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# Exponential Stability for Neutral Stochastic Differential Delay Equations with Markovian Switching and Nonlinear Impulsive Effects

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**Abstract:** In this paper, the problems on the exponential stability in $p$-th ($p \geq 2$)-moment and the almost sure exponential stability for neutral stochastic differential delay equation with Markovian switching and impulses are analyzed. By establishing an impulsive delay integral inequality, the Lyapunov theorem on the exponential stability in $p$-th ($p \geq 2$)-moment is given. Then, by using the Borel-Cantelli lemma, the almost sure exponential stability theorem is also proved. Two major advantages of these two results are that the differentiability or continuity of the delay function is not required, and that while considering the concerned problem, the difficulty coming from the simultaneous presence of the neutral item, the impulsive disturbance and the stochastic perturbations is overcome. An example is provided to examine the effectiveness and potential of the theoretic results obtained.

**Keywords:** Almost sure exponential stability, exponential stability, impulses, Markovian switching, neutral stochastic differential delay equation.

## 1. INTRODUCTION

Neutral stochastic differential delay equation is regarded as a special case for stochastic differential delay equation. This equation has been found in many applications, such as biology mechanics, physics, medicine and economics [1-4]. One of the important issues in the study of neutral stochastic differential delay equation is the automatic control, with consequent emphasis being placed on the stability analysis. While considering the stability analysis for neutral stochastic differential delay equation, the state and the neutral item are chronically seen as an ensemble part, which can cause some difficulties. Over the past few decades, the Lyapunov function approach, the Lyapunov-Krasovskii functional approach, the Razumikhin-type theorem and the fixed point theorem have been developed in the stability analysis for such equation, see [2,6-8] and the references cited therein.

Markovian jump systems are one special hybrid dynamical systems consisting of a family of subsystems driven by differential and difference equations, and a logical rule such as a Markov chain that models the switching mechanism between these subsystems. Hybrid systems driven by continuous-time Markov chains have been widely employed to model real-life systems including battle management command, control and communications systems, failure prone manufacturing, microelectronic circuit design verification, power generation and distribution, population demographic dynamics, and macroeconomics of national economy, see [9]. Stability analysis of stochastic differential equation with Markovian switching has been widely conducted by using some prevalent methods, such as the Lyapunov function approach, the Lyapunov-Krasovskii functional, the Razumikhin-type theorem, the comparison theorem and the differential delay inequality, etc., see [10-18] and the reference therein. However, due to the simultaneous presence of the neutral item, the stochastic perturbation and the Markovian switching, stability analysis of neutral stochastic differential delay equation with Markovian switching has not been investigated yet. The exponential stability in moment and the almost sure asymptotic stability for neutral stochastic differential equation with time delay and Markovian jumping parameters have been discussed by using the Lyapunov function approach in [19], and the results obtained are concerned with the constant delay case and the time-varying delay case with a restrictive condition that its derivative value is less than one. Note that for neutral stochastic differential equation, the extension from the constant delay to the time-varying delay is not easily to be accomplished without this restrictive condition. Recently, in order to remove this restrictive condition, in [21], Chen *et al*. have studied the exponential stability in moment for neutral stochastic systems with time-varying delay and Markovian switch-

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ing by using the integral inequality.

Many evolution processes are characterized by the fact that they experience an abrupt change of their state at certain moments, such as threshold phenomena in biology, bursting rhythm models in medicine, optimal control models in economics, and frequency modulated systems, etc. These abrupt changes are of short duration and may be described by impulsive differential equation. The basic theory of impulsive differential equation has been significantly developed [22]. The stability analysis of impulsive differential equation and stochastic impulsive differential equation has been extensively investigated, and the methods and the results are much more mature, see [23-32] and the references therein. However, the stability analysis of neutral differential equation with impulses has seldom been studied since the presence of the neutral item can cause some difficulties. In [33], by establishing a singular impulsive delay differential inequality and using the Lyapunov function, some sufficient conditions ensuring the globally exponential stability for a class of nonlinear impulsive neutral differential equation with time-varying delay are obtained. However, to the best of our knowledge, there is no work which is concerned with the exponential stability in $p$-th $(p \geq 2)$-moment and the almost sure exponential stability for neutral stochastic differential delay equation (NSDDE) with impulses. Thus, in this paper, we will make the first attempt to consider such problems.

The main difficulty in studying the exponential stability in $p$-th $(p \geq 2)$-moment and the almost sure exponential stability for neutral stochastic differential delay equation with impulses comes from the impulsive effects. Although the stability of neutral differential delay equation with impulses were investigated in [34-39], the proposed methods and the obtained results cannot be used since the neutral item, the stochastic perturbation and the impulsive effects simultaneously exist. In [3-8,19-21], the exponential stability in $p$-th $(p \geq 2)$-moment and the almost sure exponential stability for neutral stochastic differential delay equation with Markovian switching have been considered by using the Lyapunov function, the Lyapunov-Krasovskii functional approach, the Razumikhin-type theorem and the fixed point theorem, which are not used to study the concerned problems due to the existence of impulses. Thus, how to consider the exponential stability in $p$-th $(p \geq 2)$-moment and the almost sure exponential stability for neutral stochastic differential delay equation with Markovian switching and impulses (NSDDEwMSI) becomes the main motivation of this paper.

In this paper, an impulsive delay integral inequality is first established, and then the Lyapunov theorem on the exponential stability in $p$-th $(p \geq 2)$-moment for NSDDEwMSI is obtained. By using the Borel-Cantelli lemma, the almost sure exponential stability is also shown. Finally, one example examines the effectiveness of the results obtained. The contribution in this paper is listed from three aspects: 1) the exponential stability in $p$-th $(p \geq 2)$-moment and the almost sure exponential stability for NSDDEwMSI are investigated, which has seldom been reported in the available literature; 2) the delay integral inequality, the Lyapunov function and the stochastic analysis are incorporated to overcome the difficulty stemming from the simultaneous existence of the neutral term, the stochastic perturbation and the nonlinear impulsive effects; 3) the time-varying delay is required to be a bounded function, which means that the obtained results in this paper are not only fit for the slow time-varying delay, but also for the fast time-varying delay.

