks can not realize synchronization only by themselves, and some effective control strategies have been proposed including impulsive control [19], intermittent control [20], adaptive control [21–23]. In synchronization schemes, the feature of projective synchronization is that it can be synchronized in proportion. The hybrid controllers are adopted in [24,25] to discuss the projective synchronization. Evidently, the complex controllers are inconvenient and undesirable in the practical applications.

With the development of modern science and technology, mathematician Pitts and neurologist Mcculloch first proposed the concept of artificial neural network (ANN) in 1943 [26]. ANN is a network system with parallel computing capability in which many processing units are connected to each other according to a certain topological structure [27–32]. The advantage of NNs is that they can process continuous analog signals and chaotic, incomplete information [33,34]. The research results show that the fractional calculus model can more accurately describe the dynamic behavior of the actual systems [1–6], and is beneficial to describe the memory and genetic properties of neurons [13,14].

As is known to all, there are numerous types of delay, such as time-varying delay [35,36], discrete delay [37], distributed delay [38], leakage delay [39–41], etc. The occurrence of these delays usually causes oscillation, bifurcation and instability of the power system. Therefore, the study of dynamic systems with time delays has become a hot topic in the theoretical and application fields. However, to the best of our knowledge, the problem of QPS for FOCVNNs with mixed delays has not been found in the existing literature.

In the practical applications, NNs are related to complex signals. In order to solve this problem, scholars have

Manuscript received May 8, 2021; revised June 30, 2021; accepted July 18, 2021. Recommended by Associate Editor Ohmin Kwon under the direction of Editor Jessie (Ju H.) Park. This work was jointly supported by the Natural Science Foundation of Anhui Province (1908085MA01), the Natural Science Foundation of the Higher Education Institutions of Anhui Province (KJ2019A0573, KJ2019A0557), and the Top Young Talents Program of Higher Learning Institutions of Anhui Province (gxyq2019048).

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proposed CVNNs. The two feasible approaches have been used to analyze the dynamical behaviors of CVNNs: one is to handle the considered FOCVNNs as a compact entirety [42–44], the other is to decompose the CVNNs into two RVNNs [45–47]. In this paper, we use complex function theory to discuss the stability or synchronization of FOCVNNs without decomposing complex-valued systems into two real-valued systems, which not only greatly decrease the difficulty of theoretical analysis, but also reduce the complexity of derived results.

It is well known that the Lyapunov direct method is the most effective method to analyze the stability of system in [48], where observer-based chaos synchronization using Legendre polynomials has been presented and applied to secure communications. In [49], a secure communication system based on chaos synchronization using brain emotional learning-based intelligent controller is presented.

By the aforementioned discussions, the main objective of this paper is to investigate the problem of QPS of Caputo type FOCVNNs with mixed delays. By constructing an appropriate Lyapunov function, using the fractional Razumikhin theorem, the properties of the Mittag-Leffler function, and some inequality analysis techniques, using linear feedback controllers and adaptive feedback controllers, several sufficient criteria ensuring the QPS for the concerned network models are derived. The main contributions of this paper can be summarized as follows:

- In the existing QPS literature, the FOCVNNs models almost focus on discrete delays. In this paper, we consider the FOCVNNs with mixed delays including discrete delay and the distributed delay, thus the considered model in this article is more general and less conservative.
- Without decomposing the CVNNs into two RVNNs for analysis, the CVNNs is handled directly as a whole, which greatly reduces the difficulty of theoretical analysis and the complexity of calculation.
- Adopting the linear feedback controller and adaptive feedback controller respectively, two algebraic criteria of QPS of the FOCVNNs with mixed delays are obtained, which are easy to check and judge the synchronization. Moreover, the upper bound of synchronization error is estimated. The comparison between the two cases of controllers is presented by numerical simulation.

2. PRELIMINARIES AND MODEL DESCRIPTION

This section introduces some definitions, lemmas and Caputo type FOCVNNs with mixed delays.

Definition 1 [50]**:** The fractional integral of order p for a function $m(t) \in C[[0,+\infty),R]$ is defined as

$$_0D_t^{-p}m(t) = \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}m(s)ds,$$

where $p > 0$, and $\Gamma(\cdot)$ is the gamma function defined as

$$\Gamma(p) = \int_0^{+\infty} e^{-t}t^{p-1}dt.$$

Definition 2 [50]**:** The Caputo fractional derivative of order p for a function $m(t) \in C[[0,+\infty),R]$ is defined as

$${}_{t_0}^{C}D_t^p f(t) = \frac{1}{\Gamma(k-p)}\int_{t_0}^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p-k+1}}d\tau,$$

where $t \ge t_0$, and k is a positive integer such that $k-1 < p < k$.

Definition 3 [50]**:** The two-parameter Mittag-Leffler function is defined as

$$E_{\upsilon,\omega}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\upsilon+\omega)},\ \upsilon > 0,\ \omega > 0,\ z \in \mathbb{C}.$$

The one- parameters Mittag-Leffler function is defined as

$$E_\upsilon = E_{\upsilon,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\upsilon+1)},\ \upsilon > 0,\ z \in \mathbb{C}.$$

Lemma 1 [39]**:** The following inequality holds if $m(t) \in \mathbb{C}$ is a continuous analytic function, then

$${}_{t_0}^{C}D_t^p m(t)\overline{m(t)} \le m(t){}_{t_0}^{C}D_t^p\overline{m(t)} + \overline{m(t)}{}_{t_0}^{C}D_t^p m(t),$$

where $t \ge t_0,\ 0 < p < 1$.

Lemma 2 [39]**:** For any two complex numbers η and μ, the inequality holds

$$\eta\overline{\mu} + \overline{\eta}\mu \le \gamma\eta\overline{\eta} + \frac{1}{\gamma}\mu\overline{\mu},$$

where $\gamma > 0$.

Lemma 3 [42]**:** Let $t \ge t_0$, then $E_p(\varpi(t-t_0)^p)$ is monotonically non-increasing and $0 \le E_p(\varpi(t-t_0)^p) \le 1$, where $\varpi \le 0$.

Lemma 4 [51]**:** Let $U(t)$ and $V(t)$ be two nonnegative continuous functions, and satisfy

$${}_{t_0}^{C}D_t^p(V(t)+U(t)) \le -\sigma V(t) + \rho,$$

where $0 < p < 1$, $\sigma > 0$, $\rho > 0$, then

$$V(t) \le (V(t_0)+U(t_0)-\frac{\rho}{\sigma})E_p(-\sigma(t-t_0)^p) + \frac{\rho}{\sigma},$$

where $t \ge t_0 + \left(\frac{\Gamma(p)}{\sigma}\right)^{\frac{1}{1-p}}$.

Lemma 5 [53]**:** For nondecreasing and differentiable function $m(t)$ on $t \in [t_0,+\infty)$, then

$${}_{t_0}^{C}D_t^p(m(t)-\lambda)^2 \le 2(m(t)-\lambda){}_{t_0}^{C}D_t^p m(t),$$

where $0 < p < 1$, λ is any constant.

Lemma 6 [42]**:** For $\forall\beta \in \mathbb{C}$, the following inequality holds:

$$\beta+\overline{\beta} \leq 2|\beta|,$$

where $\mathbb{C}$ denotes complex field.

In this article, we discuss a class of FOCVNNs with mixed delays as follows:

$$\begin{aligned}{}_{t_0}^{C}D_t^p x_i(t) =& -d_i x_i(t)+\sum_{k=1}^{n} a_{ik} f_k(x_k(t)) \\ &+\sum_{k=1}^{n} b_{ik} g_k(x_k(t-\tau_1)) \\ &+\sum_{k=1}^{n} m_{ik}\int_{t-\tau_2}^{t} h_k(x_k(s))ds+J_i(t), \qquad (1)\end{aligned}$$

where $0<p<1$, $x_i(t)=(x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbb{C}^n$ represents the state variable of the ith neuron, $f(x)$, $g(x)$, $h(x)$ denote the activation functions without and with delay respectively, $d_i \in \mathbb{C}$ denote the connection weight, a_{ik}, b_{ik}, $m_{ik} \in \mathbb{C}$ respectively expression the connection weight of the kth neuron to the ith neuron at time t, $t-\tau_1$ and $t-\tau_2$, respectively, where τ_1, τ_2 is non-negative constant transmission delay, J_i is bias. The initial state with system (1) is

$$x_i(s)=\phi_i(s),\ t\in[-\gamma,0),\ \gamma\in\max\{\tau_1,\tau_2\},$$

where $\|\phi(t)\| = \sup_{s\in[-\gamma,0]} \|\phi(s)\|$, $i=1, 2, \cdots, n$. To investigate the synchronization, the response system is

$$\begin{aligned}{}_{t_0}^{C}D_t^p y_i(t) =& -d_i y_i(t)+\sum_{k=1}^{n} a_{ik} f_k(y_k(t)) \\ &+\sum_{k=1}^{n} b_{ik} g_k(y_k(t-\tau_1)) \\ &+\sum_{k=1}^{n} m_{ik}\int_{t-\tau_2}^{t} h_k(y_k(s))ds \\ &+J_i(t)+v_i(t), \qquad (2)\end{aligned}$$

where $y_i(t)=(y_{i1}(t), y_{i2}(t), \cdots, y_{in}(t))^T \in \mathbb{C}^n$ represents the state variable of the response system (2), others are the same as for the drive system (1), $v_i(t)$ is the controller. The initial state with system (2) is

$$y_i(s)=\varphi_i(s),\ t\in[-\gamma,0),\ \gamma\in\max\{\tau_1,\tau_2\},$$

where $\|\phi(t)\| = \sup_{s\in[-\gamma,0]} \|\phi(s)\|,\ i=1,2,\cdots,n.$

Definition 4 [51]**:** System (1) and system (2) achieve QPS, provided that for any initial values $x_i(s)=\phi_i(s)\in\mathbb{C}$, $y_i(s)=\varphi_i(s)\in\mathbb{C}$, there exists a small error bound $\delta>0$ such that

$$\lim_{t\to\infty}|y_i(t)-\alpha x_i(t)|\leq\delta,$$

where $t\geq t_0$ and $\alpha\in\mathbb{C}$ is the projective coefficient. Especially, systems (1) and (2) achieve quasi-complete synchronized if $\alpha=1$ and achieve quasi-anti synchronized if $\alpha=-1$.

Let the synchronization error between system (1) and system (2) be $u_i(t)=y_i(t)-\alpha x_i(t)$, then

$$\begin{aligned}&{}_{t_0}^{C}D_t^p u_i(t) \\ &= -d_i u_i(t)+\sum_{k=1}^{n} a_{ik}\left[f_k(y_k(t))-f_k(\alpha x_k(t))\right] \\ &\quad+\sum_{k=1}^{n} a_{ik}\left[f_k(\alpha x_k(t))-\alpha f_k(x_k(t))\right] \\ &\quad+\sum_{k=1}^{n} b_{ik}\left[g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))\right] \\ &\quad+\sum_{k=1}^{n} b_{ik}\left[g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))\right] \\ &\quad+\sum_{k=1}^{n} m_{ik}\int_{t-\tau_2}^{t}[h_k(y_k(s))-h_k(\alpha x_k(s))]ds \\ &\quad+\sum_{k=1}^{n} m_{ik}\int_{t-\tau_2}^{t}[h_k(\alpha x_k(s))-\alpha h_k(x_k(s))]ds \\ &\quad+(1-\alpha)J_i(t)+v_i(t). \qquad (3)\end{aligned}$$

The initial state with system (3) is

$$u_i(s)=\psi_i(s)=\varphi_i(s)-\phi_i(s),\ t\in[-\gamma,0),$$

where $\gamma\in\max\{\tau_1,\tau_2\}$, $i=1,\cdots,n$.

Assumption 1: Assume that $f(x), g(x)$ and $h(x)$ are activation functions, p, $q\in\mathbb{C}$ such that

$$\begin{aligned}&|f(p)-f(q)|\leq\Lambda_1|p-q|, \\ &|g(p)-g(q)|\leq\Lambda_2|p-q|, \\ &|h(p)-h(q)|\leq\Lambda_3|p-q|,\end{aligned}$$

where Λ_1, Λ_2, $\Lambda_3>0$ are Lipschitz constants.

Assumption 2: For any $\theta\in\mathbb{C}$, the real numbers l_1, l_2, l_3, $l_4>0$ exist such that

$$|f(\theta)|\leq l_1,\ |g(\theta)|\leq l_2,\ |h(\theta)|\leq l_3,\ |J(\theta)|\leq l_4.$$

3. MAIN RESULTS

In this section, we mainly construct the Lyapunov functions, by using the fractional Razumikhin theorem, the properties of Mittag-Leffler function and some skills of inequality. The sufficient criteria of QPS for Caputo type FONNs with mixed delays are derived under the linear feedback controller and adaptive feedback controller respectively.

The linear feedback controller is proposed as follows:

$$v_i(t)=-k_i(y_i(t)-\alpha x_i(t)), \qquad (4)$$

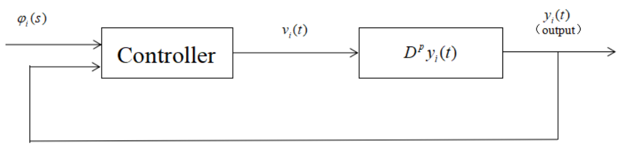

Fig. 1. Control framework of linear feedback controller.

where $k_i \in \mathbb{C}$ is the gain. Fig. 1 represents the control framework of linear feedback controller.

For convenience, we introduce the following denotations

$$\begin{aligned}
\varpi &= \min_{1\le i\le n}\Big\{d_i+k_i+\overline{d_i}+\overline{k_i}-l_4^2-n\Lambda_1^2-n\Lambda_3^2\tau_2^2\\
&\quad-\sum_{k=1}^{n}2\left(a_{ik}\overline{a_{ik}}+b_{ik}\overline{b_{ik}}+m_{ik}\overline{m_{ik}}\right)\Big\},\\
\vartheta &= 2\left(1+\alpha\overline{\alpha}\right)\left(l_1^2+l_2^2+l_3^2\tau_2^2\right)+(1-\alpha)\overline{(1-\alpha)},\\
\eta &= \max_{1\le i\le n}\Lambda_2^2\sum_{k=1}^{n}\left(b_{ki}\overline{b_{ki}}\right)^{\frac{1}{2}},\\
\mu &= \min_{1\le i\le n}\Big\{d_i+\overline{d_i}+2k_i^*-l_4-2\sum_{k=1}^{n}\Big[\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}\\
&\quad+\left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}}+\left(m_{ik}\overline{m_{ik}}\right)^{\frac{1}{2}}\Big]-\Lambda_1^2\sum_{k=1}^{n}\left(a_{ki}\overline{a_{ki}}\right)^{\frac{1}{2}}\\
&\quad-\Lambda_3^2\tau_2^2\sum_{k=1}^{n}\left(m_{ki}\overline{m_{ki}}\right)^{\frac{1}{2}}\Big\},\\
\delta &= 2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(1+\alpha\overline{\alpha}\right)\Big[\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}l_1^2+\left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}}l_2^2\\
&\quad+\left(m_{ik}\overline{m_{ik}}\right)^{\frac{1}{2}}l_3^2\tau_2^2\Big]+l_4(1-\alpha)\overline{(1-\alpha)}.
\end{aligned}$$

Theorem 1: Under Assumptions 1 and 2, if k_i satisfies the inequality $\varpi > n\Lambda_2^2\varsigma,\ \varsigma > 1$, then the drive system (1) is QPS with the response system (2) under the linear feedback controller (4). In addition, the error bound can be estimated by $\sqrt{\frac{\vartheta}{\varpi-n\Lambda_2^2\varsigma}}$.