**Notations:** In this paper, $R^n$ and $R^{m\times n}$ are the $n$-dimension Euclidean space and the set of the $m\times n$-dimension real matrix, respectively. For an $n$-dimension vector $x = \mathrm{col}[x_1,\ x_2,\ \ldots,\ x_n]$ with the norm $|x|^2 = \sum_{i=1}^n x_i^2$. $(\Omega,\ \mathcal{F},\ \{\mathcal{F}\}_{t\geq t_0},\ \mathcal{P})$ represents a complete probability space with a filtration $\{\mathcal{F}\}_{t\geq t_0}$ satisfying the usual conditions. Let $B(t) = \mathrm{col}[B_1(t),\ B_2(t),\ \ldots,\ B_m(t)]$ be an $m$-dimensional Brownian motion defined on $(\Omega,\ \mathcal{F},\ \{\mathcal{F}\}_{t\geq t_0},\ \mathcal{P})$ satisfying $\mathcal{E}\{B_i(t)\} = 0$ $(i = 1,\ 2,\ \ldots,\ m)$, $\mathcal{E}\{B_i(t)B_j(t)\} = t$ for $i = j$ and $\mathcal{E}\{B_i(t)B_j(t)\} = 0$ for $i \neq j,\ i,\ j = 1,\ 2,\ \ldots,\ m$, where $\mathcal{E}\{\cdot\}$ stands for the expectation operator. For an appropriate dimensional matrix $A$, $A^T$ represents its transpose. $|\cdot|$ also denotes the trace norm for matrices. For $\bar{\tau} > 0$, let $\mathcal{PC} \equiv \mathcal{PC}([t_0-\bar{\tau}, t_0]; R^n)$ represent the family of all almost surely bounded, and continuous functions everywhere except for an infinite number of points $s$ at which $\xi(s)$ and $\xi(s^-) = \lim_{r\to s^-}\xi(r)$ exists and $\xi(s^+) = \lim_{r\to s^+}\xi(r) = \xi(s)$ from $[t_0-\bar{\tau},\ t_0]$ into $R^n$ and as usual, equipped with $\sup_{\theta\in[t_0-\bar{\tau},t_0]}|\xi(\theta)| < +\infty$ for any $\xi \in \mathcal{PC}$. $\mathcal{PC}^b_{\mathcal{F}_{t_0}} \equiv \mathcal{PC}^b_{\mathcal{F}_{t_0}}([t_0-\bar{\tau},\ t_0]; R^n)$ denotes the family of all $\mathcal{F}_{t_0}$-measurable and $\mathcal{PC}$-valued random variables $\xi = \{\xi(\theta) : t_0-\bar{\tau} \leq \theta \leq t_0\}$ with $\mathcal{E}\{\sup_{\theta\in[t_0-\bar{\tau},t_0]}|\xi(\theta)|^p\} < +\infty$ $(p \geq 1)$, for any $\xi \in \mathcal{PC}^b_{\mathcal{F}_{t_0}}$. $H(a-0)$ denotes the left-hand limit of the function $H(\cdot)$ at $a$, i.e., $H(a-0) = \lim_{u\to 0^-} H(a+u)$.

## 2. PROBLEM STATEMENT AND PRELIMINARIES

Let $\{r(t),\ t \geq t_0\}$ be a right-continuous Markov chain on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ taking values in a finite state space $\mathcal{N} = \{1, 2, \ldots, N\}$ with the generator $\Gamma = \{\gamma_{ij}\}_{N\times N}$, where $\mathcal{P}\{r(t+\Delta) = j|r(t) = i\} = \gamma_{ij}\Delta + o(\Delta)$ $(i \neq j)$ and $\mathcal{P}\{r(t+\Delta) = j|r(t) = i\} = 1+\gamma_{ii}\Delta + o(\Delta)$ $(i = j)$. where $\lim_{\Delta\to 0^+}\frac{o(\Delta)}{\Delta} = 0$. $\gamma_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$.

We consider the following NSDDEwMSI:

$$\begin{cases} d[x(t)-\mathcal{D}(t,x_\tau(t),r(t))] \\ \ = f(t,x(t),x_\tau(t),r(t))dt \\ \qquad + g(t,x(t),x_\tau(t),r(t))dB(t),\ t \geq t_0,\ t \neq t_k, \\ x(t) = I_k(x(t^-)),\ t = t_k,\ k = 1,\ 2,\ \ldots, \end{cases} \tag{1}$$

with the initial value $\{x(\theta) : t_0 - \bar{\tau} \le \theta \le t_0\} = \xi \in \mathcal{PC}([t_0 - \bar{\tau}, t_0]; R^n)$, where $x(t) = \mathrm{col}[x_1(t), x_2(t), \ldots, x_n(t)] \in R^n$ and $x_\tau(t) = x(t-\tau(t))$ with $\tau(\cdot) : [t_0, +\infty) \to [0, \bar{\tau}]$ $(\bar{\tau} > 0)$. $\mathcal{D}(\cdot, \cdot, \cdot) : [t_0, +\infty) \times R^n \times \mathcal{N} \to R^n$ is the neutral vector, $f(\cdot,\cdot,\cdot,\cdot) : [t_0, +\infty) \times R^n \times R^n \times \mathcal{N} \to R^n$ is the drift coefficient vector, $g(\cdot,\cdot,\cdot,\cdot) : [t_0, +\infty) \times R^n \times R^n \times \mathcal{N} \to R^{n\times m}$ is the diffusion coefficient matrix. The impulsive function $I_k(\cdot) : R^n \to R^n$ $(k = 1, 2, \ldots)$. The impulsive instant sequence $\{t_k\}_{k=1}^{+\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $\lim_{k\to+\infty} t_k = +\infty$. $x(t_k) = x(t_k^+) = \lim_{t\to t_k^+} x(t)$ and $x(t_k^-) = \lim_{t\to t_k^-} x(t)$. Let $x(t, t_0, \xi, r(t_0))$ denote the solution of (1). $x(t) = x(t, t_0, \xi, r(t_0))$.

**Hypothesis I:** There exists a positive constant $L$ such that for any $t \ge t_0$, $x, y, \bar{x}, \bar{y} \in R^n$, $i \in \mathcal{N}$, $|f(t,x,y,i) - f(t,\bar{x},\bar{y},i)| \vee |g(t,x,y,i) - g(t,\bar{x},\bar{y},i)| \le L(|x-\bar{x}| + |y-\bar{y}|)$, $f(t,0,0,i) = 0$, and $g(t,0,0,i) = 0$.

**Hypothesis II:** There exists $\kappa_i \in (0,1)$ $(i \in \mathcal{N})$ such that for any $t \ge t_0$, $|\mathcal{D}(t,x,i) - \mathcal{D}(t,y,i)| \le \kappa_i |x-y|$, $\forall x, y \in R^n$, and $\mathcal{D}(t,0,i) = 0$. $\kappa = \max_{i\in\mathcal{N}}\{\kappa_i\} \in (0,1)$.

**Hypothesis III:** There exist some positive constants $\beta_k$ $(k = 1,2,\ldots)$ such that $|I_k(x) - I_k(y)| \le \beta_k |x-y|$, $\forall$ $x$, $y \in R^n$, and $I_k(0) = 0$.

**Hypothesis IV:** For the impulsive time sequence $\{t_k\}$, $t_k - t_{k-1} \ge \bar{\tau}$, $k = 1, 2, \ldots$.

**Remark 1:** Similar to the method in [2] (see pp. 204 in Theorem 2.2), under Hypotheses I-III, the existence and uniqueness of the solution to system (1) can be examined. Hypothesis IV can guarantee the exponential decay of NSDDE with Markovian switching on $[t_k, t_{k+1})$ $(k = 0, 1, 2, \ldots)$.

**Lemma 1:** Let $\gamma > 0$, $\bar{\gamma} > 0$, $\hat{\lambda}_i > 0$ $(i = 0, 1, 2, 3)$ with $\hat{\lambda}_1 + \frac{\hat{\lambda}_2}{\gamma} < 1$ and $\bar{\gamma} < \gamma$. Let $\tau(\cdot)$ be a bounded Borel measurable map from $[t_0, +\infty)$ into $[0, \bar{\tau}]$ $(\bar{\tau} > 0)$, and let $y(\cdot)$ be a nonnegative function on $[t_0 - \bar{\tau}, +\infty)$. If inequality

$$y(t) \le \begin{cases} \hat{\lambda}_0 e^{-\gamma(t-t_k)} + \hat{\lambda}_1 y_\tau(t) + \hat{\lambda}_2 \int_{t_k}^{t} e^{-\gamma(t-s)} y_\tau(s) ds, \\ \quad t \in [t_k, t_{k+1}), \\ \hat{\lambda}_3 e^{-\bar{\gamma}(t-t_k)}, \quad t \in [t_k - \bar{\tau}, t_k], \end{cases} \tag{2}$$

holds for any $k = 0, 1, 2, \ldots$, where $y_\tau(t) = y(t - \tau(t))$, then

$$y(t) \le M' e^{-\mu(t-t_k)},\ t \in [t_k - \bar{\tau}, t_{k+1}),\ k = 0, 1, 2, \ldots, \tag{3}$$

where $\mu = \min\{\bar{\gamma}, \bar{\mu}\}$, $\bar{\mu} \in (0, \gamma)$ is a solution of equation $\hat{\lambda}_1 e^{\bar{\mu}\bar{\tau}} + \frac{\hat{\lambda}_2 e^{\bar{\mu}\bar{\tau}}}{\gamma - \bar{\mu}} = 1$, and $M' = \max\{\hat{\lambda}_3, \frac{\hat{\lambda}_0(\gamma-\bar{\mu})}{\hat{\lambda}_2 e^{\bar{\mu}\bar{\tau}}}\} > 0$.