Proof: Consider the following Lyapunov function

$$V_1(t)=\sum_{i=1}^{n}u_i(t)\overline{u_i(t)}. \tag{5}$$

By Lemma 1, we can obtain the p-order Caputo derivative of $V_1(t)$ as follows:

$$\begin{aligned}
&{}_{t_0}^{C}D_t^{p}V_1(t)\\
&\le\sum_{i=1}^{n}\left[u_i(t)D^p\overline{u_i(t)}+\overline{u_i(t)}D^pu_i(t)\right]\\
&=-\sum_{i=1}^{n}\left(d_i+k_i+\overline{d_i}+\overline{k_i}\right)u_i(t)\overline{u_i(t)}\\
&\quad+\sum_{i=1}^{n}\left[\overline{(1-\alpha)J_i(t)}u_i(t)+(1-\alpha)J_i(t)\overline{u_i(t)}\right]\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{a_{ik}}u_i(t)\overline{\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)}\\
&\quad+a_{ik}\overline{u_i(t)}\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)\Big]\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{a_{ik}}u_i(t)\overline{\left(f_k(\alpha x_k(t))-\alpha f_k(x_k(t))\right)}\\
&\quad+a_{ik}\overline{u_i(t)}\left(f_k(\alpha x_k(t))-\alpha f_k(x_k(t))\right)\Big]\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{b_{ik}}u_i(t)\Big(\overline{g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))}\Big)\\
&\quad+b_{ik}\overline{u_i(t)}\left(g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))\right)\Big]\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{b_{ik}}u_i(t)\Big(\overline{g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))}\Big)\\
&\quad+b_{ik}\overline{u_i(t)}\left(g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))\right)\Big]\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{\left(h_k(y_k(s))-h_k(\alpha x_k(s))\right)}ds\\
&\quad+m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}\left(h_k(y_k(s))-h_k(\alpha x_k(s))\right)ds\Big]\\
&\quad+\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{\left(h_k(\alpha x_k(s))-\alpha h_k(x_k(s))\right)}ds\\
&\quad+m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}\left(h_k(\alpha x_k(s))-\alpha h_k(x_k(s))\right)ds\Big].
\end{aligned}\tag{6}$$

According to Lemma 2 and Assumption 1, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{a_{ik}}u_i(t)\overline{\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)}\\
&\quad+a_{ik}\overline{u_i(t)}\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)\Big]\\
&\le\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}\\
&\quad+\overline{\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)}\\
&\quad\times\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)\Big]\\
&\le\sum_{i=1}^{n}\sum_{k=1}^{n}\left[a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}+\Lambda_1^2u_k(t)\overline{u_k(t)}\right]\\
&\le\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}+n\sum_{i=1}^{n}\Lambda_1^2u_i(t)\overline{u_i(t)}.
\end{aligned}\tag{7}$$

Similarly,

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{b_{ik}}u_i(t)\left(\overline{g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))}\right)\\
&\quad+b_{ik}\overline{u_i(t)}\left(g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))\right)\Big]\\
&\le\sum_{i=1}^{n}\sum_{k=1}^{n}b_{ik}\overline{b_{ik}}u_i(t)\overline{u_i(t)}\\
&\quad+n\Lambda_3^2\sum_{i=1}^{n}e_i(t-\tau_1)\overline{e_i(t-\tau_1)},
\end{aligned}\tag{8}$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{(h_k(y_k(s))-h_k(\alpha x_k(s)))}ds\right.\\
&\qquad\left.+\, m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}(h_k(y_k(s))-h_k(\alpha x_k(s)))\,ds\right]\\
&\le \sum_{i=1}^{n}\sum_{k=1}^{n} m_{ik}\overline{m_{ik}}u_i(t)\overline{u_i(t)} + n\sum_{i=1}^{n}\Lambda_3^2\tau_2^2 u_i(t)\overline{u_i(t)}. \qquad (9)
\end{aligned}
$$

According to Lemma 2 and Assumption 2, we get

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\overline{a_{ik}}u_i(t)\overline{(f_k(\alpha x_k(t))-\alpha f_k(x_k(t)))}\right.\\
&\qquad\left.+a_{ik}\overline{u_i(t)}\,(f_k(\alpha x_k(t))-\alpha f_k(x_k(t)))\right]\\
&\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left[a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}+\overline{(f_k(\alpha x_k(t))-\alpha f_k(x_k(t)))}\right.\\
&\qquad\left.\times\,(f_k(\alpha x_k(t))-\alpha f_k(x_k(t)))\right]\\
&\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left[a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}+2\left(f_k(\alpha x_k(t))\overline{f_k(\alpha x_k(t))}\right.\right.\\
&\qquad\left.\left.+\alpha\overline{\alpha}f_k(x_k(t))\overline{f_k(x_k(t))}\right)\right]\\
&\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left[a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}+2\left(l_1^2+\alpha\overline{\alpha}l_1^2\right)\right]\\
&=\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}+2(1+\alpha\overline{\alpha})l_1^2. \qquad (10)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\overline{b_{ik}}u_i(t)\left(\overline{g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))}\right)\right.\\
&\qquad\left.+b_{ik}\overline{u_i(t)}\,(g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1)))\right]\\
&\le \sum_{i=1}^{n}\sum_{k=1}^{n}b_{ik}\overline{b_{ik}}u_i(t)\overline{u_i(t)}+2\,(1+\alpha\overline{\alpha})\,l_2^2, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n}\left[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{(h_k(\alpha x_k(s))-\alpha h_k(x_k(s)))}ds\right.\\
&\qquad\left.+\, m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}(h_k(\alpha x_k(s))-\alpha h_k(x_k(s)))\,ds\right]\\
&\le \sum_{i=1}^{n}\sum_{k=1}^{n}m_{ik}\overline{m_{ik}}u_i(t)\overline{u_i(t)}+2\,(1+\alpha\overline{\alpha})\,l_3^2\tau_2^2, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\left[\overline{(1-\alpha)J_i(t)}u_i(t)+(1-\alpha)\,J_i(t)\overline{u_i(t)}\right]\\
&\le (1-\alpha)\,\overline{(1-\alpha)}+\sum_{i=1}^{n}l_4^2u_i(t)\overline{u_i(t)}. \qquad (13)
\end{aligned}
$$

Submitting (7)-(13) into (6), by fractional Razumikhin theorem [44], we can get

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^pV_1(t)\\
&\le -\sum_{i=1}^{n}\left(d_i+k_i+\overline{d_i}+\overline{k_i}-l_4^2-n\Lambda_1^2-n\Lambda_3^2\tau_2^2\right.\\
&\qquad\left.-\sum_{k=1}^{n}2\left(a_{ik}\overline{a_{ik}}+b_{ik}\overline{b_{ik}}+m_{ik}\overline{m_{ik}}\right)\right)u_i(t)\overline{u_i(t)}\\
&\qquad+n\sum_{i=1}^{n}\Lambda_2^2u_i(t-\tau_1)\overline{u_i(t-\tau_1)}+2\,(1+\alpha\overline{\alpha})\\
&\qquad\times\left(l_1^2+l_2^2+l_3^2\,\tau_2^2\right)+(1-\alpha)\,\overline{(1-\alpha)}\\
&\le -\varpi V_1(t)+n\Lambda_2^2V_1(t-\tau_1)+\vartheta\\
&\le -\left(\varpi-n\Lambda_2^2\varsigma\right)V_1(t)+\vartheta, \qquad (14)
\end{aligned}
$$

where $\varsigma>1$. According to Lemma 4 and (14), there exists $t_1=t_0+\left(\frac{\Gamma(p)}{\varpi-n\Lambda_2^2\varsigma}\right)^{\frac{1}{1-p}}$, such that

$$
\begin{aligned}
V_1(t)\le&\left(V_1(t_0)-\frac{\vartheta}{\varpi-n\Lambda_2^2\varsigma}\right)\\
&\times E_p\left(-\left(\varpi-n\Lambda_2^2\varsigma\right)(t-t_0)^p\right)\\
&+\frac{\vartheta}{\varpi-n\Lambda_2^2\varsigma},\ t\ge t_1.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|u(t)\|\\
&\le\sqrt{AE_p\left[-\left(\varpi-n\Lambda_2^2\varsigma\right)(t-t_0)^p\right]+\frac{\vartheta}{\varpi-n\Lambda_2^2\varsigma}}, \qquad (15)
\end{aligned}
$$

where $t\ge t_1$, $A=V_1(t_0)-\frac{\vartheta}{\varpi-n\Lambda_2^2\varsigma}$. Finally, from Lemma 3 and (15), it could be found that

$$
\lim_{t\to+\infty}\|u(t)\|\le\sqrt{\frac{\vartheta}{\varpi-n\Lambda_2^2\varsigma}}.
$$

Hence, systems (1) and (2) are QPS under the linear feedback controller (4). The proof of Theorem 1 is completed.

Next, the adaptive controller is proposed

$$
\begin{cases}
v_i(t)=-k_i(t)\,(y_i(t)-\alpha x_i(t)),\\
{}_{t_0}^{C}D_t^pk_i(t)=\rho_ie_i(t)\overline{e_i(t)},
\end{cases} \qquad (16)
$$

where $k_i(t)$ is the adaptive feedback strength, ρ_i is a positive constant. Fig. 2 shows the control framework of adaptive feedback controller.

Theorem 2: Under Assumptions 1 and 2, if k_i^* satisfies the inequality $\mu>\eta\xi$, where $\xi>1$, then the drive system (1) is QPS with the response system (2) under the

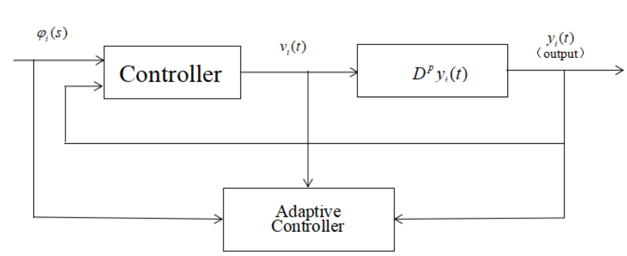

Fig. 2. Control framework of adaptive feedback controller.

adaptive controller (16). In addition, the error bound can be estimated by $\sqrt{\frac{\delta}{\mu-\eta\xi}}$.

Proof: Consider the following Lyapunov function:

$$\begin{aligned} V_2(t) &= V_{21}(t)+V_{22}(t) \\ &= \sum_{i=1}^{n} u_i(t)\overline{u_i(t)}+\sum_{i=1}^{n}\frac{1}{\rho_i}(k_i(t)-k_i^*)^2. \end{aligned} \tag{17}$$

By Lemmas 1 and 5, the p-order Caputo derivative of $V_2(t)$ can be estimated as follows:

$$\begin{aligned} &{}_{t_0}^{C}D_t^p V_2(t) \\ &\le \sum_{i=1}^{n}\left[u_i(t)D^p\overline{u_i(t)}+\overline{u_i(t)}D^p u_i(t)\right] \\ &\quad +\sum_{i=1}^{n}\frac{2}{\rho_i}(k_i(t)-k_i^*)D^p k_i(t) \\ &\le -\sum_{i=1}^{n}\left(d_i+\overline{d_i}+2k_i^*\right)u_i(t)\overline{u_i(t)} \\ &\quad +\sum_{i=1}^{n}\left[\overline{(1-\alpha)J_i(t)}u_i(t)\ +(1-\alpha)J_i(t)\overline{u_i(t)}\right] \\ &\quad +\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{a_{ik}}u_i(t)\overline{(f_k(y_k(t))-f_k(\alpha x_k(t)))} \\ &\quad +a_{ik}\overline{u_i(t)}\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)\Big] \\ &\quad +\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{a_{ik}}u_i(t)\overline{(f_k(\alpha x_k(t))-\alpha f_k(x_k(t)))} \\ &\quad +a_{ik}\overline{u_i(t)}\left(f_k(\alpha x_k(t))-\alpha f_k(x_k(t))\right)\Big] \\ &\quad +\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{b_{ik}}u_i(t)\Big(\overline{g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))}\Big) \\ &\quad +b_{ik}\overline{u_i(t)}\left(g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))\right)\Big] \\ &\quad +\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{b_{ik}}u_i(t)\Big(\overline{g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))}\Big) \\ &\quad +b_{ik}\overline{u_i(t)}\left(g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))\right)\Big] \\ &\quad +\sum_{i=1}^{n}\sum_{k=1}^{n}[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{(h_k(y_k(s))-h_k(\alpha x_k(s)))}ds \\ &\quad +m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}\left(h_k(y_k(s))-h_k(\alpha x_k(s))\right)ds\Big] \\ &\quad +\sum_{i=1}^{n}\sum_{k=1}^{n}[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{(h_k(\alpha x_k(s))-\alpha h_k(x_k(s)))}ds \\ &\quad +m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}\left(h_k(\alpha x_k(s))-\alpha h_k(x_k(s))\right)ds\Big]. \end{aligned} \tag{18}$$

According to Lemma 6 and Assumption 1, we derive

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{a_{ik}}u_i(t)\overline{(f_k(y_k(t))-f_k(\alpha x_k(t)))} \\ &\quad +a_{ik}\overline{u_i(t)}\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)\Big] \\ &\le 2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}u_i(t)\overline{u_i(t)}\right)^{\frac{1}{2}}[f_k(y_k(t)) \\ &\quad -f_k(\alpha x_k(t))\overline{(f_k(y_k(t))-f_k(\alpha x_k(t)))}\Big]^{\frac{1}{2}} \\ &\le 2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}\Big[\frac{1}{2}\left(u_i(t)\overline{u_i(t)}\right)+\frac{1}{2} \\ &\quad \times\left(f_k(y_k(t))-f_k(\alpha x_k(t))\right)\overline{(f_k(y_k(t))-f_k(\alpha x_k(t)))}\Big] \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}\left(u_i(t)\overline{u_i(t)}+\Lambda_1^2 u_k(t)\overline{u_k(t)}\right) \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)} \\ &\quad +\Lambda_1^2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ki}\overline{a_{ki}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)}. \end{aligned} \tag{19}$$

Similarly,

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{b_{ik}}u_i(t)\left(\overline{g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))}\right) \\ &\quad +b_{ik}\overline{u_i(t)}\left(g_k(y_k(t-\tau_1))-g_k(\alpha x_k(t-\tau_1))\right)\Big] \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)}+\Lambda_2^2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(b_{ki}\overline{b_{ki}}\right)^{\frac{1}{2}} \\ &\quad \times u_i(t-\tau_1)\overline{u_i(t-\tau_1)}, \end{aligned} \tag{20}$$

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{k=1}^{n}[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{(h_k(y_k(s))-h_k(\alpha x_k(s)))}ds \\ &\quad +m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}\left(h_k(y_k(s))-h_k(\alpha x_k(s))\right)ds\Big] \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(m_{ik}\overline{m_{ik}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)} \\ &\quad +\Lambda_3^2\tau_2^2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(m_{ki}\overline{m_{ki}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)}. \end{aligned} \tag{21}$$

According to Lemma 6 and Assumption 2, we derive

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{a_{ik}}u_i(t)\overline{(f_k(\alpha x_k(t))-\alpha f_k(x_k(t)))} \\ &\quad +a_{ik}\overline{u_i(t)}\left(f_k(\alpha x_k(t))-\alpha f_k(x_k(t))\right)\Big] \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}\Big[u_i(t)\overline{u_i(t)} \\ &\quad +\overline{(f_k(\alpha x_k(t))-\alpha f_k(x_k(t)))} \\ &\quad \times\left(f_k(\alpha x_k(t))-\alpha f_k(x_k(t))\right)] \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}\left[u_i(t)\overline{u_i(t)}+2\left(1+\alpha\overline{\alpha}\right)l_1^2\right] \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)} \\ &\quad +2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}(1+\alpha\overline{\alpha})l_1^2. \end{aligned} \tag{22}$$