**Proof:** Letting $H(\bar{\mu}) = \hat{\lambda}_1 e^{\bar{\mu}\bar{\tau}} + \frac{\hat{\lambda}_2 e^{\bar{\mu}\bar{\tau}}}{\gamma-\bar{\mu}} - 1$. Thus, $H(0)H(\gamma - 0) < 0$ holds. That is, there exists a positive constant $\bar{\mu} \in (0, \gamma)$ such that $H(\bar{\mu}) = 0$. For any $\varepsilon > 0$, letting $M'_\varepsilon := \max\{(\hat{\lambda}_3 + \varepsilon)e^{-\bar{\gamma}\bar{\tau}}, (\hat{\lambda}_0 + \varepsilon)(\gamma - \mu)/(\hat{\lambda}_2 e^{\mu\tau})\} > 0$. Now, we only claim that (2) implies

$$y(t) \le M'_\varepsilon e^{-\mu(t-t_k)}, \quad t \in [t_k - \bar{\tau}, t_{k+1}), \tag{4}$$

where $k = 0, 1, 2, \ldots$.

Note that for any $t \in [t_k - \bar{\tau}, t_k]$, (4) holds. If (4) does not hold for any $t \in (t_k, t_{k+1})$, there exists a $t \in (t_k, t_{k+1})$, such that $y(t) \ge M'_\varepsilon e^{-\mu(t-t_k)}$. Set $t_k^* = \inf\{t \in (t_k, t_{k+1}) : y(t) \ge M'_\varepsilon e^{-\mu(t-t_k)}\}$. Moreover, we have $y(t) \le M'_\varepsilon e^{-\mu(t-t_k)}$, $t \in [t_k - \bar{\tau}, t_k^*)$ and

$$y(t_k^*) = M'_\varepsilon e^{-\mu(t_k^* - t_k)}. \tag{5}$$

However, (2) implies

$$\begin{aligned} y(t_k^*) &\le \hat{\lambda}_0 e^{-\gamma(t_k^*-t_k)} + \hat{\lambda}_1 y_\tau(t_k^*) \\ &\quad + \hat{\lambda}_2 \int_{t_k}^{t_k^*} e^{-\gamma(t_k^*-s)} y_\tau(s) ds \\ &\le \left[\hat{\lambda}_0 - \frac{M'_\varepsilon \hat{\lambda}_2 e^{\mu\bar{\tau}}}{\gamma - \mu}\right] e^{-\gamma t_k^*} \\ &\quad + M'_\varepsilon \left[\hat{\lambda}_1 + \frac{\hat{\lambda}_2}{\gamma-\mu}\right] e^{\mu\bar{\tau} - \mu t_k^*}. \end{aligned} \tag{6}$$

In view of the definitions of $\mu$, $H(\bar{\mu})$ and $M'_\varepsilon$, it obtains that $\hat{\lambda}_1 e^{\mu\bar{\tau}} + \frac{\hat{\lambda}_2 e^{\mu\bar{\tau}}}{\gamma-\mu} \le 1$, and $\hat{\lambda}_0 - \frac{M'_\varepsilon \hat{\lambda}_2 e^{\mu\bar{\tau}}}{\gamma-\mu} \le \hat{\lambda}_0 - \frac{\hat{\lambda}_2 e^{\mu\bar{\tau}}}{\gamma-\mu}(\hat{\lambda}_0 + \varepsilon)\frac{\gamma-\mu}{\hat{\lambda}_2 e^{\mu\bar{\tau}}} < 0$. Hence, it implies from (6) that $y(t_k^*) < M'_\varepsilon e^{-\mu(t_k^*-t_k)}$ which contradicts (5). Then, (4) holds. As $\varepsilon \to 0^+$ in (4), inequality (3) is derived. □

**Remark 2:** In Lemma 1, on every subinterval $[t_k, t_{k+1})$ $(k = 0, 1, 2, \ldots)$, $y(t)$ is controlled by the function of exponential decay, which is presented in (3). In [30,33], the differential delay inequality is established to analyze the stability for delay differential equations with impulses. For NSDDE, such inequality is not easily established since the neutral term and the stochastic perturbation simultaneously exists. Motivated by [30,33], one delay integral inequality is established in Lemma 1 to overcome such problem. This lemma can be used to investigate the stability for NSDDE with impulses, see Theorem 1.

## 3. MAIN RESULTS

**Theorem 1:** Under Hypotheses I-IV, assume that there exist a function $V(t,x,i) \in C^{2,1}([t_k, t_{k+1}) \times R^n \times \mathcal{N}; [0, +\infty))$ $(k = 0, 1, 2, \ldots)$, some constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\lambda_{1i} > 0$, $\lambda_{2i} > 0$ $(i \in \mathcal{N})$, and $p \ge 2$ such that for any $\tilde{x}$, $x$, $y \in R^n$,

$$\alpha_1 |\tilde{x}|^p \le V(t, \tilde{x}, i) \le \alpha_2 |\tilde{x}|^p, \tag{7}$$

$$\mathcal{L}V(t,x,y,i) \le -\lambda_{1i}|x|^p + \lambda_{2i}|y|^p, \tag{8}$$

for all $t \ge t_0$, $t \ne t_k$ $(k = 1, 2, \ldots)$ and $i \in \mathcal{N}$. If inequalities $\lambda_2 < \lambda_1 \left[\frac{\alpha_1(1-\kappa)^p}{\alpha_2(1+\kappa)^{p-1}} - \kappa\right]$, and $\lambda < v_0$ are satisfied, where $\lambda_1 = \min_{i\in\mathcal{N}}\{\lambda_{1i}\}$, $\lambda_2 = \max_{i\in\mathcal{N}}\{\lambda_{2i}\}$,

$\lambda = \sup_{k=1,2,\ldots}\left\{\frac{\ln\rho_k}{t_k-t_{k-1}}\right\}$, $\rho_k = \max\{\max\{\beta_k^p e^{\nu_0\bar{\tau}},\ 1\}, [(\beta_k^p+\kappa e^{\nu_0\bar{\tau}})(\lambda_1-\nu_0\alpha_2(1+\kappa)^{p-1})]/[(1+\kappa)^{p-1}(\kappa\lambda_1+\lambda_2)e^{\nu_0\bar{\tau}}]\}$ $(k=1,2,\ldots)$ and $\nu_0 \in \left(0,\frac{\lambda_1}{\alpha_2(1+\kappa)^{p-1}}\right)$ is the unique solution of the algebraic equation $\kappa e^{\nu\bar{\tau}} + \alpha_2(\lambda_1\kappa+\lambda_2)(1+\kappa)^{p-1}e^{\nu\bar{\tau}}/[\lambda_1\alpha_1(1-\kappa)^{p-1}-\nu\alpha_1\alpha_2(1-\kappa^2)^{p-1}]=1$. Then, NSDDEwMSI (1) is exponentially stable in $p$-th ($p\geq 2$)-moment.