Similarly,

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{b_{ik}}u_i(t)\left(\overline{g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))}\right)$$
$$+b_{ik}\overline{u_i(t)}\left(g_k(\alpha x_k(t-\tau_1))-\alpha g_k(x_k(t-\tau_1))\right)\Big]$$
$$\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)}+2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}}$$
$$\times(1+\alpha\overline{\alpha})l_2^2, \tag{23}$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\Big[\overline{m_{ik}}u_i(t)\int_{t-\tau_2}^{t}\overline{(h_k(\alpha x_k(s))-\alpha h_k(x_k(s)))}ds$$
$$+m_{ik}\overline{u_i(t)}\int_{t-\tau_2}^{t}\left(h_k(\alpha x_k(s))-\alpha h_k(x_k(s))\right)ds\Big]$$
$$\le \sum_{i=1}^{n}\sum_{k=1}^{n}\left(m_{ik}\overline{m_{ik}}\right)^{\frac{1}{2}}u_i(t)\overline{u_i(t)}+2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(m_{ik}\overline{m_{ik}}\right)^{\frac{1}{2}}$$
$$\times(1+\alpha\overline{\alpha})l_3^2\tau_2^2, \tag{24}$$

$$\sum_{i=1}^{n}\left[\overline{(1-\alpha)J_i(t)}u_i(t)+(1-\alpha)J_i(t)\overline{u_i(t)}\right]$$
$$\le 2\sum_{i=1}^{n}\left(J_i(t)\overline{J_i(t)}\right)^{\frac{1}{2}}\left((1-\alpha)\overline{(1-\alpha)}u_i(t)\overline{u_i(t)}\right)^{\frac{1}{2}}$$
$$\le l_4\sum_{i=1}^{n}u_i(t)\overline{u_i(t)}+l_4(1-\alpha)\overline{(1-\alpha)}. \tag{25}$$

Submitting (19)-(25) into (18), by fractional Razumikhin theorem [52], we can get

$${}_{t_0}^{C}D_t^{p}V_2(t)$$
$$=-\sum_{i=1}^{n}\Bigg\{d_i+\overline{d_i}+2k_i^*-l_4-2\sum_{k=1}^{n}\Big[\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}$$
$$+\left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}}+\left(m_{ik}\overline{m_{ik}}\right)^{\frac{1}{2}}\Big]-\Lambda_1^2\sum_{k=1}^{n}\left(a_{ki}\overline{a_{ki}}\right)^{\frac{1}{2}}$$
$$-\Lambda_3^2\tau_2^2\sum_{k=1}^{n}\left(m_{ki}\overline{m_{ki}}\right)^{\frac{1}{2}}\Bigg\}u_i(t)\overline{u_i(t)}$$
$$+\Lambda_2^2\sum_{i=1}^{n}\sum_{k=1}^{n}\left(b_{ki}\overline{b_{ki}}\right)^{\frac{1}{2}}u_i(t-\tau_1)\overline{u_i(t-\tau_1)}$$
$$+2\sum_{i=1}^{n}\sum_{k=1}^{n}(1+\alpha\overline{\alpha})\Big[\left(a_{ik}\overline{a_{ik}}\right)^{\frac{1}{2}}l_1^2+\left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}}l_2^2$$
$$+\left(m_{ik}\overline{m_{ik}}\right)^{\frac{1}{2}}l_3^2\tau_2^2\Big]+l_4(1-\alpha)\overline{(1-\alpha)}$$
$$\le-(\mu-\eta\xi)V_{21}(t)+\delta, \tag{26}$$

where $\xi>1$, according to Lemma 4 and (26), there exists $t_2=t_0+\left(\frac{\Gamma(p)}{\mu-\eta\xi}\right)^{\frac{1}{1-p}}$ such that

$$V_{21}(t)\le\left(V_{21}(t_0)+V_{22}(t_0)-\frac{\delta}{\mu-\eta\xi}\right)$$
$$\times E_p\left(-(\mu-\eta\xi)(t-t_0)^p\right)+\frac{\delta}{\mu-\eta\xi},$$
$$t\ge t_2.$$

Then,

$$\|u(t)\|\le\sqrt{BE_p\left(-(\mu-\eta\xi)(t-t_0)^p\right)+\frac{\delta}{\mu-\eta\xi}}, \tag{27}$$

where $t\ge t_2$, $B=V_{21}(t_0)+V_{22}(t_0)-\frac{\delta}{\mu-\eta\xi}$. Finally, from Lemma 3 and (27), it could be found that

$$\lim_{t\to+\infty}\|u(t)\|\le\sqrt{\frac{\delta}{\mu-\eta\xi}}.$$

Therefore, systems (1) and (2) are QPS under the controller (16).

Remark 1: When $\tau_1=0$, $\tau_2=0$, system (1) degenerates into a FOCVNNs, whose QPS has been studied in [42]. When $\tau_1\neq0$, $\tau_2=0$, system (1) degenerates into a FOCVNNs with delay, the QPS has been studied in [43]. Therefore, the model used in this paper is more general.

Remark 2: The problem of QPS of fractional-order neural networks with adaptive controllers has not been discussed. Therefore, the research of this paper is meaningful.

Remark 3: In most of the existing literature, it is customary to decompose the complex-valued system into two real-valued systems for discussion, such as [39–41]. However, in this paper, the complex-valued system is discussed as a compact whole, which greatly reduces the difficulty of theoretical analysis and the complexity of calculation. Therefore, the method used in this paper is of more research significance.

Remark 4: In [24,25], the hybrid controllers are used, which is inconvenient and undesirable in applications. In this paper we adopt linear controllers and adaptive controllers to discuss the QPS of FOCVNNs with mixed delays, where the used methods are very convenient and applicable.

4. NUMERICAL SIMULATIONS

In this section, two numerical examples are given to verify the above results.

Example 1: Consider the following 2-dimensional FOCVNNMD as the drive system

$${}_{t_0}^{C}D_t^{p}x_i(t)=-d_ix_i(t)+\sum_{k=1}^{2}a_{ik}f_k(x_k(t))$$
$$+\sum_{k=1}^{2}b_{ik}g_k(x_k(t-\tau_1))$$
$$+\sum_{k=1}^{2}m_{ik}\int_{t-\tau_2}^{t}h_k(x_k(s))ds$$
$$+J_i(t),\ i=1,\ 2. \tag{28}$$

The corresponding response system is described as

$$\begin{aligned}{}_{t_0}^{C}D_t^{p} y_i(t) = & -d_i y_i(t) + \sum_{k=1}^{2} a_{ik} f_k(y_k(t)) \\ & + \sum_{k=1}^{2} b_{ik} g_k(y_k(t-\tau_1)) \\ & + \sum_{k=1}^{2} m_{ik} \int_{t-\tau_2}^{t} h_k(x_k(s))ds \\ & + J_i(t) + v_i(t),\ i = 1,\ 2. \end{aligned} \tag{29}$$

where $p = 0.7$, $x_i(t) = m_i(t) + iq_i(t)$, $m_i(t), q_i(t) \in R$, $\tau_1 = 1$, $\tau_2 = 0.5$,

$$\begin{aligned} & C = \mathrm{diag}(c_1, c_2) = (1+i, 1+i), \\ & J_i(t) = (J_1(t), J_2(t))^T = (0,0), \\ & A = (a_{ij})_{2\times 2} = \begin{pmatrix} 0.2+0.2i & -0.1-0.1i \\ 0.1+0.1i & -0.2-0.2i \end{pmatrix}, \\ & B = (b_{ij})_{2\times 2} = \begin{pmatrix} -0.5-0.5i & -1-i \\ -2-2i & 1+i \end{pmatrix}, \\ & M = (m_{ij})_{2\times 2} = \begin{pmatrix} 3+3i & -1-i \\ 1+i & 2+2i \end{pmatrix}. \end{aligned}$$

The linear feedback controller $v_i(t)$ is designed as (4), choose $k_1 = 25+25i$, $k_2 = 30+30i$, $\varsigma = 1.2$ The initial conditions of system (28) and (29) are selected as $\phi_1(s) = -0.3+0.4i$, $\phi_2(s) = 0.7-0.6i$, $\varphi_1 = 0.9+0.4i$, $\varphi_2(s) = 0.8-0.5i$, $s \in [-1,0)$. $f(x(t)) = g(x(t)) = h(x(t)) = \frac{1-e^{-m_i(t)}}{1+e^{-m_i(t)}} + i\frac{1}{1+e^{-q_i(t)}}$. By calculation, we have $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$, $l_1 = l_2 = l_3 = \sqrt{2}$, $l_4 = 0$, $\alpha = 0.3+0.8i$,

$$\begin{aligned}\varpi = & \min_{1\le i\le n} \Big\{ d_i + k_i + \overline{d_i} + \overline{k_i} - l_4^2 - n\Lambda_1^2 - n\Lambda_3^2\tau_2^2 \\ & - \sum_{k=1}^{n} 2\big(a_{ik}\overline{a_{ik}} + b_{ik}\overline{b_{ik}} + m_{ik}\overline{m_{ik}}\big)\Big\} = 9.3,\end{aligned}$$

and $9.3 = \varpi > n\Lambda_2^2\varsigma = 2.4$, then the condition and assumption of Theorem 1 are satisfied. Moreover, the error bounded is estimated as follows:

$$\sqrt{\frac{\vartheta}{\varpi - n\Lambda_2^2\varsigma}} \approx 1.56.$$

Fig. 3 shows the state trajectory of the error system when $\alpha = 0.3+0.8i$, Fig. 4 shows the trajectory of the error norm $\|u(t)\|$ when $\alpha = 0.3+0.8i$, illustrating that the (28)-(29) can achieve QPS under the condition of Theorem 1. The phase trajectories of real and imaginary part of system (28) are shown in Figs. 5 and 6, respectively.

Example 2: About the 2-dimensional FOCVNNMD given by (28) and (29), we consider the QPS of Theorem 2. Take the same system parameters as Example 1. The adaptive controller $v_i(t)$ is designed as (17), where the initial value of $k_i(t)$ are set as $k_1(0) = 1+0.2i$, $k_2(0) = 0.2+i$, $k_1^* = 25$, $k_2^* = 20$, $\alpha = 0.4+0.5i$, $\xi = 1.5$.

The initial conditions of system (28) and (29) are chosen as $\phi_1(s) = -0.3+0.4i$, $\phi_2(s) = 0.2-0.7i$, $\varphi_1 =$

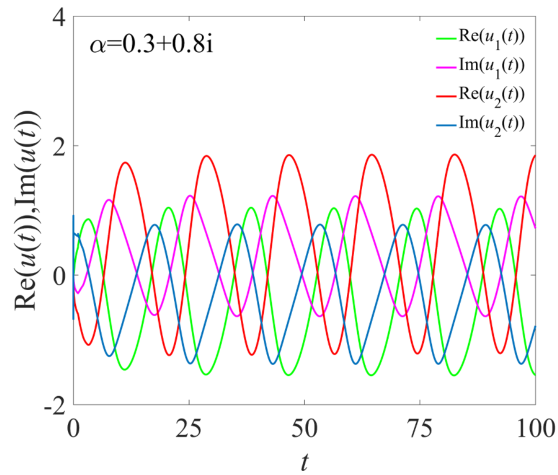

Fig. 3. The state trajectory of the error system when $\alpha = 0.3+0.8i$.

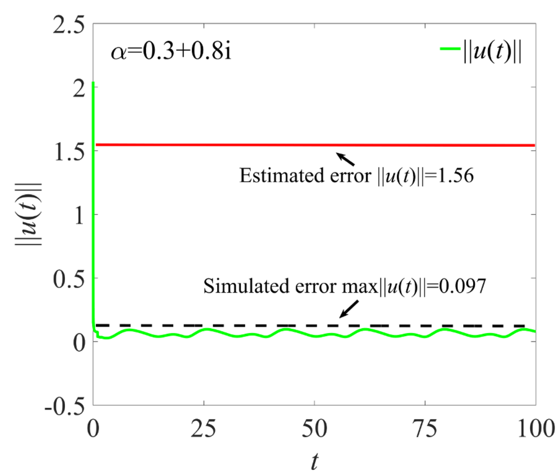

Fig. 4. The trajectory of the error norm $\|u(t)\|$ when $\alpha = 0.3+0.8i$.

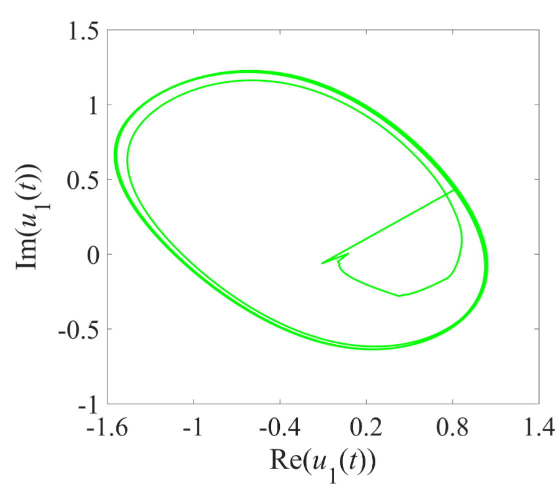

Fig. 5. The phase trajectories of real part and imaginary part of system (28).

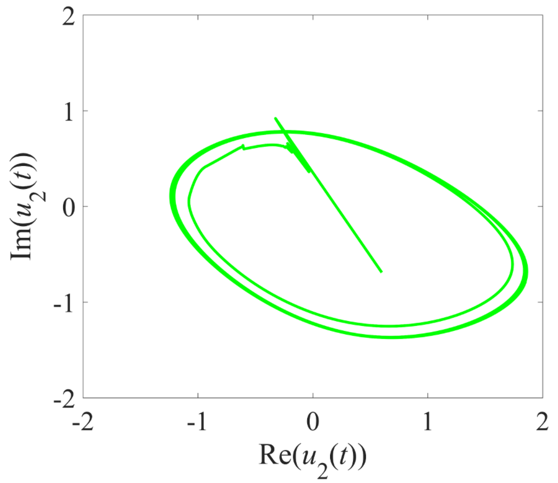

Fig. 6. The phase trajectories of real part and imaginary part of system (28).

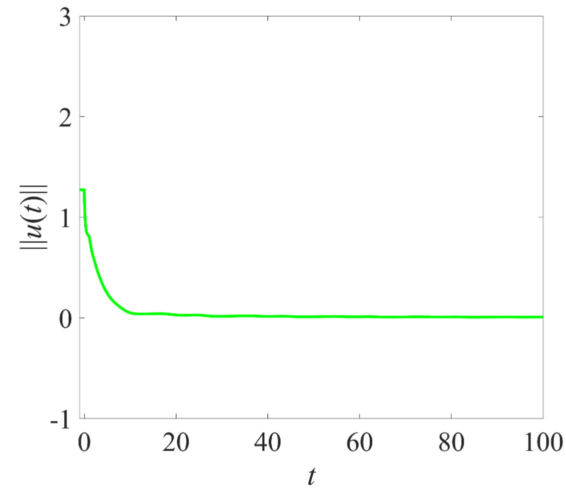

Fig. 8. The trajectory of the error norm $\|u(t)\|$ when $\alpha = 1$.

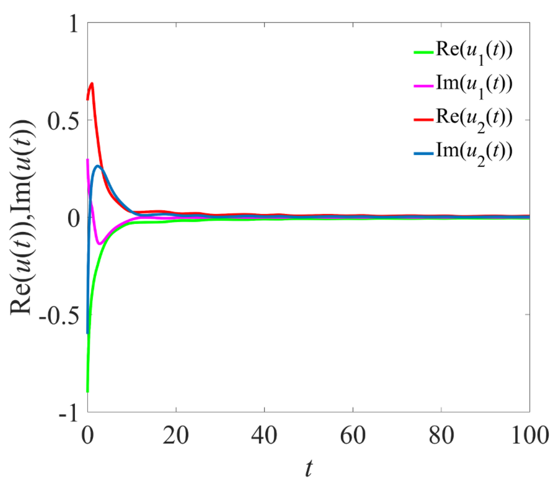

Fig. 7. The state trajectory of the error system when $\alpha = 1$.

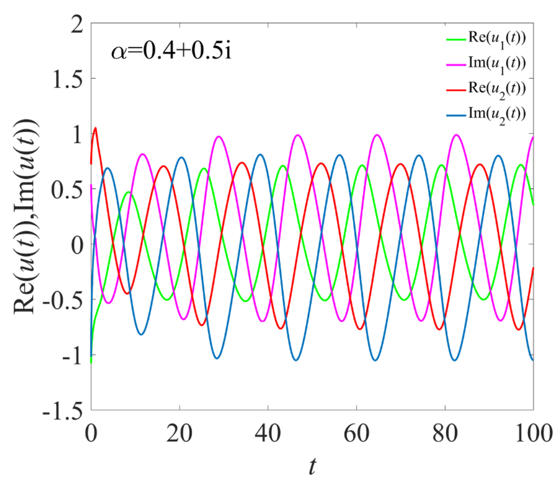

Fig. 9. The state trajectory of the error system when $\alpha = 0.4 + 0.5i$.