**Proof:** For any $t\in[t_0,+\infty)$, from (8), we have

$$\mathcal{L}V(t,x,y,i) \leq -\lambda_1|x|^p+\lambda_2|y|^p. \tag{9}$$

For any $t\in[t_0,t_1)$ and $r(t)\in\mathcal{N}$, by applying Itô formula to the Lyapunov function $\exp(\frac{\lambda_1(t-t_0)}{\alpha_2(1+\kappa)^{p-1}})V(t,x(t)-\mathcal{D}(t,x_\tau(t),r(t)),r(t))$, and taking the mathematical expectation in turn, it yields from inequality (9) and Lemma 4.3 in [19] that

$$\begin{aligned}&\mathcal{E}\{|x(t)-\mathcal{D}(t,x_\tau(t),r(t))|^p\}\\ &\quad\leq \frac{\mathcal{E}\{V(t_0,x(t_0)-\mathcal{D}(t_0,x_\tau(t_0),r(t_0)),r(t_0))\}}{\alpha_1}\\ &\qquad\times\exp\left(-\frac{\lambda_1(t-t_0)}{\alpha_2(1+\kappa)^{p-1}}\right)\\ &\qquad+\frac{\kappa\lambda_1+\lambda_2}{\alpha_1}\\ &\qquad\times\int_{t_0}^{t}\exp\left(-\frac{\lambda_1(t-s)}{\alpha_2(1+\kappa)^{p-1}}\right)\mathcal{E}\{|x_\tau(s)|^p\}ds.\end{aligned}\tag{10}$$

Using Lemma 4.5 in [19], we obtain

$$\mathcal{E}\{|x(t)|^p\}\leq\kappa\mathcal{E}\{|x_\tau(t)|^p\}+\frac{\mathcal{E}\{|x(t)-\mathcal{D}(t,x_\tau(t),r(t))|^p\}}{(1-\kappa)^{p-1}},\tag{11}$$

for any $t\in[t_0,t_1)$.
Substituting (11) into (10) yields that for any $t\in[t_0,t_1)$,

$$\begin{aligned}&\mathcal{E}\{|x(t)|^p\}\\ &\quad\leq\frac{\tilde{M}}{\alpha_1(1-\kappa)^{p-1}}\exp\left(-\frac{\lambda_1(t-t_0)}{\alpha_2(1+\kappa)^{p-1}}\right)\\ &\qquad+\kappa\mathcal{E}\{|x_\tau(t)|^p\}+\frac{\kappa\lambda_1+\lambda_2}{\alpha_1(1-\kappa)^{p-1}}\\ &\qquad\times\int_{t_0}^{t}\exp\left(-\frac{\lambda_1(t-s)}{\alpha_2(1+\kappa)^{p-1}}\right)\mathcal{E}\{|x_\tau(s)|^p\}ds,\end{aligned}\tag{12}$$

where $\tilde{M}\geq\mathcal{E}\{V(t_0,x(t_0)-\mathcal{D}(t_0,x_\tau(t_0),r(t_0)),r(t_0))\}$. For any $t\in[t_0-\bar{\tau},t_0]$, it is seen that

$$\mathcal{E}\{|x(t)|^p\}\leq\frac{\tilde{M}}{\alpha_1(1-\kappa)^{p-1}}\exp\left(-\frac{\lambda_1(t-t_0)}{\alpha_2(1+\kappa)^{p-1}}\right).\tag{13}$$

By virtue of Lemma 1, then from (12)-(13), it implies that for any $t\in[t_0,t_1)$,

$$\mathcal{E}\{|x(t)|^p\}\leq M_0\tilde{M}e^{-\nu_0(t-t_0)},\tag{14}$$

where $M_0=(\max\{1,\ [\alpha_1(1-\kappa)^{p-1}(\lambda_1-\alpha_2(1+\kappa)^{p-1}\nu_0)]/[\alpha_2(1+\kappa)^{p-1}(\lambda_1\kappa+\lambda_2)e^{\nu_0\tau}]\})/(\alpha_1(1-\kappa)^{p-1})>0$.
Furthermore, it yields from (14) that

$$\mathcal{E}\{|x(t_1)|^p\}\leq\beta_1^p\mathcal{E}\{|x(t_1^-)|^p\}\leq\beta_1^pM_0\tilde{M}e^{-\nu_0(t_1-t_0)}.\tag{15}$$

Thus, for any $t\in[t_1-\bar{\tau},t_1]$, we have

$$\mathcal{E}\{|x(t)|^p\}\leq\max\{\beta_1^pe^{\nu_0\bar{\tau}},1\}M_0\tilde{M}e^{-\nu_0(t-t_0)}.\tag{16}$$

By Hypotheses III-IV and (15)-(16), it gives

$$\begin{aligned}&\mathcal{E}\{V(t_1,x(t_1)-\mathcal{D}(t_1,x_\tau(t_1),r(t_1)),r(t_1))\}\\ &\leq\alpha_2(1+\kappa)^{p-1}(\beta_1^p\mathcal{E}\{|x(t_1^-)|^p\}+\kappa\mathcal{E}\{|x_\tau(t_1)|^p\})\\ &\leq\alpha_2(1+\kappa)^{p-1}(\beta_1^p+\kappa e^{\nu_0\bar{\tau}})M_0\tilde{M}e^{-\nu_0(t_1-t_0)}.\end{aligned}\tag{17}$$

For any $t\in[t_1,t_2)$ and $r(t)\in\mathcal{N}$, similar to the derivation processes (12)-(13), we have

$$\begin{aligned}&\mathcal{E}\{|x(t)|^p\}\\ &\leq\begin{cases}\dfrac{\mathcal{E}\{V(t_1,x(t_1)-\mathcal{D}(t_1,x_\tau(t_1),r(t_1)),r(t_1))\}}{\alpha_1(1-\kappa)^{p-1}}\\ \quad\times\exp\left(-\dfrac{\lambda_1(t-t_1)}{\alpha_2(1+\kappa)^{p-1}}\right)\\ \quad+\kappa\mathcal{E}\{|x_\tau(t)|^p\}+\dfrac{\kappa\lambda_1+\lambda_2}{\alpha_1(1-\kappa)^{p-1}}\\ \quad\times\displaystyle\int_{t_1}^{t}\exp\left(-\frac{\lambda_1(t-s)}{\alpha_2(1+\kappa)^{p-1}}\right)\mathcal{E}\{|x_\tau(s)|^p\}ds,\\ \quad t\in[t_1,t_2),\\ \max\{\beta_1^pe^{\nu_0\bar{\tau}},1\}M_0\tilde{M}e^{-\nu_0(t-t_0)},\\ \quad t\in[t_1-\bar{\tau},t_1].\end{cases}\end{aligned}\tag{18}$$