$-1.2+0.7i$, $\varphi_2(s) = 0.8-1.3i$, $s \in [-1,0)$. By calculation, we have

$$\eta = \max_{1\le i\le n} \Lambda_2^2 \sum_{k=1}^{n} \left(b_{ki}\overline{b_{ki}}\right)^{\frac{1}{2}} \approx 2.83,$$

$$\mu = \min_{1\le i\le n} \Bigg\{ d_i + \overline{d_i} + 2k_i^* - l_4 - 2\sum_{k=1}^{n} \Big[(a_{ik}\overline{a_{ik}})^{\frac{1}{2}} + \left(b_{ik}\overline{b_{ik}}\right)^{\frac{1}{2}} + (m_{ik}\overline{m_{ik}})^{\frac{1}{2}} \Big] - \Lambda_1^2 \sum_{k=1}^{n} (a_{ki}\overline{a_{ki}})^{\frac{1}{2}} - \Lambda_3^2 \tau_2^2 \sum_{k=1}^{n} (m_{ki}\overline{m_{ki}})^{\frac{1}{2}} \Bigg\} \approx 22.70,$$

$22.70 = \mu > \eta\xi = 4.245$, then the condition and assumptions of Theorem 2 are satisfied. Moreover, the error bounded is estimated as follows:

$$\sqrt{\frac{\delta}{\mu - \eta\xi}} \approx 2.49.$$

Figs. 7 and 8 respectively show the state trajectory of the error system and the trajectory of the error norm $\|u(t)\|$ when $\alpha = 1$, illustrating that the (28)-(29) can achieve quasi-complete synchronization under the condition of Theorem 2. Figs. 9 and 10 respectively show the state trajectory of the error system and the trajectory of the error norm $\|u(t)\|$ when $\alpha = 0.4+0.5i$, illustrating that the (28)-(29) can achieve QPS synchronization under the condition of Theorem 2.

Remark 5: Examples 1 and 2 show that the synchronization time of linear controller is about 0.0033, while that of the adaptive controller is about 0.00014. As we

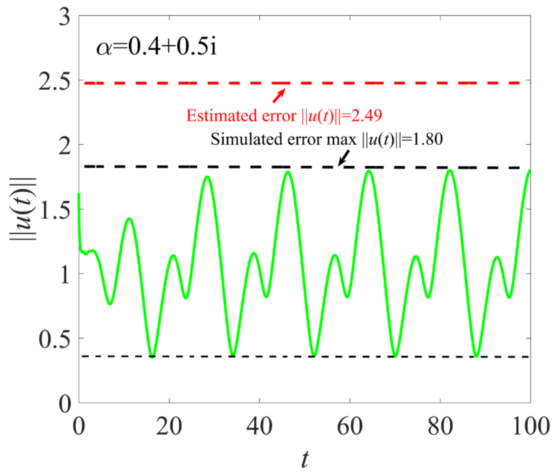

Fig. 10. The trajectory of the error norm $\|u(t)\|$ when $\alpha = 0.4+0.5i$.

all know, the shorter the synchronization time, the better. Therefore, the synchronization effect is better under adaptive control.

Remark 6: The MATLAB numerical simulations in examples 1 and 2 show that the synchronization error under the linear controller tends to 0, and the simulation error upper bound is much smaller than the estimated error upper bound. However, the synchronization error of adaptive controller tends to 1. Therefore, the synchronization convergence of linear feedback controller is better.

5. CONCLUSIONS

In this paper, by constructing the Lyapunov function, using the fractional Razumikhin theorem, the properties of the Mittag-Leffler function and some inequality techniques, the sufficient conditions for the QPS of the FOCVNNs with mixed delays are obtained. The feasibility of the results are verified by two numerical simulation examples.

REFERENCES

[1] R. L. Bagley and P. J. Torvik, "A theoretical basis for the application of fractional calculus to viscoelasticity," *Journal of Rheology*, vol. 27, no. 3, pp. 201-210, 1983.

[2] X. Li and X. Tian, "Fractional order thermo-viscoelastic theory of biological tissue with dual phase lag heat conduction model," *Applied Mathematical Modelling*, vol. 95, pp. 612-622, 2021.

[3] R. L. Magin, "Fractional calculus models of complex dynamics in biological tissues," *Computers Mathematics with Applications*, vol. 59, no. 5, pp. 1586-1593, 2010.

[4] F. B. Yousef, A. Yousef, and C. Maji, "Effects of fear in a fractional-order predator-prey system with predator density-dependent prey mortality," *Chaos, Solitons and Fractals*, vol. 145, p. 110711, 2021.

[5] M. Das and G. P. Samanta, "A delayed fractional order food chain model with fear effect and prey refuge," *Mathematics and Computers in Simulation*, vol. 178, pp. 218-245, 2020.

[6] E. K. Lenzi, M. dos Santos, M. K. Lenzi, D. S. Vieira, and L. R. da Silva, "Solutions for a fractional diffusion equation: Anomalous diffusion and adsorption-desorption processes," *Journal of King Saud University-Science*, vol. 28, no. 1, pp. 3-6, 2016.

[7] J. Teng, H. Zhang, C. Tang, and H. Lin, "Novel molecular level insights into forward osmosis membrane fouling affected by reverse diffusion of draw solutions based on thermodynamic mechanisms," *Journal of Membrane Science*, vol. 620, p. 118815, 2021.

[8] P. Ghamisi, M. S. Couceiro, J. A. Benediktsson, and N. Ferreira, "An efficient method for segmentation of images based on fractional calculus and natural selection," *Expert Systems with Applications*, vol. 39, no. 16, pp. 12407-12417, 2012.

[9] A. Gomez-Echavarrla, J. P. Ugarte, and C. Tobon, "The fractional fourier transform as a biomedical signal and image processing tool: A review," *Biocybernetics and Biomedical Engineering*, vol. 40, pp. 1081-1093, 2020.

[10] V. E. Bondarenko, "Information processing, memories, and synchronization in chaotic neural network with the time delay," *Complexity*, vol. 11, pp. 39-52, 2005.

[11] G. A. Anastassiou, "Fractional neural network approximation," *Computers Mathematics with Applications*, vol. 64, pp. 1655-1676, 2012.

[12] P. Muthukumar and P. Balasubramaniam, "Feedback synchronization of the fractional order reverse butterfly-shaped chaotic system and its application to digital cryptography," *Nonlinear Dynamics*, vol. 74, no. 4, pp. 1169-1181, 2013.

[13] L. Chen, R. Wu, J. Cao, and J. B. Liu, "Stability and synchronization of memristor-based fractional-order delayed neural networks," *Neural Networks*, vol. 71, pp. 37-44, 2015.

[14] X. Wu and H. Bao, "Finite time complete synchronization for fractional-order multiplex networks," *Applied Mathematics and Computation*, vol. 377, p. 125188, 2020.

[15] H. Zhang, X. Y. Wang, and X. Lin, "Topology identification and module phase synchronization of neural network with time delay," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, pp. 885-892, 2017.

[16] Z. Yao, P. Zhou, Z. Zhu, and J. Ma, "Phase synchronization between a light-dependent neuron and a thermosensitive neuron," *Neurocomputing*, vol. 423, pp. 518-534, 2021.

[17] Q. Gan, X. Rui, and X. Kang, "Synchronization of chaotic neural networks with mixed time delays," *Communications in Nonlinear Science Numerical Simulation*, vol. 16, pp. 966-974, 2011.

[18] L. Yang and J. Jiang, "The role of coupling-frequency weighting exponent on synchronization of a power network," *Physica A*, vol. 464, pp. 115-122, 2016.

[19] G. Arthi and N. Brindha, "On finite-time stability of nonlinear fractional-order systems with impulses and multi-state time delays," *Results in Control and Optimization*, vol. 2, p. 100010, 2021.