From Lemma 1, it gives that for $t\in[t_1,t_2)$, $\mathcal{E}\{|x(t)|^p\}\leq\max\{\max\{\beta_1^pe^{\nu_0\bar{\tau}},\ 1\}M_0\tilde{M}e^{-\nu_0(t_1-t_0)},\ [\mathcal{E}\{V(t_1,x(t_1)-\mathcal{D}(t_1,x_\tau(t_1),r(t_1)),r(t_1))\}(\lambda_1-\alpha_2(1+\kappa)^{p-1}\nu_0)]/(\alpha_2(1+\kappa)^{p-1}(\lambda_1\kappa+\lambda_2)e^{\nu_0\bar{\tau}})\}\leq\rho_1M_0\tilde{M}e^{-\nu_0(t-t_0)}$. Suppose that for all $m=1,2,\ldots,k$, inequality

$$\mathcal{E}\{|x(t)|^p\}\leq\rho_0\rho_1\cdots\rho_{m-1}M_0\tilde{M}e^{-\nu_0(t-t_0)}\tag{19}$$

hold for any $t\in[t_{m-1},t_m)$, where $\rho_0=1$. Furthermore, it yields from (19) that $\mathcal{E}\{|x(t_k)|^p\}\leq\beta_k^p\rho_0\rho_1\cdots\rho_{k-1}M_0\tilde{M}e^{-\nu_0(t_k-t_0)}$. Thus, for any $t\in[t_k-\bar{\tau},t_k]$, it yields $\mathcal{E}\{|x(t)|^p\}\leq\max\{\beta_k^pe^{\nu_0\bar{\tau}},\ 1\}\rho_0\rho_1\cdots\rho_{k-1}M_0\tilde{M}e^{-\nu_0(t_2-t_0)}$. Similarly, when $t=t_k$, $\mathcal{E}\{V(t_k,\ x(t_k)-\mathcal{D}(t_k,\ x_\tau(t_k),\ r(t_k)),\ r(t_k))\}\leq\alpha_2\mathcal{E}\{|x(t_k)-\mathcal{D}(t_k,\ x_\tau(t_k),\ r(t_k))|^p\}\leq\max\{\beta_k^pe^{\nu_0\bar{\tau}},\ 1\}\rho_0\rho_1\cdots\rho_{k-1}\alpha_2(1+\kappa)^{p-1}(\beta_2^p+\kappa e^{\nu_0\tau})M_0\tilde{M}e^{-\nu_0(t_k-t_0)}$. Then,

$$\mathcal{E}\{|x(t)|^p\} \le \begin{cases} \frac{\mathcal{E}\{V(t_k, x(t_k) - \mathcal{D}(t_k, x_\tau(t_k), r(t_k)), r(t_k))\}}{\alpha_1(1-\kappa)^{p-1}} \\ \quad \times \exp\left(-\frac{\lambda_1(t-t_k)}{\alpha_2(1+\kappa)^{p-1}}\right) \\ \quad + \kappa\mathcal{E}\{|x_\tau(t)|^p\} + \frac{\kappa\lambda_1+\lambda_2}{\alpha_1(1-\kappa)^{p-1}} \\ \quad \times \int_{t_k}^{t} \exp\left(-\frac{\lambda_1(t-s)}{\alpha_2(1+\kappa)^{p-1}}\right)\mathcal{E}\{|x_\tau(s)|^p\}ds, \\ \quad t \in [t_k, t_{k+1}), \\ \max\{\beta_k^p e^{\nu_0\bar{\tau}}, 1\}\rho_0\rho_1\cdots\rho_{k-1}M_0\tilde{M}e^{-\nu_0(t-t_0)}, \\ \quad t \in [t_k - \bar{\tau}, t_k]. \end{cases}$$

Using Lemma 1, it follows that for any $t \in [t_k - \bar{\tau}, t_{k+1})$ ($k = 0,\ 1,\ 2,\ \ldots$), $\mathcal{E}\{|x(t)|^p\} \le \max\{\max\{\beta_k^p e^{\nu_0\bar{\tau}}, 1\}\rho_0\rho_1\cdots\rho_{k-1}M_0\tilde{M}e^{-\nu_0(t_k-t_0)}, \prod_{i=1}^{k-1}\rho_i(\beta_k^p + \kappa e^{\nu_0\bar{\tau}})^{p-1}[(\lambda_1 - \alpha_2(1+\kappa)^{p-1}\nu_0)M_0\tilde{M}e^{-\nu_0(t_k-t_0)}]/[(1+\kappa)^{p-1}(\lambda_1\kappa + \lambda_2)e^{\nu_0\bar{\tau}}]\}e^{-\nu_0(t-t_k)} = \rho_0\rho_1\cdots\rho_{k-1}\rho_k M_0\tilde{M}e^{-\nu_0(t-t_0)}$.

By the mathematical induction, it is checked that

$$\mathcal{E}\{|x(t)|^p\} \le M_0\tilde{M}\prod_{i=0}^{k}\rho_i e^{-\nu_0(t-t_0)}, \tag{20}$$

holds for any $t \in [t_k - \bar{\tau},\ t_{k+1})$ ($k = 0,\ 1,\ 2,\ \ldots$). Noting that $\rho_k \le e^{\lambda(t_k - t_{k-1})}$ ($k = 1, 2, \ldots$) is derived from (20), it implies that $\mathcal{E}\{|x(t)|^p\} \le \rho_0\rho_1\cdots\rho_{k-1}\rho_k M_0\tilde{M}e^{-\nu_0(t-t_0)} \le M_0\tilde{M}e^{-(\nu_0-\lambda)(t-t_0)}$, for any $t \in [t_k - \bar{\tau},\ t_{k+1})$ ($k = 0,\ 1,\ 2, \ldots$). The proof of this theorem is completed. □

**Remark 3:** In Theorem 1, the exponential stability in $p$-th ($p \ge 2$)-moment for NSDDEwMSI (1) has been investigated, and some sufficient conditions have been proposed. From (7) and (8), it can be shown that NSDDEwMSI (1) has exponential decay in $p$-th ($p \ge 2$)-moment with decay rate $\nu_0$ on every impulsive interval $[t_k,\ t_{k+1})$ ($k = 0, 1, 2, \ldots$). For every mode on Markovian switching in (1), one corresponding Lyapunov function $V(t,\ \tilde{x}, i)$ is constructed for (1) with inequality (7) being satisfied, which is easily realized. Then, the Lyapunov Itô operator are expected to be estimated in term of inequality (8). Thus, $\lambda_2 < \lambda_1[\alpha_1(1-\kappa)^p/(\alpha_2(1+\kappa)^{p-1}) - \kappa]$ is obtained by using Lemma 1. Inequality $\lambda < \nu_0$ is satisfied to determine the impulsive strength and the length of impulsive interval. $\rho_k$ ($k = 1, 2, \ldots$) can be determined from the impulsive strength at the impulsive instant, which has given in Hypothesis III. Then, the length of every impulsive interval $t_k - t_{k-1}$ ($k = 1, 2, \ldots$) is derived from $\lambda < \nu_0$.

**Remark 4:** The exponential stability in $p$-th ($p \ge 2$)-moment for NSDDE without impulses has been discussed by using some excellent methodologies such as the Lyapunov function approach [5], the Lyapunov functional approach [6], the Lyapunov-Razumikhin theorem [7] and the fixed point theorem [8]. These methodologies cannot directly used to consider our concerned problem in this paper since the neutral term, the stochastic perturbation and the nonlinear impulsive effects simultaneously exist. The approach for analyzing the stability for deterministic neutral delay differential equations with impulses cannot be easily generalized for considering the exponential stability in $p$-th ($p \ge 2$)-moment for (1), see [10,29,30,33-35].

**Theorem 2:** Suppose that all conditions of Theorem 1 are satisfied with $\sup_{k=1,2,\ldots}\{t_k - t_{k-1}\} < +\infty$, then NSDDEwMSI (1) is almost surely exponentially stable.