[20] S. Cai and M. Hou, "Quasi-synchronization of fractional-order heterogeneous dynamical networks via aperiodic intermittent pinning control," *Chaos, Solitons and Fractals*, vol. 146, p. 110901, 2021.

[21] Y. Y. Chen, R. Huang, Y. Ge, and Y. Zhang, "Spherical formation tracking control of nonlinear second-order agents with adaptive neural flow estimate," *IEEE Transactions on Neural Networks and Learning Systems*, pp. 1-12, 2021. DOI: 10.1109/TNNLS.2021.3071317

[22] Y. Y. Chen, K. Chen, and A. Astolfi, "Adaptive formation tracking control for first-order agents in a time-varying flowfield," *IEEE Transactions on Automatic Control*, p. 1, 2021. DOI: 10.1109/TAC.2021.3074900

[23] Y. Y. Chen, K. Chen, and A. Astolfi, "Adaptive formation tracking control for directed networked vehicless in a time-varying flowfield," *Journal of Guidance, Control, and Dynamics*, vol. 44, no. 10, 2021.

[24] Y. Zhang and S. Deng, "Finite-time projective synchronization of fractional-order complex-valued memristor-based neural networks with delay," *Chaos, Solitons and Fractals*, vol. 128, pp. 176-190, 2019.

[25] J. Yu, C. Hu, H. Jiang, and X. Fan, "Projective synchronization for fractional neural networks," *Neural Networks*, vol. 49, pp. 87-95, 2014.

[26] W. S. Mcculloch and W. Pitts, "A logical calculus of the ideas immanent in nervous activity," *The Bulletin of Mathematical Biophysics*, vol. 5, no. 4, pp. 115-133, 1943.

[27] M. A. Islas, J. J. Rubio, S. Muñiz, G. Ochoa, J. Pacheco, J. A. Meda-Campaña, D. Mujica-Varga, C. Aguilar-Ibañez, G. J. Gutierrez, and A. Zacarias, "A fuzzy logic model for hourly electrical power demand modeling," *Electronics*, vol. 10, no. 4, p. 448, 2021.

[28] J. J. Rubio, "SOFMLS: Online self-organizing fuzzy modified least-squares network," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 6, pp. 1296-1309, 2009.

[29] H. S. Chiang, M. Y. Chen, and Y. J. Huang, "Wavelet-based EEG processing for epilepsy detection using fuzzy entropy and associative petri net," *IEEE Access*, vol. 7, pp. 103255-103262, 2019.

[30] J. J. Rubio, "Stability analysis of the modified Levenberg-Marquardt Algorithm for the artificial neural network training," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 32, no. 8, pp. 3510-3524, 2021.

[31] L. A. Soriano, E. Zamora, J. M. Vazquez-Nicolas, G. Hernández, J. A. B. Madrigal, and D. Balderas, "PD control compensation based on a cascade neural network applied to a robot manipulator, "*Frontiers in Neurorobotics*, vol. 14, 2020.

[32] F. Furlán , E. Rubio, H. Sossa, and V. Ponce, "CNN based detectors on planetary environments: A performance evaluation," *Frontiers in Neurorobotics*, vol. 14, p. 85, 2020.

[33] Z. T. Huang, Q. G. Yang, and X. S. Luo, "Exponential stability of impulsive neural networks with time-varying delays," *Chaos Solitons and Fractals*, vol. 35, no. 4, pp. 770-780, 2008.

[34] Y. Li, Y. Q. Chen, and I. Podlubny, "Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized mittag-leffler stability," *Computers Mathematics with Applications*, vol. 59, no. 5, pp. 1810-1821, 2010.

[35] H. Zhang, M. Ye, and J. Cao, and A. Alsaedi, "Synchronization control of Riemann-Liouville fractional competitive network systems with time-varying delay and different time scales," *International Journal of Control, Automation, and Systems*, vol. 16, pp. 1404-1414, 2018.

[36] M. Zarefard and S. Effati, "Adaptive synchronization between two non-identical BAM neural networks with unknown parameters and time-varying delays," *International Journal of Control, Automation, and Systems*, vol. 15, pp. 1877-1887, 2017.

[37] S. B. Stojanovic, D. L. Debeljkovic, and M. A. Misic, "Finite-time stability for a linear discrete-time delay systems by using discrete convolution: An LMI approach," *International Journal of Control, Automation, and Systems*, vol. 14, no. 4, pp. 1144-1151, 2016.

[38] E. Kaslik, M. Neamu, and L. F. Vesa, "Global stability analysis of an unemployment model with distributed delay," *Mathematics and Computers in Simulation*, vol. 185, no. 4, pp. 535-546, 2021.

[39] W. Zhang, H. Zhang, J. Cao, H. M. Zhang, and D. Chen, "Synchronization of delayed fractional-order complex-valued neural networks with leakage delay," *Physica A: Statistical Mechanics and its Applications*, vol. 556, p. 124710, 2020.

[40] C. Xu, M. Liao, P. Li, and S. Yuan, "Impact of leakage delay on bifurcation in fractional-order complex-valued neural networks," *Chaos, Solitons and Fractals*, vol. 142, p. 110535, 2021.

[41] X. You, S. Dian, R. Guo, and S. Li, "Exponential stability analysis for discrete-time quaternion-valued neural networks with leakage delay and discrete time-varying delays," *Neurocomputing*, vol. 430, pp. 71-81, 2021.

[42] S. Yang, J. Yu, C. Hu, and H. Jiang, "Quasi-projective synchronization of fractional-order complex-valued recurrent neural networks," *Neural Networks*, vol. 104, pp. 104-113, 2018.

[43] X. You, Q. Song, and Z. Zhao, "Existence and finite-time stability of discrete fractional-order complex-valued neural networks with time delays," *Neural Networks*, vol. 123, pp. 248-260, 2020.

[44] Y. Xu and W. Li, "Finite-time synchronization of fractional-order complex-valued coupled systems," *Physica A*, vol. 549, p. 123903, 2020.

[45] H. Zhang, M. Ye, and R. Ye, and J. Cao, "Synchronization stability of Riemann-Liouville fractional delay-coupled complex neural networks," *Physica A: Statistical Mechanics and its Applications*, vol. 508, pp. 155-165, 2018.

[46] E. Arslan, G. Narayanan, M. S. Ali, S. Arik, and S. Saroha, "Controller design for finite-time and fixed-time stabilization of fractional-order memristive complex-valued BAM neural networks with uncertain parameters and time-varying delays," *Neural Networks*, vol. 130, pp. 60-74, 2020.

[47] X. Yang, C. Li, T. Huang, Q. Song, and J. Huang, "Synchronization of fractional-order memristor-based complex-valued neural networks with uncertain parameters and time delays," *Chaos, Solitons and Fractals*, vol. 110, pp. 105-123, 2018.

[48] S. Khorashadizadeh and M. H. Majidi, "Synchronization of two different chaotic systems using Legendre polynomials with applications in secure communications," *Frontiers of Information Technology and Electronic Engineering*, vol. 19, pp. 1180-1190, 2018.

[49] M. Samimi, M. H. Majidi, and S. Khorashadizadeh, "Secure communication based on chaos synchronization using brain emotional learning," *International Journal of Electronics and Communications*, vol. 127, p. 153424, 2020.

[50] A. A. Kilbas, and H. M. Srivastava, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V. Amsterdam, vol. 204, no. 49-52, pp. 2453-2461, 2006.

[51] H. L. Li, C. Hu, J. Cao, H. Jiang, and A. Alsaedi, "Quasi-projective and complete synchronization of fractional-order complex-valued neural networks with time delays," *Neural Networks*, vol. 118, pp. 102-109, 2019.

[52] D. Baleanu, S. J. Sadati, R. Ghaderi, A. Ranjbar, and F. Jarad, "Razumikhin stability theorem for fractional systems with delay," *Abstract and Applied Analysis*, vol. 2010, p. 124812, 2010.

[53] H. L. Li, J. Cao, H. Jiang, and A. Alsaedi, "Finite-time synchronization of fractional-order complex networks via hybrid feedback control," *Neurocomputing*, vol. 320, pp. 69-75, 2018.

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