**Proof:** For $\varsigma = \inf_{k\in N}\{t_k - t_{k-1}\}$, we choose $\delta$ with $0 < \delta < \varsigma$ sufficiently small such that $\delta < t_{k+1} - t_k$ ($k = 0$, 1, 2, $\ldots$). For the fixed $\delta > 0$, let $k_\delta = \left[\left[\frac{t_k - t_{k-1}}{\delta}\right]\right] \in N$, where $[[X]]$ is the maximum integer no more than $X$. Then, $k_\delta \le \frac{\Delta_{\sup}}{\delta}$, where $\Delta_{\sup} = \sup_{k=1,2,\ldots}\{t_k - t_{k-1}\}$. For any $t \in [t_k,\ t_{k+1})$ ($k = 0,\ 1,\ 2,\ \ldots$), there exists some $i$ with $1 \le i \le k_\delta + 1$ such that $t_k + (i-1)\delta \le t < t_{k+1} + i\delta$. Hence, for any $t \in [t_k, t_{k+1})$ ($k = 0,\ 1,\ 2,\ \ldots$), we have

$$\mathcal{E}\left[\sup_{t_k \le t < t_{k+1}} |x(t - \mathcal{D}(t, x_\tau(t), r(t))|^p\right] \le \sum_{i=1}^{k_\delta+1}\mathcal{E}\left[\sup_{t_k+(i-1)\delta \le t < t_{k+1}+i\delta} |x(t) - \mathcal{D}(t, x_\tau(t), r(t))|^p\right]. \tag{21}$$

For every $i$ satisfying $1 \le i \le k_\delta + 1$ ($k_\delta \in N$), it yields

$$\begin{aligned} &\mathcal{E}\left[\sup_{t_k+(i-1)\delta \le t < t_{k+1}+i\delta} |x(t) - \mathcal{D}(t, x_\tau(t), r(t))|^p\right] \\ &\le 3^{p-1}\mathcal{E}\{|x(t_k + (i-1)\delta) - \mathcal{D}(t_k + (i-1)\delta, \\ &\quad x_\tau(t_k + (i-1)\delta), r(t_k + (i-1)\delta))|^p\} \\ &\quad + 3^{p-1}\mathcal{E}\left[\int_{t_k+(i-1)\delta}^{t_k+i\delta} |f(s, x(s), x_\tau(s), r(s))|ds\right]^p \\ &\quad + 3^{p-1}\mathcal{E}\left[\sup_{t_k+(i-1)\delta \le t < t_{k+1}+i\delta}\left|\int_{t_k+(i-1)\delta}^{t} g(s, x(s), \right.\right. \\ &\quad \left.\left. x_\tau(s), r(s))dB(s)\right|^p\right]. \end{aligned} \tag{22}$$

From Remark 2 and Hypothesis II, it implies

$$\begin{aligned} &\mathcal{E}\{|x(t_k + (i-1)\delta) - \mathcal{D}(t_k + (i-1)\delta, \\ &\quad x_\tau(t_k + (i-1)\delta), r(t_k + (i-1)\delta))|^p\} \\ &\le 2^{p-1}[1 + \kappa^p]M_0\tilde{M}e^{-\mu_0(t_k - t_0)}. \end{aligned} \tag{23}$$

By Hypothesis I and Remark 2, it gives

$$\begin{aligned} &\mathcal{E}\left[\int_{t_k+(i-1)\delta}^{t_k+i\delta} |f(s, x(s), x_\tau(s), r(s))|ds\right]^p \\ &\le 2^p\delta^p L^p M_0\tilde{M}e^{-\mu_0(t_k-t_0)}. \end{aligned} \tag{24}$$

By using Hypothesis I and Remark 2 as well as the Burkholder-Davis-Gundy inequality [9], it follows

$$\mathcal{E}\left[\sup_{t_k+(i-1)\delta \le t < t_{k+1}+i\delta}\left|\int_{t_k+(i-1)\delta}^{t} g(s, x(s), \right.\right.$$

$$x_\tau(s), r(s))dB(s)\Big|^p\Big]$$
$$\leq C_p\mathcal{E}\left\{\int_{t_k+(i-1)\delta}^{t_k+i\delta} |g(s,x(s),x_\tau(s),r(s))|^p ds\right\}$$
$$\leq 2C_p\delta L^p M_0\tilde{M}e^{-\mu_0(t_k-t_0)}, \quad (25)$$

where $C_p > 0$ is a constant.
Substituting (23)-(25) into (22) implies that

$$\mathcal{E}[\sup_{t_k+(i-1)\delta\leq t<t_{k+1}+i\delta} |x(t)-\mathcal{D}(t,x_\tau(t),r(t))|^p]$$
$$\leq \tilde{K}e^{-\mu_0(t_k-t_0)}, \quad (26)$$

for any $i = 1, 2, \ldots, k_\delta+1$, where $\tilde{K} > 0$.
By using inequalities (21) and (26), we have

$$\mathcal{E}[\sup_{t_k\leq t<t_{k+1}} |x(t-\mathcal{D}(t,x_\tau(t),r(t))|^p]$$
$$\leq (k_\delta+1)\tilde{K}e^{-\mu_0(t_k-t_0)}. \quad (27)$$

Therefore, for any $\varepsilon > 0$ satisfying $\varepsilon \in (0,\mu_0)$, it follows from (27) that $\mathcal{P}\{\sup_{t_k+(i-1)\delta\leq t<t_{k+1}+i\delta} |x(t) - \mathcal{D}(t,x_\tau(t),r(t))|^p > (k_\delta+1)\tilde{K}e^{-(\mu_0-\varepsilon)(t_k-t_0)}\} \leq e^{-\varepsilon(t_k-t_0)}$. Note that $t_k \to +\infty$ as $k \to +\infty$. By using the well-known Borel-Cantelli lemma, it yields $\sup_{t_k+(i-1)\delta\leq t<t_{k+1}+i\delta} |x(t) - \mathcal{D}(t,x_\tau(t),r(t))|^p \leq (k_\delta+1)\tilde{K}e^{-(\mu_0-\varepsilon)(t_k-t_0)}$, a.s.

Letting $\varepsilon \to 0^+$, it follows that for any $t \in [t_k,t_{k+1})$ $(k = 0, 1, 2, \ldots)$, $|x(t) - \mathcal{D}(t,x_\tau(t),r(t))|^p \leq (k_\delta + 1)\tilde{K}'e^{-\mu_0(t-t_0)}$, a.s. where $\tilde{K}' = \tilde{K}e^{\mu_0\Delta_{\sup}} > 0$.

Furthermore, for any $t \in [t_0,+\infty)$, we have

$$|x(t)-\mathcal{D}(t,x_\tau(t),r(t))|^p \leq (k_\delta+1)\tilde{K}'e^{-\mu_0(t-t_0)}, \quad \text{a.s.} \quad (28)$$

For any $\varepsilon_0 \in (0, \min\{\mu_0, -\frac{p}{\bar{\tau}}\ln(\kappa)\})$, we can choose $\chi > 0$ sufficiently large such that $l = \frac{\kappa^p(1+\chi)^{p-1}e^{\varepsilon_0\bar{\tau}}}{\chi^{p-1}} < 1$. Thus, for any $T > 0$, by using inequality (28), it gives $\sup_{t_0\leq t\leq T}[e^{\varepsilon_0(t-t_0)}|x(t)|^p] \leq (k_\delta+1)\tilde{K}'(1+\chi)^{p-1} + l\mathcal{E}\{\sup_{\theta\in[t_0-\bar{\tau},t_0]}|\xi(\theta)|^p\} + l\sup_{t_0\leq t\leq T}[e^{\varepsilon_0(t-t_0)}|x(t)|^p]$, which implies that $\sup_{0\leq t\leq T}[e^{\varepsilon_0(t-t_0)}|x(t)|^p] \leq [(k_\delta+1)\tilde{K}'(1+\chi)^{p-1} + l\mathcal{E}\{\sup_{\theta\in[t_0-\bar{\tau},t_0]}|\xi(\theta)|^p\}]/(1-l)$. Letting $T \to +\infty$, it yields $\sup_{t_0\leq t<+\infty}[e^{\varepsilon_0(t-t_0)}|x(t)|^p] \leq [(k_\delta+1)\tilde{K}'(1+\chi)^{p-1}+l\mathcal{E}\{\sup_{\theta\in[t_0-\bar{\tau},t_0]}|\xi(\theta)|^p\}]/(1-l)$, which implies $\lim\sup_{t\to+\infty}\frac{\log(|x(t)|)}{t} \leq -\frac{\varepsilon_0}{p}$. □

**Remark 5:** In [24], the almost surely exponential stability for impulsive stochastic differential equations has been investigated by using the Borel-Cantelli lemma. Based on Theorem 1, the almost surely exponential stability for NSDDEwMSI (1) can be guaranteed by using the Borel-Cantelli lemma. The reason why this theorem can be obtained is that apart from using the result given in Theorem 1, Hypotheses I-III are satisfied.

In the following, we will investigate the following neutral stochastic linear delay differential equations with Markovian switching and nonlinear impulsive effects.

$$\begin{cases} d[x(t)-\mathcal{D}(r(t))x_\tau(t)] \\ \quad = [A(r(t))x(t)+C(r(t))x_\tau(t)]dt \\ \qquad +H(r(t))x(t)dB(t),\ t\geq t_0,\ t\neq t_k, \\ x(t) = B_kx(t^-),\ t=t_k,\ k=1,\ 2,\ \ldots, \end{cases} \quad (29)$$

with the initial value $\{x(\theta) : t_0 - \bar{\tau} \leq \theta \leq t_0\} = \xi \in \mathcal{PC}([t_0-\bar{\tau}, t_0]; R^n)$, where $x(t) \in R^n$ with $\tau(\cdot) : [t_0,+\infty) \to [0,\bar{\tau}]$ $(\bar{\tau} > 0)$ being a bounded delay function. $B(t)$ is one dimension standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq t_0}, \mathcal{P})$. When $r(t) = i \in \mathcal{N}$, $A(r(t))$, $C(r(t))$, $H(r(t))$ and $\mathcal{D}(r(t))$ are denoted by $A_i$, $C_i$, $H_i$ and $\mathcal{D}_i$, respectively. $\kappa = \max_{i\in\mathcal{N}}\{|\mathcal{D}_i|\} \in (0,1)$. $B_k$ denotes the impulsive strength matrix at the impulsive instant with $|B_k| \neq 1$ $(k=1,2,\ldots)$. When $V(t,x,i) = q_i|x|^2$, we have the following corollary.

**Corollary 1:** For systems (29), there exists some positive constants $\lambda_1$, $\lambda_2$, $\lambda$ and $\nu_0$ such that inequalities (11) and (12) are satisfied with $\lambda_1 = \min_{i\in\mathcal{N}}\{\lambda_{1i}\}$, $\lambda_2 = \max_{i\in\mathcal{N}}\{\lambda_{2i}\}$, $\lambda = \sup_{k=1,2,\ldots}\{\frac{\ln\rho_k}{t_k-t_{k-1}}\}$, $\rho_k = \max_{k=1,2,\ldots}\{\max\{|B_k|^2e^{\nu_0\bar{\tau}}, 1\}, [(|B_k|^2 + \kappa e^{\nu_0\bar{\tau}})(\lambda_1 - \nu_0\alpha_2(1+\kappa))]/[(1+\kappa)(\kappa\lambda_1+\lambda_2)]\}$, and $\nu_0 \in (0, \lambda_1/(\alpha_2(1+\kappa)))$ is a unique solution of the algebraic equation $\kappa e^{\nu\bar{\tau}} + [\alpha_2(\lambda_1\kappa+\lambda_2)(1+\kappa)]e^{\nu\bar{\tau}}/(\lambda_1\alpha_1(1-\kappa) - \nu\alpha_1\alpha_2(1-\kappa^2)) = 1$, where $\lambda_{1i} = q_i[\lambda_{\max}(A_i^T + A_i) + |C_i| + |D_i^TA_i| + \mathrm{trace}(H_iH_i^T) + \sum_{j=1}^N \gamma_{ij}q_j] < 0$, $\lambda_{2i} = q_i[|C_i| + |D_i^TA_i| + 2|D_i^TC_i|]$. Then, the exponential stability in mean square and the almost surely exponential stability for neutral stochastic linear delay differential equations with Markovian switching and nonlinear impulsive effects are guaranteed, respectively.

One algorithm for Corollary 1 are given as follows:

**Step 1:** Calculate the parameters $\lambda_{1i}$, $\lambda_{2i}$ $(i\in\mathcal{N})$. Moreover, calculate out $\lambda_1$, $\lambda_2$ and $\nu_0$;

**Step 2:** Choose matrices $B_k$ in (29) such that $\|B_k\| \in (0,1)$. Calculate out the parameters $\rho_k$ $(k=1,2,\ldots)$;

**Step 3:** Determine the impulsive time instants $\{t_k\}_{k=1}^\infty$ such that inequalities $\lambda_2 < \lambda_1[\alpha_1(1-\kappa)^p/(\alpha_2(1+\kappa)^{p-1}) - \kappa]$, and $\lambda < \nu_0$ are both satisfied.

## 4. EXAMPLE

$B(t)$ denotes a scalar Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq t_0}, \mathcal{P})$. We consider one coupled system consisting of a mass-spring-damper (MSD) and a pendulum [40] under the random external forces and impulsive effects, which are presented by

$$\begin{cases} M\ddot{z}(t)+C\dot{z}(t)+Kz(t)+m\ddot{z}(t-\tau) \\ \quad +h(t,z(t),\dot{z}(t),z(t-\tau),\dot{z}(t-\tau),r(t),\dot{B}(t))=0, \\ \quad t\geq t_0,\ \ t\neq t_k, \\ z(t_k)=\delta z(t_k^-),\ \dot{z}(t_k)=\delta\dot{z}(t_k^-),\ k=1,\ 2,\ \ldots, \end{cases} \quad (30)$$

where $M$, $C$, $K$, and $m$ denote the mass, stiffness, damping of MSD, and the mass of a pendulum, respectively. $\tau = 1.0$ is the time delay, $\mathcal{B}(t)$ is a scalar white noise, $h(t, z(t), \dot{z}(t), z(t-\tau), \dot{z}(t-\tau), r(t), \dot{B}(t)) = -a(r(t))z(t-\tau) - b(r(t))\dot{z}(t-\tau) - [c(r(t))z(t) + d(r(t))\dot{z}(t)]\dot{B}(t)$ represents the random external force, $z(t)$, $\dot{z}(t)$ and $\ddot{z}(t)$ describe the position, velocity, and acceleration of the MSD at time $t$. The impulsive instant sequence $\{t_k\}_{k=1}^{\infty}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. $z(t_k) = z(t_k^+) = \lim_{t\to t_k^+} z(t)$, $z(t_k^-) = \lim_{t\to t_k^-} z(t)$, $\dot{z}(t_k) = \dot{z}(t_k^+) = \lim_{t\to t_k^+} \dot{z}(t)$, and $\dot{z}(t_k^-) = \lim_{t\to t_k^-} \dot{z}(t)$. $r(t)$ denotes the Markov chain with its state $\mathcal{N} = \{1\ 2\}$ and its generator $\Gamma = (\gamma_{ij})_{2\times 2}$ ($\gamma_{11} = -2$, $\gamma_{12} = 2$, $\gamma_{21} = 3$, and $\gamma_{22} = -3$).

Letting $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$. Then, system (30) can be presented as NSDDEwMSI.

$$\begin{cases} d[x(t) - \mathcal{D}x(t-\tau)] \\ \quad = f(t, x(t), x(t-\tau), r(t))dt \\ \qquad + g(t, x(t), x(t-\tau), r(t))dB(t), \\ \qquad t \geq t_0,\ t \neq t_k, \\ x(t_k) = B_k x(t_k^-),\ k = 1,\ 2,\ \ldots, \end{cases} \tag{31}$$

where $x(t) = \mathrm{col}[x_1(t)\ \ x_2(t)]$, $f(t, x(t), x(t-\tau), k) = Ax(t) + C(k)x(t-\tau)$, $g(t, x(t), x(t-\tau), k) = H(k)x(t)$, and

$$\mathcal{D} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{m}{M} \end{bmatrix},\ B_k = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{C}{M} \end{bmatrix},$$

$$C(1) = \begin{bmatrix} 0 & 0 \\ \frac{a(1)}{M} & \frac{b(1)}{M} \end{bmatrix},\ C(2) = \begin{bmatrix} 0 & 0 \\ \frac{a(2)}{M} & \frac{b(2)}{M} \end{bmatrix},$$

$$H(1) = \begin{bmatrix} 0 & 0 \\ \frac{c(1)}{M} & \frac{d(1)}{M} \end{bmatrix},\ H(2) = \begin{bmatrix} 0 & 0 \\ \frac{c(2)}{M} & \frac{d(2)}{M} \end{bmatrix}.$$

For systems (31), we define the Lyapunov function $V(t, x - \mathcal{D}y, k) = q_k|x - \mathcal{D}y|^2$ ($k = 1,\ 2$) with $q_1 = 1.0$ and $q_2 = 0.8$. Clearly, $\alpha_1 = \min_{k=1,2}\{q_k\} = 0.8$ and $\alpha_2 = \max_{k=1,2}\{q_k\} = 1.0$. Furthermore, the infinitesimal operators are given as $\mathcal{L}V(t, x, y, 1) = 2q_1[x - Dy]^T[Ax(t) + C(1)x(t-\tau)] + q_1\mathrm{trace}[H(1)H^T(1)]|x(t)|^2 + \sum_{j=1}^{2}\gamma_{1j}q_j|x - Dy|^2 \leq -\lambda_{11}|x(t)|^2 + \lambda_{21}|x(t-\tau)|^2$, and $\mathcal{L}V(t, x, y, 2) = 2q_2[x - Dy]^T[Ax(t) + C(2)x(t-\tau)] + q_2\mathrm{trace}[H(2)H^T(2)]|x(t)|^2 + \sum_{j=1}^{2}\gamma_{2j}q_j|x - Dy|^2 \leq -\lambda_{12}|x(t)|^2 + \lambda_{22}|x(t-\tau)|^2$, where $\lambda_{11} = -q_1[\lambda_{\max}(A^T + A) + \lambda_{\max}(C(1)) + \lambda_{\max}(D^TA) + \mathrm{trace}[H(1)H^T(1) - 0.2(1 - m/M)]$, $\lambda_{21} = q_1[\lambda_{\max}(D^TA) + \lambda_{\max}(-D^TC(1) - C^T(1)D) + 0.2(m/M)(1 - m/M)]$, $\lambda_{12} = -q_2[\lambda_{\max}(A^T + A) + \lambda_{\max}(C(2)) + \lambda_{\max}(D^TA) + \mathrm{trace}[H(2)H^T(2) + 0.6(1 + m/M)]$, and $\lambda_{22} = q_2[\lambda_{\max}(D^TA) + \lambda_{\max}(-D^TC(2) - C^T(2)D) + 0.6(m/M)(1 + m/M)]$.

When $M = 10$, $C = 20$, $K = 10$, $m = 1.0$, $a(1) = 2$, $a(2) = -2$, $b(1) = 1$, $b(2) = -2$, $c(1) = -3$, $c(2) = -2$, $d(1) = -2$, $d(2) = 1$ and $\delta = 0.8$ in (30), we have $\lambda_1 = 0.568$ and $\lambda_2 = 0.1804$. Moreover, $\lambda_2 < \lambda_1[\alpha_1(1 - m/M)^2/(\alpha_2(1 + m/M)) - m/M]$ holds. Meanwhile, $\nu_0 = 0.1445$ and $\rho_k = 1.1457$ ($k = 1,\ 2,\ \ldots$). Letting $t_k = 0.5 + 1.1k$ ($k = 1, 2, \ldots$). Then, $\lambda = 0.1236 < \nu_0 = 0.1445$. Consequently, from Corollary 1, it can be concluded that NSDDEwMSI (31) is exponentially stable in mean square and almost surely exponentially stable. When the initial value $\mathrm{col}[x_1(t)\ \ x_2(t)] = \mathrm{col}[-1\ \ 2]$ ($t \in [-1.0, 0]$) and $r(0) = 1$, Figs. 1 and 2 illustrate the exponential stability in mean square and the almost surely exponential stability for NSDDEwMSI (31), respectively.

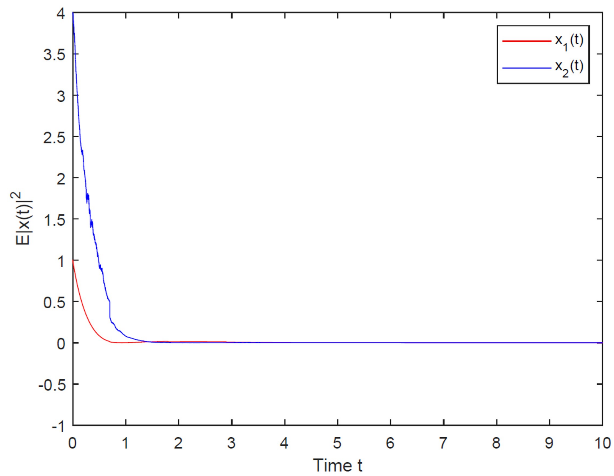


Fig. 1. Dynamical trajectory in the mean square sense for (31).

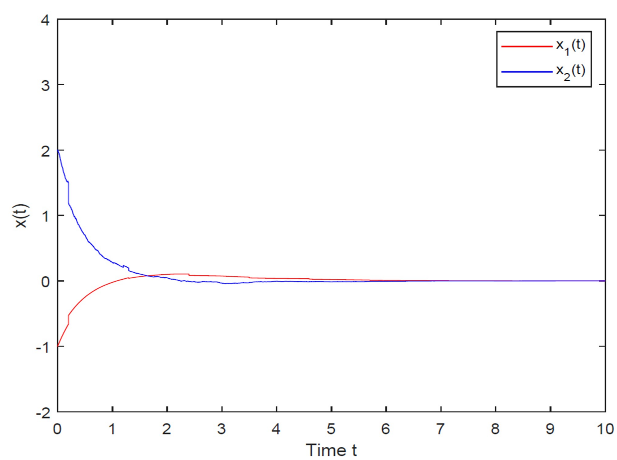


Fig. 2. Dynamical trajectory in almost sure sense for (31).

## 5. CONCLUSION

In this paper, the exponential stability in $p$ ($p \geq 2$)-moment and the almost surely exponential stability for neutral stochastic differential delay equations with Markovian switching and nonlinear impulsive effects have been investigated. When the globally Lipschitz condition is satisfied for the drift term and the diffusion term, the contractive condition is fulfilled for the neutral term, by using the generalized integral-type Halanay inequality, the Lyapunov function and the theory of stochastic analysis, these two stochastic stabilities are considered. One example is provided to illustrate the effectiveness of the theoretical results derived.

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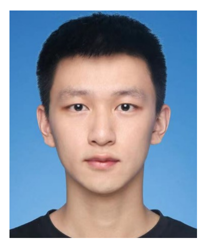

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